CHAPTER

10 Fourier Transform Properties

The time and frequency domains are alternative ways of representing signals. The Fourier transform is the mathematical relationship between these two representations. If a signal is modified in one domain, it will also be changed in the other domain, although usually not in the same way. For example, it was shown in the last chapter that *convolving* time domain signals results in their frequency spectra being *multiplied*. Other mathematical operations, such as addition, scaling and shifting, also have a matching operation in the opposite domain. These relationships are called *properties* of the Fourier Transform, how a mathematical change in one domain results in a mathematical change in the other domain.

Linearity of the Fourier Transform

The Fourier Transform is *linear*, that is, it possesses the properties of *homogeneity* and *additivity*. This is true for all four members of the Fourier transform family (Fourier transform, Fourier Series, DFT, and DTFT).

Figure 10-1 provides an example of how homogeneity is a property of the Fourier transform. Figure (a) shows an arbitrary time domain signal, with the corresponding frequency spectrum shown in (b). We will call these two signals: $x[\ ]$ and $X[\ ]$, respectively. *Homogeneity* means that a change in amplitude in one domain produces an identical change in amplitude in the other domain. This should make intuitive sense: when the amplitude of a time domain waveform is changed, the amplitude of the sine and cosine waves making up that waveform must also change by an equal amount.

In mathematical form, if $x[\ ]$ and $X[\ ]$ are a Fourier Transform pair, then $kx[\ ]$ and $kX[\ ]$ are also a Fourier Transform pair, for any constant k. If the frequency domain is represented in *rectangular* notation, $kX[\ ]$ means that both the real part and the imaginary part are multiplied by k. If the frequency domain is represented in *polar* notation, $kX[\ ]$ means that the magnitude is multiplied by k, while the phase remains unchanged.

章

10

傅里叶变换的性质

时域和频域是信号表示的两种不同方式。傅里叶变换正是连接这两种表示方式的数学关系。当信号在一个域中被修改时，它在另一个域中也会发生相应变化，尽管通常方式不同。例如，上一章已证明，时域信号的*卷积*会导致其频谱被*乘以*。其他数学运算如加法、缩放和移位，在对应域中也存在匹配操作。这些关系被称为傅里叶变换的*特性*，即一个域中的数学变化如何在另一个域中产生相应的数学变化。

傅里叶变换的线性

傅里叶变换是*线性的*，即它具有*同质性*和*可加性*的特性。傅里叶变换家族的四个成员（傅里叶变换、傅里叶级数、DFT和 DTFT）都是如此。

图10-1展示了傅里叶变换的同质性特征。图(a)呈现任意时域信号，对应的频谱如图(b)所示。我们将这两个信号分别命名为$x[\ ]$和$X[\ ]$。所谓*同质性*，是指时域信号的振幅变化在频域中会呈现完全相同的振幅变化。这种特性直观上很容易理解：当时域波形的振幅发生变化时，构成该波形的正弦和余弦波的振幅也必然发生等量变化。

在数学形式中，如果$x[\ ]$和$X[\ ]$是傅里叶变换对，那么对于任意常数k，$k\,x[\ ]$和$k\,X[\ ]$也是傅里叶变换对。如果频率域用*矩形*表示法表示，$k\,X[\ ]$表示实部和虚部都乘以k。如果频率域用*极坐标*表示法表示，$k\,X[\ ]$表示幅值乘以k，而相位保持不变。

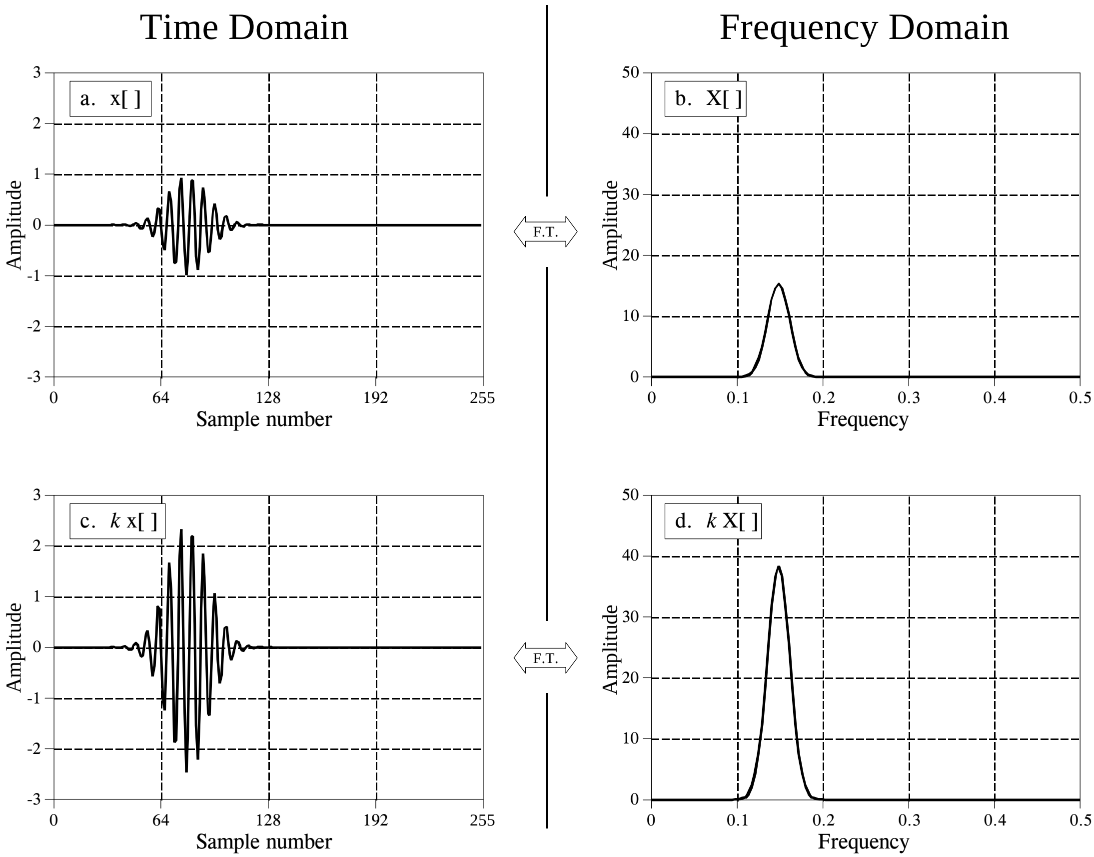


FIGURE 10-1
Homogeneity of the Fourier transform. If the amplitude is changed in one domain, it is changed by the same amount in the other domain. In other words, *scaling* in one domain corresponds to *scaling* in the other domain.

Additivity of the Fourier transform means that *addition* in one domain corresponds to *addition* in the other domain. An example of this is shown in Fig. 10-2. In this illustration, (a) and (b) are signals in the time domain called $x_1[\ ]$ and $x_2[\ ]$, respectively. Adding these signals produces a third time domain signal called $x_3[\ ]$, shown in (c). Each of these three signals has a frequency spectrum consisting of a real and an imaginary part, shown in (d) through (i). Since the two time domain signals *add* to produce the third time domain signal, the two corresponding spectra *add* to produce the third spectrum. Frequency spectra are added in rectangular notation by adding the real parts to the real parts and the imaginary parts to the imaginary parts. If: $x_1[n] + x_2[n] = x_3[n]$, then: $ReX_1[f] + ReX_2[f] = ReX_3[f]$ and $ImX_1[f] + ImX_2[f] = ImX_3[f]$. Think of this in terms of cosine and sine waves. All the cosine waves add (the real parts) and all the sine waves add (the imaginary parts) with no interaction between the two.

Frequency spectra in polar form cannot be directly added; they must be converted into rectangular notation, added, and then reconverted back to

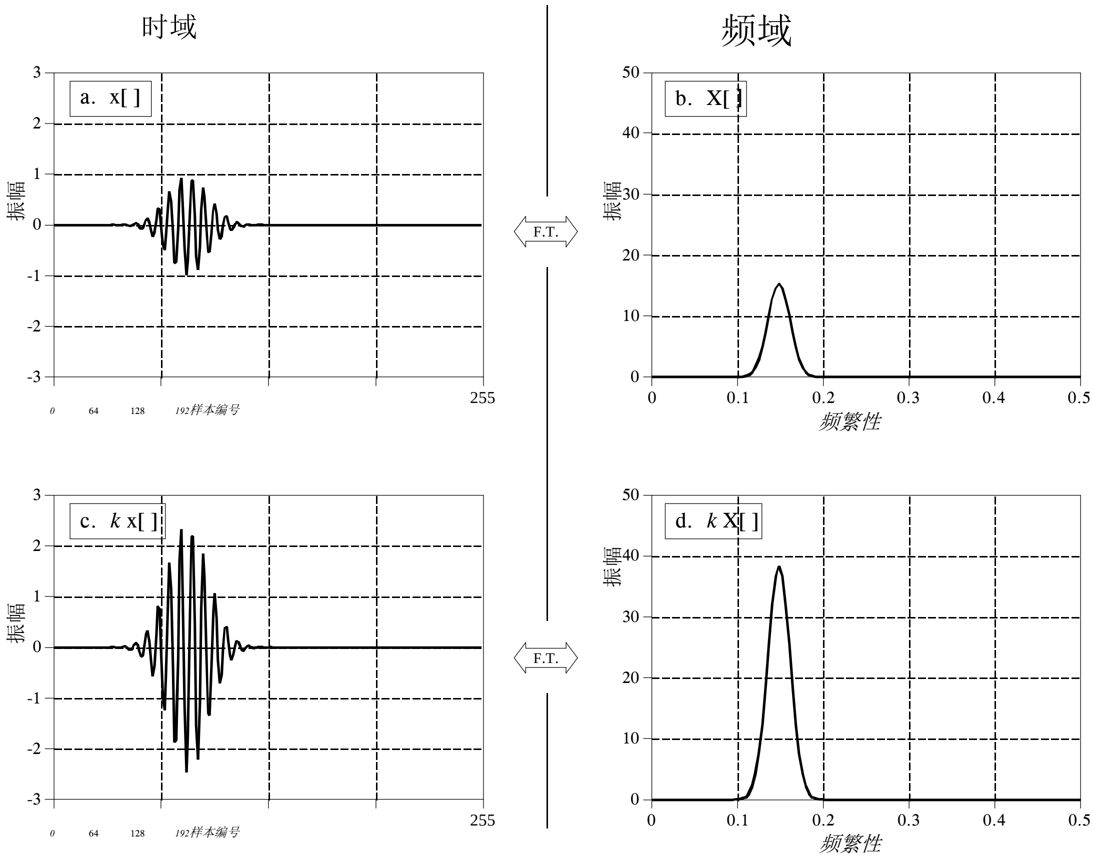


图10-1
傅里叶变换的均匀性。如果在一个域中改变振幅，那么在另一个域中也会以相同的量改变。换言之，一个域中的*缩放*对应于另一个域中的*缩放*。

*傅里叶变换的可加性*意味着一个域中的*加法*对应另一个域中的*加法*。图10-2展示了这一原理的示例。图中(a)和(b)分别是时域中的信号$x_1[\ ]$和$x_2[\ ]$。将这两个信号相加会产生第三个时域信号$x_3[\ ]$，如(c)所示。这三个信号的频谱均由实部和虚部组成，分别如(d)至(i)所示。由于两个时域信号*相加*得到第三个时域信号，对应的频谱也*相加*得到第三个频谱。在矩形表示法中，频谱的相加方式是实部相加实部，虚部相加虚部。如果：$x_1[n] + x_2[n] = x_3[n]$，那么：$Re X_1[f] + Re X_2[f] = Re X_3[f]$ 且 $Im X_1[f] + Im X_2[f] = Im X_3[f]$。用余弦波和正弦波来理解这个关系：所有余弦波相加（实部），所有正弦波相加（虚部），两者之间没有相互作用。

极坐标形式的频谱不能直接相加；必须先转换为直角坐标形式，相加后再转换回极坐标形式。

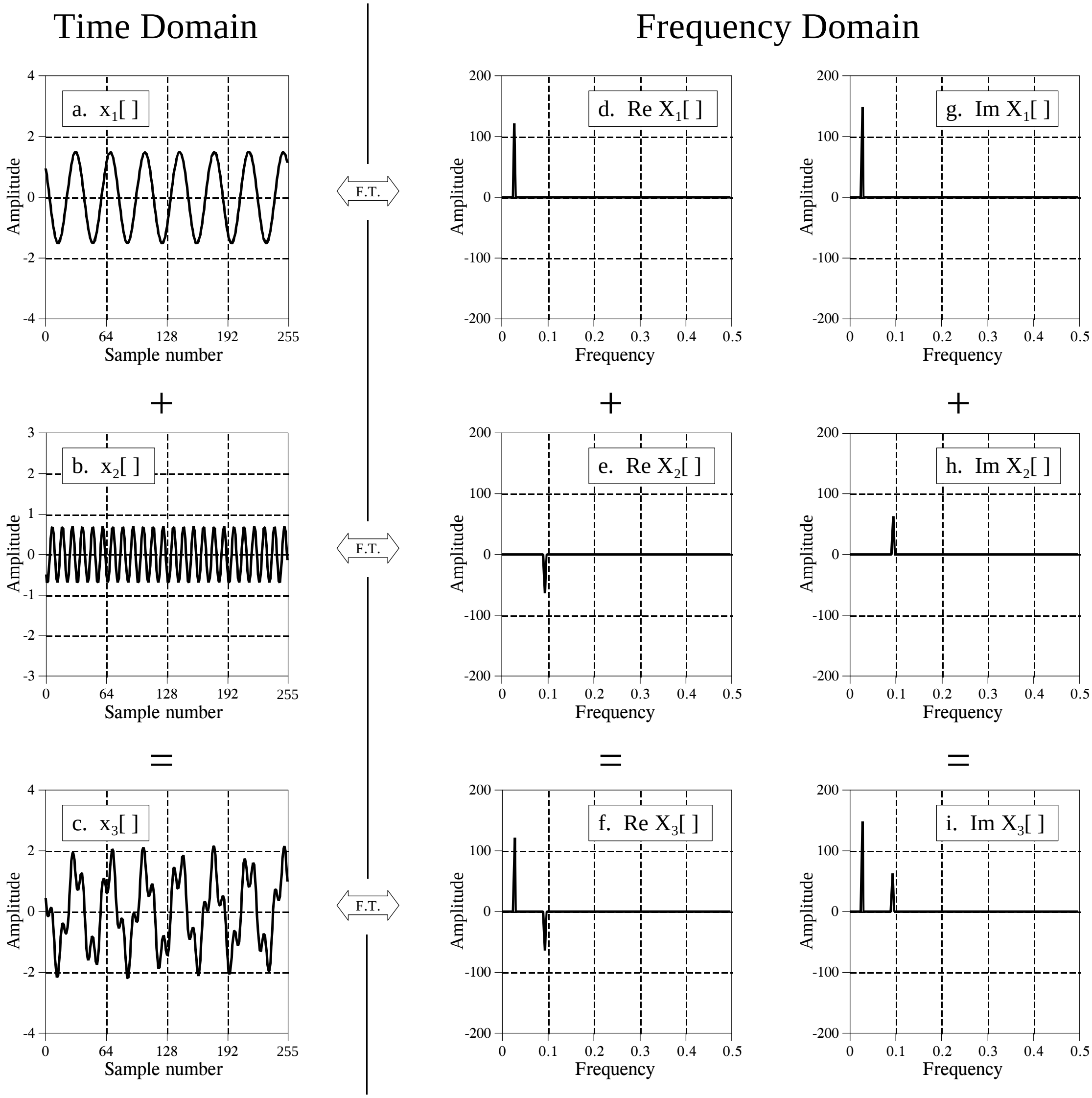


FIGURE 10-2
Additivity of the Fourier transform. Adding two or more signals in one domain results in the corresponding signals being added in the other domain. In this illustration, the time domain signals in (a) and (b) are added to produce the signal in (c). This results in the corresponding real and imaginary parts of the frequency spectra being added.

polar form. This can also be understood in terms of how sinusoids behave. Imagine adding two sinusoids having the same frequency, but with different amplitudes (A_1 and A_2) and phases (ϕ_1 and ϕ_2). If the two phases happen to be same ($\phi_1 = \phi_2$), the amplitudes will add ($A_1 + A_2$) when the sinusoids are added. However, if the two phases happen to be exactly opposite ($\phi_1 = -\phi_2$), the amplitudes will *subtract* ($A_1 - A_2$) when the sinusoids are added. The point is, when sinusoids (or spectra) are in polar form, they *cannot* be added by simply adding the magnitudes and phases.

