SOLUTIONS MANUAL

to accompany

Digital Signal Processing: A Computer-Based Approach

Fourth Edition

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Chapter 10 – Part 1

10.1 For this problem, we have $N = 71$ and $\omega_s - \omega_p = 0.04\pi$ and we assume that $\delta_p = \delta_s$.

(a) Using Kaiser's formula of Eq. (10.3):

$$\delta_s = 10^{\left(\frac{\left[N(14.6)\left(\omega_s - \omega_p\right)/2\pi\right] - 13}{-20}\right)} = 10^{\left(\frac{\left[71(14.6)(0.04\pi)/2\pi\right] - 13}{-20}\right)} = 0.0206.$$

$$\alpha_s = -20\log_{10}\left(\delta_s\right) = 33.7320 \text{ dB}.$$

(b) Using Bellanger's formula of Eq. (10.4):

$$\delta_s = \left(0.1?10^{\left(\frac{-3(N+1)\left(\omega_s - \omega_p\right)/2\pi}{2}\right)}\right)^{1/2} = \left(0.1?10^{\left(\frac{-72?3?0.04\pi/2\pi}{2}\right)}\right)^{1/2} = 0.0263.$$

$$\alpha_s = -20\log_{10}\left(\delta_s\right) = 31.6 \text{ dB}.$$

(c) Using Hermann's formula of Eq. (10.5), we first find that:

$$F\left(\delta_s, \delta_s\right) = b_1 + b_2\left(\log_{10}\delta_s - \log_{10}\delta_s\right) = b_1 = 11.01217.$$

Therefore:

$$D_\infty\left(\delta_s, \delta_s\right) = N\left(\omega_s - \omega_p\right)/2\pi + F\left(\delta_s, \delta_s\right)\left(\omega_s - \omega_p\right)^2/4\pi^2$$

$$= 71(0.04\pi)/(2\pi) + 11.01217(0.04\pi)^2/(2\pi)^2 = 1.4244.$$

Solving for:

$$D_\infty\left(\delta_s\right) = \left[a_1\left(\log_{10}\delta_s\right)^2 + a_2\left(\log_{10}\delta_s\right) + a_3\right]\left(\log_{10}\delta_s\right) -$$

$$\left[a_4\left(\log_{10}\delta_s\right)^2 + a_5\left(\log_{10}\delta_s\right) + a_6\right]$$

$$= a_1\left(\log_{10}\delta_s\right)^3 + a_2\left(\log_{10}\delta_s\right)^2 + a_3\left(\log_{10}\delta_s\right) -$$

$$a_4\left(\log_{10}\delta_s\right)^2 - a_5\left(\log_{10}\delta_s\right) - a_6$$

$$= a_1\left(\log_{10}\delta_s\right)^3 + \left(a_2 - a_4\right)\left(\log_{10}\delta_s\right)^2 + \left(a_3 - a_5\right)\left(\log_{10}\delta_s\right) - a_6.$$

Let $x = \left(\log_{10}\delta_s\right)$, and thus

$$D_\infty(\delta_s) = 0.005309x^3 + 0.06848x^2 - 1.0702x - 0.4278 = 1.4244.$$

Solving this, we get: $x = -21.5147,\ 10.2049,\ -1.589.$

The most reasonable solution is the last one, which yields:

$$\delta_s = 10^x = 10^{-1.5890} = 0.0258,$$
$$\alpha_s = -20\log_{10}(\delta_s) = 31.78 \text{ dB}.$$

10.2 For this problem, we have $N = 71$ and $\omega_s - \omega_p = 0.04\pi = \Delta\omega$. Using Eq. (10.45):

$$\alpha_s = 2.285(\Delta\omega)N + 8 = 2.285(0.04\pi)71 + 8 = 28.3871 \text{ dB}.$$

10.3 (a) From Eq. (10.17): $H_{HP}(e^{j\omega}) = \begin{cases} 0, & |\omega| < \omega_c, \\ 1, & \omega_c \le |\omega| \le \pi, \end{cases}$

$$h_{HP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{HP}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_c}^{\pi} = \frac{1}{2\pi}\left[\frac{e^{-j\omega_c n}}{jn} - \frac{e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n}}{jn} - \frac{e^{j\omega_c n}}{jn}\right]$$

$$= \frac{\sin(\pi n)}{\pi n} - \frac{\sin(\omega_c n)}{\pi n}.$$

Using the properties of the sinc function, we arrive at: $h_{HP}[0] = 1 - \dfrac{\omega_c}{\pi}.$

Therefore: $h_{HP}[n] = \begin{cases} 1 - \dfrac{\omega_c}{\pi}, & n = 0, \\ -\dfrac{\sin(\omega_c n)}{\pi n}, & \text{otherwise.} \end{cases}$

(b) From Eq. (10.18): $H_{BP}(e^{j\omega}) = \begin{cases} 0, & |\omega| < \omega_{c1}, \\ 1, & \omega_{c1} \le |\omega| \le \omega_{c2}, \\ 0, & \omega_{c2} \le |\omega| \le \pi, \end{cases}$

$$h_{BP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{BP}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c2}}^{-\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c1}}^{\omega_{c2}}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c1} n}}{jn} - \frac{e^{-j\omega_{c2} n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c2} n}}{jn} - \frac{e^{j\omega_{c1} n}}{jn}\right]$$

$$= \frac{\sin(\omega_{c2}n)}{\pi n} - \frac{\sin(\omega_{c1}n)}{\pi n}.$$

(c) From Eq. (10.20): $H_{BS}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_{c1}, \\ 0, & \omega_{c1} \le |\omega| \le \omega_{c2}, \\ 1, & \omega_{c2} \le |\omega| \le \pi. \end{cases}$

$$h_{BS}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{BS}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_{c2}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c2}}^{\pi} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c1}}^{\omega_{c1}}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_c n}}{jn} - \frac{e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n}}{jn} - \frac{e^{j\omega_c n}}{jn}\right]\frac{1}{2\pi}\left[\frac{e^{-j\omega_{c1}n}}{jn} - \frac{e^{-j\omega_{c1}n}}{jn}\right]$$

$$= \frac{\sin(\pi n)}{\pi n} - \frac{\sin(\omega_{c2}n)}{\pi n} + \frac{\sin(\omega_{c1}n)}{\pi n}.$$

Using the properties of the sinc function, we arrive at: $h_{BS}[0] = 1 - \frac{\omega_{c2} - \omega_{c1}}{\pi}$.

Therefore: $h_{HP}[n] = \begin{cases} 1 - \frac{\omega_{c2} - \omega_{c1}}{\pi}, & n = 0, \\ \frac{\sin(\omega_{c1}n)}{\pi n} - \frac{\sin(\omega_{c2}n)}{\pi n}, & \text{otherwise.} \end{cases}$

10.4 The ideal L-band digital filter $H_{ML}(z)$ has a frequency response given by:

$H_{ML}(e^{j\omega}) = A_k$, for $\omega_{k-1} \le \omega \le \omega_k$, $1 \le k \le L$.

It can be considered as sum of L ideal bandpass filters with cutoff frequencies at:

$\omega_{c1}^{k} = \omega_{k-1}$ and $\omega_{c2}^{k} = \omega_k$, where $\omega_{c1}^{0} = 0$ and $\omega_{c2}^{L} = \pi$

From Eqn. (10.19) the impulse response of an ideal bandpass filter is given by:

$$h_{BP}[n] = \frac{\sin(\omega_{c2}n)}{\pi n} - \frac{\sin(\omega_{c1}n)}{\pi n}.$$

Therefore: $h_{BP}^{k}[n] = \frac{\sin(\omega_k n)}{\pi n} - \frac{\sin(\omega_{k-1}n)}{\pi n}$..

Hence: $h_{ML}[n] = \sum_{k=1}^{L} h_{BP}^{k}[n] = \sum_{k=1}^{L} A_k\left(\frac{\sin(\omega_k n)}{\pi n} - \frac{\sin(\omega_{k-1}n)}{\pi n}\right)$

$$= A_1\left(\frac{\sin(\omega_1 n)}{\pi n} - \frac{\sin(0n)}{\pi n}\right) + \sum_{k=2}^{L-1} A_k \frac{\sin(\omega_k n)}{\pi n}$$

$$- \sum_{k=2}^{L-1} A_k\left(\frac{\sin(\omega_k n)}{\pi n} - \frac{\sin(\omega_{k-1}n)}{\pi n}\right) + A_L\left(\frac{\sin(\omega_L n)}{\pi n} - \frac{\sin(\omega_{L-1}n)}{\pi n}\right)$$

$$= A_1 \frac{\sin(\omega_1 n)}{\pi n} + \sum_{k=2}^{L-1} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=2}^{L-1} A_k \frac{\sin(\omega_{k-1} n)}{\pi n} - A_L \frac{\sin(\omega_{L-1} n)}{\pi n}$$

$$= \sum_{k=1}^{L-1} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=2}^{L} A_k \frac{\sin(\omega_{k-1} n)}{\pi n}.$$

Since $\omega_L = \pi$, $\sin(\omega_L n) = 0$. We can add a term of $A_L \frac{\sin(\omega_L n)}{\pi n}$ to the first sum in the above expression and change the index range of the second sum, resulting in:

$$h_{ML}[n] = \sum_{k=1}^{L} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=1}^{L-1} A_{k+1} \frac{\sin(\omega_k n)}{\pi n}.$$

Finally, since $A_{L+1} = 0$, we can add a term: $A_{L+1} \frac{\sin(\omega_L n)}{\pi n}$ to the second sum. This leads to:

$$h_{ML}[n] = \sum_{k=1}^{L} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=1}^{L} A_{k+1} \frac{\sin(\omega_k n)}{\pi n} = \sum_{k=1}^{L} (A_k - A_{k+1}) \frac{\sin(\omega_k n)}{\pi n}.$$

10.5 The impulse response for the Hilbert Transformer is given by:

$$H_{HT}(e^{j\omega}) = \begin{cases} j, & -\pi < \omega < 0, \\ -j, & 0 < \omega < \pi \end{cases}$$

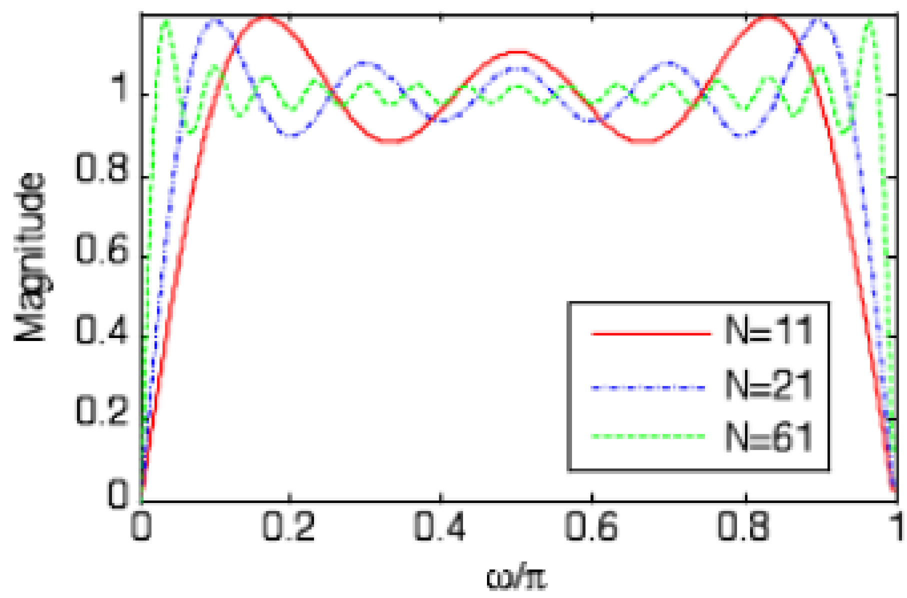
Therefore: $h_{HT}[n] = \frac{1}{2\pi} \int_{-\pi}^{0} H_{HT}(e^{j\omega}) e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{0}^{\pi} H_{HT}(e^{j\omega}) e^{j\omega n} d\omega$

$$= \frac{1}{2\pi} \int_{-\pi}^{0} j e^{j\omega n} d\omega - \frac{1}{2\pi} \int_{0}^{\pi} j e^{j\omega n} d\omega = \frac{2}{2\pi n} \big(1 - \cos(\pi n)\big) = \frac{2\sin^2(\pi n/2)}{\pi n}, \quad n \neq 0.$$

For $n = 0$, $h_{HT}[0] = \frac{1}{2\pi} \int_{-\pi}^{0} j\, d\omega - \frac{1}{2\pi} \int_{0}^{\pi} j\, d\omega = 0.$

Hence: $h_{HT}[n] = \begin{cases} 0, & \text{if } n = 0, \\ \dfrac{2\sin^2(\pi n/2)}{\pi n}, & \text{if } n \neq 0. \end{cases}$

Since $h_{HT}[n] = -h_{HT}[-n]$, and the length of the truncated impulse response is odd, it is a Type 3 linear-phase FIR filter.



From the frequency response plots given above, we observe the presence of ripples at the band edges due to the Gibbs phenomenon caused by the truncation of the impulse response.

10.6 First, we note that: $\mathcal{H}\{x[n]\} = \sum_{k=-\infty}^{\infty} h_{HT}[n-k]x[k]$..

Hence, $F\{\mathcal{H}\{x[n]\}\} = H_{HT}(e^{j\omega})X(e^{j\omega}) = \begin{cases} jX(e^{j\omega}), & -\pi < \omega < 0, \\ -jX(e^{j\omega}), & 0 < \omega < \pi. \end{cases}$

(a) Let $y[n] = \mathcal{H}\{\mathcal{H}\{\mathcal{H}\{\mathcal{H}\{\mathcal{H}\{\mathcal{H}\{x[n]\}\}\}\}\}\}$.
Hence:

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$$Y(e^{j\omega}) = \begin{cases} (j)^6 X(e^{j\omega}), & -\pi < \omega < 0, \\ (-j)^6 X(e^{j\omega}), & 0 < \omega < \pi, \end{cases} = -X(e^{j\omega}).$$

Therefore, $y[n] = -x[n]$.

(b) Define $g[n] = \mathcal{H}\{x[n]\}$, and $h^*[n] = x[n]$. Then:

$\sum_{=-\infty}^{\infty} \mathcal{H}\{x[\ ]\}\, x[\ ] = \sum_{=-\infty}^{\infty} g[\ ]h^*[\ ]$.

But from Parseval's relation in Table 3.4: $\sum_{=-\infty}^{\infty} g[\ ]h^*[\ ] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})G(e^{j\omega})\,d\omega$.

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Therefore: $\sum_{=-\infty}^{\infty} \mathcal{H}\{x[\ ]\}\, x[\ ] = \frac{1}{2\pi}\int_{\pi}^{\pi} H_{HT}(e^{j\omega})X(e^{j\omega})X(e^{-j\omega})\,d\omega$

where $H_{HT}(e^{j\omega}) = \begin{cases} j, & -\pi < \omega < 0, \\ -j, & 0 < \omega < \pi. \end{cases}$

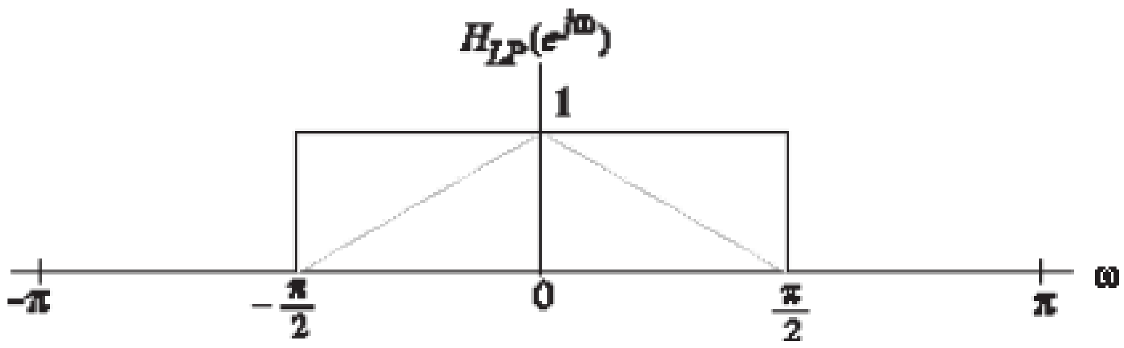
Since the integrand $H_{HT}(e^{j\omega})X(e^{j\omega})X(e^{-j\omega})$ is an odd function of ω,

$$\int_{\pi}^{\pi} H_{HT}(e^{j\omega})X(e^{j\omega})X(e^{-j\omega})\,d\omega = 0.$$

As a result: $\sum_{=-\infty}^{\infty} H\{x[\ ]\}\, x[\ ] = 0.$

10.7 ? Given the frequency response for the ideal low pass filter: $H_{LP}(z) = \sum_{n=0}^{N} h_{LP}[n]\, z^{-n}$.