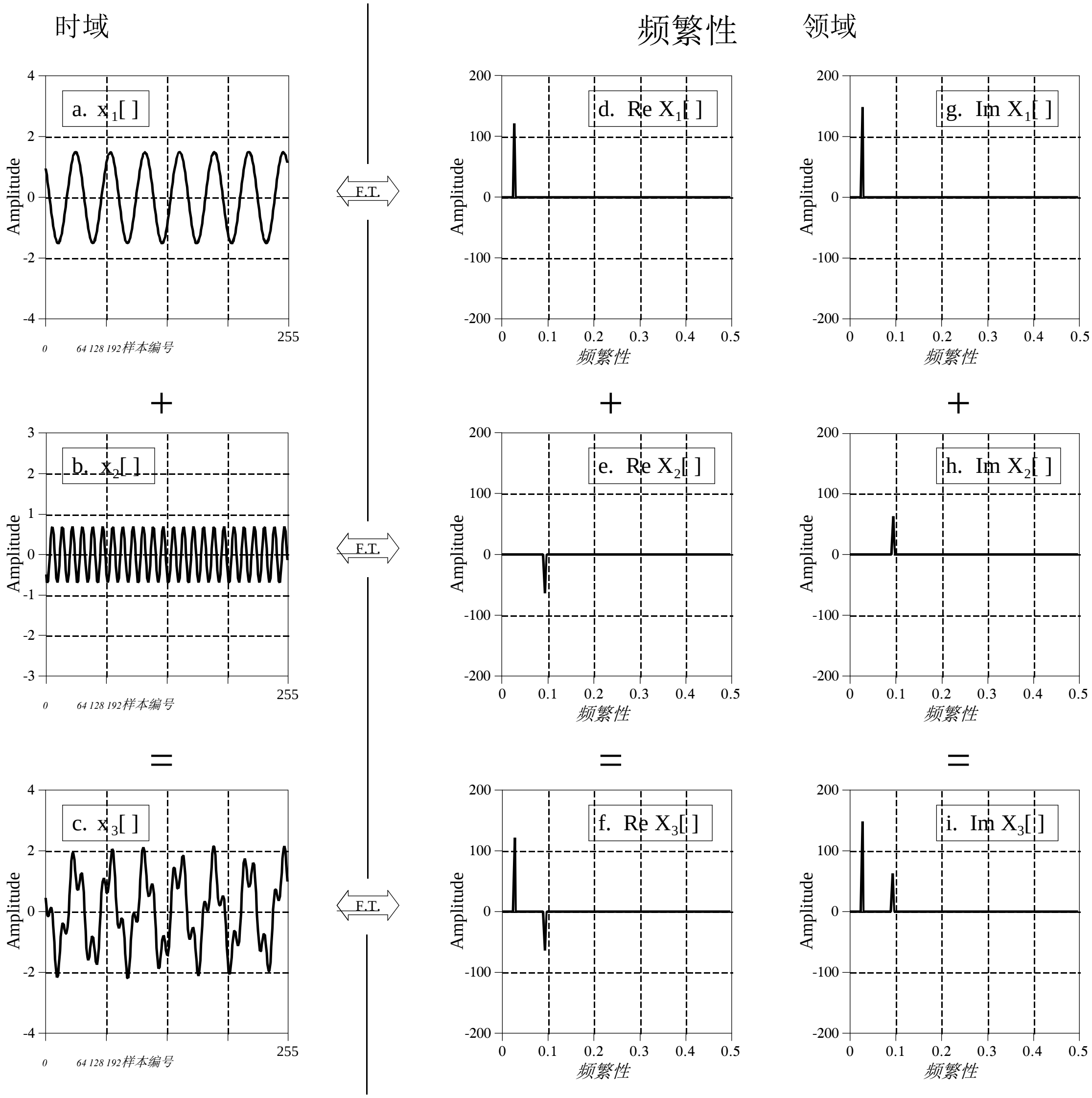


图10-2
傅里叶变换的可加性。在一个域中叠加两个或多个信号时，对应信号在另一域中也会叠加。如图所示，将(a)和(b)中的时域信号相加可得到(c)中的信号，这导致频谱的实部和虚部相应叠加。

极坐标形式。这也可以从正弦波的行为方式来理解。想象一下，将两个具有相同频率但振幅（A_1和A_2）和相位（φ_1和φ_2）不同的正弦波相加。如果两个相位恰好相同（$\varphi_1=\varphi_2$），则正弦波相加时振幅会相加（A_1+A_2）。然而，如果两个相位恰好完全相反（$\varphi_1=\varphi_2$），则正弦波相加时振幅会*相减*（A_1-A_2）。关键在于，当正弦波（或频谱）处于极坐标形式时，它们*无法*通过简单地相加幅值和相位来实现叠加。

In spite of being linear, the Fourier transform is *not* shift invariant. In other words, a shift in the time domain *does not* correspond to a shift in the frequency domain. This is the topic of the next section.

Characteristics of the Phase

In mathematical form: if $x[n] \leftrightarrow Mag\,X[f]$ & $Phase\,X[f]$, then a shift in the time domain results in: $x[n+s] \leftrightarrow Mag\,X[f]$ & $Phase\,X[f] + 2\pi s f$, (where f is expressed as a fraction of the sampling rate, running between 0 and 0.5). In words, a shift of s samples in the time domain leaves the magnitude unchanged, but adds a linear term to the phase, $2\pi s f$. Let's look at an example of how this works.

Figure 10-3 shows how the phase is affected when the time domain waveform is shifted to the left or right. The magnitude has not been included in this illustration because it isn't interesting; it is not changed by the time domain shift. In Figs. (a) through (d), the waveform is gradually shifted from having the peak centered on sample 128, to having it centered on sample 0. This sequence of graphs takes into account that the DFT views the time domain as *circular*; when portions of the waveform exit to the right, they reappear on the left.

The time domain waveform in Fig. 10-3 is symmetrical around a vertical axis, that is, the left and right sides are mirror images of each other. As mentioned in Chapter 7, signals with this type of symmetry are called *linear phase*, because the phase of their frequency spectrum is a *straight line*. Likewise, signals that don't have this left-right symmetry are called *nonlinear phase*, and have phases that are something other than a straight line. Figures (e) through (h) show the phase of the signals in (a) through (d). As described in Chapter 7, these phase signals are *unwrapped*, allowing them to appear without the discontinuities associated with keeping the value between π and $-\pi$.

When the time domain waveform is shifted to the right, the phase remains a straight line, but experiences a *decrease* in slope. When the time domain is shifted to the left, there is an *increase* in the slope. This is the main property you need to remember from this section; a shift in the time domain corresponds to changing the slope of the phase.

Figures (b) and (f) display a unique case where the phase is entirely zero. This occurs when the time domain signal is *symmetrical* around sample *zero*. At first glance, this symmetry may not be obvious in (b); it may appear that the signal is symmetrical around sample 256 (i.e., $N/2$) instead. Remember that the DFT views the time domain as circular, with sample zero inherently connected to sample N-1. Any signal that is symmetrical around sample zero will also be symmetrical around sample $N/2$, and vice versa. When using members of the Fourier Transform family that do not view the time domain as periodic (such as the DTFT), the symmetry must be around sample zero to produces a zero phase.

傅里叶变换是线性的，但*不是*平移不变的。换言之，时域的平移*并不*对应于频域的平移。这是下一节的主题。

相的特性

在数学形式中：如果$x[n]$.*马格* $X[f]$ &*相位* $X[f]$，那么在
时域结果为：$x[n+s]$.*马格* $X[f]$ &*相位* $X[f] +2\pi sf$（其中f表示采样率的分数，取值范围在0到0.5之间）。换句话说，时域中s个样本的偏移会使幅值保持不变，但会在相位上增加一个线性项$2\pi sf$。让我们来看一个示例来说明其工作原理。

图10-3展示了时域波形向左或向右移动时相位的变化情况。图中未包含幅值信息，因为这并不重要——时域移动不会改变幅值。从图(a)到图(d)可以看出，波形逐渐从以第128个采样点为中心的峰值，转变为以第0个采样点为中心的峰值。这一系列图表的绘制基于DFT（离散傅里叶变换）将时域视为*循环*的特性：当波形部分向右延伸时，它们会在左侧重新出现。

图10-3中的时域波形围绕垂直轴对称，即左右两侧互为镜像。如第七章所述，具有这种对称性的信号被称为*线性相位*，因为其频谱相位呈*直线*形态。反之，不具备左右对称性的信号则称为*非线性相位*，其相位形态并非直线。图(e)至(h)展示了(a)至(d)中信号的相位特征。正如第七章所述，这些相位信号经过*解缠绕处理*后，可呈现无 π 与- π 间值不连续的形态。

当时域波形向右移动时，相位保持为一条直线，但斜率*减小*。当时域向左移动时，斜率*增大*。这是本节需要记住的主要属性：时域的移动对应于相位斜率的变化。

图(b)和(f)展示了一个相位完全为零的独特情况。当时域信号在采样点*零点*周围呈现*对称性*时，就会出现这种情况。乍看之下，这种对称性在(b)中可能并不明显；信号似乎是对称于采样点256（即$N/2$）的。需要记住的是，DFT将时域视为环形结构，采样点零点与采样点N-1本质上是相连的。任何对称于采样点零点的信号，同样会对称于采样点$N/2$，反之亦然。当使用不将时域视为周期性的傅里叶变换族成员（如 DTFT）时，必须对称于采样点零点才能产生零相位。

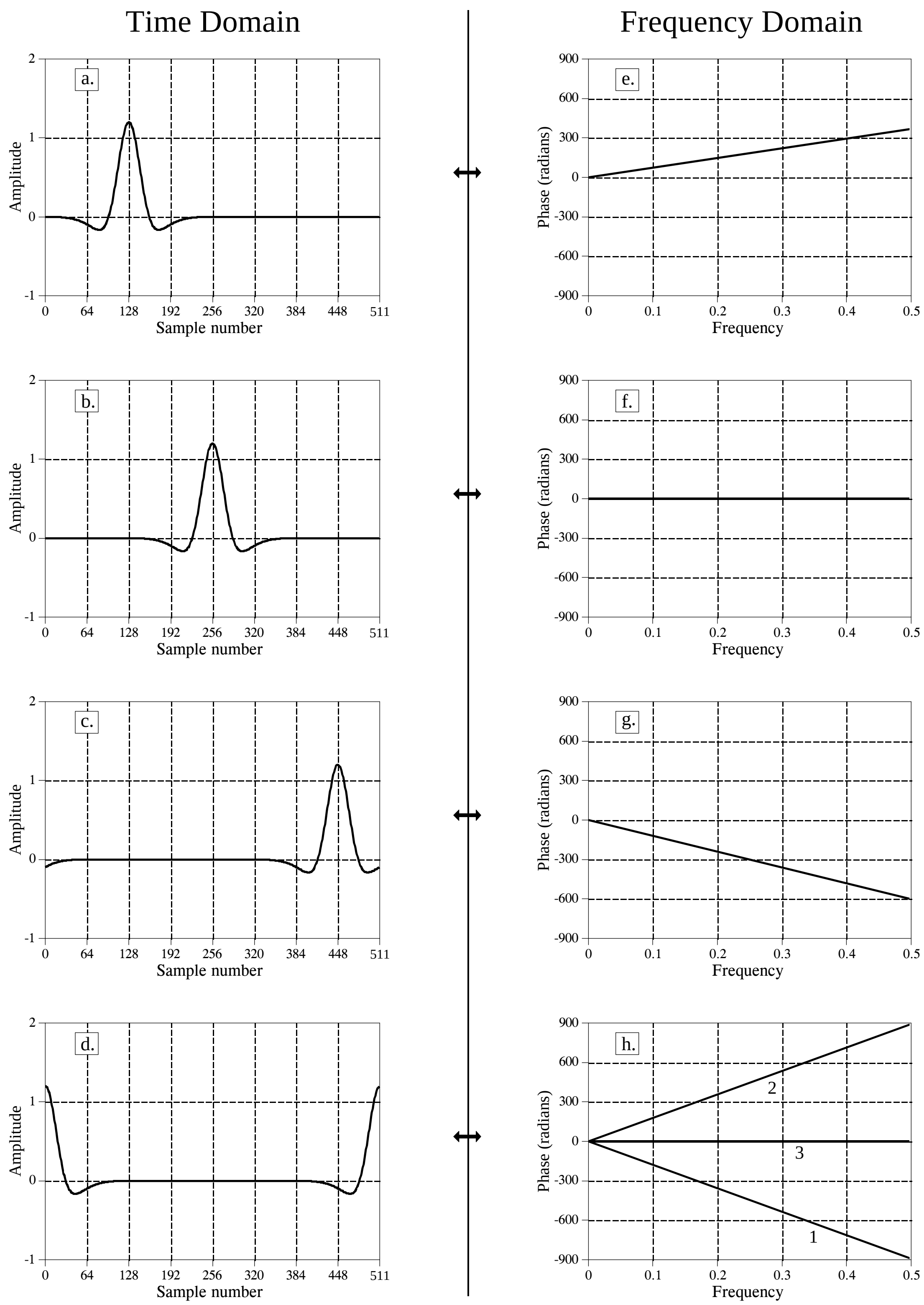


FIGURE 10-3
Phase changes resulting from a time domain shift.

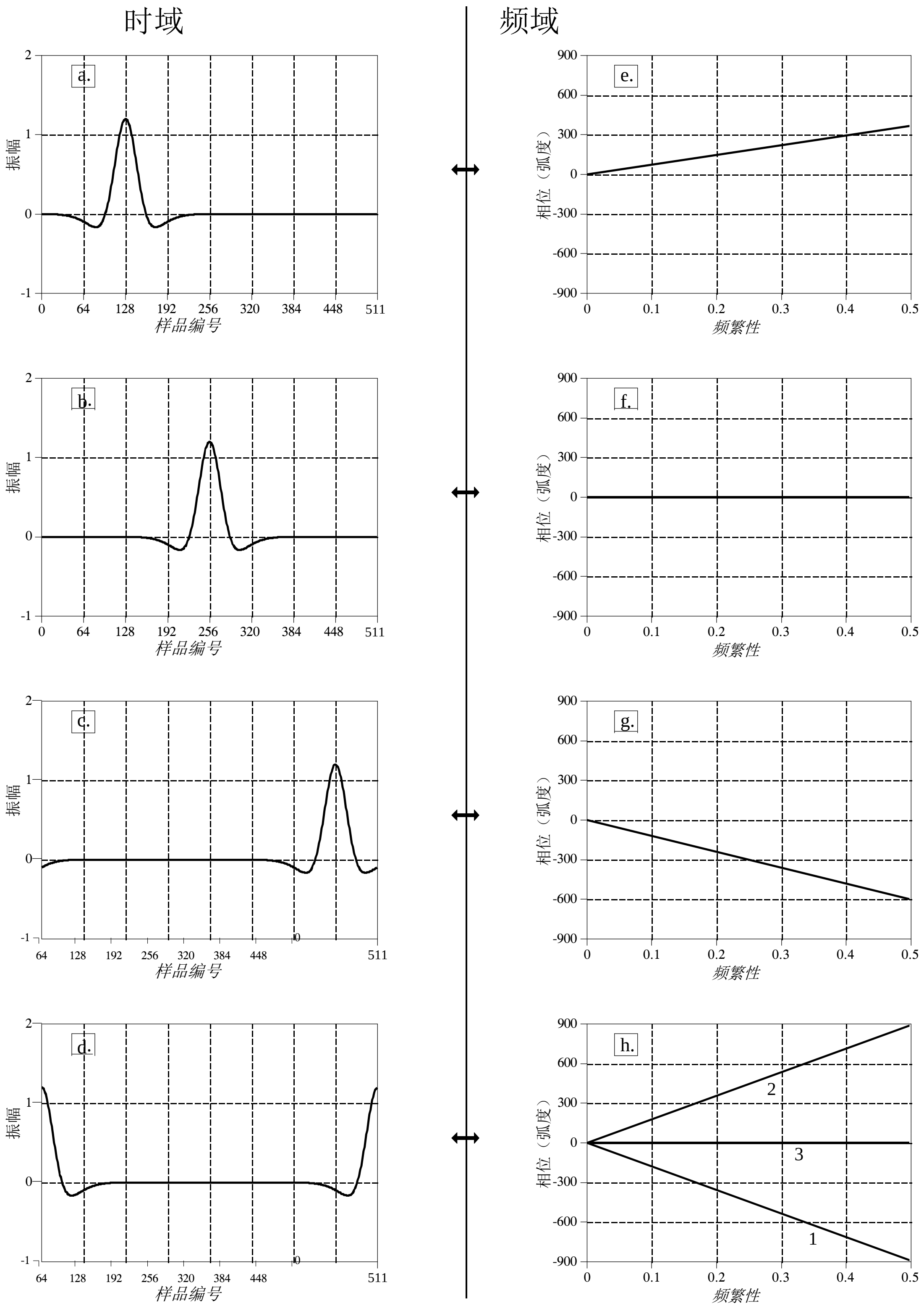


图10-3
由时域偏移引起的相位变化。

Figures (d) and (h) shows something of a riddle. First imagine that (d) was formed by shifting the waveform in (c) slightly more to the right. This means that the phase in (h) would have a slightly more negative slope than in (g). This phase is shown as line 1. Next, imagine that (d) was formed by starting with (a) and shifting it to the left. In this case, the phase should have a slightly more positive slope than (e), as is illustrated by line 2. Lastly, notice that (d) is symmetrical around sample $N/2$, and should therefore have a zero phase, as illustrated by line 3. Which of these three phases is correct? They all are, depending on how the π and 2π phase ambiguities (discussed in Chapter 8) are arranged. For instance, every sample in line 2 differs from the corresponding sample in line 1 by an integer multiple of 2π, making them equal. To relate line 3 to lines 1 and 2, the π ambiguities must also be taken into account.

To understand why the phase behaves as it does, imagine shifting a waveform by *one* sample to the right. This means that all of the sinusoids that compose the waveform must also be shifted by *one* sample to the right. Figure 10-4 shows two sinusoids that might be a part of the waveform. In (a), the sine wave has a very low frequency, and a one sample shift is only a small fraction of a full cycle. In (b), the sinusoid has a frequency of one-half of the sampling rate, the highest frequency that can exist in sampled data. A one sample shift at this frequency is equal to an entire 1/2 cycle, or π radians. That is, when a shift is expressed in terms of a phase change, it becomes *proportional* to the frequency of the sinusoid being shifted.

For example, consider a waveform that is symmetrical around sample zero, and therefore has a zero phase. Figure 10-5a shows how the phase of this signal changes when it is shifted left or right. At the highest frequency, one-half of the sampling rate, the phase increases by π for each one sample shift to the left, and decreases by π for each one sample shift to the right. At zero frequency there is no phase shift, and all of the frequencies between follow in a straight line.

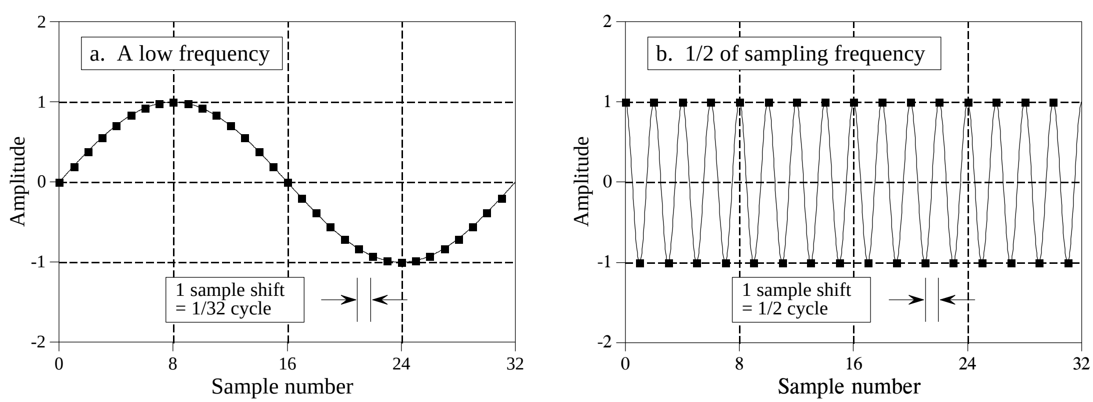


FIGURE 10-4
The relationship between samples and phase. Figures (a) and (b) show low and high frequency sinusoids, respectively. In (a), a one sample shift is equal to 1/32 of a cycle. In (b), a one sample shift is equal to 1/2 of a cycle. This is why a shift in the waveform changes the phase more at high frequencies than at low frequencies.

图(d)和(h)展示了一个谜题。首先设想(d)是通过将(c)中的波形略微向右偏移形成的。这意味着(h)中的相位斜率会比(g)中略微更负，如图1所示。接着设想(d)是从(a)开始向左偏移形成的。此时相位斜率应比(e)中略微更正，如图2所示。最后注意到(d)在采样点$N/2$处对称，因此应具有零相位，如图3所示。这三个相位中哪一个正确？根据π和2 π相位模糊度（第8章讨论）的排列方式，它们都正确。例如，图2中每个采样点与图1中对应采样点相差2 π的整数倍，因此相等。要将图3与图1、图2关联起来，还必须考虑π模糊度的影响。

要理解相位为何呈现特定行为，可以这样想象：将波形向右移动一个采样点。这意味着构成该波形的所有正弦波也必须向右移动一个采样点。图10-4展示了波形中可能包含的两个正弦波。在(a)中，正弦波频率极低，单个采样点的移动仅相当于半个周期。而在(b)中，正弦波频率达到采样率的一半——这是采样数据中可存在的最高频率。此时单个采样点的移动相当于完整的1/2周期，即π弧度。也就是说，当用相位变化来表示移动时，其数值与被移动正弦波的频率*成正比*。

例如，考虑一个关于采样零点对称的波形，因此具有零相位。图10-5a显示了当该信号向左或向右移动时，其相位的变化情况。在最高频率（即采样率的一半）处，相位每向左移动一个采样点增加π，每向右移动一个采样点减少π。在零频率处没有相位偏移，所有中间频率都沿直线分布。

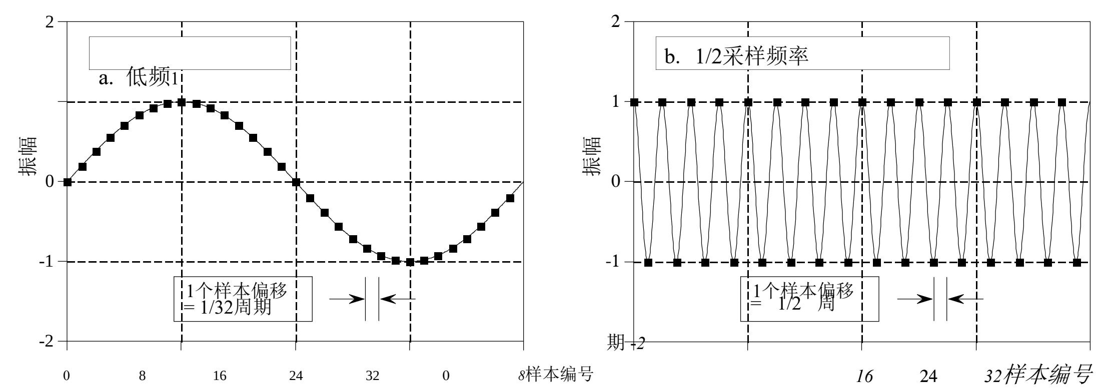


图10-4
样本与相位的关系。图(a)和(b)分别显示低频和高频正弦波。在(a)中，一个样本偏移相当于1/32个周期。在(b)中，一个样本偏移相当于1/2个周期。这就是为什么波形偏移在高频时比低频时对相位的影响更大。

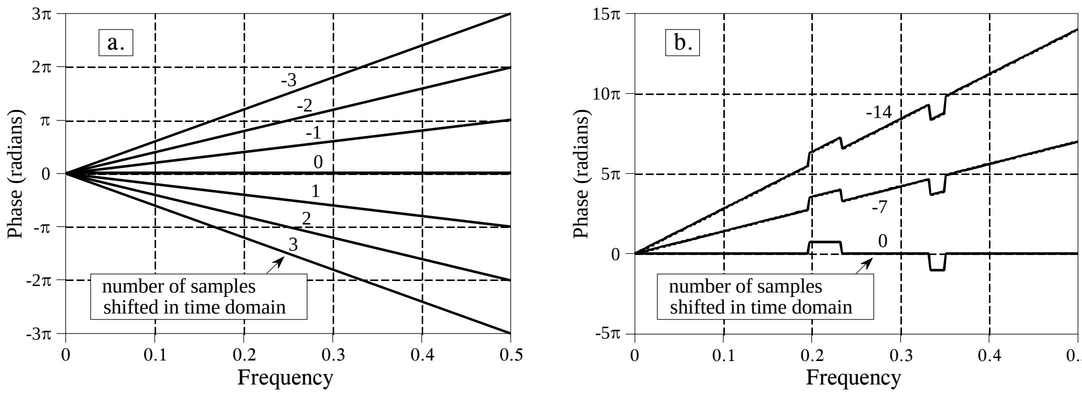


FIGURE 10-5
Phases resulting from time domain shifting. For each sample that a time domain signal is shifted in the positive direction (i.e., to the right), the phase at frequency 0.5 will decrease by π radians. For each sample shifted in the negative direction (i.e., to the left), the phase at frequency 0.5 will increase by π radians. Figure (a) shows this for a linear phase (a straight line), while (b) is an example using a nonlinear phase.

All of the examples we have used so far are *linear* phase. Figure 10-5b shows that *nonlinear* phase signals react to shifting in the same way. In this example the nonlinear phase is a straight line with two rectangular pulses. When the time domain is shifted, these nonlinear features are simply superimposed on the changing slope.

What happens in the *real* and *imaginary parts* when the time domain waveform is shifted? Recall that frequency domain signals in rectangular notation are nearly impossible for humans to understand. The real and imaginary parts typically look like random oscillations with no apparent pattern. When the time domain signal is shifted, the wiggly patterns of the real and imaginary parts become even more oscillatory and difficult to interpret. Don't waste your time trying to understand these signals, or how they are changed by time domain shifting.

Figure 10-6 is an interesting demonstration of what information is contained in the *phase*, and what information is contained in the *magnitude*. The waveform in (a) has two very distinct features: a rising edge at sample number 55, and a falling edge at sample number 110. Edges are very important when information is encoded in the *shape* of a waveform. An edge indicates *when* something happens, dividing whatever is on the left from whatever is on the right. It is time domain encoded information in its purest form. To begin the demonstration, the DFT is taken of the signal in (a), and the frequency spectrum converted into polar notation. To find the signal in (b), the phase is replaced with random numbers between -π and π, and the inverse DFT used to reconstruct the time domain waveform. In other words, (b) is based only on the information contained in the *magnitude*. In a similar manner, (c) is found by replacing the magnitude with small random numbers before using the inverse DFT. This makes the reconstruction of (c) based solely on the information contained in the *phase*.

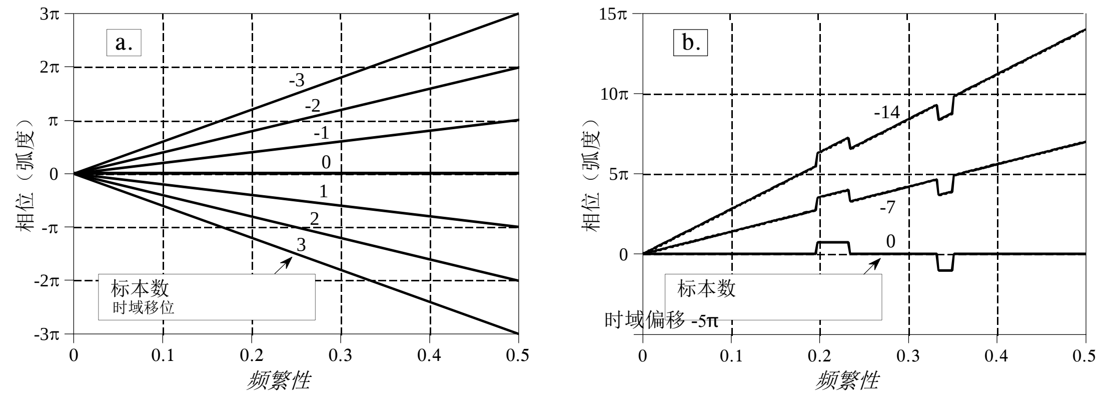
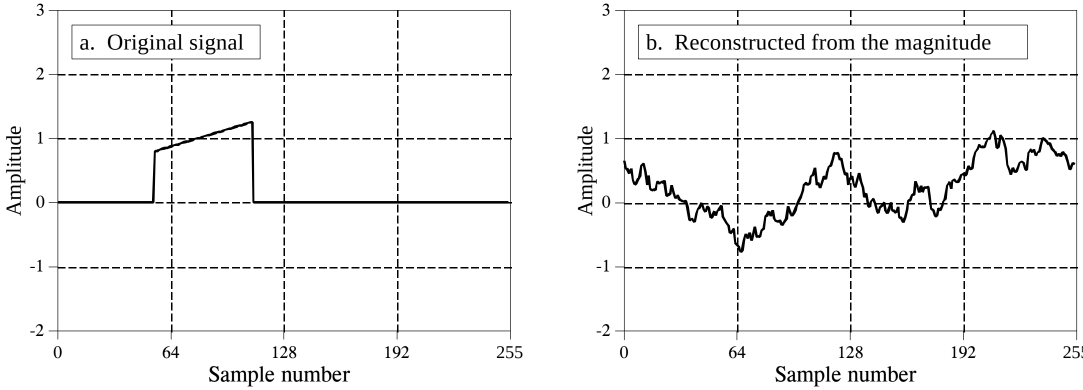


图10-5
由时域移位产生的相位。对于每个时域信号正向（即向右）移位的样本，0.5频率处的相位将减少π弧度。对于每个负向（即向左）移位的样本，0.5频率处的相位将增加π弧度。图(a)展示了线性相位（直线）的情况，而(b)是使用非线性相位的示例。

到目前为止，我们所用的所有例子都是*线性*相位。图10-5b显示，*非线性*相位信号对移位的反应方式相同。在这个例子中，非线性相位是一条带有两个矩形脉冲的直线。当时域发生移位时，这些非线性特征就简单地叠加在变化的斜率上。

当时域波形发生偏移时，*实部*和*虚部*会发生什么变化？需要提醒的是，用矩形符号表示的频域信号对人类来说几乎难以理解。实部和虚部通常呈现随机振荡特征，缺乏明显规律。当时域信号发生偏移时，实部和虚部的波纹图案会变得更加剧烈且难以解读。别浪费时间去琢磨这些信号，或是它们如何被时域偏移改变。

图10-6生动展示了*相位*与*幅值*各自承载的信息特征。图(a)的波形具有两个显著特征：第55个采样点的上升沿和第110个采样点的下降沿。当信息以波形*形态*编码时，波沿至关重要——它们标记*事件发生时刻*，将左侧信号与右侧信号分隔开来，堪称时域编码信息的最纯粹形态。演示开始时，对图(a)信号进行DFT变换，并将频谱转换为极坐标表示。为获得图(b)信号，将相位替换为-π至π之间的随机数值，再通过逆DFT重建时域波形。简而言之，图(b)仅基于*幅值*所含信息。类似地，图(c)则是通过在使用逆DFT前将幅值替换为随机小数值而生成。这使得仅基于*相位*中所含信息的(c)的重建。



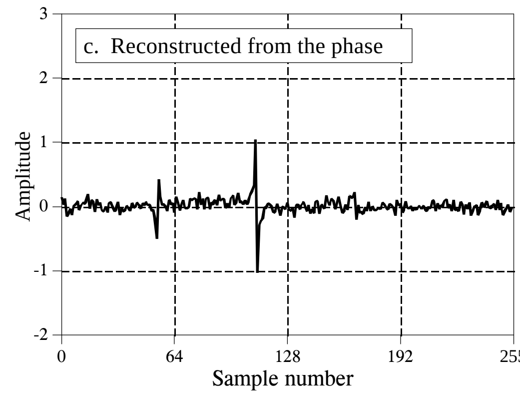


FIGURE 10-6
Information contained in the phase. Figure (a) shows a pulse-like waveform. The signal in (b) is created by taking the DFT of (a), replacing the *phase* with random numbers, and taking the Inverse DFT. The signal in (c) is found by taking the DFT of (a), replacing the *magnitude* with random numbers, and taking the Inverse DFT. The location of the *edges* is retained in (c), but not in (b). This shows that the phase contains information on the location of events in the time domain signal.