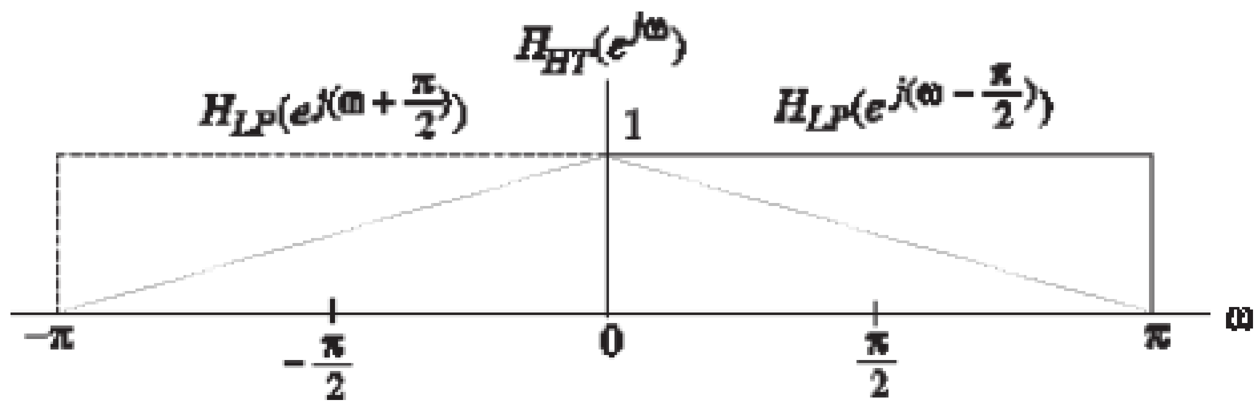
Its frequency response is shown in Figure (a) below.



(a)

A plot of the frequency response of: $H_{HT}(e^{j\omega}) = H_{LP}(e^{j(\omega-\frac{\pi}{2})}) + H_{LP}(e^{j(\omega+\frac{\pi}{2})})$

is shown in Figure (b) below.



(b)

It is evident from this figure that $H_{HT}(e^{j\omega})$ is the frequency response of an ideal Hilbert transformer. Therefore, we have:

$$H_{HT}(e^{j\omega}) = H_{LP}(e^{j(\omega+\frac{\pi}{2})}) + H_{LP}(e^{j(\omega-\frac{\pi}{2})})$$

$$= \sum_{n=0}^{N} h_{LP}[n]\, e^{-jn(\omega+\frac{\pi}{2})} + \sum_{n=0}^{N} h_{LP}[n]\, e^{-jn(\omega-\frac{\pi}{2})}$$

$$= \sum_{n=0}^{N} h_{LP}[n] e^{-jn\omega}\left(e^{-jn\pi/2} + e^{jn\pi/2}\right) = \sum_{n=0}^{N} 2h_{LP}[n] e^{-jn\omega} \cos\left(\frac{n\pi}{2}\right).$$

For n odd, $\cos(n\omega/2) = 0$ and we can drop all odd terms in the above expression. Let $N = 2M$ with N even and let $r = 2n$. Then, we can rewrite the above equation as:

$$H_{HT}(e^{j\omega}) = \sum_{r=0}^{M} 2h_{LP}[2r] \cos(r\pi) e^{-j2r\omega}.$$

The corresponding transfer function of the Hilbert transformer is therefore given by:

$$H_{HT}(z) = \sum_{n=0}^{M} 2h_{LP}[2n]\cos(n\pi)\, z^{-2n} = \sum_{n=0}^{M} 2(-1)^n h_{LP}[2n]\, z^{-2n}.$$

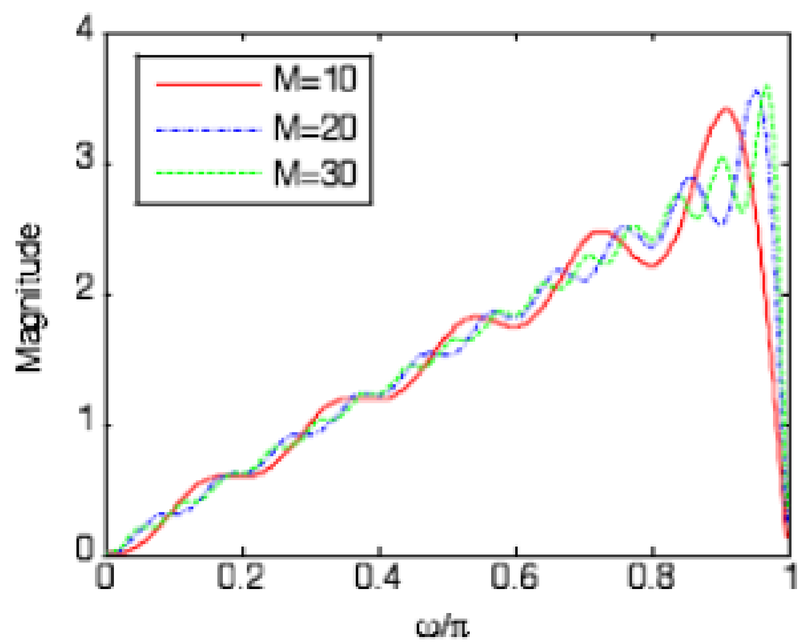
10.8 We know that the frequency response of the ideal differentiator is given by $H_{DIF}(e^{j\omega}) = j\omega.$

Hence: $h_{DIF}[n] = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} j\,\omega e^{j\omega n} d\omega = \dfrac{j}{2\pi}\displaystyle\int_{-\pi}^{\pi}\omega e^{j\omega n} d\omega = \dfrac{j}{2\pi}\left[\dfrac{\omega e^{j\omega n}}{jn} + \dfrac{e^{j\omega n}}{n^2}\right]_{-\pi}^{\pi}.$

Therefore: $h_{DIF}[n] = \dfrac{\cos(\pi n)}{n} - \dfrac{\sin(\pi n)}{\pi n^2} = \dfrac{\cos(\pi n)}{n}, \quad n \neq 0.$

For $n = 0$: $h_{DIF}[0] = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} j\,\omega d\omega = 0.$ Hence: $h_{DIF}[n] = \begin{cases} 0, & n = 0, \\ \dfrac{\cos(\pi n)}{n}, & |n| > 0. \end{cases}$

Since $h_{DIF}[n] = -h_{DIF}[-n]$, the truncated impulse response is a Type 3 linear-phase FIR filter. The magnitude responses of the above differentiator for several values of M are given below:



10.9 Given that N = 2M + 1, we can write the impulse response of the high pass filter as:

$$\hat{h}_{HP}[n] = \begin{cases} 1 - \dfrac{\omega_c}{\pi}, & \text{for } n = M, \\ -\dfrac{\sin\left(\omega_c(n-m)\right)}{\pi(n-m)}, & \text{if } n \neq M, 0 \leq n < N, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we check for the delay complementary property:

$$\hat{H}_{HP}(z) + \hat{H}_{LP}(z) = \sum_{n=-\infty}^{\infty}\hat{h}_{HP}[n]z^{-n} + \sum_{n=-\infty}^{\infty}\hat{h}_{LP}[n]z^{-n} = \sum_{n=0}^{N-1}\hat{h}_{HP}[n]z^{-n} + \sum_{n=0}^{N-1}\hat{h}_{LP}[n]z^{-n}$$

$$= \sum_{n=0}^{N-1}\left(\hat{h}_{HP}[n] + \hat{h}_{LP}[n]\right)z^{-n}.$$

But: $\hat{h}_{HP}[n] + \hat{h}_{LP}[n] = \hat{h}_{HP}[n] + \hat{h}_{LP}[n] = \begin{cases} 0, & 0 \le n \le N-1,\ n \ne M, \\ 1, & n=M. \end{cases}$,

Hence, $\hat{H}_{HP}(z) + \hat{H}_{LP}(z) = z^{-M}$, and the two filters are delay-complementary.

10.10 The frequency response of the zero-phase ideal linear passband lowpass filter is:

$$H_{LLP}(e^{j\omega}) = \begin{cases} |\omega|, & |\omega| < \omega_c, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the impulse response is computed as:

$$h_{LLP}[n] = \frac{1}{2\pi}\left\{ -\int_{-\omega_c}^{0} \omega e^{j\omega n} d\omega + \int_{0}^{\omega_c} \omega e^{j\omega n} d\omega \right\}$$

$$= \frac{1}{2\pi}\left\{ -\left[\frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right]_{-\omega_c}^{0} + \left[\frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right]_{0}^{\omega_c} \right\}$$

$$= \frac{1}{2\pi}\left\{ \frac{\omega_c e^{j\omega_c n} - \omega_c e^{-j\omega_c n}}{jn} + \frac{e^{j\omega_c n} + e^{-j\omega_c n} - 2}{n^2} \right\}$$

$$= \frac{\omega_c}{\pi n}\sin(\omega_c n) + \frac{\cos(\omega_c n) - 1}{\pi n^2}.$$

10.11 The frequency response of the zero-phase ideal band-limited differentiator is:

$$H_{BLDIF}(e^{j\omega}) = \begin{cases} \omega, & |\omega| < \omega_c, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the impulse response can be found as follows:

$$h_{BLDIF}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \omega e^{j\omega n} d\omega = \frac{1}{2\pi}\left\{ \left[\frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right]_{-\omega_c}^{\omega_c} \right\}$$

$$= \frac{1}{2\pi}\left\{ \frac{\omega_c e^{j\omega_c n} + \omega_c e^{-j\omega_c n}}{jn} + \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{n^2} \right\}$$

$$= -j\frac{\omega_c}{\pi n}\cos(\omega_c n) + j\frac{1}{\pi n^2}\sin(\omega_c n).$$

10.12 The frequency response of a causal ideal notch filter can thus be expressed as:

$$H_{notch}(e^{j\omega}) = H_{notch}(\omega)\, e^{j\theta(\omega)},$$

where $H_{notch}(\omega)$ is the amplitude response which can be expressed as:

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$$H_{notch}(\omega) = \begin{cases} 1, & 0 \le \omega \le \omega_o, \\ -1, & \omega_o < \omega < \pi. \end{cases}$$

It follows then that $H_{notch}(\omega)$ is related to the amplitude response $H_{LP}(\omega)$ of the ideal lowpass filter with a cutoff at ω_o through $H_{notch}(\omega) = \pm[2H_{LP}(\omega) - 1]$. Hence, the impulse response of the ideal notch filter is given by $h_{notch}[n] = \pm[2h_{LP}[n] - \delta[n]]$, where:

$$h_{LP}[n] = \frac{\sin(\omega_o n)}{\pi n}, \quad -\infty < n < \infty.$$

Therefore: $$h_{notch}[n] = \begin{cases} \pm 1, & n = 0, \\ \pm\dfrac{2\sin(\omega_o n)}{\pi n}, & \text{otherwise.} \end{cases}$$

10.13 First, consider another filter with a frequency response $G(e^{j\omega})$ given by

$$G(e^{j\omega}) = \begin{cases} 0, & 0 \le \omega \le \omega_p, \\ \dfrac{-\pi}{2\Delta\omega}\sin\left(\dfrac{\pi(\omega - \omega_p)}{\Delta\omega}\right) & \omega_p < \omega \le \omega_s, \\ \dfrac{-\pi}{2\Delta\omega}\sin\left(\dfrac{\pi(\omega + \omega_p)}{\Delta\omega}\right) & -\omega_s \le \omega \le -\omega_p, \\ 0, & \text{elsewhere.} \end{cases}$$

Clearly: $$G(e^{j\omega}) = \frac{dH_{LP}(e^{j\omega})}{d\omega}.$$

Finding the impulse response of the secondary function:

$$g[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})e^{j\omega n}\,d\omega$$

$$= \frac{-\pi}{8\pi\Delta\omega j}\left\{\int_{\omega_p}^{\omega_s} e^{j\left(\frac{\pi(\omega-\omega_p)}{\Delta\omega}\right)} e^{j\omega n}\,d\omega - \int_{\omega_p}^{\omega_s} e^{-j\left(\frac{\pi(\omega-\omega_p)}{\Delta\omega}\right)} e^{j\omega n}\,d\omega \right.$$

$$\left. + \int_{-\omega_s}^{-\omega_p} e^{j\left(\frac{\pi(\omega+\omega_p)}{\Delta\omega}\right)} e^{j\omega n}\,d\omega - \int_{-\omega_s}^{-\omega_p} e^{-j\left(\frac{\pi(\omega+\omega_p)}{\Delta\omega}\right)} e^{j\omega n}\,d\omega \right\}$$

$$= \frac{-1}{j8\Delta\omega}\left\{ e^{-\frac{j\pi\omega_p}{\Delta\omega}}\left[\frac{e^{j\omega_s\left(n+\frac{\pi}{\Delta\omega}\right)} - e^{j\omega_p\left(n+\frac{\pi}{\Delta\omega}\right)}}{j\left(n+\frac{\pi}{\Delta\omega}\right)}\right] - e^{-\frac{j\pi\omega_p}{\Delta\omega}}\left[\frac{e^{j\omega_s\left(n-\frac{\pi}{\Delta\omega}\right)} - e^{j\omega_p\left(n-\frac{\pi}{\Delta\omega}\right)}}{j\left(n-\frac{\pi}{\Delta\omega}\right)}\right]\right\}$$

$$+e^{\frac{j\pi\omega_p}{\Delta\omega}}\left(\frac{e^{-j\omega_p\left(n+\frac{\pi}{\Delta\omega}\right)}-e^{-j\omega_s\left(n+\frac{\pi}{\Delta\omega}\right)}}{j\left(n+\frac{\pi}{\Delta\omega}\right)}\right]-e^{-\frac{j\pi\omega_p}{\Delta\omega}}\left(\frac{e^{-j\omega_p\left(n-\frac{\pi}{\Delta\omega}\right)}-e^{-j\omega_s\left(n-\frac{\pi}{\Delta\omega}\right)}}{j\left(n-\frac{\pi}{\Delta\omega}\right)}\right]\Bigg)$$

$$=\frac{-1}{4\,\Delta\omega j}\left(\frac{-\sin(\omega_s n)-\sin(\omega_p n)}{\left(n+\frac{\pi}{\Delta\omega}\right)}-\frac{-\sin(\omega_s n)-\sin(\omega_p n)}{\left(n-\frac{\pi}{\Delta\omega}\right)}\right)$$

$$=\frac{\left(\sin(\omega_s n)+\sin(\omega_p n)\right)}{4\,\Delta\omega j}\left(\frac{-2\pi/\Delta\omega}{n^2-\frac{\pi^2}{\Delta\omega^2}}\right)$$

$$=\frac{\sin(\omega_c n)\cos(\Delta\omega n/2)}{\Delta\omega j}\left(\frac{1}{\pi\left(1-(\Delta\omega/\pi)^2n^2\right)}\right).$$

Using the properties of the DTFT, we know that: $h_{LP}[n]=\frac{j}{n}g[n]$.

Therefore: $h_{LP}[n]=\left(\frac{\cos(\Delta\omega n/2)}{1-(\Delta\omega/\pi)^2n^2}\right)\left(\frac{\sin(\omega_c n)}{\pi n}\right).$

10.14 From Eq. (10.9): $\Phi_R=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_t\left(e^{j\omega}\right)-H_d\left(e^{j\omega}\right)\right|^2 d\omega,$,

where $\quad H_t\left(e^{j\omega}\right)=\sum_{n=-M}^{M}h_t[n]e^{-j\omega n}.$

Using Parseval's relation, we can write: $\Phi_R=\sum_{n=-\infty}^{\infty}\left|h_t[n]-h_d[n]\right|^2$

$$=\sum_{n=-M}^{M}\left|h_t[n]-h_d[n]\right|^2+\sum_{n=-\infty}^{-M-1}h_d^2[n]+\sum_{n=M+1}^{\infty}h_d^2[n].$$

Therefore: $\Phi_{Haan}=\sum_{n=-\infty}^{\infty}\left|h_d[n]?w_{Hann}[n]-h_d[n]\right|^2$

$$=\sum_{n=-M}^{M}\left|h_d[n]\left(\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi n}{2M+1}\right)\right)-h_d[n]\right|^2+\sum_{n=-\infty}^{-M-1}h_d^2[n]+\sum_{n=M+1}^{\infty}h_d^2[n]$$

Hence: $\Phi_{Excess} = \Phi_R - \Phi_{Haan} = \sum_{n=-M}^{M} \left| h_d[n] ? w_R[n] - h_d[n] \right|^2$

$$- \sum_{n=-M}^{M} \left| h_d[n] ? \left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi n}{2M+1} \right) \right) - h_d[n] \right|^2$$

$$= - \sum_{n=-M}^{M} \left| \frac{h_d[n]}{2}\cos\left(\frac{2\pi n}{2M+1} \right) - \frac{h_d[n]}{2} \right|^2 = \frac{1}{2}(1+2M)\left| \cos\left(\frac{2\pi M}{2M+1} \right) - 1 \right|^2 .$$

10.15 For a Hamming window: $\Phi_{Hamm} = \sum_{n=-\infty}^{\infty} \left| h_d[n] ? w_{Hamm}[n] - h_d[n] \right|^2$. Recall

$$\Phi_R = \sum_{n=-\infty}^{\infty} \left| h_t[n] - h_d[n] \right|^2 .$$

Therefore: $\Phi_{Excess} = \Phi_R - \Phi_{Hamm} = - \sum_{n=-M}^{M} \left| h_d[n] \left\{ 0.46\cos\left(\frac{2\pi n}{2M+1} \right) - 0.46 \right\} \right|^2$

$$= - \sum_{n=-M}^{M} \left| 0.46 h_d[n] \left\{ \cos\left(\frac{2\pi n}{2M+1} \right) - 1 \right\} \right|^2 = 0.46(2M+1)\left| \cos\left(\frac{2\pi M}{2M+1} \right) - 1 \right|^2 .$$

10.16 The responses of an ideal lowpass filter with a cutoff frequency $\omega_c = \pi/2$ are:

$$h_{HB}[n] = \frac{\sin(\omega_c n)}{\pi n} = \frac{\sin(\pi n/2)}{\pi n}, \qquad H_{HB}(e^{j\omega}) = \begin{cases} 1, & -\pi/2 < \omega < \pi/2, \\ 0, & \text{otherwise.} \end{cases}$$

The frequency responses of the Hilbert Transformer (from Eq. (10.24)) and the ideal discrete-time Differentiator (from Eq. (10.26)) are as follows:

$$H_{HT}(e^{j\omega}) = \begin{cases} j, & -\pi < \omega < 0, \\ -j, & 0 < \omega < \pi \end{cases} \qquad H_{DIF}(e^{j\omega}) = j\omega .$$