The result? The locations of the edges are clearly present in (c), but totally absent in (b). This is because an edge is formed when many sinusoids *rise* at the same location, possible only when their *phases* are coordinated. In short, much of the information about the shape of the time domain waveform is contained in the *phase*, rather than the *magnitude*. This can be contrasted with signals that have their information encoded in the frequency domain, such as audio signals. The magnitude is most important for these signals, with the phase playing only a minor role. In later chapters we will see that this type of understanding provides strategies for designing filters and other methods of processing signals. Understanding how information is represented in signals is always the first step in successful DSP.

Why does left-right symmetry correspond to a zero (or linear) phase? Figure 10-7 provides the answer. Such a signal can be decomposed into a left half and a right half, as shown in (a), (b) and (c). The sample at the center of symmetry (zero in this case) is divided equally between the left and right halves, allowing the two sides to be perfect mirror images of each other. The magnitudes of these two halves will be *identical*, as shown in (e) and (f), while the phases will be opposite in sign, as in (h) and (i). Two important concepts fall out of this. First, every signal that is symmetrical between the left and right will have a linear phase *because* the nonlinear phase of the left half exactly cancels the nonlinear phase of the right half.

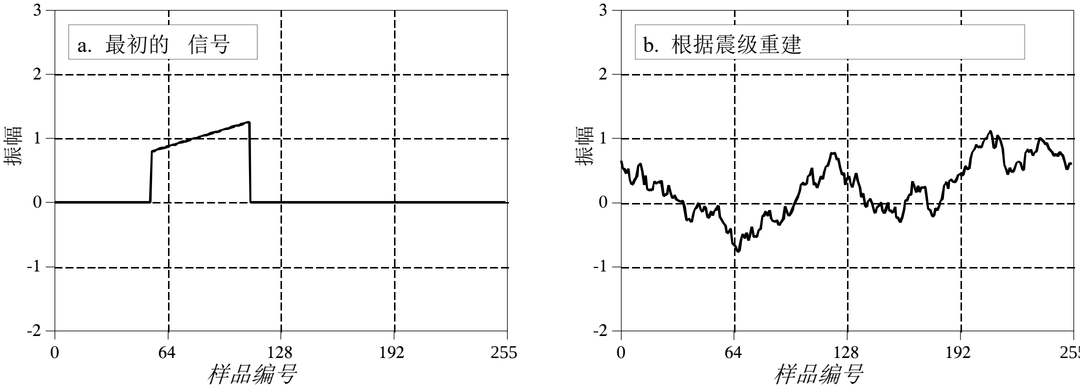
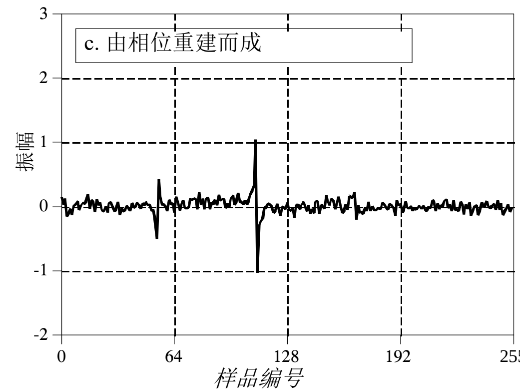


图10-6
相位信息的呈现方式。图(a)展示了脉冲波形。图(b)中的信号是通过对(a)进行DFT变换，将*相位*替换为随机数后进行逆DFT得到的。图(c)中的信号则是通过对(a)进行DFT变换，将*幅值*替换为随机数后进行逆DFT获得的。图(c)保留了*波峰*的位置特征，而图(b)则未能保留。这表明相位信息包含了时域信号中事件位置的特征信息。



结果如何？边缘的位置在(c)中清晰可见，但在(b)中完全消失。这是因为当多个正弦波*在同一位置上升*时才会形成边缘，而这种情况只有当它们的*相位*协调一致时才可能发生。简而言之，时域波形的大部分信息都包含在*相位*中，而非*幅度*。这与音频信号等在频域编码信息的信号形成鲜明对比——对于这类信号，幅度才是关键，相位仅起次要作用。在后续章节中我们将看到，这种理解方式为设计滤波器和其他信号处理方法提供了策略。理解信号中信息的表征方式，始终是成功进行数字信号处理的第一步。

为什么左右对称性对应零相位（或线性相位）？图10-7给出了答案。这种信号可以分解为左半部分和右半部分，如(a)、(b)和(c)所示。对称中心（此处为零点）的样本被平均分配到左右两半，使得两侧成为彼此的完美镜像。如(e)和(f)所示，两半的幅值将*完全相同*，而相位则如(h)和(i)所示符号相反。由此可得出两个重要结论：首先，所有左右对称的信号都具有线性相位，*因为*左半部分的非线性相位会完全抵消右半部分的非线性相位。

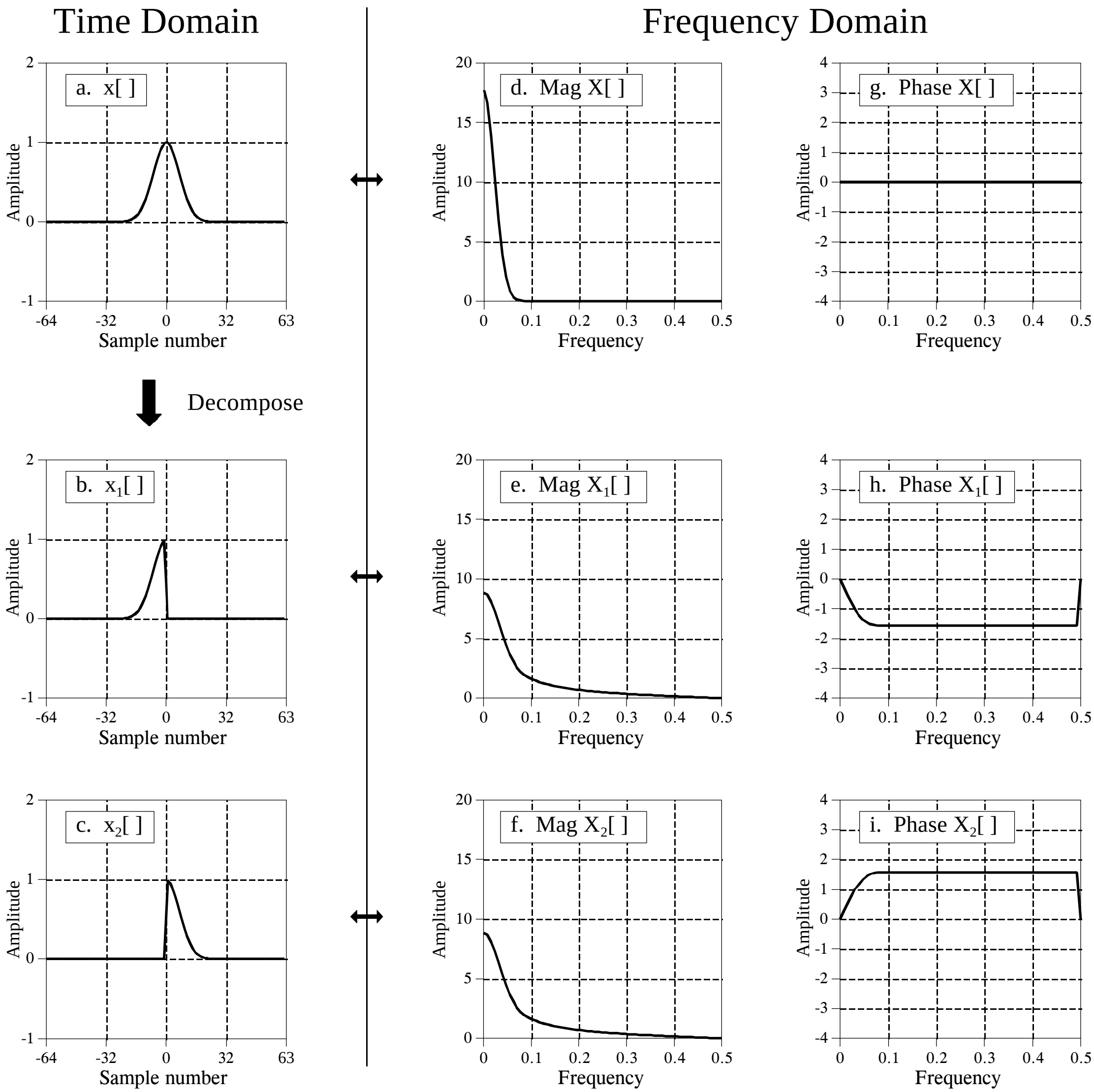


FIGURE 10-7
Phase characteristics of left-right symmetry. A signal with left-right symmetry, shown in (a), can be decomposed into a right half, (b), and a left half, (c). The magnitudes of the two halves are identical, (e) and (f), while the phases are the negative of each other, (h) and (i).

Second, imagine flipping (b) such that it becomes (c). This left-right flip in the time domain does nothing to the magnitude, but changes the sign of every point in the phase. Likewise, changing the sign of the phase flips the time domain signal left-for-right. If the signals are continuous, the flip is around zero. If the signals are discrete, the flip is around sample zero *and* sample $N/2$, simultaneously.

Changing the sign of the phase is a common enough operation that it is given its own name and symbol. The name is **complex conjugation**, and it is

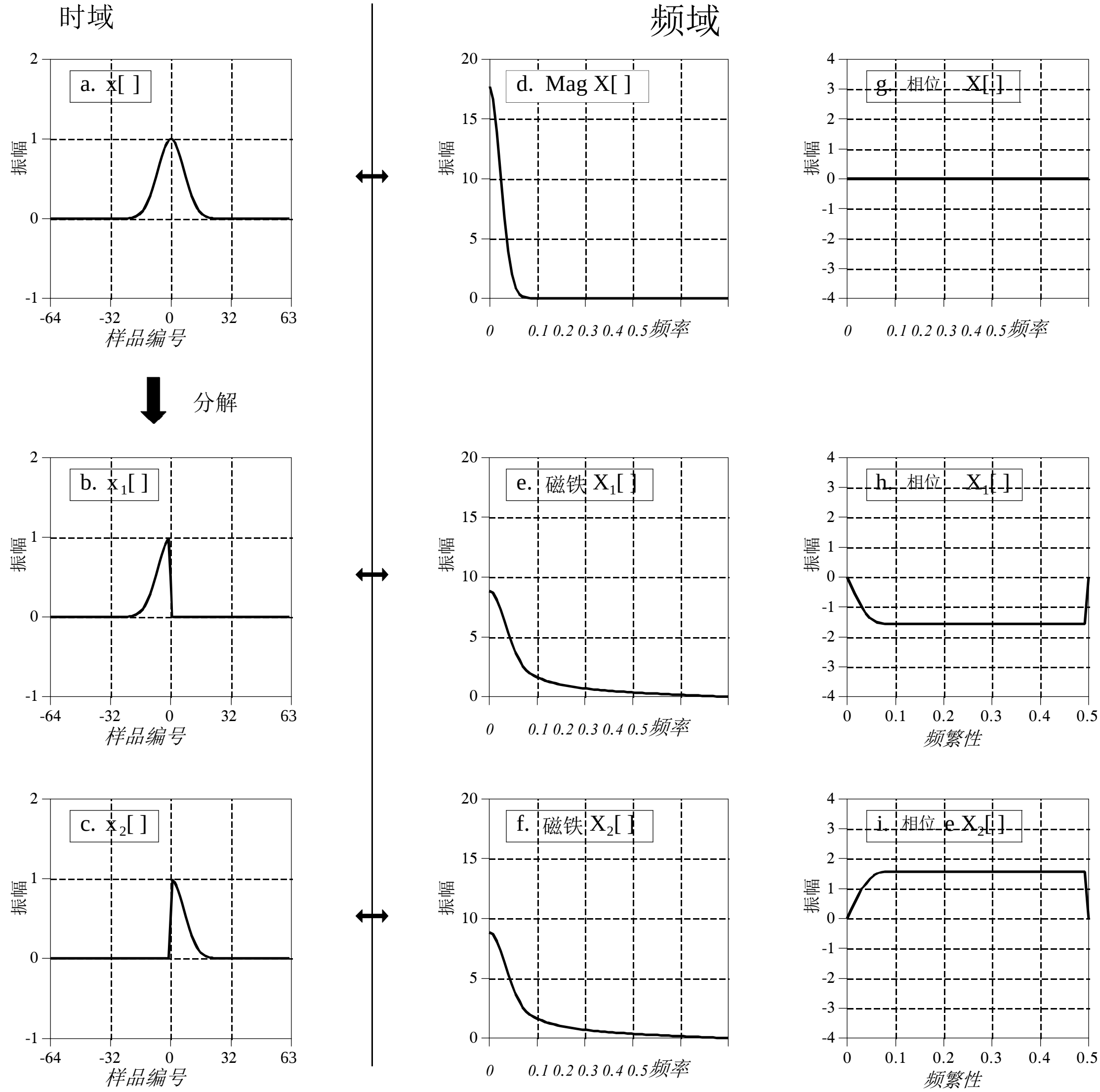


图10-7
左右对称信号的相位特性。如图(a)所示，具有左右对称性的信号可分解为右半部分(b)和左半部分(c)。两部分的幅值相同(e)和(f)，而相位互为负值(h)和(i)。

其次，想象将(b)翻转为(c)。这种时域中的左右翻转不会改变幅值，但会改变相位中每个点的符号。同样地，改变相位符号会将时域信号从左翻转到右。如果信号是连续的，翻转发生在零点附近；如果信号是离散的，翻转则同时发生在采样点零点*和*采样点$N/2$附近。

改变相位符号是一种非常常见的操作，因此它被赋予了自己的名称和符号。该名称为**复共轭**，并且它

represented by placing a star to the upper-right of the variable. For example, if $X[f]$ consists of $Mag\,X[f]$ and $Phase\,X[f]$, then $X^*[f]$ is called the *complex conjugate* and is composed of $Mag\,X[f]$ and $-Phase\,X[f]$. In rectangular notation, the complex conjugate is found by leaving the real part alone, and changing the sign of the imaginary part. In mathematical terms, if $X[f]$ is composed of $Re\,X[f]$ and $Im\,X[f]$, then $X^*[f]$ is made up of $Re\,X[f]$ and $-Im\,X[f]$.

Here are several examples of how the complex conjugate is used in DSP. If $x[n]$ has a Fourier transform of $X[f]$, then $x[-n]$ has a Fourier transform of $X^*[f]$. In words, flipping the time domain left-for-right corresponds to changing the sign of the phase. As another example, recall from Chapter 7 that correlation can be performed as a convolution. This is done by flipping one of the signals left-for-right. In mathematical form, $a[n] * b[n]$ is convolution, while $a[n] * b[-n]$ is correlation. In the frequency domain these operations correspond to $A[f] \times B[f]$ and $A[f] \times B^*[f]$, respectively. As the last example, consider an arbitrary signal, $x[n]$, and its frequency spectrum, $X[f]$. The frequency spectrum can be changed to *zero phase* by multiplying it by its complex conjugate, that is, $X[f] \times X^*[f]$. In words, whatever phase $X[f]$ happens to have will be canceled by adding its opposite (remember, when frequency spectra are multiplied, their phases are added). In the time domain, this means that $x[n] * x[-n]$ (a signal convolved with a left-right flipped version of itself) will have left-right symmetry around sample zero, regardless of what $x[n]$ is.

To many engineers and mathematicians, this kind of manipulation *is* DSP. If you want to be able to communicate with this group, get used to using their language.

Periodic Nature of the DFT

Unlike the other three Fourier Transforms, the DFT views *both* the time domain and the frequency domain as *periodic*. This can be confusing and inconvenient since most of the signals used in DSP are *not* periodic. Nevertheless, if you want to use the DFT, you must conform with the DFT's view of the world.

Figure 10-8 shows two different interpretations of the time domain signal. First, look at the upper signal, the time domain viewed as *N* points. This represents how digital signals are typically acquired in scientific experiments and engineering applications. For instance, these 128 samples might have been acquired by sampling some parameter at regular intervals of *time*. Sample 0 is distinct and separate from sample 127 because they were acquired at *different* times. From the way this signal was formed, there is no reason to think that the samples on the left of the signal are even related to the samples on the right.