In order to get derive the impulse response of the Hilbert Transformer, we note that:

$$H_{HT}(e^{j\omega}) = jH_{HB}(e^{j(\omega-\pi/2)}) - jH_{HB}(e^{j(\omega+\pi/2)}).$$

Therefore, using the properties of the DTFT, we get:

$$h_{HT}[n] = jh_{HB}[n]e^{jn(\pi/2)} - jh_{HB}[n]e^{-jn(\pi/2)}$$

$$= 2h_{HB}[n]\left(\frac{1}{2j}e^{jn(\pi/2)} - \frac{1}{2j}e^{-jn(\pi/2)} \right) = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2} \right)\sin\left(\frac{n\pi}{2} \right).$$

In order to derive the impulse response of the Differentiator, we note that:

$$H_{DIF}(e^{j\omega}) = j\omega H_{HB}(e^{j(\omega-\pi/2)}) + j\omega H_{HB}(e^{j(\omega+\pi/2)}).$$

Therefore, using the properties of the DTFT, and in particular: $\frac{?}{?n} x[n] \; ? \; j\omega X(e^{j\omega})$

We get (by use of the double angle formula):

$$h_{DIF}[n] = \frac{?}{?n}\left(h_{HB}[n]e^{jn(\pi/2)} \right) + \frac{?}{?n}\left(h_{HB}[n]e^{-jn(\pi/2)} \right)$$

$$= 2\frac{?}{?n}\left\{h_{HB}[n]\left(\frac{1}{2}e^{jn(\pi/2)} + \frac{1}{2}e^{-jn(\pi/2)}\right)\right\} = 2\frac{?}{?n}\left\{h_{HB}[n]\cos\left(\frac{\pi n}{2}\right)\right\}$$
$$= \frac{?}{?n}\left\{\frac{2\sin(\pi n/2)\cos(\pi n/2)}{\pi n}\right\} = \frac{?}{?n}\left\{\frac{\sin(\pi n)}{\pi n}\right\} = \frac{\cos(\pi n)}{\pi n} - \frac{\sin(\pi n)}{\pi^2 n^2}.$$

10.17 For each problem, the codes used to generate the plots are given below, assuming that ω_c is the average of the stopband and passband frequencies:

```
fRange = -M:M;
idealLPF = (wc/pi)*sinc((wc/pi)*fRange);
fNum = idealLPF.*hann(L)';
[h,w] = freqz(fNum,1,512);
plot(w/pi,20*log10(abs(h)));grid;
xlabel(    '\omega/\pi'        ); ylabel(        'Gain, dB'       );
```

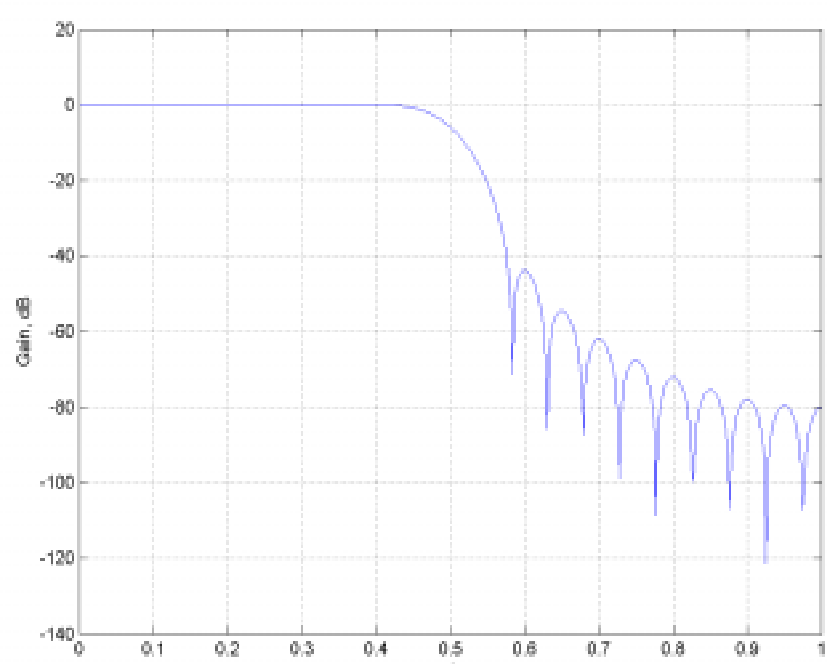
(a) Given: $\omega_p = 0.42\pi$, $\omega_s = 0.58\pi$, $\delta_p = 0.002$, $\delta_s = 0.008$.

Thus: $\Delta\omega = 0.16\pi$, $\alpha_s = -20\log_{10}\delta_s = 41.93$ dB.

From Table 10.2, we see that for fixed-window functions, we can achieve the minimum stopband attenuation by using Hann, Hamming, or Blackman windows. Hann will have the lowest filter length:

Since $M = \dfrac{3.11\pi}{0.16\pi} = 19.43$, $N_{Haan} = \lceil 2M + 1 \rceil = 40$.

The corresponding frequency response, generated with the code, is shown below:



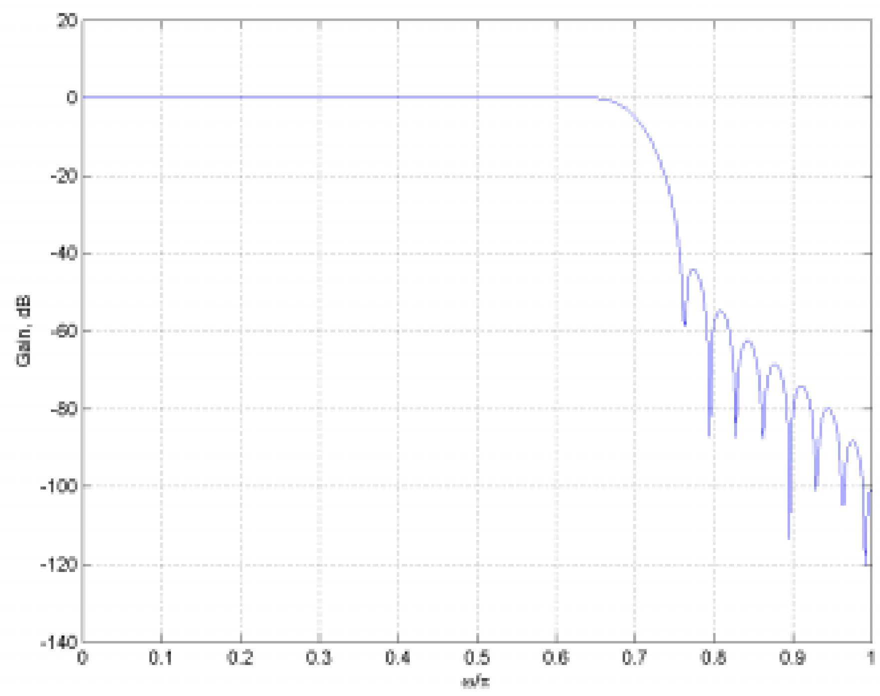
(b) Given: $\omega_p = 0.65\pi$, $\omega_s = 0.76\pi$, $\delta_p = 0.002$, $\delta_s = 0.004$.

Thus: $\Delta\omega = 0.11\pi$, $\alpha_s = -20\log_{10}\delta_s = 47.95$ dB.

From Table 10.2, we see that for fixed-window functions, we can achieve the minimum stopband attenuation by using Hann, Hamming, or Blackman windows. Hann will have the lowest filter length:

Since $M = \dfrac{3.11\pi}{0.16\,\pi} = 28.27$, $N_{Haan} = \lceil 2M + 1 \rceil = 59.$

The corresponding frequency response, generated with the code, is shown below:



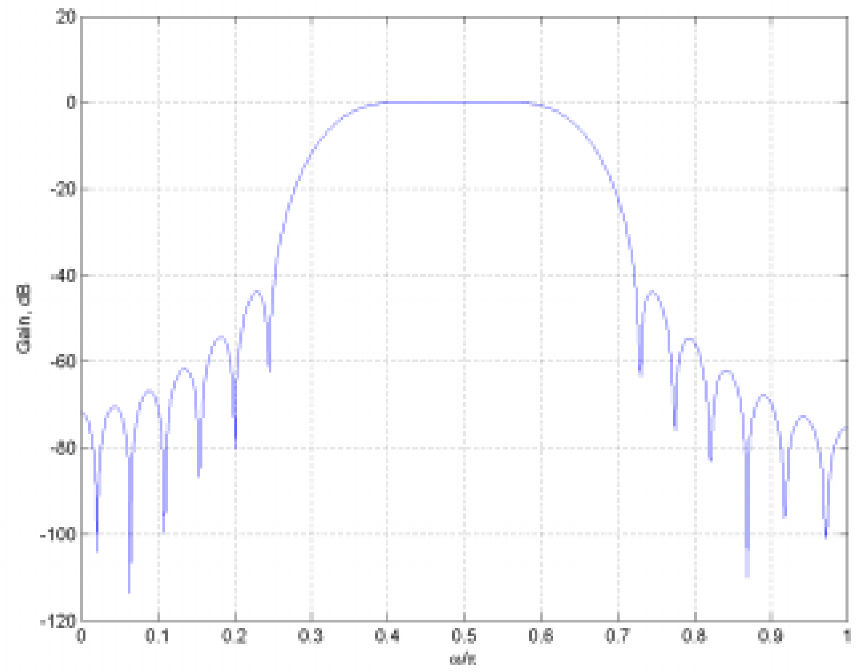
10.18 The code for the plotting of the frequency response is as follows:

```
fRange = -M:M;
idealBPF = (wc2/pi)*sinc((wc2/pi)*fRange)                    ...
   -(wc1/pi)*sinc((wc1/pi)*fRange);
fNum = idealBPF.*hann(L)';
[h,w] = freqz(fNum,1,512);
plot(w/pi,20*log10(abs(h)));grid;
xlabel(     '\omega/\pi'        ); ylabel(        'Gain, dB'       );
```

We are given that: $\omega_{p1} = 0.4\,\pi, \omega_{p2} = 0.55\pi, \omega_{s1} = 0.25\pi, \omega_{s2} = 0.75\pi,$
$\delta_p = 0.02$, $\delta_{s1} = 0.006$, $\delta_{s2} = 0.008.$

We can compute the following: $\Delta\omega_1 = 0.15\,\pi$, $\Delta\omega_2 = 0.2\,\pi$,
$\alpha_{s1} = -20\log_{10}\delta_{s1} = 44.43$ dB, $\alpha_{s2} = -20\log_{10}\delta_{s2} = 41.93$ dB.

From Table 10.2, we see that the Hann window will have minimum length and meet the minimum stopband attenuation. We can use the smaller stopband, so that it definitely meets these more restrictive criteria: $M = \dfrac{3.11\pi}{0.15\,\pi} = 20.733.$
Therefore $N = 43.$ The frequency response is given below:



10.19 The code for this problem is given below:

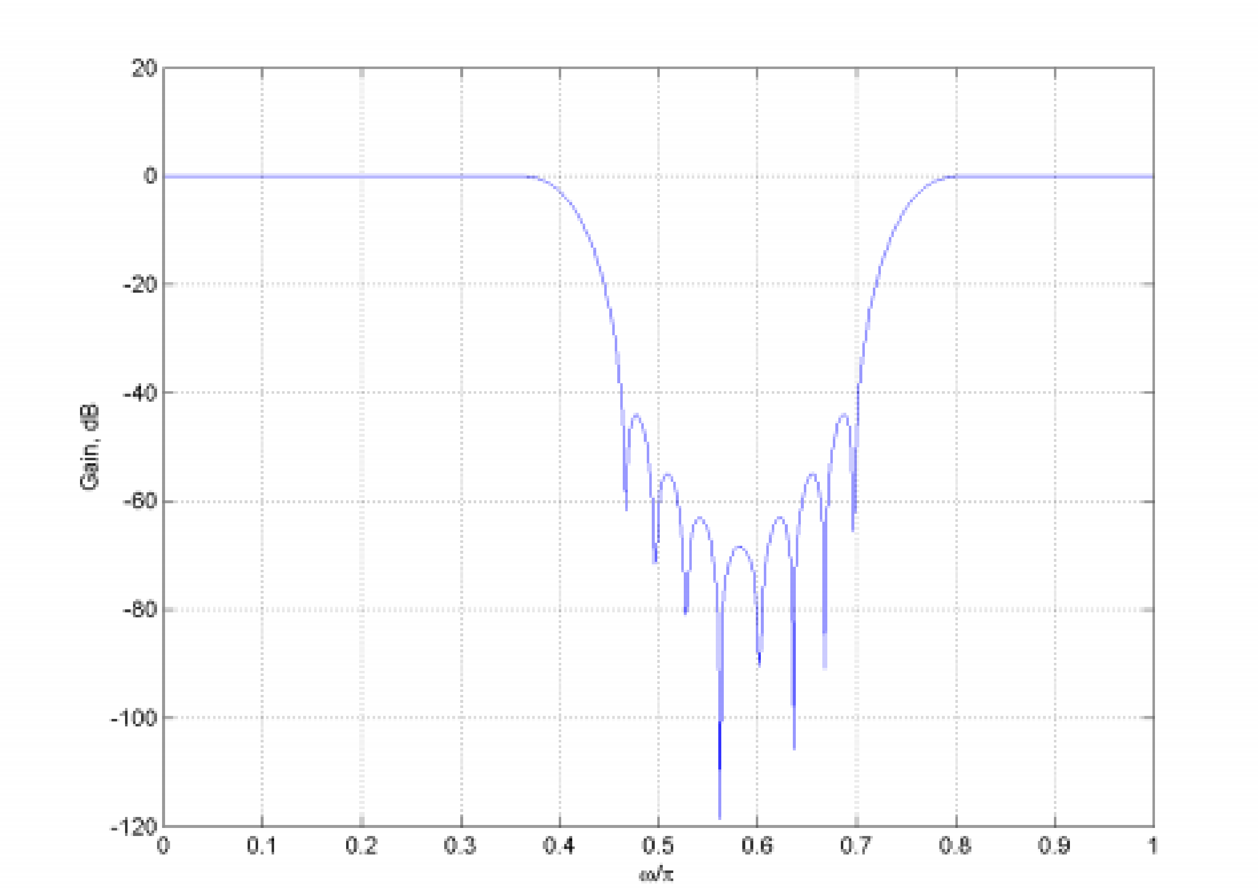
```
wp1 = 0.4*pi;
wp2 = 0.55*pi;
ws1 = 0.25*pi;
ws2 = 0.75*pi;
dp = 0.002;
ds1 = 0.006;
ds2 = 0.008;

as1 = -20*log10(ds1);
as2 = -20*log10(ds2);
diff1 = wp1 - ws1;
diff2 = ws2 - wp2;
diff = min(diff1,diff2);
wc1 = (wp1+ws1)/2;
wc2 = (wp2+ws2)/2;

M = 3.11*pi/diff;
L = 2*ceil(M) + 1;
[as1 as2 M L]
M = ceil(M);
fRange = -M:M;
idealBPF = (wc2/pi)*sinc((wc2/pi)*fRange)                    ...
   -(wc1/pi)*sinc((wc1/pi)*fRange);
fNum = idealBPF.*hann(L)';
[h,w] = freqz(fNum,1,512);
plot(w/pi,20*log10(abs(h)));grid;
xlabel(    '\omega/\pi'        ); ylabel(        'Gain, dB'       );
```

We are given that: $\omega_{p1} = 0.33\pi$, $\omega_{p2} = 0.8\pi$, $\omega_{s1} = 0.5\pi$, $\omega_{s2} = 0.7\pi$,
$\delta_{p1} = 0.008$, $\delta_{p2} = 0.01$, $\delta_s = 0.03$.

We can compute: $\Delta\omega_1 = 0.17\pi$, $\Delta\omega_2 = 0.2\pi$, $\alpha_s = -20\log_{10}\delta_s = 41.93$ dB.

From Table 10.2, we see that the Hann window will have minimum length and meet the minimum stopband attenuation. Again, can use the smaller stopband, so that it definitely meets the more restrictive criteria: $M = \dfrac{3.11\pi}{0.17\pi} = 31.1$.

Therefore $N = 65$. The frequency response is given below:



10.20 The raised cosine window is given as follows:

$$w_{GC}[n] = \left[\alpha + \beta\cos\left(\frac{2\pi n}{2M+1}\right) + \gamma\cos\left(\frac{4\pi n}{2M+1}\right)\right] w_R[n]$$

$$= \left[\alpha + 2\beta\left(e^{j\left(\frac{2\pi n}{2M+1}\right)} + e^{-j\left(\frac{2\pi n}{2M+1}\right)}\right) + 2\gamma\left(e^{j\left(\frac{4\pi n}{2M+1}\right)} + e^{-j\left(\frac{4\pi n}{2M+1}\right)}\right)\right] w_R[n].$$

Hence: $\Psi_{GC}(e^{j\omega}) = \alpha\Psi_R(e^{j\omega}) + 2\beta\Psi_R\left(e^{j\left(\omega - \frac{2\pi}{2M+1}\right)}\right) + 2\beta\Psi_R\left(e^{j\left(\omega + \frac{2\pi}{2M+1}\right)}\right)$

$$+2\gamma\Psi_R\left(e^{j\left(\omega - \frac{4\pi}{2M+1}\right)}\right) + 2\gamma\Psi_R\left(e^{j\left(\omega + \frac{4\pi}{2M+1}\right)}\right).$$

For the Hann window: $\alpha = 0.5$, $\beta = 0.5$, and $\gamma = 0$. Hence:

$$\Psi_{Haan}(e^{j\omega}) = 0.5\,\Psi_R(e^{j\omega}) + \Psi_R\left(e^{j\left(\omega - \frac{2\pi}{2M+1}\right)}\right) + 2\beta\Psi_R\left(e^{j\left(\omega + \frac{2\pi}{2M+1}\right)}\right)$$

$$= 0.5\frac{\sin\left(\frac{(2M+1)\omega}{2}\right)}{\sin(\omega/2)} + \frac{\sin\left((2M+1)\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)}$$

$$+ \frac{\sin\left((2M+1)\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)}.$$

For the Hamming window, $\alpha = 0.54$, $\beta = 0.46$, and $\gamma = 0$. Hence:

$$\Psi_{Hamming}(e^{j\omega}) = 0.54\Psi_R(e^{j\omega}) + 0.92\Psi_R\left(e^{j\left(\omega - \frac{2\pi}{2M+1}\right)}\right) + 0.92\Psi_R\left(e^{j\left(\omega + \frac{2\pi}{2M+1}\right)}\right)$$

$$= 0.54\frac{\sin\left(\frac{(2M+1)\omega}{2}\right)}{\sin(\omega/2)} + 0.92\frac{\sin\left((2M+1)\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)}$$

$$+0.92\frac{\sin\left((2M+1)\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)}.$$

For the Blackmann window $\alpha = 0.42$, $\beta = 0.5$, and $\gamma = 0.08$.

$$\Psi_{Blackman}(e^{j\omega}) = 0.42\Psi_R(e^{j\omega}) + \Psi_R\left(e^{j\left(\omega - \frac{2\pi}{2M+1}\right)}\right) + \Psi_R\left(e^{j\left(\omega + \frac{2\pi}{2M+1}\right)}\right)$$

$$+ 0.16\Psi_R\left(e^{j\left(\omega - \frac{4\pi}{2M+1}\right)}\right) + 0.16\Psi_R\left(e^{j\left(\omega - \frac{4\pi}{2M+1}\right)}\right)$$

$$= 0.42\frac{\sin\left(\frac{(2M+1)\omega}{2}\right)}{\sin(\omega/2)} + \frac{\sin\left((2M+1)\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)}$$

$$+ \frac{\sin\left((2M+1)\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)} + 0.16\frac{\sin\left((2M+1)\left(\frac{\omega}{2} - \frac{2\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} - \frac{2\pi}{2M+1}\right)}$$

$$+0.16\frac{\sin\left((2M+1)\left(\frac{\omega}{2} - \frac{2\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} - \frac{2\pi}{2M+1}\right)}.$$

10.21 Given that: $H(z) = z^{-D} \approx \sum_{n=0}^{N} h[n]z^{-n} = h[0] + h[1]z^{-1} + h[2]z^{-2} + \quad + h[N]z^{-N}$.

(a) We see that if: $\hat{x}_a(t) = \sum_{k=-N_1}^{N_2} P_k(t)x[n+k]$,

?

Then: $P_k(t) = \prod_{\substack{l=-N_1 \\ l \neq k}}^{N_2} \left(\frac{t - t_l}{t_k - t_l}\right)$ for $-N_1 \leq k \leq N_2$.

Here, we have: $\hat{H}(z) = \sum_{n=0}^{N} h[n]z^{-n}$.

And the solution follows if: $P_k(t) = h[n], N_1 = 0, N_2 = N, k = n, t = D, t \ = k$, and $t_k = n$.

Therefore, we have: $h[n] = \prod_{\substack{k=0 \\ k \neq n}}^{N} \frac{D - k}{n - k}$ for $0 \leq n \leq N$.

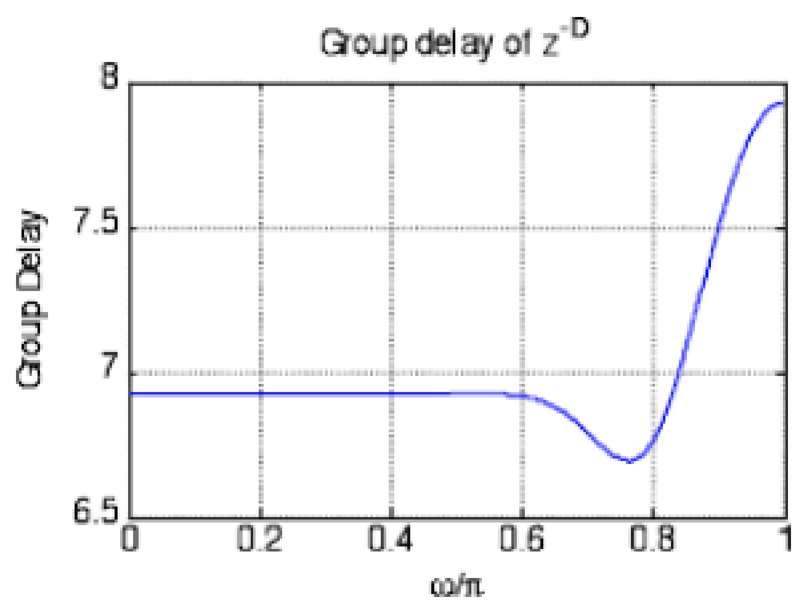
?

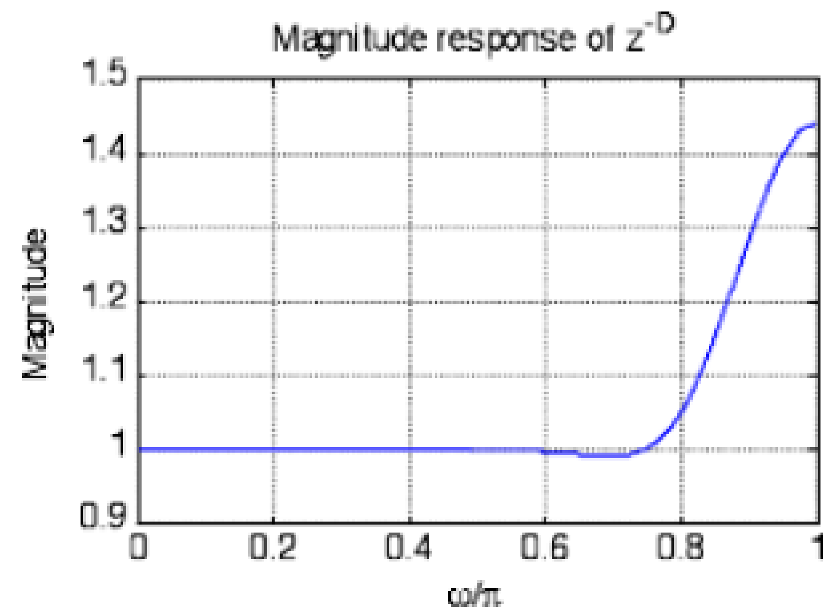
(b) Given that: $N = 21, D = 90/13, L = 22$

$$H(z) \approx \sum_{n=0}^{21} h[n]z^{-n}, \quad \text{where} \quad h[n] = \prod_{\substack{k=0 \\ k \neq n}}^{21} \frac{90/13 - k}{n - k}.$$

```
% Problem #10.21
D = 90/13;
N = 21;
for   n = 0:N,
        for   k = 0:N,
                if   n ~= k,
           tmp(n+1,k+1) = (D-k)/(n-k);
                else
           tmp(n+1,k+1) = 1;
                end
        end
end
h = prod(tmp');
[Gd,W] = grpdelay(h,1,512);
[H, w] = freqz(h,1,512);

figure(1);
plot(W/pi, Gd);
xlabel(     '\omega/\pi'        );
ylabel(     'Group Delay'       );
title(      'Group delay of z^-^D'          );
grid;
```

```
figure(2);
plot(w/pi, (abs(H)));
xlabel(     '\omega/\pi'         );
ylabel(     'Magnitude'       );
title(       'Magnitude response of z^-^D'                   );
grid;
```





10.22 The frequency response of an ideal comb filter is given as:

$$H_{comb}(e^{j\omega}) = \begin{cases} 0, & \omega = k\omega_0 \quad 0 \le k \le M, \\ 1, & \text{otherwise.} \end{cases}$$

(a) To show this note that: $x[n] = s[n] + \sum_{k=0}^{M} A_k \sin(k\omega_o n + \phi_k) = s[n] + r[n]$,

where $s[n]$ is the desired signal and $r[n] = \sum_{k=0}^{M} A_k \sin(k\omega_o n + \phi_k)$ is the harmonic interference with fundamental frequency ω_o. Therefore:

$$r[n-D] = \sum_{k=0}^{M} A_k \sin[k\omega_o(n-D) + \phi_k] = \sum_{k=0}^{M} A_k \sin(k\omega_o n + \phi_k - 2\pi k) = r[n].$$

(b) We first compute the output as follows:

$$y[n] = x[n] - x[n-D] \qquad = s[n] + r[n] - s[n-D] - r[n-D]$$
$$= s[n] + r[n] - s[n-D] - r[n] \quad = s[n] - s[n-D].$$

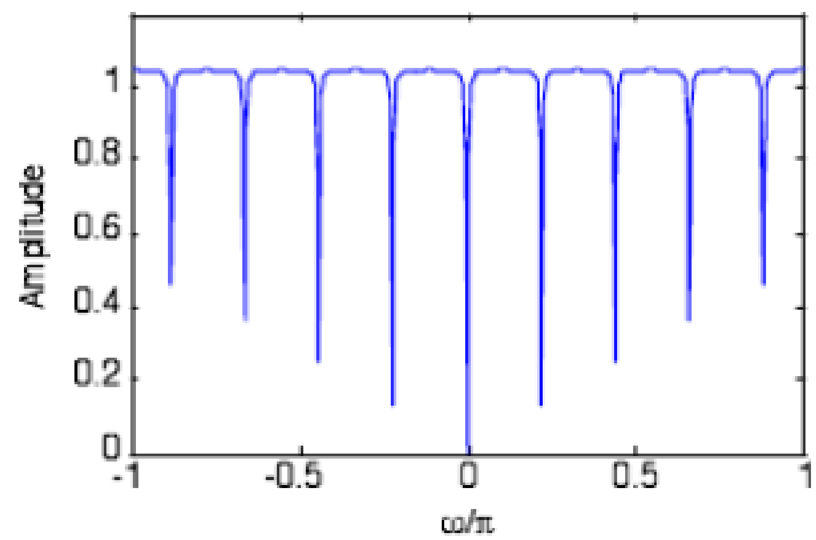
Hence, $y[n]$ does not contain any harmonic disturbances.

(c) Given the modified filter: $H_c(z) = \dfrac{1 - z^{-D}}{1 - \rho^D z^{-D}}$.

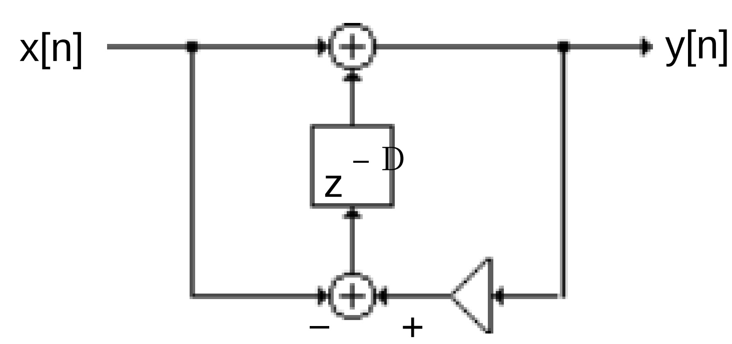
Thus: $H_c(e^{j\omega}) = \dfrac{1-e^{-jD\omega}}{1-\rho^D e^{-jD\omega}} = \dfrac{(1-\cos(D\omega))+j\sin(D\omega)}{(1-\rho^D\cos(D\omega))+j\rho^D\sin(D\omega)}$.

Then: $\left|H_c(e^{j\omega})\right| = \sqrt{\dfrac{2(1-\cos(D\omega))}{1-2\rho^D\cos(D\omega)+\rho^{2D}}}$.

A plot of $\left|H_c(e^{j\omega})\right|$ for $\omega_o = 0.22\pi$ and $\rho = 0.99$ is shown below:



(d) The realization is shown below, and includes only a single delay and two additions.



10.23 Using the results of Problems 10.21 and 10.22 $H_c(z) = \dfrac{P(z)}{Q(z)} = \dfrac{1-N(z)}{1-0.9^D N(z)}$,

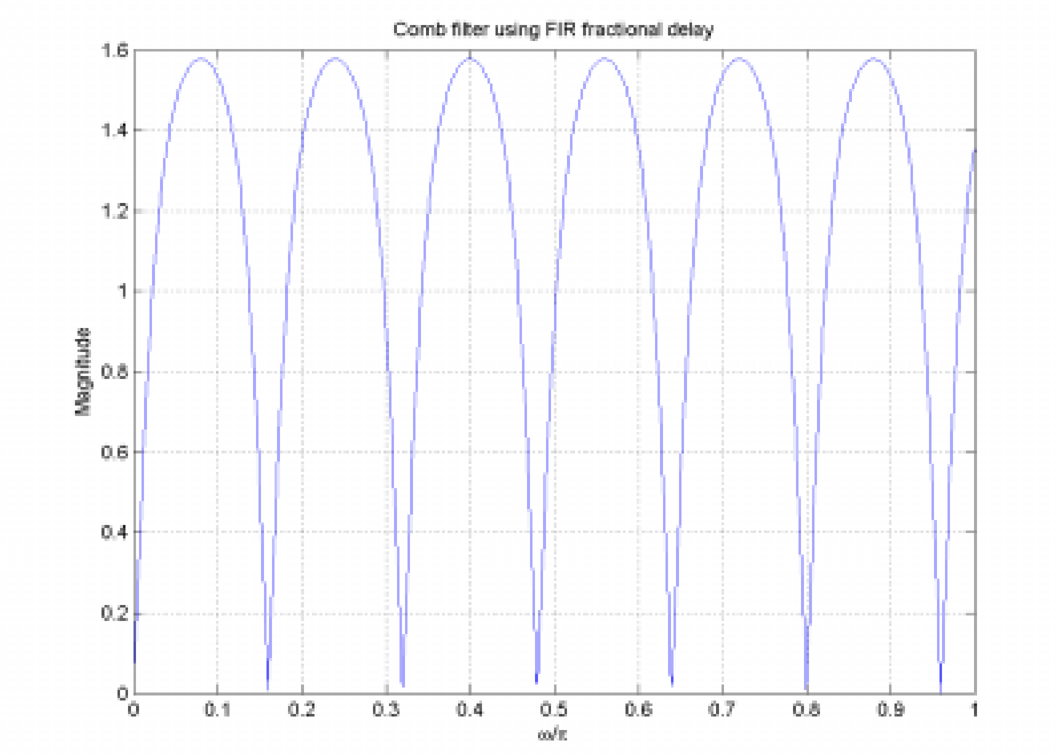
with $D = 2\pi/0.16\,\pi$, N = 20, and $\rho = 0.9$

The code for plotting the results is shown below:

```
% Problem #10.23
D = 2/.16;
N = 20;
rho = 0.9;
for    n = 0:N,
       for    k = 0:N,
              if    n ~= k,
         tmp(n+1,k+1) = (D-k)/(n-k);
              else
         tmp(n+1,k+1) = 1;
              end
       end
```

```
end
h = prod(tmp');
[H,w] = freqz(h,1,1024);
Hc = (1-H)./(1-(rho^D)*H);
x = sqrt((2*(1-cos(D*w)))./(1-
2*(rho^D)*cos(D*w)+rho^(2*D)));
%plot(w/pi, abs(Hc));
plot(w/pi, x); grid;
xlabel(     '\omega/\pi'          );
ylabel(     'Magnitude'       );
title(       'Comb filter using FIR fractional delay'                              );
```

The resulting plot is shown next:



10.24 Using the results of Problems 9.34 and 10.22:

$$H_c(z) = \frac{P(z)}{Q(z)} = \frac{D(z) - z^{-11}D(z^{-1})}{D(z) - 0.9^D z^{-11}D(z^{-1})},$$

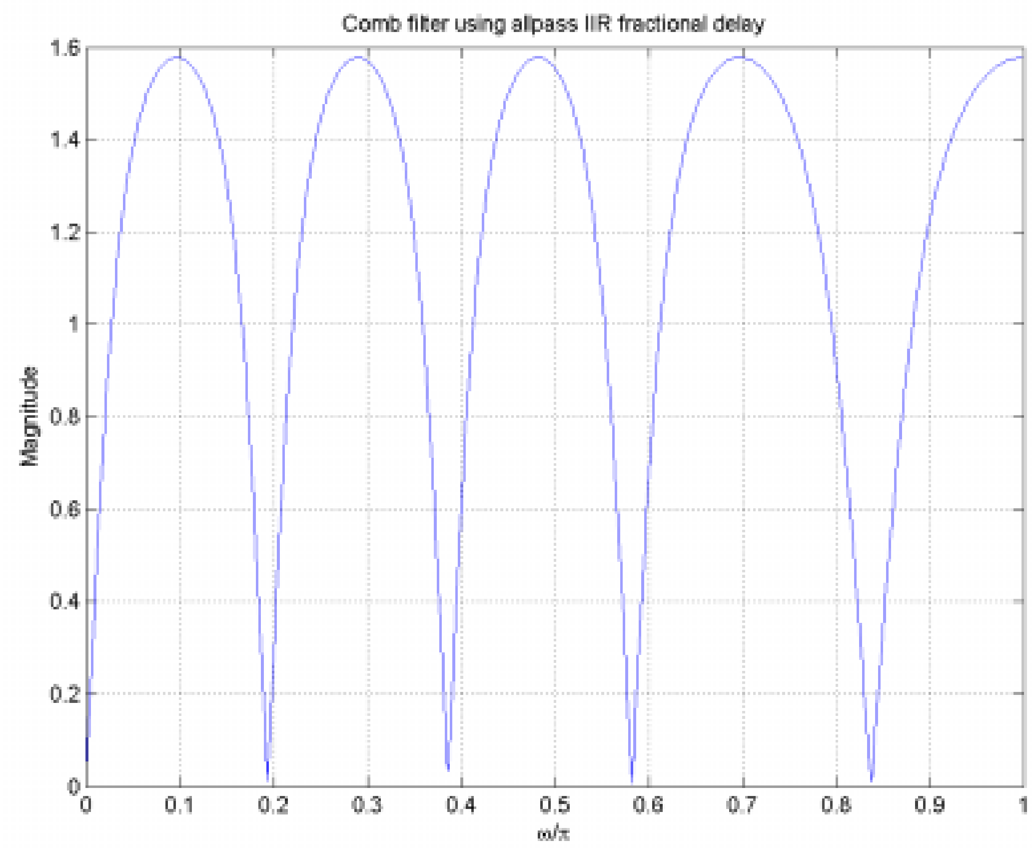
with $D = 2\pi/0.16\,\pi$, N = 10, and $\rho = 0.9$.