Unfortunately, the DFT doesn't see things this way. As shown in the lower figure, the DFT views these 128 points to be a single period of an infinitely long periodic signal. This means that the left side of the acquired signal is

通过在变量右上方放置星号来表示。例如，如果$X[f]$由*幅值*$X[f]$和*相位*$X[f]$组成，则$X^*[f]$被称为*复共轭*，由*幅值*$X[f]$和-*相位*$X[f]$构成。在矩形表示法中，复共轭通过保留实部不变并改变虚部符号来确定。用数学语言表述，如果$X[f]$由*实部*$X[f]$和*虚部*$X[f]$组成，则$X^*[f]$由*实部*$X[f]$和 $-ImX[f]$.

以下是复共轭在数字信号处理中应用的几个例子。如果$x[n]$的傅里叶变换为$X[f]$，那么$x[-n]$的傅里叶变换为$X^*[f]$。换言之，将时域从左到右翻转相当于改变相位的符号。再举一个例子，回顾第7章可知相关性可以通过卷积实现。这是通过将其中一个信号从左到右翻转来完成的。数学形式上，$a[n]$t $b[n]$是卷积，而$a[n]$t $b[-n]$是相关性。在频域中，这些操作分别对应$A[f]\times B[f]$和$A[f]\times B^*[f]$。作为最后一个例子，考虑一个任意信号$x[n]$及其频谱$X[f]$。通过将其与复共轭相乘，频谱可变为*零相位*，即$X[f]\times X^*[f]$。换言之，无论$X[f]$具有何种相位，通过添加其相反相位即可抵消（需注意频谱相乘时相位相加）。在时域中，这意味着$x[n]$t $x[-n]$（与自身左右翻转版本卷积的信号）在采样点零点处具有左右对称性，无论$x[n]$为何。

对许多工程师和数学家来说，这种操作*就是* DSP。如果你想和这群人交流，就得习惯使用他们的语言。

DFT的周期性

与其他三种傅里叶变换不同，DFT将*时间域和*频率域都视为*周期性*。这可能会令人困惑和不便，因为DSP中使用的大多数信号*都不是*周期性的。然而，如果你想使用DFT，你必须遵循DFT的世界观。

图10-8展示了时域信号的两种不同解读方式。首先观察上方信号，即以N个采样点呈现的时域图。这代表了科学实验和工程应用中数字信号的典型采集方式。例如，这128个样本可能是通过以固定*时间*间隔对某个参数进行采样获得的。样本0与样本127存在明显区分，因为它们的采集时间*不同*。从信号生成方式来看，左侧样本与右侧样本之间甚至不存在任何关联性。

遗憾的是，密度泛函理论（DFT）并不认同这一观点。如下方示意图所示，DFT将这128个数据点视为无限长周期信号的一个周期。这意味着所采集信号的左侧部分

connected to the right side of a duplicate signal. Likewise, the right side of the acquired signal is connected to the left side of an identical period. This can also be thought of as the right side of the acquired signal wrapping around and connecting to its left side. In this view, sample 127 occurs next to sample 0, just as sample 43 occurs next to sample 44. This is referred to as being **circular**, and is identical to viewing the signal as being *periodic*.

The most serious consequence of time domain periodicity is **time domain aliasing.** To illustrate this, suppose we take a time domain signal and pass it through the DFT to find its frequency spectrum. We could immediately pass this frequency spectrum through an Inverse DFT to reconstruct the original time domain signal, but the entire procedure wouldn't be very interesting. Instead, we will modify the frequency spectrum in some manner before using the Inverse DFT. For instance, selected frequencies might be deleted, changed in amplitude or phase, shifted around, etc. These are the kinds of things routinely done in DSP. Unfortunately, these changes in the frequency domain can create a time domain signal that is too long to fit into

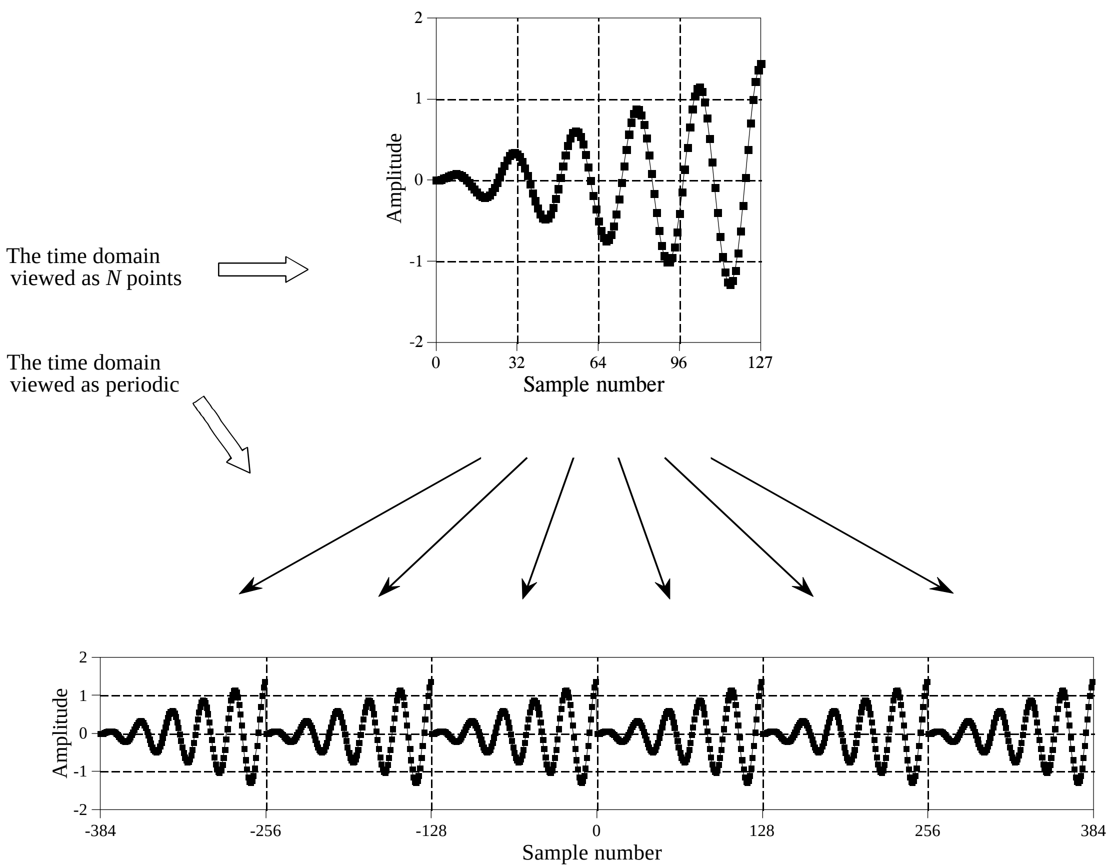


FIGURE 10-8
Periodicity of the DFT's time domain signal. The time domain can be viewed as N samples in length, shown in the upper figure, or as an infinitely long periodic signal, shown in the lower figure.

与重复信号的右侧相连。同样地，右侧的
采集信号与同一周期的左侧相连。这种现象也可理解为采集信号的右侧绕过并连接到其左侧。从这个角度看，样本127紧邻样本0，就像样本43紧挨着样本44一样。这种现象被称为**循环**，与将信号视为*周期性* 的观察方式完全一致。

时域周期性最严重的后果是**时域混叠现象。**举个例子，假设我们获取一个时域信号并用DFT转换得到其频谱。虽然可以直接将频谱通过逆DFT还原原始时域信号，但整个过程缺乏实际意义。因此，我们通常会在使用逆DFT前对频谱进行处理——比如删除特定频率、调整幅度或相位、进行频移等操作。这些处理在数字信号处理中是常规操作。但要注意，这类频域调整可能会导致时域信号过长，无法完整呈现原始时域特征。

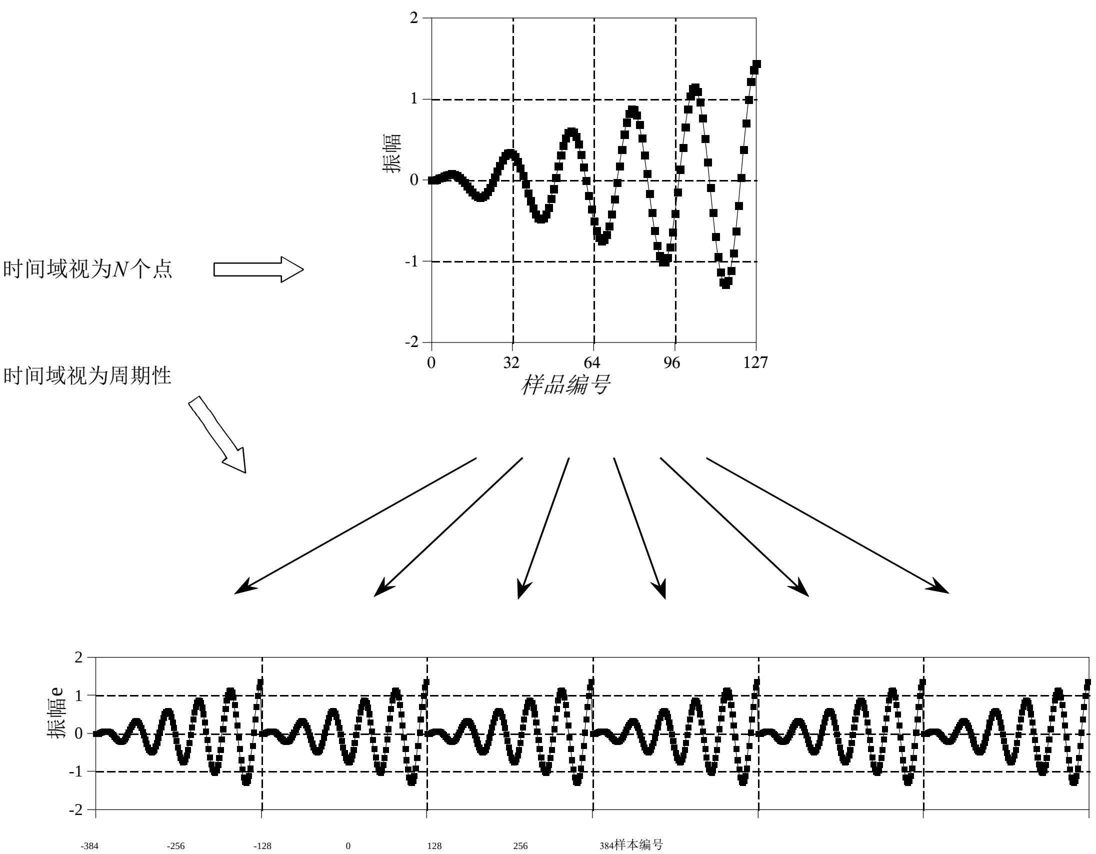


图10-8
DFT时域信号的周期性。时域可视为长度为*N*的采样点（如上图所示），或视为无限长的周期性信号（如下图所示）。

a single period. This forces the signal to spill over from one period into the adjacent periods. When the time domain is viewed as *circular*, portions of the signal that overflow on the right suddenly seem to reappear on the left side of the signal, and vice versa. That is, the overflowing portions of the signal *alias* themselves to a new location in the time domain. If this new location happens to already contain an existing signal, the whole mess adds, resulting in a loss of information. Circular convolution resulting from frequency domain multiplication (discussed in Chapter 9), is an excellent example of this type of aliasing.

Periodicity in the frequency domain behaves in much the same way, but is more complicated. Figure 10-9 shows an example. The upper figures show the magnitude and phase of the frequency spectrum, viewed as being composed of $N/2+1$ samples spread between 0 and 0.5 of the sampling rate. This is the simplest way of viewing the frequency spectrum, but it doesn't explain many of the DFT's properties.

The lower two figures show how the DFT views this frequency spectrum as being periodic. The key feature is that the frequency spectrum between 0 and 0.5 appears to have a *mirror image* of frequencies that run between 0 and -0.5. This mirror image of **negative frequencies** is slightly different for the magnitude and the phase signals. In the magnitude, the signal is flipped left-for-right. In the phase, the signal is flipped left-for-right, *and* changed in sign. As you recall, these two types of symmetry are given names: the magnitude is said to be an **even** signal (it has *even* symmetry), while the phase is said to be an **odd** signal (it has *odd* symmetry). If the frequency spectrum is converted into the real and imaginary parts, the *real part* will always be *even*, while the *imaginary part* will always be *odd*.

Taking these negative frequencies into account, the DFT views the frequency domain as periodic, with a period of 1.0 times the sampling rate, such as -0.5 to 0.5, or 0 to 1.0. In terms of sample numbers, this makes the length of the frequency domain period equal to N, the same as in the time domain.

The periodicity of the frequency domain makes it susceptible to **frequency domain aliasing**, completely analogous to the previously described time domain aliasing. Imagine a time domain signal that corresponds to some frequency spectrum. If the time domain signal is modified, it is obvious that the frequency spectrum will also be changed. If the modified frequency spectrum cannot fit in the space provided, it will push into the adjacent periods. Just as before, this aliasing causes two problems: frequencies aren't where they should be, and overlapping frequencies from different periods add, destroying information.

Frequency domain aliasing is more difficult to understand than time domain aliasing, since the periodic pattern is more complicated in the frequency domain. Consider a single frequency that is being forced to move from 0.01 to 0.49 in the frequency domain. The corresponding negative frequency is therefore moving from -0.01 to -0.49. When the positive frequency moves

单个周期的信号会溢出到相邻周期。当时间域呈现*循环*结构时，右侧溢出的信号部分会突然出现在左侧，反之亦然。这意味着信号的溢出部分会在时间域中*产生混叠* 现象。若此时新位置已有信号存在，就会形成信息叠加，导致数据丢失。这种由频域乘法（详见第九章）引发的循环卷积现象，正是混叠效应的典型例证。

频域中的周期性表现方式与之类似，但更为复杂。图10-9展示了具体示例。上图展示了频谱的幅值和相位，这些数据可视为由N/2+1个采样点构成，这些采样点分布在采样率的0到0.5区间内。这是观察频谱最直观的方式，但无法解释离散傅里叶变换（DFT）的诸多特性。

下两个图展示了DFT如何将该频谱视为周期性的。关键特征在于0到0.5之间的频谱似乎具有0到-0.5之间频率的*镜像* 。这种**负频率**的镜像在幅值信号和相位信号中略有不同：在幅值信号中，信号左右翻转；在相位信号中，信号左右翻转*且* 符号改变。如你所知，这两种对称性有特定名称：幅值信号被称为**偶**信号（具有*偶* 对称性），而相位信号被称为**奇**信号（具有*奇*对称性）。若将频谱转换为实部和虚部，*实部*始终为*偶*，而*虚部*始终为*奇* 。

考虑到这些负频率，DFT将频域视为周期性的，周期为采样率的1.0倍，例如-0.5到0.5，或0到1.0。就样本数量而言，这使得频域周期的长度等于N，与时域相同。

频域的周期性特性使其容易产生**频域混叠现象**，这与之前描述的时域混叠现象如出一辙。想象一个对应特定频谱的时域信号，当对其进行修改时，频谱必然随之改变。若修改后的频谱无法适应现有空间，就会向相邻频段挤占。这种混叠现象同样会引发两个问题：一是频率位置出现偏差，二是不同频段的频率相互叠加，导致信息丢失。

频域混叠比时域混叠更难理解，因为频域中的周期性模式更为复杂。考虑一个在频域中被迫从0.01移动到0.49的单一频率。因此，对应的负频率从-0.01移动到-0.49。当正频率移动时