The code for plotting the results is shown below:

```
% Problem #10.24
D = 2/0.16;
rho = 0.9;
N = min(10,floor(D));
for   k = 1:N,
        for   n = 0:N,
       p(n+1) = (D-N+n)/(D-N+k+n);
        end
   d(k) = ((-1)^k)*nchoosek(N,k)*prod(p);
end
[H,w] = freqz(fliplr(d)-d, fliplr(d)-(rho^D).*d ,
512);
plot(w/pi, abs(H));grid;
```

```
xlabel(     '\omega/\pi'        );
ylabel(     'Magnitude'      );
title(      'Comb filter using allpass IIR fractional
delay'    );
```

The resulting plot is shown next:



10.25 We start with the expression for the low pass filter, derived from Figure 10.13(a):

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & -\omega_p \le \omega \le \omega_p, \\ 1 - \left(\dfrac{\omega - \omega_p}{\omega_s - \omega_p}\right), & \omega_p < \omega \le \omega_s, \\ 1 + \left(\dfrac{\omega + \omega_p}{\omega_s - \omega_p}\right), & -\omega_s < \omega \le \omega_p, \\ 0, & \text{elsewhere.} \end{cases}$$

For $n \neq 0$, we can directly apply the inverse DTFT:

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\int_{\omega_p}^{\omega_p} e^{j\omega n}\,d\omega + \int_{\omega_p}^{\omega_s}\left(1 - \frac{\omega - \omega_p}{\Delta\omega}\right)e^{j\omega n}\,d\omega + \int_{-\omega_s}^{-\omega_p}\left(1 + \frac{\omega + \omega_p}{\Delta\omega}\right)e^{j\omega n}\,d\omega\right]$$

$$= \frac{1}{2\pi}\left[\int_{\omega_s}^{\omega_s} e^{j\omega n}\,d\omega - \int_{\omega_p}^{\omega_s}\frac{\omega - \omega_p}{\Delta\omega}e^{j\omega n}\,d\omega + \int_{-\omega_s}^{-\omega_p}\frac{\omega + \omega_p}{\Delta\omega}e^{j\omega n}\,d\omega\right]$$

$$= \frac{1}{2\pi}\left\{\frac{2\sin(\omega_s n)}{\pi n} - \frac{1}{\Delta\omega}\left[\frac{(\omega-\omega_p)e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2}\right]_{\omega_p}^{\omega_s} + \frac{1}{\Delta\omega}\left[\frac{(\omega+\omega_p)e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2}\right]_{-\omega_s}^{-\omega_p}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{2\sin(\omega_s n)}{\pi n} - \frac{1}{\Delta\omega}\left[\frac{\Delta\omega\, e^{j\omega_s n}}{jn} + \frac{e^{j\omega_s n} - e^{j\omega_p n}}{n^2} + \frac{-\Delta\omega e^{-j\omega_s n}}{jn} - \frac{e^{-j\omega_p n} - e^{-j\omega_s n}}{n^2}\right]\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{2\sin(\omega_s n)}{\pi n} - \frac{2}{\Delta\omega}\left[\frac{\Delta\omega\sin(\omega_s n)}{n} + \frac{\cos(\omega_s n) - \cos(\omega_p n)}{n^2}\right]\right\}$$

$$= \frac{1}{\Delta\omega}\left\{\frac{\cos(\omega_p n)}{\pi n^2} - \frac{\cos(\omega_s n)}{\pi n^2}\right\}$$

$$= \frac{1}{\Delta\omega}\left\{\frac{\cos((\omega_c - \Delta\omega/2)n)}{\pi n^2} - \frac{\cos((\omega_c + \Delta\omega/2)n)}{\pi n^2}\right\}$$

$$= \frac{2\sin(\Delta\omega n/2)}{\Delta\omega n}\,\frac{\sin(\omega_c n)}{\pi n}.$$

Next, for we evaluate the integral directly:

$$h_{LP}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})\,d\omega = \frac{1}{2\pi}\,\frac{2(\omega_s+\omega_p)}{2} = \frac{\omega_c}{\pi}.$$

Hence, we verify the result in Figure 10.13(b):

$$h_{LP}[n] = \begin{cases} \dfrac{\omega_c}{\pi}, & \text{if } n = 0, \\ \dfrac{2\sin(\Delta\omega n/2)}{\Delta\omega n}\,\dfrac{\sin(\omega_c n)}{\pi n}, & \text{if } n \neq 0. \end{cases}$$

An alternate approach to solving this problem is to consider the frequency response:

$$G(e^{j\omega}) = \frac{d\,H_{LP}(e^{j\omega})}{d\omega} = \begin{cases} 0, & -\omega_p \le \omega \le \omega_p, \\ -\dfrac{1}{\Delta\omega}, & \omega_p < \omega < \omega_s, \\ \dfrac{1}{\Delta\omega}, & -\omega_s < \omega < -\omega_p, \\ 0, & \text{elsewhere.} \end{cases}$$

Its inverse DTFT is given by

$$g[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\omega_s}^{-\omega_p}\frac{1}{\Delta\omega}e^{j\omega n}\,d\omega - \frac{1}{2\pi}\int_{\omega_p}^{\omega_s}\frac{1}{\Delta\omega}e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi\,\Delta\omega}\left\{\left.\frac{e^{j\omega n}}{jn}\right|_{-\omega_s}^{-\omega_p} - \left.\frac{e^{j\omega n}}{jn}\right|_{\omega_p}^{\omega_s}\right\} = \frac{1}{j\pi n\,\Delta\omega}\Big(\cos(\omega_p n) - \cos(\omega_s n)\Big).$$

Thus: $h_{LP}[n] = \frac{j}{n} g[n] = \frac{1}{\pi n^2 \Delta\omega}\left(\cos(\omega_p n) - \cos(\omega_s n)\right)$

$$= \frac{1}{\pi n^2 \Delta\omega}\left[\cos\left(\left(\omega_c - \frac{\Delta\omega}{2}\right)n\right) - \cos\left(\left(\omega_s + \frac{\Delta\omega}{2}\right)n\right)\right]$$

$$= \frac{2\sin(\Delta\omega n/2)}{\Delta\omega n} ? \frac{\sin(\omega_c n)}{\pi n}, \quad \text{for } n \neq 0.$$

And for $n = 0$, $h_{LP}[n] = \frac{\omega_c}{\pi}$.

10.26 Consider the case when the transition region is approximated by a second order spline. In this case, the ideal frequency response can be constructed by convolving an ideal, no-transition-band frequency response with a triangular pulse of width $\Delta\omega = \omega_s - \omega_p$, which in turn can be obtained by convolving two rectangular pulses of width $\Delta\omega/2$. In the time domain this implies that the impulse response of a filter with transition band approximated by a second order spline is given by the product of the impulse response of an ideal low pass filter with no transition region and square of the impulse response of a rectangular pulse.

Thus: $H_{LP(ideal)}[n] = \frac{\sin(\omega_c n)}{\pi n}$ and $H_{rec}[n] = \frac{\sin(\Delta\omega n/4)}{\Delta\omega n/4}$.

Hence: $H_{LP}[n] = H_{LP(ideal)}[n]\left(H_{rec}[n]\right)^2$.

Thus, for a lowpass filter with a transition region approximated by a second order spline:

$$h_{LP}[n] = \begin{cases} \frac{\omega_c}{\pi}, & \text{if } n = 0, \\ \left(\frac{\sin(\Delta\omega n/4)}{\Delta\omega n/4}\right)^2 \frac{\sin(\omega_c n)}{\pi n}, & \text{otherwise.} \end{cases}$$

Similarly the frequency response of a lowpass filter with the transition region specified by a P-th order spline can be obtained by convolving in the frequency domain an ideal filter with no transition region with P rectangular pulses of width $\Delta\omega/P$. Hence:

$$H_{LP}[n] = H_{LP(ideal)}[n]\left(H_{rec}[n]\right)^P,$$

where the rectangular pulse is of width $\Delta\omega/P$. Therefore:

$$h_{LP}[n] = \begin{cases} \frac{\omega_c}{\pi}, & \text{if } n = 0, \\ \left(\frac{\sin(\Delta\omega n/2P)}{\Delta\omega n/2P}\right)^P \frac{\sin(\omega_c n)}{\pi n}, & \text{otherwise.} \end{cases}$$

10.27 Given the mean square error: $\varepsilon(\alpha_i) = \sum_{k=-L}^{L}\left\{x[k] - x_a(k)\right\}^2$.

(a) $N = 1$ and hence, $x_a(t) = \alpha_0 + \alpha_1 t$.

For $L = 5$. we first fit the data set $\{x[k]\}$ for each of $-5 \le k \le 5$, to the polynomial $x_a(t) = \alpha_0 + \alpha_1 t$ with a minimum mean-square error at $t = -5, -4, \cdots, -1, 0, 1, \cdots, 5$, and then replace $x[0]$ with a new value $\bar{x}[0] = x(0) = \alpha_0$. The mean-square error is then given by:

$$\varepsilon(\alpha_0, \alpha_1) = \sum_{k=-5}^{5} \left(x[k] - \alpha_0 - \alpha_1 k\right)^2.$$?

We set: $\dfrac{?\,\varepsilon(\alpha_0, \alpha_1)}{?\,\alpha_0} = 0$ and $\dfrac{?\,\varepsilon(\alpha_0, \alpha_1)}{?\,\alpha_1} = 0$,

which yields: $11\alpha_0 + \alpha_1 \sum_{k=-5}^{5} k = \sum_{k=-5}^{5} x[k]$, and $\alpha_0 \sum_{k=-5}^{5} k + \alpha_1 \sum_{k=-5}^{5} k^2 = \sum_{k=-5}^{5} k\, x[k]$.

From the first equation we get: $\bar{x}[0] = \alpha_0 = \dfrac{1}{11} \sum_{k=-5}^{5} x[k]$.

In the general case we thus have $\bar{x}[n] = \alpha_0 = \dfrac{1}{11} \sum_{k=-5}^{5} x[k]$, which is a moving average filter of length 11.

(b) $N = 1$ and hence, $x_a(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2$. Here, we fit the data set $\{x[k]\}$ for each of $-5 \le k \le 5$, to the polynomial $x_a(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2$ with a minimum mean-square error at $t = -5, -4, \cdots, -1, 0, 1, \cdots, 5$, and then replace $x[0]$ with a new value $\bar{x}[0] = x_a(0) = \alpha_0$. The mean-square error is now given by:

? $$\varepsilon(\alpha_0, \alpha_1, \alpha_2) = \sum_{k=-5}^{5} \left(x[k] - \alpha_0 - \alpha_1 k - \alpha_2 k^2\right)^2.$$

We set:

$$\frac{?\,\varepsilon(\alpha_0, \alpha_1, \alpha_2)}{?\,\alpha_0} = 0, \quad \frac{?\,\varepsilon(\alpha_0, \alpha_1, \alpha_2)}{?\,\alpha_1} = 0, \quad \frac{?\,\varepsilon(\alpha_0, \alpha_1, \alpha_2)}{?\,\alpha_2} = 0,$$

which yields:

$$11\alpha_0 + 110\alpha_2 = \sum_{k=-5}^{5} x[k], \quad 110\alpha_1 = \sum_{k=-5}^{5} k\, x[k], \quad 110\alpha_0 + 1958\alpha_2 = \sum_{k=-5}^{5} k^2 x[k].$$

From the first and the third equations we then get:

$$\alpha_0 = \frac{1958 \sum_{k=-5}^{5} x[k] - 110 \sum_{k=-5}^{5} k^2 x[k]}{(1958 \times 11) - (110)^2} = \frac{1}{429} \sum_{k=-5}^{5} \left(89 - 5k^2\right) x[k].$$

Hence, here we replace $x[n]$ with a new value $x'[n] = \alpha_0$ which is a weighted combination of the original data set:

$$x'[n] = \frac{1}{429}\sum_{k=-5}^{5}\left(89 - 5k^2\right)x[n-k]$$

$$= \frac{1}{429}(-36\,x[n+5] + 9\,x[n+4] + 44\,x[n+3] + 69\,x[n+2] + 84\,x[n+1]$$
$$+69\,x[n+2] + 84\,x[n+1] + 89\,x[n] + 84\,x[n-1]$$
$$+69\,x[n-2] + 44\,x[n-3] + 9\,x[n-4] - 36\,x[n-5]).$$

(c) The impulse response of the FIR filter of Part (a) is given by

$$h_1[n] = \frac{1}{11}\{1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1\},$$

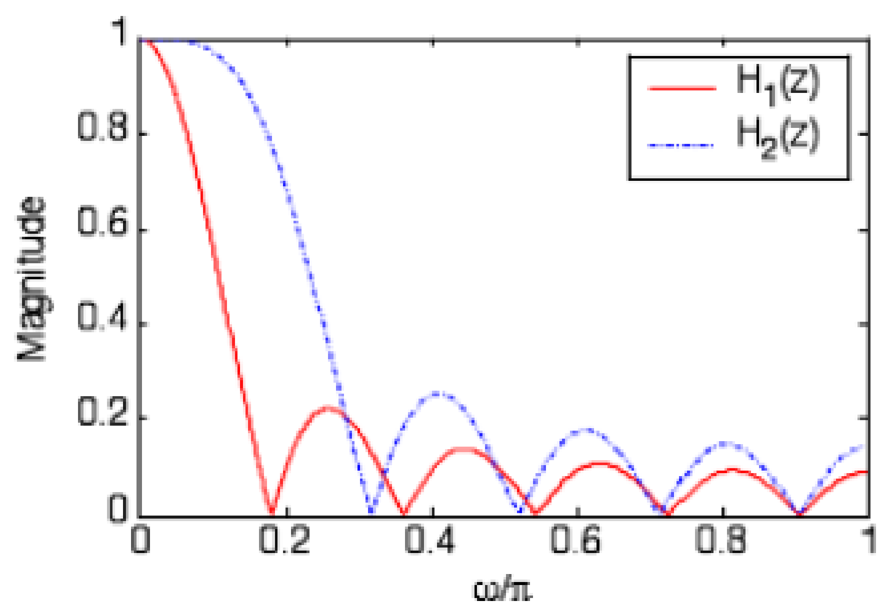
whereas, the impulse response of the FIR filter of Part (b) is given by:

$$h_2[n] = \frac{1}{429}\{-36,\ 9,\ 44,\ 69,\ 84,\ 89,\ 84,\ 69,\ 44,\ 9,\ -36\}.$$

The corresponding frequency responses are given by:

$$H_1(e^{j\omega}) = \frac{1}{11}\sum_{k=-5}^{5} e^{-j\omega k}, \qquad H_2(e^{j\omega}) = \frac{1}{429}\sum_{k=-5}^{5}\left(89 - 5k^2\right)e^{-j\omega k}.$$

A plot of the magnitude responses of these two filters are shown below from which it can be seen that the filter of Part (b) has a wider passband and thus provides smoothing over a larger frequency range than the filter of Part (a).



10.28 Spencer's smoothing formula is given as:

$$y[n] = \frac{1}{320}\{-3x[n-7] - 6x[n-6] - 5x[n-5] + 3x[n-4] + 21x[n-3] + 46x[n-2]$$
$$+67x[n-1] + 74x[n] + 67x[n+1] + 46x[n+2] + 21x[n+3] + 3x[n+4]$$

$+5x[n+5] - 6x[n-6] - 3x[n+7]\}$.

Hence: $H_3(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})}$

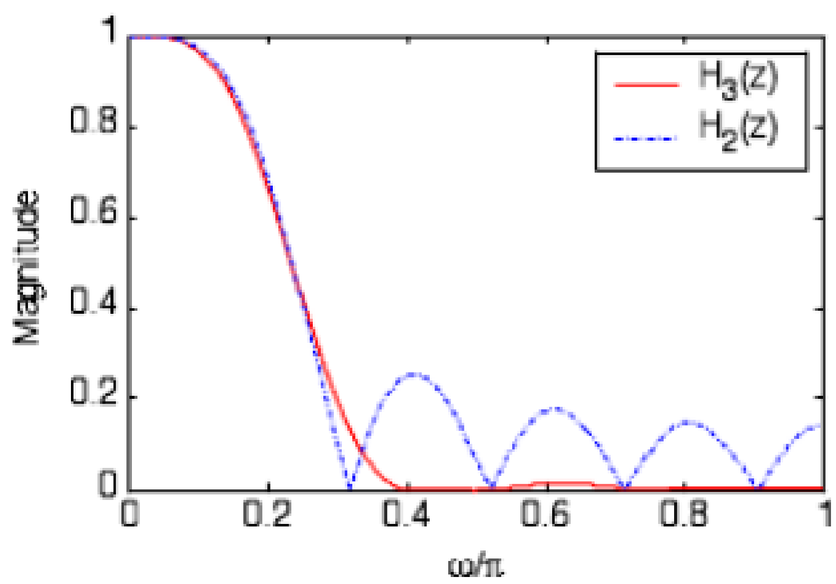
$= \dfrac{1}{320}\{-3e^{-j7\omega} - 6e^{-j6\omega} - 5e^{-j5\omega} + 3e^{-j4\omega} + 21e^{-j3\omega} 46e^{-j2\omega} + 67e^{-j\omega} + 74$

$+67e^{j\omega} + 46e^{j2\omega} + 21e^{j3\omega} + 3e^{j4\omega} - 5e^{j5\omega} - 6e^{j6\omega} - 3e^{j7\omega}\}$

$= \dfrac{1}{160}\{74 + 67\cos\omega + 46\cos(2\omega) + 21\cos(3\omega) + 3\cos(4\omega)$

$- 5\cos(5\omega) - 6\cos(6\omega) - 3\cos(7\omega)\}$.

The magnitude response of the above FIR filter $H_3(z)$ is plotted below (solid line) along with that of the FIR filter $H_2(z)$ of Part (b) of Problem 10.27 (dashed line). Note that both filters have roughly the same passband but the Spencer's filter has very large attenuation in the stopband and hence it provides better smoothing than the filter of Part (b).



10.29 The coefficients for each of $L = 3$, 4, and 5, are determined as follows:

(a) $L = 3$. $P(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$. Now $P(0) = 0$ is satisfied by the way $P(x)$ has been defined. Also to ensure $P(1) = 1$ we require $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Choose $m = 1$ and $n = 1$. Since $\left.\dfrac{dP(x)}{dx}\right|_{x=0} = 0$, hence $\left.\alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2\right|_{x=0} = 0$, implying $\alpha_1 = 0$. Also since $\left.\dfrac{dP(x)}{dx}\right|_{x=1} = 0$, hence, $\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$. Thus solving the three equations:

$\alpha_1 + \alpha_2 + \alpha_3 = 1$, $\alpha_1 = 0$, and $\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$,

we arrive at $\alpha_1 = 0$, $\alpha_2 = 3$, $\alpha_3 = -2$. Therefore, $P(x) = 3x^2 - 2x^3$.

(b) $L = 4$. Hence, $P(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4$. Choose $m = 2$ and $n = 1$ (alternatively one can choose $m = 1$ and $n = 2$ for better stopband performance). Then,

$P(1) = 1\ ?\quad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1.$ Also,

$$\left.\frac{dP(x)}{dx}\right|_{x=0} = 0\ ?\quad \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + 4\alpha_4 x^3\Big|_{x=0} = 0,$$

$$\left.\frac{d^2P(x)}{dx^2}\right|_{x=0} = 0\ ?\quad 2\alpha_2 + 6\alpha_3 x + 12\alpha_4 x^2\Big|_{x=0} = 0,$$

$$\left.\frac{dP(x)}{dx}\right|_{x=1} = 0\ ?\quad \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 = 0.$$

Solving the above four simultaneous equations we get $\alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 4,\ \alpha_4 = -3.$ Therefore, $P(x) = 4x^3 - 3x^4.$

(c) $L = 5.$ Hence, $P(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5.$ Choose $m = 2$ and $n = 2.$ Following a procedure similar to that in parts (a) and (b) we get $\alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 10,\ \alpha_4 = -15,\ \alpha_5 = 6.$

Therefore $P(x) = 10x^3 - 15x^4 + 6x^5.$

10.30 (a) $H[k] = H(e^{j\omega_k}) = H(e^{j2\pi k/M}),\ 0 \le k \le M-1.$ Thus, $h[n] = \frac{1}{M}\sum_{k=0}^{M-1} H[k] W_M^{-kn},$

where $W_M = e^{-j2\pi k/M}.$

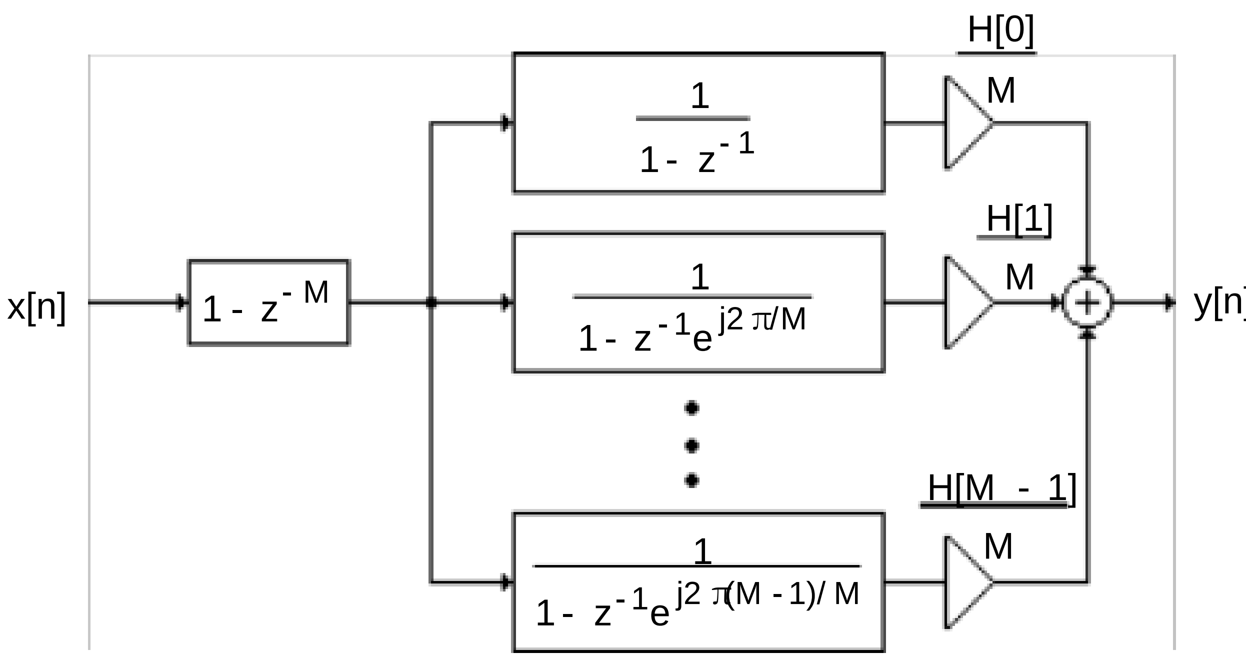
$$\text{Now, } H(z) = \sum_{n=0}^{M-1} h[n] z^{-n} = \frac{1}{M}\sum_{n=0}^{M-1}\sum_{k=0}^{M-1} H[k] W_M^{-kn} z^{-n} = \frac{1}{M}\sum_{k=0}^{M-1} H[k]\left\{\sum_{n=0}^{M-1} W_M^{-kn} z^{-n}\right\}.$$

$$\text{We can write } \sum_{n=0}^{M-1} W_M^{-kn} z^{-n} = \sum_{n=0}^{\infty} W_M^{-kn} z^{-n} - \sum_{n=M}^{\infty} W_M^{-kn} z^{-n}$$

$$= \sum_{n=0}^{\infty} W_M^{-kn} z^{-n} - W_M^{-kM} z^{-M}\sum_{n=0}^{\infty} W_M^{-kn} z^{-n} = \left(1 - z^{-M}\right)\sum_{n=0}^{\infty} W_M^{-kn} z^{-n} = \frac{1 - z^{-M}}{1 - W_M^{-k} z^{-1}}.$$

$$\text{Therefore, } H(z) = \frac{1 - z^{-M}}{M}\sum_{k=0}^{M-1}\frac{H[k]}{1 - W_M^{-k} z^{-1}}.$$

(b) A diagram of this realization is shown below:



(c) Note $H(z) = \frac{1 - z^{-M}}{M}\sum_{k=0}^{M-1}\frac{H[k]}{1 - W_M^{-k}z^{-1}} = \frac{1}{M}\sum_{k=0}^{M-1} H[k]\left\{\sum_{n=0}^{M-1} W_M^{-kn}z^{-n}\right\}$.

On the unit circle the above reduces to $H(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} H[k]\left\{\sum_{n=0}^{M-1} W_M^{-kn}e^{-j\omega n}\right\}$. For $\omega = j2\pi/M$, we then get from the above $H(e^{j2\pi/M}) = \sum_{k=0}^{M-1} H[k]\left\{\frac{1}{M}\sum_{n=0}^{M-1} W_M^{-kn}e^{-2\pi n/M}\right\}$?

$= \sum_{k=0}^{M-1} H[k]\left\{\frac{1}{M}\sum_{n=0}^{M-1} W_M^{-kn}W_M^{n}\right\} = \sum_{k=0}^{M-1} H[k]\left\{\frac{1}{M}\sum_{n=0}^{M-1} W_M^{-(k-\ )n}\right\}$. Using the identity of Eq. (5.23) of ?

text we observe that $\frac{1}{M}\sum_{n=0}^{M-1} W_M^{-(k-\ )n} = \begin{cases} 1, & \text{if } \ = k, \\ 0, & \text{otherwise.} \end{cases}$

? Hence, $H(e^{j2\pi/M}) \stackrel{?}{=} H[\ ]$.

10.31 The desired magnitude response is denoted as $|H_d(e^{j\omega})|$

? (a) For the Type 1 FIR filter, $H(e^{j\omega}) = e^{-j\frac{\omega(M-1)}{2}}|H(e^{j\omega})|$. Since in the frequency sampling approach we sample the DTFT $H(e^{j\omega})$ at M points given by $\omega = \frac{2\pi k}{M}$, $0 \le k \le M-1$, therefore $H[k] = H(e^{j2\pi k/M}) = |H_d(e^{j2\pi k/M})|e^{j2\pi k(M-1)/M}$, $0 \le k \le M-1$. Since the filter is of Type 1, $M-1$ is even, thus, $e^{j2\pi k(M-1)/2} = 1$. Moreover, $h[n]$ being real, $H(e^{j\omega}) = H^*(e^{j\omega})$. Thus, $H(e^{j\omega}) = e^{j\omega(M-1)/2}|H(e^{j\omega})|$, $\pi \le \omega < 2\pi$. Hence,

$$H[k] = \begin{cases} |H_d(e^{j2\pi k/M})|e^{-j2\pi k(M-1)/2M}, & k = 0,1,2,\cdots,\frac{M-1}{2}, \\ |H_d(e^{j2\pi k/M})|e^{-j2\pi(M-k)(M-1)/2M} & k = \frac{M+1}{2},\frac{M+3}{2},\cdots,M-1. \end{cases}$$

(b) For the Type 2 FIR filter

$$H[k] = \begin{cases} \left|H_d(e^{j2\pi k/M})\right| e^{-j2\pi k(M-1)/2M}, & k = 0,1,2,\cdots,\frac{M-1}{2}, \\ 0, & k = \frac{M}{2}, \\ \left|H_d(e^{j2\pi k/M})\right| e^{-j2\pi(M-k)(M-1)/2M}, & k = \frac{M}{2}+1,\cdots, M-1. \end{cases}$$

10.32 We design a linear phase FIR lowpass filter of length 21 using the frequency sampling approach with passband edge at $\omega_p = 0.6\pi$.
?

(a) From the length, we can determine that the frequency spacing between 2 consecutive DFT samples is given by $2\pi/21 = 0.2992$. The desired passband edge is between the frequency samples at $\omega = 2\pi(6/21)$ and $\omega = 2\pi(7/21)$, and again between $\omega = 2\pi(13/21)$ and $\omega = 2\pi(14/21)$ for the other side of the frequency axis. We then simply use the corresponding shifted impulse responses for those frequencies within the low pass range of this set: $G[k] = e^{-j(2\pi/N)((N-1)/2)k}$, and leave the rest of the coefficients as zero. Therefore, the 21-point DFT is given by:

$$H[k] = \begin{cases} e^{-j(2\pi/21)10k}, & k \in \{[0,...,6]\} \cup \{[14,...,20]\}, \\ 0, & k \in \{[7,...,13]\}. \end{cases}$$

A 21-point IDFT of the above DFT samples yields the impulse response coefficients given below in ascending powers of z^{-1}:

Columns 1 through 11

0.0414	0.0000	0.0443	0.0476	0.0000	0.0606
0.0732	0.0000	0.1399	0.2767	0.6667	

Columns 12 through 21

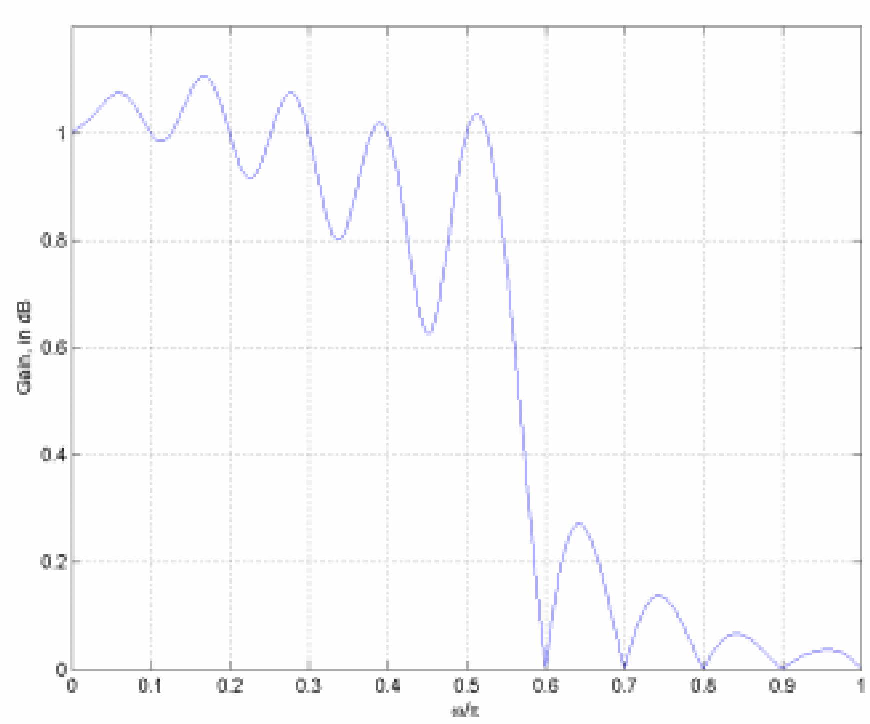
0.2767	0.1399	0.0000	0.0732	0.0606	0.0000
0.0476	0.0443	0.0000	0.0414		

(b) The program for finding and plotting the magnitude response is as follows:

```
% Problem #10.32
clear;
clc;

N = 21;
index = 1;
for   k = 0:6,
```

```
    H(index) = exp(-1i*((2*pi)/N)*((N-1)/2)*k);
    index = index + 1;
end

for    k = 7:13,
    H(index) = 0;
    index = index + 1;
end
for    k = 14:20,
    H(index) = exp(-1i*((2*pi)/N)*((N-1)/2)*k);
    index = index + 1;
end
h = ifft(H);
disp(abs(h));                        % Display coefficients
[FF, w] = freqz(h, 1, 512);
figure;
plot(w/pi, abs(FF)); axis([0 1 0 1.2]);grid;
ylabel(      'Gain, in dB'          ); xlabel(         '\omega/\pi'         );
```

The resulting plot is shown below:



10.33 We design a linear phase FIR lowpass filter of length 37 using the frequency sampling approach with passband edge at $\omega_p = 0.45\pi$.