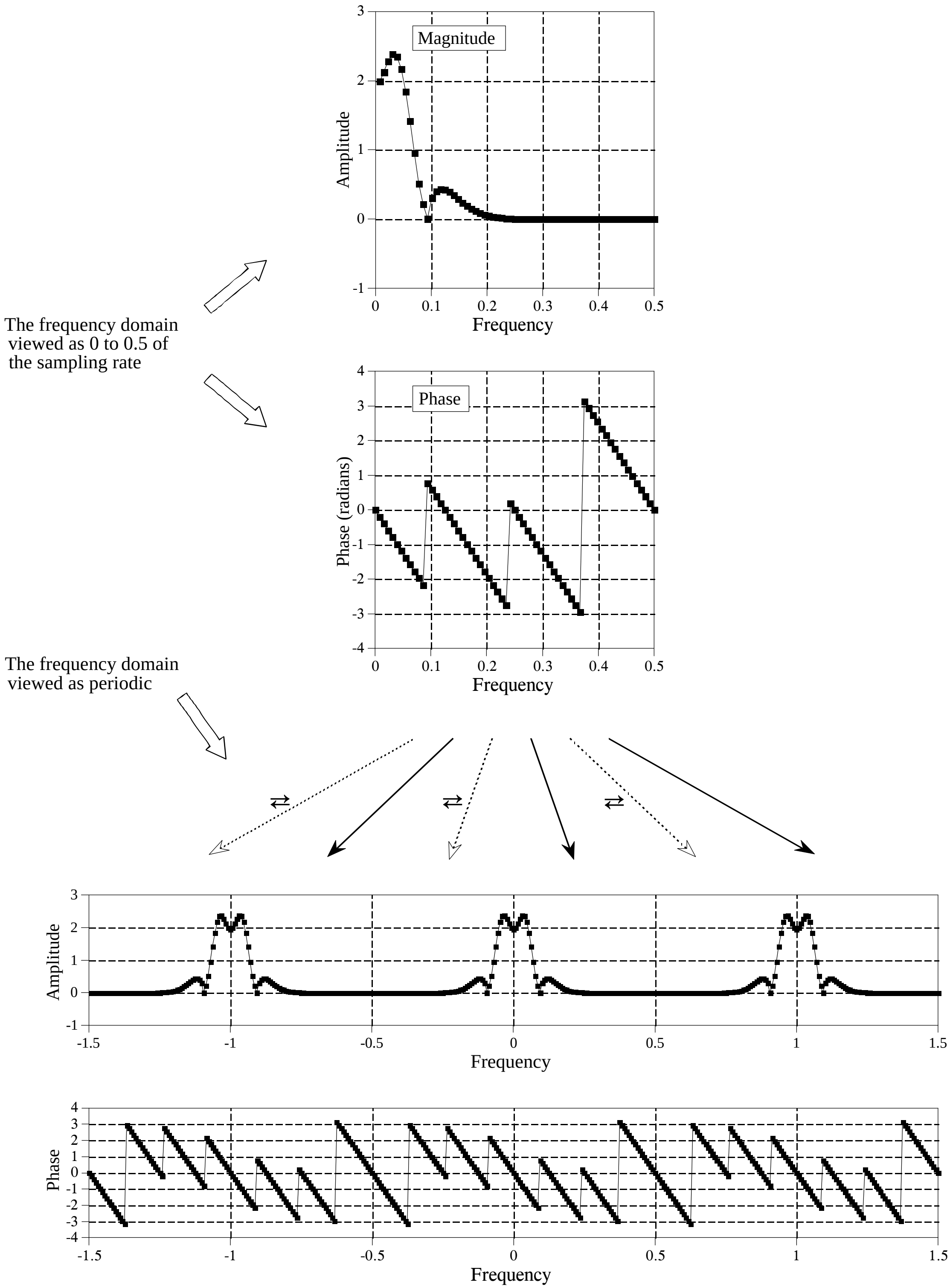


FIGURE 10-9
Periodicity of the DFT's frequency domain. The frequency domain can be viewed as running from 0 to 0.5 of the sampling rate (upper two figures), or an infinity long periodic signal with every other 0 to 0.5 segment flipped left-for-right (lower two figures).

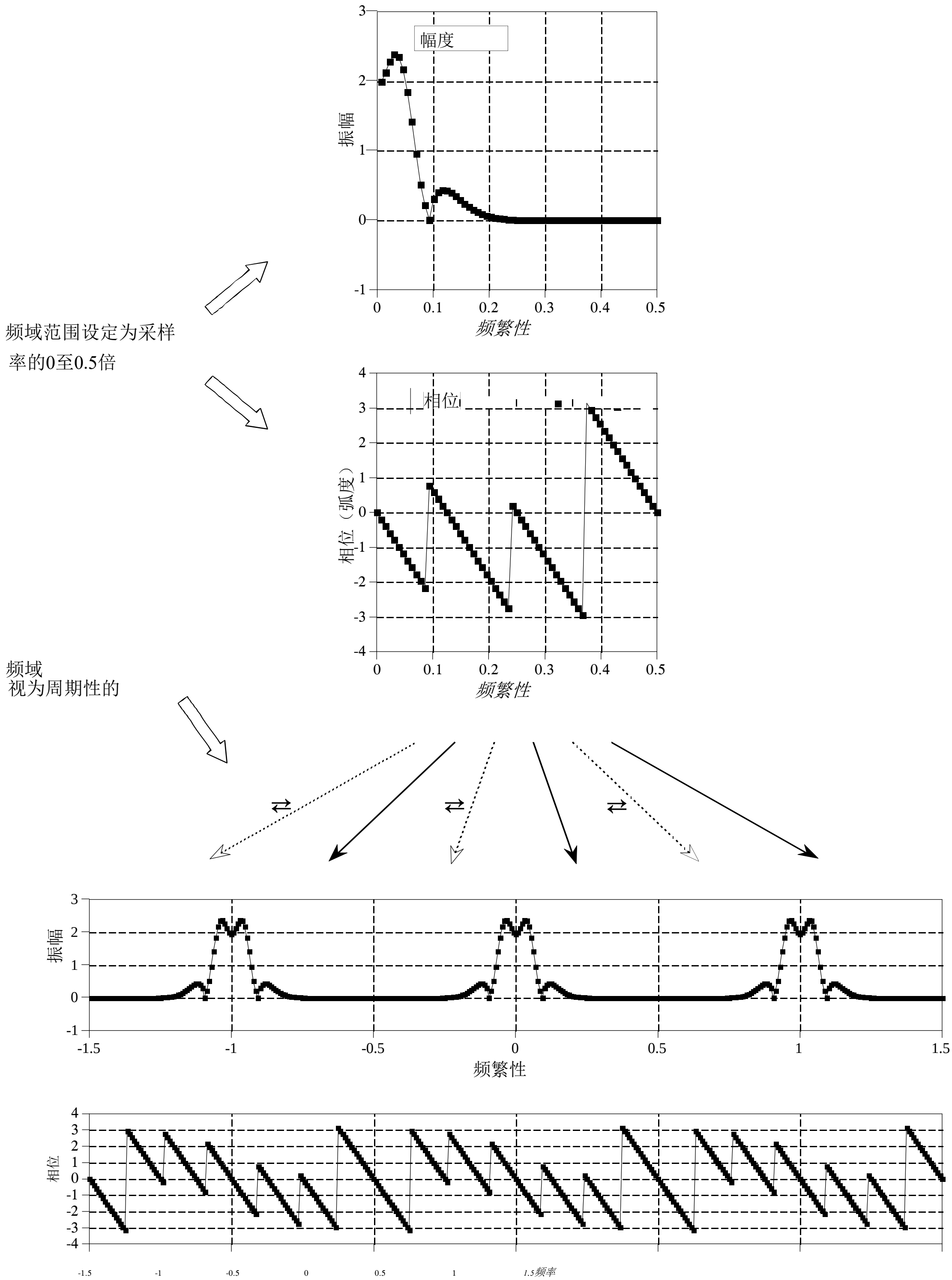


图10-9

DFT频域的周期性。频域可视为从0到采样率的0.5（上两幅图），或为一个无限长的周期信号，每隔0到0.5的区段左右翻转（下两幅图）。

across the 0.5 barrier, the negative frequency is pushed across the -0.5 barrier. Since the frequency domain is periodic, these same events are occurring in the other periods, such as between 0.5 and 1.5. A clone of the positive frequency is crossing frequency 1.5 from left to right, while a clone of the negative frequency is crossing 0.5 from right to left. Now imagine what this looks like if you can only see the frequency band of 0 to 0.5. It appears that a frequency leaving to the *right*, reappears on the *right*, but moving in the opposite direction.

Figure 10-10 illustrates how aliasing appears in the time and frequency domains when only a single period is viewed. As shown in (a), if one end of a time domain signal is too long to fit inside a single period, the protruding end will be *cut off* and *pasted* onto the other side. In comparison, (b) shows that when a frequency domain signal overflows the period, the protruding end is *folded over*. Regardless of where the aliased segment ends up, it adds to whatever signal is already there, destroying information.

Let's take a closer look at these strange things called *negative frequencies*. Are they just some bizarre artifact of the mathematics, or do they have a real world meaning? Figure 10-11 shows what they are about. Figure (a) is a discrete signal composed of 32 samples. Imagine that you are given the task of finding the frequency spectrum that corresponds to these 32 points. To make your job easier, you are told that these points represent a discrete cosine wave. In other words, you must find the frequency and phase shift (f and θ) such that $x[n] = \cos(2\pi n f/N + \theta)$ matches the given samples. It isn't long before you come up with the solution shown in (b), that is, $f = 3$ and $\theta = -\pi/4$.

If you stopped your analysis at this point, you only get 1/3 credit for the problem. This is because there are two other solutions that you have missed. As shown in (c), the second solution is $f = -3$ and $\theta = \pi/4$. Even if the idea of a *negative frequency* offends your sensibilities, it doesn't

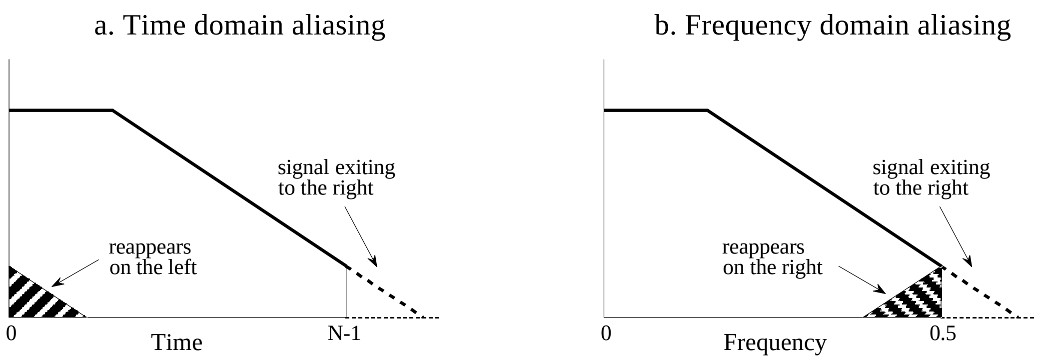


FIGURE 10-10
Examples of aliasing in the time and frequency domains, when only a single period is considered. In the time domain, shown in (a), portions of the signal that exits to the right, reappear on the left. In the frequency domain, (b), portions of the signal that exit to the right, reappear on the right as if they had been folded over.

当频率跨越0.5的阻隔时，负频率会跨越-0.5的阻隔。由于频域具有周期性，这些相同事件会在其他周期（例如0.5到1.5之间）重复发生。正频率的克隆体正从左向右跨越1.5的频率，而负频率的克隆体则从右向左跨越0.5。现在想象一下，如果只能看到0到0.5的频带会呈现什么景象。看起来一个频率向*右*方向移动后，又会以相反方向重新出现在*右*侧。

图10-10展示了在仅观察单个周期时，时域和频域中混叠现象的呈现方式。如(a)所示，当时域信号的一端过长无法容纳在单个周期内时，突出的端部会被*截断*并*粘贴*到另一端。相比之下，(b)则显示当频域信号超出周期范围时，突出端会*折叠覆盖*。无论混叠部分最终出现在何处，都会叠加到原有信号上，导致信息丢失。

让我们更仔细地看看这些被称为*负频率*的奇怪现象。它们仅仅是数学上的某种怪异产物，还是具有现实意义？图10-11展示了它们的含义。图(a)是一个由32个样本组成的离散信号。假设你被要求找出与这32个点相对应的频谱。为了简化你的任务，你被告知这些点代表一个离散余弦波。换句话说，你必须找到频率和相位偏移（f和$\boldsymbol{\theta}$），使得$x[n]$ =cos（2$\boldsymbol{\pi}$ nf/N+$\boldsymbol{\theta}$）与给定样本匹配。很快你就会得出图(b)所示的解，即f=3且$\boldsymbol{\theta}$=$\boldsymbol{\pi}$/4。

如果你在此处停止分析，你只能获得该问题1/3的学分。这是因为还有两个你遗漏的解法。如(c)所示，第二个解法是f= &3且$\boldsymbol{\theta}$=$\boldsymbol{\pi}$/4。即使*负频率*的概念让你感到不适，它也不

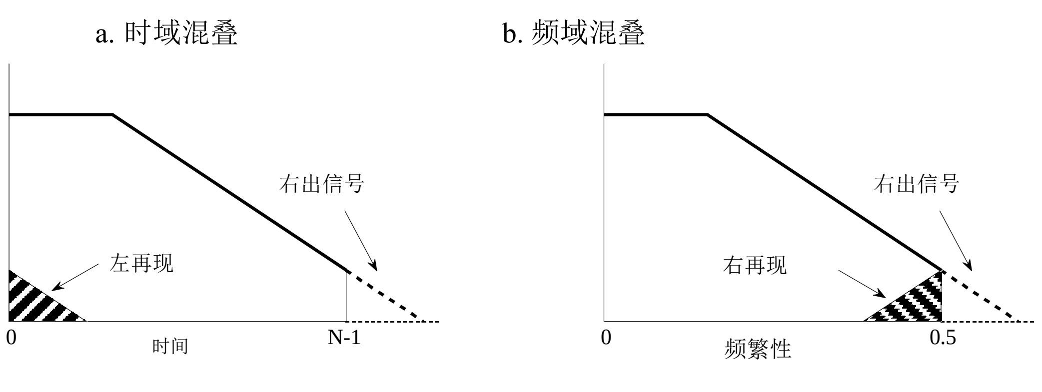


图10-10
仅考虑单个周期时，时域与频域中混叠现象的示例。在时域中（图a），信号向右传播的部分会在左侧重现。在频域中（图b），向右传播的信号部分会像折叠一样在右侧重现。

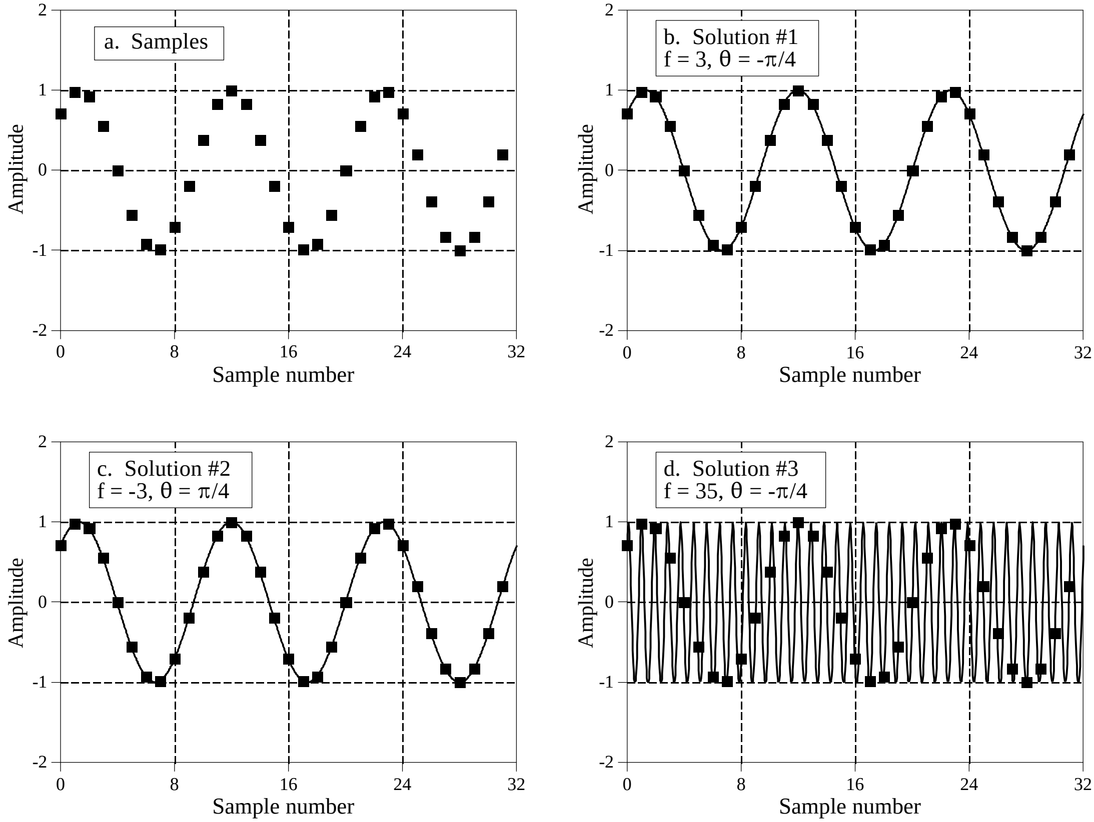


FIGURE 10-11
The meaning of negative frequencies. The problem is to find the frequency spectrum of the discrete signal shown in (a). That is, we want to find the frequency and phase of the sinusoid that passed through all of the samples. Figure (b) is a solution using a *positive* frequency, while (c) is a solution using a *negative* frequency. Figure (d) represents a family of solutions to the problem.

change the fact that it is a mathematically valid solution to the defined problem. Every *positive* frequency sinusoid can alternately be expressed as a *negative* frequency sinusoid. This applies to continuous as well as discrete signals

The third solution is not a single answer, but an infinite family of solutions. As shown in (d), the sinusoid with $f = 35$ and $\theta = -\pi/4$ passes through all of the discrete points, and is therefore a correct solution. The fact that it shows oscillation between the samples may be confusing, but it doesn't disqualify it from being an authentic answer. Likewise, $f = \pm 29$, $f = \pm 35$, $f = \pm 61$, and $f = \pm 67$ are all solutions with multiple oscillations between the points.