(a) From the length, we can determine that the frequency spacing between 2 consecutive DFT samples is given by $2\pi/37 = 0.1698$. The desired passband edge is between the frequency samples at $\omega = 2\pi(8/21)$ and $\omega = 2\pi(9/21)$, and again between $\omega = 2\pi(28/21)$ and $\omega = 2\pi(29/21)$ for the other side of the frequency axis. We then simply use the corresponding shifted impulse responses for those frequencies within the low pass range of this set:

$G[k] = e^{-j(2\pi/N)((N-1)/2)k}$, and leave the rest of the coefficients as zero. Therefore, the 37-point DFT is given by:

$$H[k] = \begin{cases} e^{-j(2\pi/37)10k}, & k \in \{[0,...,8]\} \cup \{[29,...,36]\}, \\ 0, & k \in \{[9,...,28]\}. \end{cases}$$

A 21-point IDFT of the above DFT samples yields the impulse response coefficients given below in ascending powers of z^{-1}:

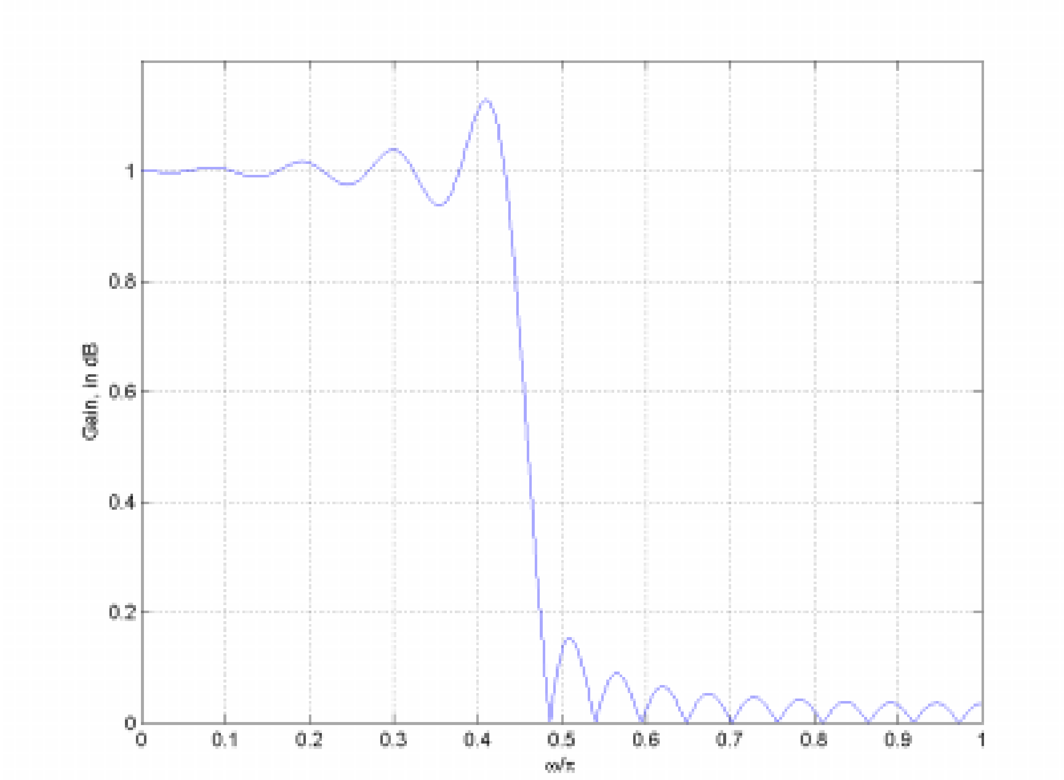
```
Columns 1 through 11

0.0203   0.0153   0.0247   0.0094   0.0285   0.0026
0.0317   0.0057   0.0344   0.0161   0.0366

Columns 12 through 22

0.0303   0.0383   0.0528   0.0396   0.0995   0.0403
0.3161   0.4595   0.3161   0.0403   0.0995

Columns 23 through 33

0.0396   0.0528   0.0383   0.0303   0.0366   0.0161
0.0344   0.0057   0.0317   0.0026   0.0285

Columns 34 through 37

0.0094   0.0247   0.0153   0.0203
```

(b) The program for finding and plotting the magnitude response is as follows:

```
% Problem #10.33
clear;
clc;

N = 37;
index = 1;
for   k = 0:8,
   H(index) = exp(-1i*((2*pi)/N)*((N-1)/2)*k);
   index = index + 1;
end

for   k = 9:28,
   H(index) = 0;
   index = index + 1;
end
for   k = 29:36,
```

```
    H(index) = exp(-1i*((2*pi)/N)*((N-1)/2)*k);
    index = index + 1;
end
length(H)
h = ifft(H);
disp(abs(h));                        % Display coefficients
[FF, w] = freqz(h, 1, 512);
figure;
plot(w/pi, abs(FF)); axis([0 1 0 1.2]);grid;
ylabel(     'Gain, in dB'          ); xlabel(         '\omega/\pi'          );
```

The resulting plot is shown below:



10.30 From Eq. (10.57) we have $H(\omega) = \sum_{k=1}^{M} c[k]\sin(\omega k)$. Now

$$H(\pi - \omega) = \sum_{k=1}^{M} c[k]\sin\big((\pi - \omega)k\big) = -\sum_{k=1}^{M} c[k]\sin(\omega k)\cos(\pi k) = \sum_{k=1}^{M} c[k](-1)^{k+1}(\omega k).$$

?

Thus, $H(\omega) = H(\pi - \omega)$ implies $\sum_{k=1}^{M} c[k]\sin(\omega k) = \sum_{k=1}^{M} c[k](-1)^{k+1}\sin(\omega k)$, or equivalently,

?

? $\sum_{k=1}^{M}\left(1 - (-1)^{k+1}\right)c[k]\sin(\omega k) = 0$, which in turn implies that $c[k] = 0$ for $k = 2,4,6,\cdots$.

But from Eq. (10.58) we have $c[k] = 2h[M-k], 1 \le k \le M$, or, $h[k] = \frac{1}{2}c[M-k]$. For k even,
?
i.e., $k = 2R$, $h[2R] = \frac{1}{2}c[M-2R] = 0$ if M is even.

10.35 By expressing $\cos(\omega n) = T_n(\cos\omega)$, where $T_n(x)$ is the n-th order Chebyshev polynomial in x, we first rewrite Eq. (10.73) in the form:

$$H(\omega) = \sum_{n=0}^{M} a[n]\cos(\omega n) = \sum_{n=0}^{M} \alpha_n \cos^n(\omega).$$

Therefore, we can rewrite Eq. (10.73) repeated below for convenience

$$P(\omega_i)\big[H(\omega_i) - D(\omega_i)\big] = (-1)^i \varepsilon, 1 \le i \le M+2,$$

? in a matrix form as

$$\begin{bmatrix} 1 & \cos(\omega_1) & \cos^M(\omega_1) & 1/P(\omega_1) \\ 1 & \cos(\omega_2) & \cos^M(\omega_2) & -1/P(\omega_2) \\ & & & \\ 1 & \cos(\omega_{M+1}) & \cos^M(\omega_{M+1}) & (-1)^M / P(\omega_{M+1}) \\ 1 & \cos(\omega_{M+2}) & \cos^M(\omega_{M+2}) & (-1)^{M+1}/P(\omega_{M+2}) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \\ \alpha_M \\ \varepsilon \end{bmatrix} = \begin{bmatrix} D(\omega_1) \\ D(\omega_2) \\ \\ D(\omega_{M+1}) \\ D(\omega_{M+2}) \end{bmatrix}.$$

Note that the coefficients $\{\alpha_i\}$ are different from the coefficients $\{a[i]\}$ of Eq. (10.73). To determine the expression of ε we use Cramer's rule arriving at $\varepsilon = \frac{\Delta_\varepsilon}{\Delta}$, where

?

$$\Delta = \det \begin{vmatrix} 1 & \cos(\omega_1) & \cos^M(\omega_1) & 1/P(\omega_1) \\ 1 & \cos(\omega_2) & \cos^M(\omega_2) & -1/P(\omega_2) \\ & & & \\ 1 & \cos(\omega_{M+1}) & \cos^M(\omega_{M+1}) & (-1)^M / P(\omega_{M+1}) \\ 1 & \cos(\omega_{M+2}) & \cos^M(\omega_{M+2}) & (-1)^{M+1}/P(\omega_{M+2}) \end{vmatrix}, \text{ and}$$

?

$$\Delta_\varepsilon = \begin{vmatrix} 1 & \cos(\omega_1) & \cos^M(\omega_1) & D(\omega_1) \\ 1 & \cos(\omega_2) & \cos^M(\omega_2) & D(\omega_2) \\ & & & \\ 1 & \cos(\omega_{M+1}) & \cos^M(\omega_{M+1}) & D(\omega_{M+1}) \\ 1 & \cos(\omega_{M+2}) & \cos^M(\omega_{M+2}) & D(\omega_{M+2}) \end{vmatrix}..$$

Expanding both determinants using the last column we get $\Delta_\varepsilon = \sum_{i=1}^{M+2} b_i D(\omega_{i+1})$ and

?

$\Delta = \sum_{i=1}^{M+2} b_i \frac{(-1)^{i-1}}{P(\omega_i)}$, where

$$b_i = \det \begin{vmatrix} 1 & \cos(\omega_1) & \cos^2(\omega_1) & \cos^M(\omega_1) \\ 1 & \cos(\omega_2) & \cos^2(\omega_2) & \cos^M(\omega_2) \\ & & & \\ 1 & \cos(\omega_{i-1}) & \cos^2(\omega_{i-1}) & \cos^M(\omega_{i-1}) \\ 1 & \cos(\omega_{i+1}) & \cos^2(\omega_{i+1}) & \cos^M(\omega_{i+1}) \\ & & & \\ 1 & \cos(\omega_{M+2}) & \cos^2(\omega_{M+2}) & \cos^M(\omega_{M+2}) \end{vmatrix}.$$

The above matrix is seen to be a Vandermonde matrix and is determinant is given by

?

$b_i = \prod_{\substack{k \neq i,\, k > i}} (\cos\omega_k - \cos\omega_i)$. Define $c_i = \dfrac{b_i}{\prod_{\substack{r=1 \\ r \neq i}}^{M+2} b_r}$. It can be shown by induction that

?

$$c_i = \prod_{\substack{n=1 \\ n \neq i}}^{M+2} \frac{1}{\cos\omega_i - \cos\omega_n}. \text{ Therefore, } \varepsilon = \frac{\sum_{i=1}^{M+2} b_i D(\omega_i)}{\sum_{i=1}^{M+2} b_i \dfrac{(-1)^i}{P(\omega_i)}}$$

$$= \frac{c_1 D(\omega_1) + c_2 D(\omega_2) + \cdots + c_{M+2} D(\omega_{M+2})}{\dfrac{c_1}{P(\omega_1)} - \dfrac{c_2}{P(\omega_2)} + \cdots + \dfrac{c_{M+2}(-1)^{M+1}}{P(\omega_{M+2})}}.$$

10.36 Using the Parks-McClellan method in Section 10.3.1:

?

$$W(\omega) = \begin{cases} 1 & \omega \in \text{passband}, \\ \dfrac{\delta_p}{\delta_s} & \omega \in \text{stopband}. \end{cases} \Rightarrow W(\omega) = \begin{cases} 1, & 0 \le \omega \le 0.4\pi, \\ 36/7, & 0.55\pi \le \omega \le \pi. \end{cases}$$

10.37 Using the Parks-McClellan method in Section 10.3.1:

$$W(\omega) = \begin{cases} 1 & \omega \in \text{passband}, \\ \dfrac{\delta_p}{\delta_s} & \omega \in \text{stopband}. \end{cases} \Rightarrow W(\omega) = \begin{cases} 4, & 0 \le \omega \le 0.65\pi, \\ 1, & 0.8\pi \le \omega \le \pi. \end{cases}$$

10.38 Using the Parks-McClellan method in Section 10.3.1:

$$W(\omega) = \begin{cases} \dfrac{\delta_p}{\delta_{s1}} & 0 \le \omega \le \omega_{s1}, \\ 1 & \omega_{p1} \le \omega \le \omega_{p2}, \\ \dfrac{\delta_p}{\delta_{s2}} & \omega_{s2} \le \omega \le \pi, \end{cases} \Rightarrow W(\omega) = \begin{cases} 1/6, & 0 \le \omega \le 0.4\pi, \\ 1, & 0.5\pi \le \omega \le 0.65\pi, \\ 1/4, & 0.8\pi \le \omega \le \pi. \end{cases}$$

10.39 From Eq. (10.55) we have $H(\omega) = \sum_{k=0}^{(N-1)/2} \tilde{b}[k]\cos(\omega k)\cos\left(\frac{\omega}{2}\right)$. Expanding the right-hand side of this equation using the trigonometric identity

$\cos A \cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$, we arrive at

$$H(\omega) = \frac{1}{2}\sum_{k=0}^{(N-1)/2} \tilde{b}[k]\cos\left(\omega\left(k+\frac{1}{2}\right)\right) + \frac{1}{2}\sum_{k=0}^{(N-1)/2} \tilde{b}[k]\cos\left(\omega\left(k-\frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\sum_{k=1}^{(N+1)/2} \tilde{b}[k-1]\cos\left(\omega\left(k-\frac{1}{2}\right)\right) + \frac{1}{2}\sum_{k=0}^{(N-1)/2} \tilde{b}[k]\cos\left(\omega\left(k-\frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\tilde{b}[0]\cos\left(\frac{\omega}{2}\right) + \frac{1}{2}\sum_{k=1}^{(N-1)/2}\left(\tilde{b}[k]+\tilde{b}[k-1]\right)\cos\left(\omega\left(k-\frac{1}{2}\right)\right) + \frac{1}{2}\tilde{b}\left[\frac{N-1}{2}\right]\cos\left(\omega\left(\frac{N}{2}\right)\right).$$

Comparing the above expression with Eq. (10.53) we observe $b[1] = \frac{1}{2}\tilde{b}[1] + \tilde{b}[0]$,

$b[k] = \frac{1}{2}\left(\tilde{b}[k]+\tilde{b}[k-1]\right)$, $2 \le k \le \frac{N-1}{2}$, $b\left[\frac{N+1}{2}\right] = \frac{1}{2}\tilde{b}\left[\frac{N-1}{2}\right]$, which are seen to be the same as given in Eq. (10.56).

10.40 From Eq. (10.57) we have $H(\omega) = \sum_{k=0}^{(N/2)-1} \tilde{c}[k]\cos(\omega k)\sin(\omega)$. Expanding the right-hand side of this equation using the trigonometric identity $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$,

we arrive at $H(\omega) = \frac{1}{2}\sum_{k=0}^{(N/2)-1} \tilde{c}[k]\sin\left(\omega(k+1)\right) - \frac{1}{2}\sum_{k=0}^{(N/2)-1} \tilde{c}[k]\sin\left(\omega(k-1)\right)$

$$= \frac{1}{2}\tilde{c}[0]\sin(\omega) - \frac{1}{2}\tilde{c}[0]\sin(-\omega) + \frac{1}{2}\tilde{c}[1]\sin(2\omega) + \frac{1}{2}\tilde{c}[2]\sin(3\omega) - \frac{1}{2}\tilde{c}[2]\sin(\omega)$$

$$+ \frac{1}{2}\tilde{c}[3]\sin(4\omega) - \frac{1}{2}\tilde{c}[3]\sin(2\omega) + \frac{1}{2}\tilde{c}[4]\sin(5\omega) - \frac{1}{2}\tilde{c}[4]\sin(3\omega) + \cdots$$

$$+ \frac{1}{2}\tilde{c}\left[\frac{N}{2}-2\right]\sin\left(\omega\left(\frac{N}{2}-1\right)\right) - \frac{1}{2}\tilde{c}\left[\frac{N}{2}-2\right]\sin\left(\omega\left(\frac{N}{2}-3\right)\right) + \frac{1}{2}\tilde{c}\left[\frac{N}{2}-1\right]\sin\left(\omega\left(\frac{N}{2}\right)\right) - \frac{1}{2}\tilde{c}\left[\frac{N}{2}-1\right]\sin\left(\omega\left(\frac{N}{2}-2\right)\right).$$

Comparing the above expression with Eq. (10.57) we observe $c[1] = \tilde{c}[0] - \frac{1}{2}\tilde{c}[2]$,

$c[k] = \frac{1}{2}\left(\tilde{c}[k-1] - \tilde{c}[k+1]\right)$, $2 \le k \le \frac{N}{2} - 2$, and $c\left[\frac{N}{2}-1\right] = \frac{1}{2}\tilde{c}\left[\frac{N}{2}-2\right]$, $c\left[\frac{N}{2}\right] = \frac{1}{2}\tilde{c}\left[\frac{N}{2}-1\right]$.

10.41 From Eq. (10.57) we have $H(\omega) = \sum_{k=0}^{(N-1)/2} \tilde{d}[k]\cos(\omega k)\sin\left(\frac{\omega}{2}\right)$ Expanding the right-hand side of this equation using the trigonometric identity $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$,

we arrive at $H(\omega) = \frac{1}{2}\sum_{k=0}^{(N-1)/2} \hat{d}[k]\sin\left(\omega\left(k+\frac{1}{2}\right)\right) - \frac{1}{2}\sum_{k=0}^{(N-1)/2} \hat{d}[k]\sin\left(\omega\left(k-\frac{1}{2}\right)\right)$

$$= \frac{1}{2}\sum_{k=1}^{(N+1)/2} \hat{d}[k]\sin\left(\omega\left(k-\frac{1}{2}\right)\right) - \frac{1}{2}\sum_{k=0}^{(N-1)/2} \hat{d}[k]\sin\left(\omega\left(k-\frac{1}{2}\right)\right)$$

$$= -\frac{1}{2}\hat{d}[0]\sin\left(-\frac{\omega}{2}\right) + \frac{1}{2}\sum_{k=1}^{(N-1)/2}\left(\hat{d}[k-1] - \hat{d}[k]\right)\sin\left(\omega\left(k-\frac{1}{2}\right)\right) + \frac{1}{2}\hat{d}\left[\frac{N+1}{2}-1\right]\sin\left(\omega\left(\frac{N+1}{2}-\frac{1}{2}\right)\right)$$

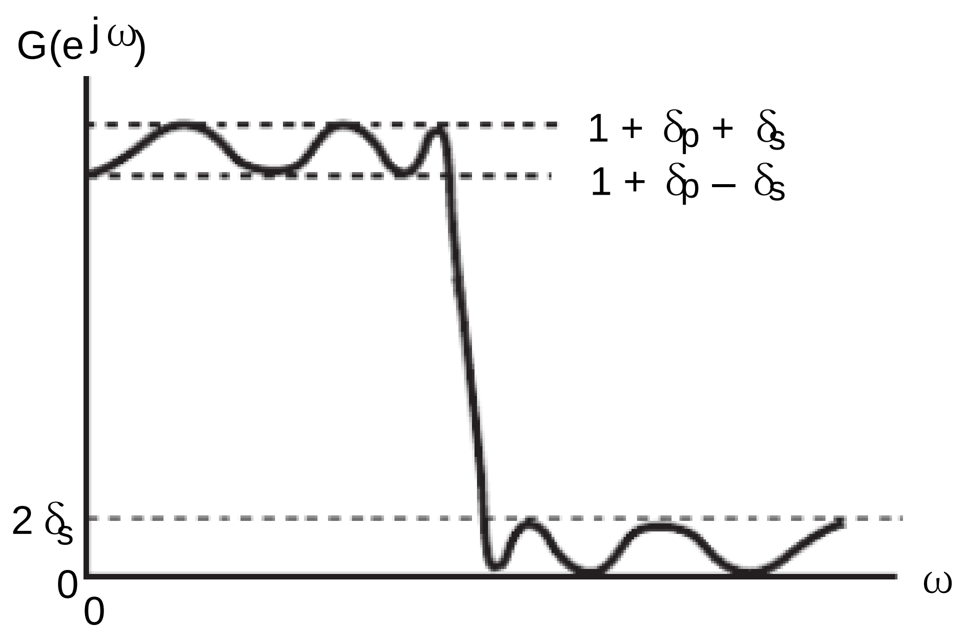
$$= \frac{1}{2}\hat{d}[0]\sin\left(\frac{\omega}{2}\right) + \frac{1}{2}\sum_{k=1}^{(N-1)/2}\left(\hat{d}[k-1] - \hat{d}[k]\right)\sin\left(\omega\left(k-\frac{1}{2}\right)\right) + \frac{1}{2}\hat{d}\left[\frac{N-1}{2}\right]\sin\left(\omega\left(\frac{N}{2}\right)\right).$$