Each of these three solutions corresponds to a different section of the frequency spectrum. For discrete signals, the first solution corresponds to frequencies between 0 and 0.5 of the sampling rate. The second solution

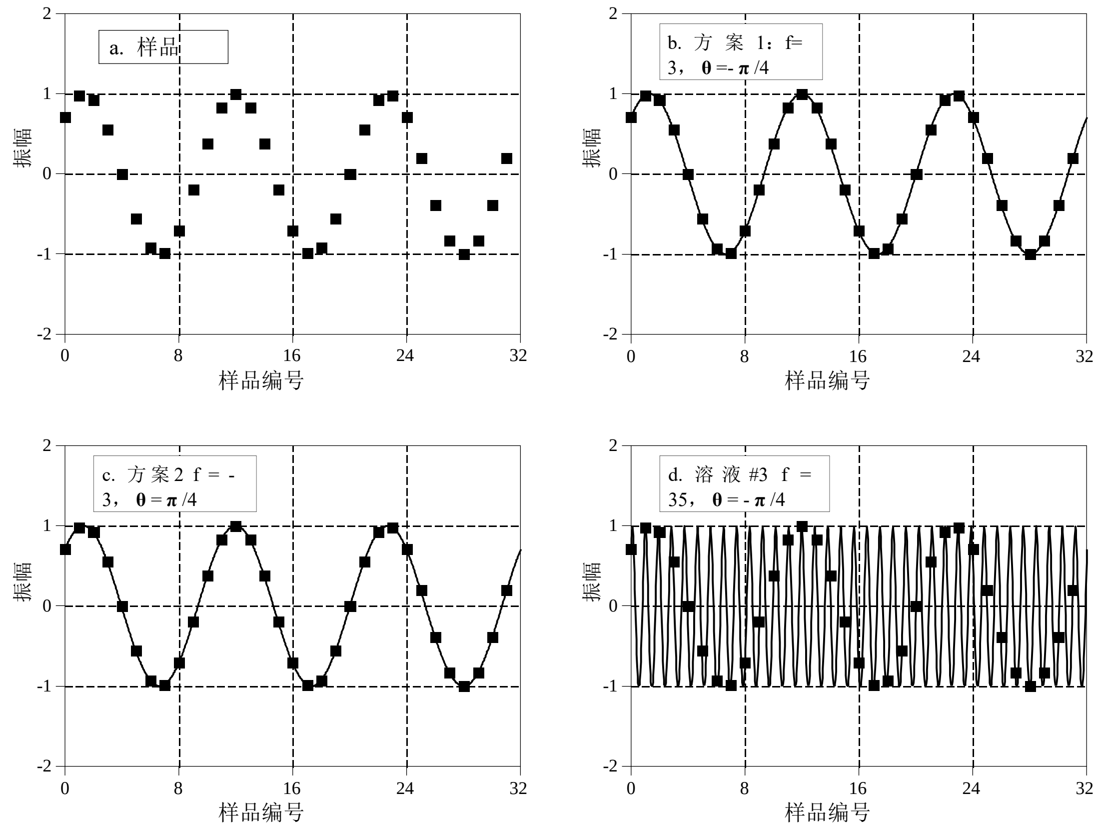


图10-11

负频率的含义。问题在于求解图(a)所示离散信号的频谱。具体来说，我们需要确定经过所有采样点的正弦波的频率和相位。图(b)展示了使用*正频率*的解法，而图(c)则展示了使用*负频率*的解法。图(d)呈现了该问题的一系列解法。

改变其为所定义问题的数学有效解这一事实。每个*正*频率正弦波都可以交替表示为*负*频率正弦波。这适用于连续信号和离散信号。

第三种解并非单一答案，而是一个无限的解族。如(d)所示，当f=35且θ= $-\pi/4$时，正弦波会穿过所有离散点，因此是正确解。虽然它在采样点之间呈现振荡可能令人困惑，但这并不影响其作为真实答案的资格。同理，f=±29、f=±35、f=±61以及f=±67都是在点间呈现多次振荡的解。

这三种解法分别对应频谱的不同区间。对于离散信号，第一种解法对应采样率的0至0.5倍频率范围。第二种解法

results in frequencies between 0 and -0.5. Lastly, the third solution makes up the infinite number of duplicated frequencies below -0.5 and above 0.5. If the signal we are analyzing is continuous, the first solution results in frequencies from zero to positive infinity, while the second solution results in frequencies from zero to negative infinity. The third group of solutions does not exist for continuous signals.

Many DSP techniques do not require the use of negative frequencies, or an understanding of the DFT's periodicity. For example, two common ones were described in the last chapter, *spectral analysis*, and the *frequency response* of systems. For these applications, it is completely sufficient to view the time domain as extending from sample 0 to N-1, and the frequency domain from zero to one-half of the sampling frequency. These techniques can use a simpler view of the world because they never result in portions of one period moving into another period. In these cases, looking at a single period is just as good as looking at the entire periodic signal.

However, certain procedures can *only* be analyzed by considering how signals overflow between periods. Two examples of this have already been presented, *circular convolution* and *analog-to-digital conversion*. In circular convolution, multiplication of the frequency spectra results in the time domain signals being convolved. If the resulting time domain signal is too long to fit inside a single period, it overflows into the adjacent periods, resulting in *time domain aliasing*. In contrast, analog-to-digital conversion is an example of *frequency domain aliasing*. A nonlinear action is taken in the time domain, that is, changing a continuous signal into a discrete signal by sampling. The problem is, the spectrum of the original analog signal may be too long to fit inside the discrete signal's spectrum. When we force the situation, the ends of the spectrum protrude into adjacent periods. Let's look at two more examples where the periodic nature of the DFT is important, *compression & expansion* of signals, and *amplitude modulation*.

Compression and Expansion, Multirate methods

As shown in Fig. 10-12, a *compression* of the signal in one domain results in an *expansion* in the other, and vice versa. For continuous signals, if $X(f)$ is the Fourier Transform of $x(t)$, then $1/k \times X(f/k)$ is the Fourier Transform of $x(kt)$, where k is the parameter controlling the expansion or contraction. If an event happens *faster* (it is compressed in time), it must be composed of *higher* frequencies. If an event happens *slower* (it is expanded in time), it must be composed of *lower* frequencies. This pattern holds if taken to either of the two extremes. That is, if the time domain signal is compressed so far that it becomes an *impulse*, the corresponding frequency spectrum is expanded so far that it becomes a *constant value*. Likewise, if the time domain is expanded until it becomes a constant value, the frequency domain becomes an impulse.

Discrete signals behave in a similar fashion, but there are a few more details. The first issue with discrete signals is *aliasing*. Imagine that the

第三种解法对应-0.5至0之间的频率范围。最后，第三种解法填补了-0.5以下和0.5以上无限多的重复频率。若分析信号为连续信号，第一种解法对应从零到正无穷的频率，第二种解法对应从零到负无穷的频率。第三类解法在连续信号中并不存在。

许多数字信号处理技术并不需要使用负频率，也不必理解离散傅里叶变换的周期性。例如，上一章介绍的两种常见技术——*频谱分析*和系统*频率响应*——就完全不需要考虑这些。对于这类应用场景，将时域视为从采样点0延伸到N-1，频域则从零延伸到采样频率的一半就足够了。这些技术之所以能采用更简单的视角，是因为它们不会导致一个周期的部分内容进入另一个周期。在这种情况下，观察单个周期的效果与观察整个周期性信号完全一样。

然而，某些处理过程*只能*通过分析信号在周期间的溢出情况来解析。我们已介绍过两个典型例子：*环形卷积*和*模数转换*。在环形卷积中，频谱相乘会导致时域信号发生卷积。若所得时域信号过长无法容纳在单个周期内，就会溢出到相邻周期，从而引发*时域混叠*现象。相比之下，模数转换则是*频域混叠*的典型代表——它在时域采取非线性处理方式，通过采样将连续信号转换为离散信号。问题在于，原始模拟信号的频谱可能过长，无法完全容纳在离散信号的频谱范围内。当强行处理时，频谱的两端会延伸到相邻周期。接下来我们再看两个需要考虑DFT周期特性的例子：信号的*压缩与扩展*，以及*幅度调制*。

压缩与扩张多速率方法

如图10-12所示，信号在一个域中的*压缩*会导致另一个域中的*扩展*，反之亦然。对于连续信号，若X（f）是x（t）的傅里叶变换，则$1/k \times X$（f/k）是x（kt）的傅里叶变换，其中k是控制扩展或收缩的参数。若事件发生*更快*（即在时间上被压缩），则必须由*更高*频率组成；若事件发生*更慢*（即在时间上被扩展），则必须由*更低*频率组成。这种模式在任一极端情况下都成立。也就是说，若时域信号被压缩到极点以至于成为*脉冲*，则对应的频谱会被扩展到极点以至于成为*常数值*。同样地，若将时域扩展至恒定值，则频域将变为脉冲。

离散信号的表现方式类似，但存在一些细节差异。离散信号的第一个问题是*混叠现象*。想象一下

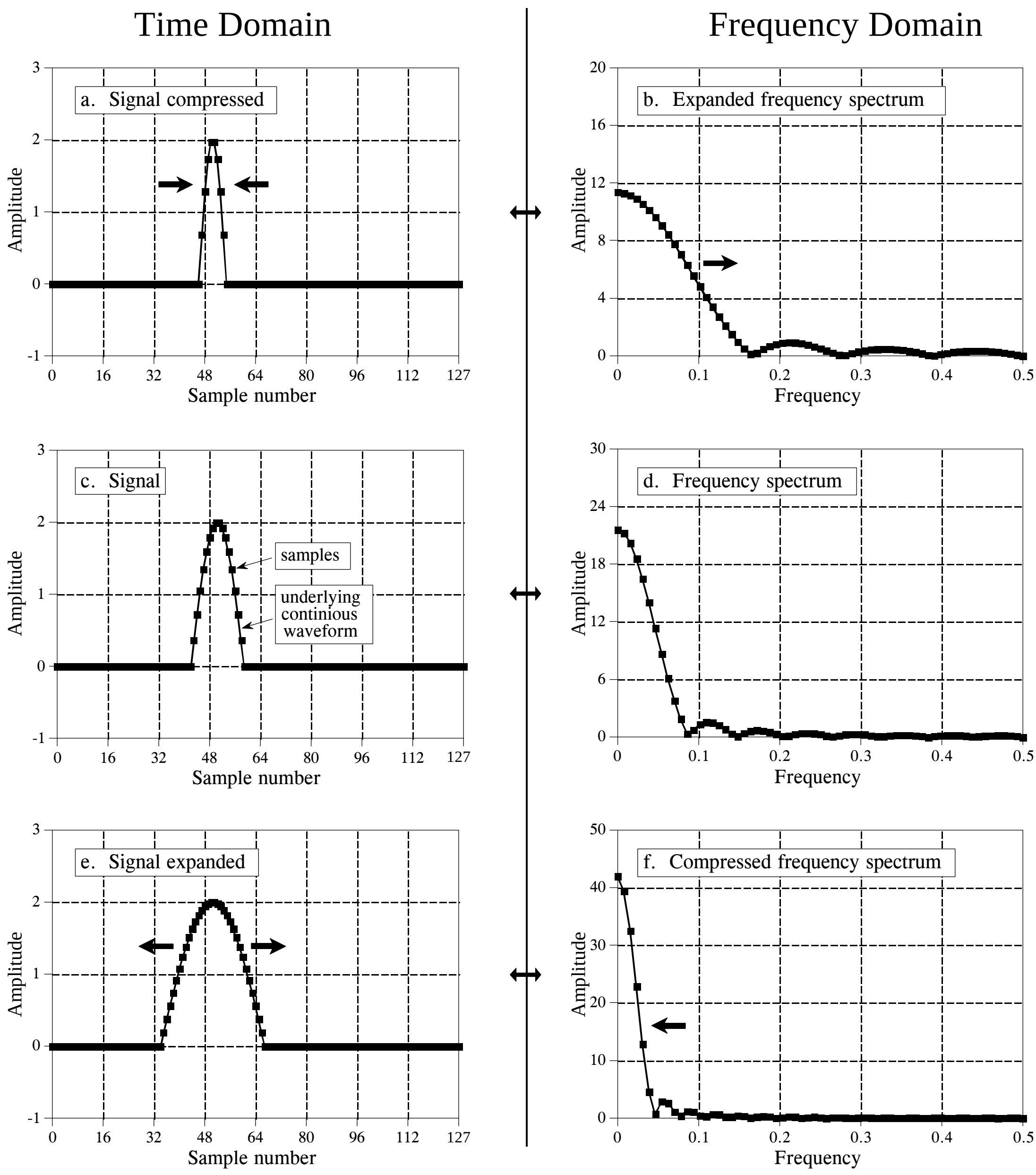


FIGURE 10-12
Compression and expansion. Compressing a signal in one domain results in the signal being expanded in the other domain, and vice versa. Figures (c) and (d) show a discrete signal and its spectrum, respectively. In (a) and (b), the time domain signal has been compressed, resulting in the frequency spectrum being expanded. Figures (e) and (f) show the opposite process. As shown in these figures, discrete signals are expanded or contracted by expanding or contracting the underlying continuous waveform. This underlying waveform is then resampled to find the new discrete signal.

pulse in (a) is compressed several times more than is shown. The frequency spectrum is expanded by an equal factor, and several of the humps in (b) are pushed to frequencies beyond 0.5. The resulting aliasing breaks the simple expansion/contraction relationship. This type of aliasing can also happen in the