Comparing the above expression with Eq. (10.61) we observe $d[1] = \hat{d}[0] - \frac{1}{2}\hat{d}[1]$,

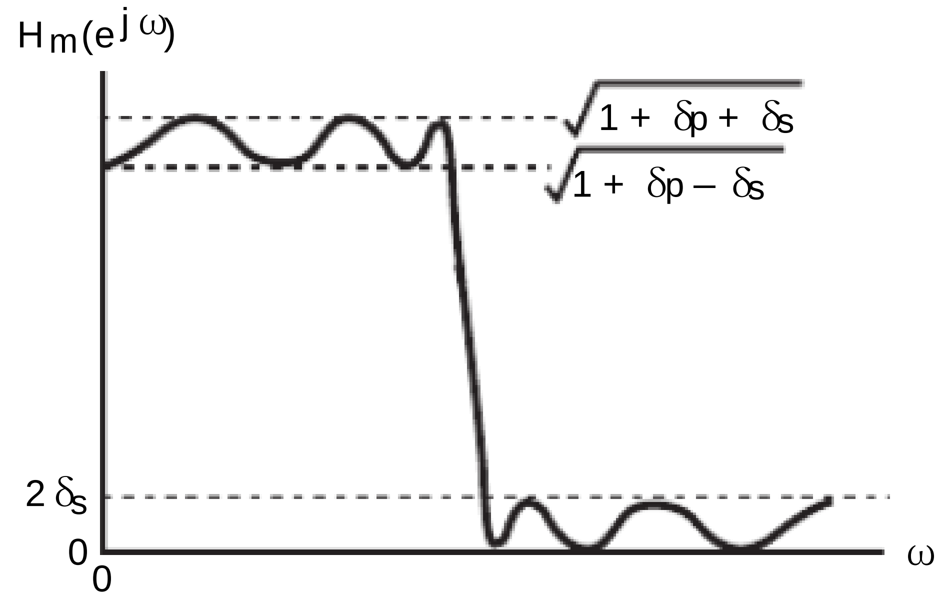
$d[k] = \frac{1}{2}\left(\hat{d}[k-1] - \hat{d}[k]\right)$ $2 \le k \le \frac{N-1}{2}$, and $d\left[\frac{N+1}{2}\right] = \hat{d}\left[\frac{N-1}{2}\right]$, which are the same as given in Eq. (10.64).

10.42 From Step 2, we have $G(e^{j\omega}) = G(\omega)e^{-jN\omega} = \delta_s^{(F)}e^{-jN\omega} + F(\omega)e^{-jN\omega}$.

The amplitude response $G(\omega)$ has been obtained by raising the amplitude response $F(\omega)$ by $\delta_s^{(F)}$ and hence, the filter $G(z)$ has double zeros in the stopband.



We may factorize as follows: $G(z) = z^{-N}H_m(z)H_m(z^{-1})$, where $H_m(z)$ is a real-coefficient minimum-phase FIR lowpass filter with half the degree of the original $H(z)$. Since, $G(\omega) \ge 0$, the amplitude response $H_m(\omega)$ of the minimum-phase filter $H_m(z)$ does not oscillate about unity in the passband. Since the original frequency response was raised by $\delta_s^{(F)}$, $H_m(\omega)$ must be normalized by a factor $\sqrt{1+\delta_s}$. Therefore,



For $H_m(\omega)$, we can see $\delta_s^{(F)} = \sqrt{\dfrac{2\delta_s}{1+\delta_s}}$ and $\delta_p^{(F)} = \sqrt{\dfrac{1+\delta_p+\delta_s}{1+\delta_s}} - 1 = \sqrt{1+\dfrac{\delta_p}{1+\delta_s}} - 1.$

10.43 Here $L = 4$. From Eq. (10.69), for a Type 1 FIR transfer function, order $N = 2L = 8$.
? Next, from Eq. (10.68), $a[k] = \hat{a}[k] = \{-5,\ \ 8,\ \ 6,\ \ 4,\ \ 2\}$. Next, using Eq. (10.52) we compute the impulse response coefficients $h[4] = a[0],\quad h[4-k] = \dfrac{a[k]}{2},\ 1 \le k \le 4$. Hence, $h[4] = -5,\ h[3] = 4,\ h[2] = 3,\ h[1] = 2,\ h[0] = 1$. Therefore,
$\{h[n]\} = \{-5,\ \ 4,\ \ 3,\ \ 2,\ \ 1,\ \ 2,\ \ 3,\ \ 4,\ \ -5\}$.

10.44 Here $L = 4$. From Eq. (10.69), for a Type 2 FIR transfer function, order $N = 2L+1 = 9$.
Next, from Eq. (10.68), $\hat{b}[k] = \hat{a}[k] = \{16,\ \ 20,\ \ 8,\ \ -8,\ \ 16\}$.
Now, using Eq. (10.56) we obtain $b[5] = \dfrac{1}{2}\hat{b}[4]$, and $b[k] = \dfrac{1}{2}\left(\hat{b}[k] + 2\hat{b}[k-1]\right),\ 1 \le k \le 4$.
Hence, $b[5] = 2,\ b[4] = \dfrac{1}{2}\left(\hat{b}[4] + 2\hat{b}[3]\right) = 4,\ b[3] = \dfrac{1}{2}\left(\hat{b}[3] + 2\hat{b}[2]\right) = 0,$
$b[2] = \dfrac{1}{2}\left(\hat{b}[2] + 2\hat{b}[1]\right) = 14,\ b[1] = \dfrac{1}{2}\left(\hat{b}[1] + 2\hat{b}[0]\right) = 14.$
Next, using Eq. (10.52) we obtain $h\left[\dfrac{N+1}{2} - k\right] = 2b[k],\ 1 \le k \le 5$. Hence,
$h[4] = \dfrac{1}{2}b[1] = 13,\ h[3] = \dfrac{1}{2}b[2] = 7,\ h[2] = \dfrac{1}{2}b[3] = 0,\ h[1] = \dfrac{1}{2}b[4] = 2,$
$h[0] = \dfrac{1}{2}b[5] = 1$. Therefore, $\{h[n]\} = \{13,\ \ 7,\ \ 0,\ \ 2,\ \ 1,\ \ 1,\ \ 2,\ \ 0,\ \ 7,\ \ 13\}$.

10.45 Here $L = 4$. From Eq. (10.69), for a Type 3 FIR transfer function, order $N = 2L = 8$. Next, from Eq. (10.68), $\hat{c}[k] = \hat{a}[k] = \{2,\ \ 0,\ \ 8,\ \ -4,\ \ 0\}$.
Now, using Eq. (10.60) we obtain
$c[1] = \hat{c}[0] - \dfrac{1}{2}\hat{c}[2] = -2,\ c[2] = \dfrac{1}{2}\left(\hat{c}[1] - \hat{c}[3]\right) = 2,\ c[3] = \dfrac{1}{2}\hat{c}[2] = 4,\ c[4] = \dfrac{1}{2}\hat{c}[3] = -4.$
Next, using Eq. (10.58) we arrive at
$h[3] = \dfrac{1}{2}c[1] = -1,\ h[2] = \dfrac{1}{2}c[2] = 1,\ h[1] = \dfrac{1}{2}c[3] = 2,\ h[0] = \dfrac{1}{2}c[4] = -2.$
Therefore,
$\{h[n]\} = \{-2,\ \ 2,\ \ 1,\ \ -1,\ \ 0,\ \ 1,\ \ -1,\ \ -2,\ \ 2\}$.

10.46 Here $L = 4$. From Eq. (10.62), for a Type 4 FIR transfer function, order $N = 2L + 1 = 9$.
Next, from Eq. (10.68), $\hat{d}[k] = \hat{a}[k] = \{6, \;\; 0, \;\; -4, \;\; -8, \;\; -12\}$.
Now, using Eq. (10.64) we obtain

$$d[1] = \hat{d}[0] - \tfrac{1}{2}\hat{d}[1] = 6,\; d[2] = \tfrac{1}{2}\left(\hat{d}[1] - \hat{d}[2]\right) = 2,\; d[3] = \tfrac{1}{2}\left(\hat{d}[2] - \hat{d}[3]\right) = 2,$$

$$d[4] = \tfrac{1}{2}\left(\hat{d}[3] - \hat{d}[4]\right) = 2,\; d[5] = \hat{d}[4] = -12.$$

Next, using Eq. (10.62) we arrive at

$$h[4] = \tfrac{1}{2}d[1] = 3,\; h[3] = \tfrac{1}{2}d[2] = 1,\; h[2] = \tfrac{1}{2}d[3] = 1,\; h[1] = \tfrac{1}{2}d[4] = 1,\; h[0] = \tfrac{1}{2}d[5] = -6.$$

Therefore,
$\{h[n]\} = \{-6, \;\; 1, \;\; 1, \;\; 1, \;\; 3, \;\; -3, \;\; -1, \;\; -1, \;\; -1, \;\; 6\}$.

10.47 It follows from Equation 10.25 that the impulse response of an ideal Hilbert Transformer is an anti-symmetric sequence. If we truncate it to a finite number of terms between $-M \le n \le M$ the impulse response is of length $(2M + 1)$ which is odd. Hence the FIR Hilbert Transformer obtained by truncation and satisfying Equation 10.86 cannot be satisfied by a Type 4 FIR filter.

10.48 (a) $X(z) = \sum_{n=0}^{N-1} x[n]z^{-n}$. Thus, $X(z) = X(z)\big|_{z^{-1} = \frac{-\alpha + z^{-1}}{1-\alpha z^{-1}}} = \sum_{n=0}^{N-1} x[n]\left(\frac{-\alpha + z^{-1}}{1-\alpha z^{-1}}\right) = \frac{P(z)}{D(z)}$,

where $P(z) = \sum_{n=0}^{N-1} p[n]\,z^{-1} = \sum x[n](1-\alpha z^{-1})^{N-1-n}(-\alpha + z^{-1})^{n}$, and

?

$$D(z) = \sum_{n=0}^{N-1} d[n]z^{-n} = (1-\alpha z^{-1})^{N-1}.$$

?

(b) $X[k] = X(z)\big|_{z=e^{j2\pi k/N}} = \frac{P(z)}{D(z)}\Big|_{z=e^{j2\pi k/N}} = \frac{P[k]}{D[k]}$, where $P[k] = P(z)\big|_{z=e^{j2\pi k/N}}$ is the

?

N-point DFT of the sequence $p[n]$ and $D[k] = D(z)\big|_{z=e^{j2\pi k/N}}$ is the N-point DFT of the sequence $d[n]$.

?

?

(c) Let $P = \left[p[0] \;\; p[1] \;\; \cdots \;\; p[N-1]\right]^T$ and $X = \left[x[0] \;\; x[1] \;\; \cdots \;\; x[N-1]\right]^T$. Without any loss of generality, assume $N = 4$ in which case $P(z) = \sum_{n=0}^{3} p[n]\,z^{-n}$

?

$$= \left(x[0] - \alpha x[1] + \alpha^2 x[2] - \alpha^3 x[3]\right)$$
$$+ \left(-3\alpha x[0] + (1+2\alpha^2)x[1] - \alpha(2+\alpha^2)x[2] + 3\alpha^2 x[3]\right)z^{-1}$$
$$+ \left(3\alpha^2 x[0] - \alpha(2+\alpha^2)x[1] + (1+2\alpha^2)x[2] - 3\alpha x[3]\right)z^{-2}$$

?

?

$+\left(-\alpha^3 x[0]+\alpha^2 x[1]-\alpha x[2]+\alpha x[3]\right)z^{-3}$. Equating like powers of z^{-1} we can write

$P = Q?X$, where $P = [p[0] \quad p[1] \quad p[2] \quad p[3]]^T$, $X = [x[0] \quad x[1] \quad x[2] \quad x[3]]^T$ and

?

$$Q = \begin{bmatrix} 1 & -\alpha & \alpha^2 & -\alpha^3 \\ -3\alpha & 1+2\alpha^2 & -\alpha(2+\alpha^2) & 3\alpha^2 \\ 3\alpha^2 & -\alpha(2+\alpha^2) & 1+2\alpha^2 & -3\alpha \\ -\alpha^3 & \alpha^2 & -\alpha^2 & 1 \end{bmatrix}.$$

It can be seen that the elements $q_{r,s}$, $0 \le r, s \le 3$, of the 4×4 matrix Q can be determined as follows:

(i) The first row is given by $q_{0,s} = (-\alpha)^s$,

(ii) The first column is given by $q_{r,0} = {}^3C_r(-\alpha)^r = \dfrac{3!}{r!(3-r)!}(-\alpha)^r$, and

(iii) the remaining elements can be obtained using the recurrence relation
$q_{r,s} = q_{r-1,s-1} - \alpha q_{r,s-1} + \alpha q_{r-1,s}$.
In the general case, we only change the computation of the elements of the first column using the relation $q_{r,0} = {}^{N-1}C_r(-\alpha)^r = \dfrac{(N-1)!}{r!(N-1-r)!}(-\alpha)^r$.

10.49 The specifications for the linear-phase IFIR filter are: $\omega_p = 0.02\pi$, $\omega_s = 0.09\pi$, $\delta_p = 0.02$, $\delta_s = 0.001$.

(a) Using Eq. (10.111) we get $L_{opt} = \dfrac{2\pi}{\omega_p+\omega_s+\sqrt{2\pi(\omega_s-\omega_p)}} = \dfrac{2\pi}{0.11\pi+\sqrt{2\pi(0.07\pi)}}$

$$= \frac{2}{0.11+\sqrt{0.07}} = 4.1308.$$

Therefore the optimal value 4, as that is the nearest smaller integer.

(b) The specifications of the shaping filter $F(z)$ and interpolator $I(z)$ are as follows:

$\omega_p^{(F)} = L_{opt}\omega_p = 0.08\pi$, $\omega_s^{(F)} = L_{opt}\omega_s = 0.36\pi$, $\delta_p^{(F)} = \dfrac{\delta_p}{2} = 0.01$, $\delta_s^{(F)} = \delta_s = 0.001$,

$\omega_p^{(I)} = 0.02\pi$, $\omega_s^{(I)} = \dfrac{2\pi}{L_{opt}} - \omega_s = 0.41\pi$, $\delta_p^{(I)} = \dfrac{\delta_p}{2} = 0.01$, $\delta_s^{(I)} = \delta_s = 0.001$.

From this we can determine the filter orders of each filter using the M-file firpmord :

```
Nf = firpmord([0.08 0.36],[1 0],[0.01 0.001])
Ni = firpmord([0.02 0.41],[1 0],[0.01 0.001])
```

which results in:

Order of F(z): 17 Order of I(z): 11

This implies that the total number of multipliers needed to implement $F(z)$ and hence, $F(z^4)$ is $R_F = \left\lceil \frac{17+1}{2} \right\rceil = 9$, and the total number of multipliers needed to implement $I(z)$ is $R_I = \left\lceil \frac{11+1}{2} \right\rceil = 6$. Therefore, the total number of multipliers needed to implement the IFIR filter is $R_{IFIR} = R_F + R_I = 9 + 6 = 15$.

Using firpmord again:

Ntotal = firpmord([0.02 0.09],[1 0],[0.02 0.001])

we get order of the firect realization is 65. Thus the total number of multipliers needed for a direct realization is 66.

(c) With the varied sparsity factors, we essentially carry out the same steps as in Part (b), using the equations set forth in the beginning portion of Part (b) to get:

For $L = L_{opt} - 1 = 3$:

Order of F(z) = 23 and order of I(z) = 6 $\Rightarrow R_{IFIR} = \left\lceil \frac{23+1}{2} \right\rceil + \left\lceil \frac{6+1}{2} \right\rceil = 12 + 4 = 16.$

For $L = L_{opt} + 1 = 5$:

Order of F(z) = 13 and order of I(z) = 16 $\Rightarrow R_{IFIR} = \left\lceil \frac{13+1}{2} \right\rceil + \left\lceil \frac{16+1}{2} \right\rceil = 7 + 9 = 16.$

For $L = L_{opt} - 2 = 2$:

Order of F(z) = 36 and order of I(z) = 1 $\Rightarrow R_{IFIR} = \left\lceil \frac{36+1}{2} \right\rceil + \left\lceil \frac{1+1}{2} \right\rceil = 19 + 2 = 21.$

For $L = L_{opt} + 2 = 6$:

Order of F(z) = 10 and order of I(z) = 22 $\Rightarrow R_{IFIR} = \left\lceil \frac{10+1}{2} \right\rceil + \left\lceil \frac{22+1}{2} \right\rceil = 6 + 12 = 18.$

This verifies that the choice of L_{opt} was indeed optimal, as there are no other choices of L which perform better (even if there may be some which perform equally well).