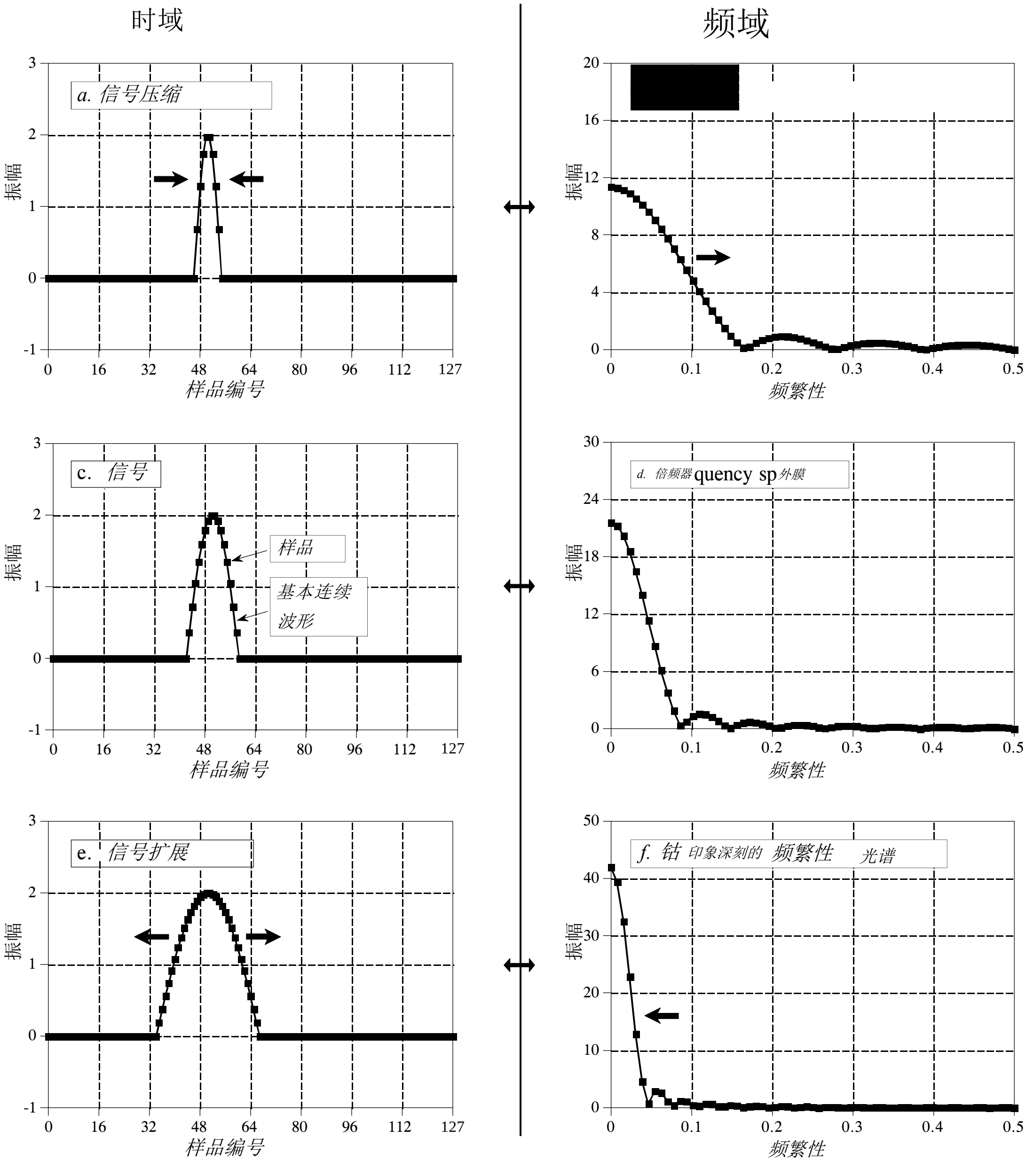


图10-12
压缩与扩展原理。当信号在一个域被压缩时，其在另一个域会相应扩展，反之亦然。图(c)和(d)分别展示了离散信号及其频谱。在(a)和(b)中，时域信号经过压缩处理，导致频谱发生扩展。图(e)和(f)则呈现了相反的过程。如图所示，离散信号通过扩展或压缩基础连续波形来实现扩展或收缩。随后对基础波形进行重采样，即可获得新的离散信号。

(a)中的脉冲被压缩了数倍于图示幅度。频谱被等比例扩展，(b)中的多个峰被推至0.5以上频率。由此产生的混叠现象破坏了简单的扩展/收缩关系。此类混叠现象也可能出现在

time domain. Imagine that the frequency spectrum in (f) is compressed much harder, resulting in the time domain signal in (e) expanding into neighboring periods.

A second issue is to define exactly what it means to compress or expand a discrete signal. As shown in Fig. 10-12a, a discrete signal is compressed by compressing the underlying *continuous* curve that the samples lie on, and then resampling the new continuous curve to find the new discrete signal. Likewise, this same process for the expansion of discrete signals is shown in (e). When a discrete signal is compressed, events in the signal (such as the width of the pulse) happen over a *fewer* number of samples. Likewise, events in an expanded signal happen over a *greater* number of samples.

An equivalent way of looking at this procedure is to keep the underlying continuous waveform the same, but resample it at a different sampling rate. For instance, look at Fig. 10-13a, a discrete Gaussian waveform composed of 50 samples. In (b), the same underlying curve is represented by 400 samples. The change between (a) and (b) can be viewed in two ways: (1) the sampling rate has been kept constant, but the underlying waveform has been expanded to be eight times wider, or (2) the underlying waveform has been kept constant, but the sampling rate has increased by a factor of eight. Methods for changing the sampling rate in this way are called **multirate** techniques. If more samples are added, it is called **interpolation**. If fewer samples are used to represent the signal, it is called **decimation**. Chapter 3 describes how multirate techniques are used in ADC and DAC.

Here is the problem: if we are given an arbitrary discrete signal, how do we know what the underlying continuous curve is? It depends on if the signal's information is encoded in the *time domain* or in the *frequency domain*. For time domain encoded signals, we want the underlying continuous waveform to be a smooth curve that passes through all the samples. In the simplest case, we might draw straight lines between the points and then round the rough corners. The next level of sophistication is to use a curve fitting algorithm, such as a spline function or polynomial fit. There is not a single "correct" answer to this problem. This approach is based on minimizing irregularities in the *time domain* waveform, and completely ignores the frequency domain.

When a signal has information encoded in the frequency domain, we ignore the time domain waveform and concentrate on the frequency spectrum. As discussed in the last chapter, a finer sampling of a frequency spectrum (more samples between frequency 0 and 0.5) can be obtained by padding the time domain signal with zeros before taking the DFT. Duality allows this to work in the opposite direction. If we want a finer sampling in the time domain (interpolation), pad the frequency spectrum with zeros before taking the Inverse DFT. Say we want to interpolate a 50 sample signal into a 400 sample signal. It's done like this: (1) Take the 50 samples and add zeros to make the signal 64 samples long. (2) Use a 64 point DFT to find the frequency spectrum, which will consist of a 33 point real part and a 33 point imaginary part. (3) Pad the right side of the frequency spectrum

在时域中，假设(f)中的频谱被进一步压缩，导致(e)中的时域信号扩展至相邻周期。

第二个关键问题在于明确离散信号压缩与扩展的具体含义。如图10-12a所示，离散信号的压缩过程是通过压缩其底层的*连续*曲线（即采样点所在曲线），再对新生成的连续曲线进行重采样以获得离散信号。类似地，离散信号扩展的处理过程如图10-12e所示。当离散信号被压缩时，信号中的事件（如脉冲宽度）会出现在*更少*的采样点上；而扩展后的信号中，事件则会分布在*更多*的采样点上。

理解这一过程的另一种等效方式是保持基础连续波形不变，但以不同的采样率进行重采样。例如，观察图10-13a所示的由50个样本组成的离散高斯波形。在(b)中，相同的底层曲线通过400个样本呈现。(a)与(b)之间的差异可通过两种方式理解：(1)采样率保持恒定，但底层波形扩展为八倍宽度；或(2)底层波形保持不变，但采样率提升八倍。这种改变采样率的方法称为**多速率**技术。若增加样本数量则称为**插值**，若减少样本数量则称为**抽取**。第三章将详细阐述多速率技术在模数转换器（ADC）和数模转换器（DAC）中的应用。

问题的核心在于：当我们接收到任意离散信号时，如何确定其背后的连续曲线形态？这取决于信号信息是编码在*时间域*还是*频率域*。对于时间域编码信号，我们期望其连续波形能形成一条平滑曲线，贯穿所有采样点。最简单的处理方式是用直线连接各点，再对粗糙的拐角进行圆滑处理。更高级的解决方案则是采用曲线拟合算法，例如样条函数或多项式拟合。这个问题没有标准答案，这种处理方式通过最小化*时间域*波形的不规则性来实现，但完全忽略了频率域的特性。

当信号在频域中包含编码信息时，我们通常忽略时域波形，专注于频谱分析。正如前一章所述，若想获得更精细的频谱采样（即在0到0.5频率区间增加采样点），可以在进行逆离散傅里叶变换（DFT）前，先在时域信号中填充零值。这种技术的对偶性原理同样适用于反向操作：若需在时域进行更精细的采样（插值处理），则需在逆离散傅里叶变换前对频谱进行零值填充。例如，当需要将50个采样点的信号插值为400个采样点时，具体操作步骤如下：(1)选取50个原始采样点，并通过添加零值使其总长度达到64个采样点；(2)使用64点离散傅里叶变换获取频谱，该频谱包含33个实部点和33个虚部点；(3)对频谱右侧进行零值填充处理。

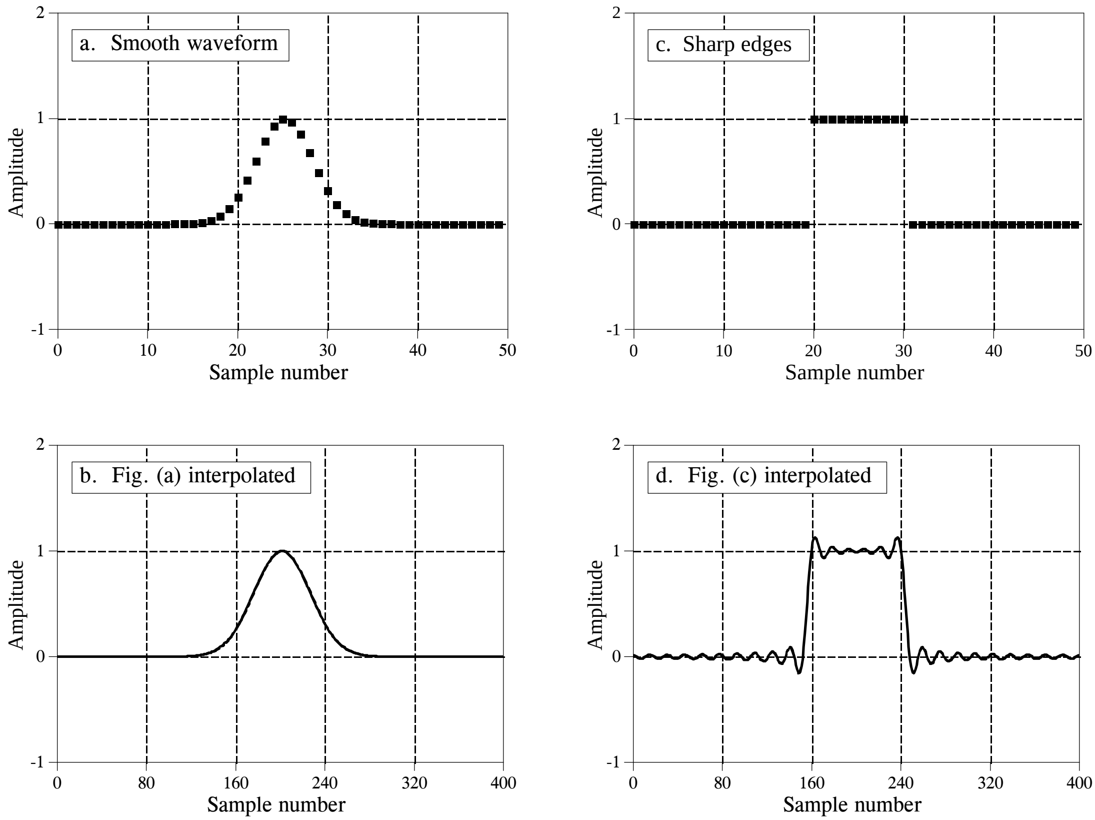


FIGURE 10-13
Interpolation by padding the frequency domain. Figures (a) and (c) each consist of 50 samples. These are interpolated to 400 samples by padding the frequency domain with zeros, resulting in (b) and (d), respectively. (Figures (b) and (d) are discrete signals, but are drawn as continuous lines because of the large number of samples).

(both the real and imaginary parts) with 224 zeros to make the frequency spectrum 257 points long. (4) Use a 512 point Inverse DFT to transform the data back into the time domain. This will result in a 512 sample signal that is a high resolution version of the 64 sample signal. The first 400 samples of this signal are an interpolated version of the original 50 samples.

The key feature of this technique is that the interpolated signal is composed of *exactly* the same frequencies as the original signal. This may or may not provide a well-behaved fit in the time domain. For example, Figs. 10-13 (a) and (b) show a 50 sample signal being interpolated into a 400 sample signal by this method. The interpolation is a smooth fit between the original points, much as if a curve fitting routine had been used. In comparison, (c) and (d) show another example where the time domain is a mess! The oscillatory behavior shown in (d) arises at edges or other discontinuities in the signal. This also includes any discontinuity between sample zero and N-1, since the time domain is viewed as being circular. This overshoot at discontinuities is called the *Gibbs effect*, and is discussed in Chapter 11. Another frequency domain interpolation technique is presented in Chapter 3, adding zeros between the time domain samples and low-pass filtering.

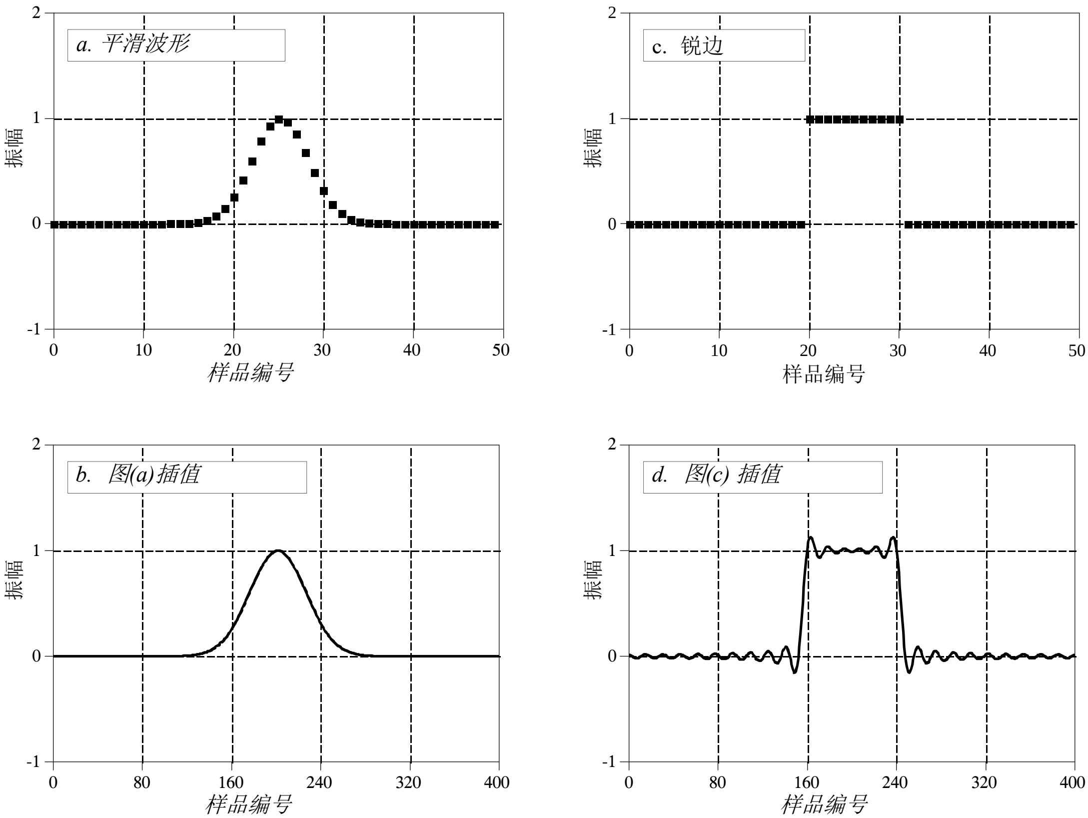


图10-13
通过在频域填充零值进行插值处理。图(a)和(c)各包含50个样本，通过在频域填充零值插值至400个样本，分别得到(b)和(d)。（图(b)和(d)为离散信号，但由于样本数量庞大，故以连续曲线形式呈现）。

（包括实部和虚部）添加224个零点，使频谱达到257个点。(4) 采用512点逆离散傅里叶变换（DFT）将数据还原至时域，最终得到512个采样点的信号，该信号是原始64采样点信号的高分辨率版本。其中前400个采样点为对原始50个采样点的插值处理结果。

该技术的核心特征在于插值信号由*完全*相同的频率构成。这种特性可能在时域产生良好的拟合效果，也可能不尽如人意。例如，图10-13(a)和(b)展示了通过该方法将50个采样点的信号插值为400个采样点的示例。插值过程如同使用曲线拟合程序般，实现了原始数据点之间的平滑衔接。相比之下，(c)和(d)则展示了时域呈现混乱状态的另一案例！(d)中出现的振荡现象源于信号边缘或其他不连续点。由于时域被视为循环结构，这种现象也包括采样点零与*N-1*之间的不连续性。这种在不连续点处的过冲现象被称为*吉布斯效应*，将在第11章详细讨论。第三章还介绍了另一种频域插值技术，该技术通过在时域采样点间添加零点并结合低通滤波实现。

Multiplying Signals (Amplitude Modulation)

An important Fourier transform property is that *convolution* in one domain corresponds to *multiplication* in the other domain. One side of this was discussed in the last chapter: time domain signals can be convolved by multiplying their frequency spectra. Amplitude modulation is an example of the reverse situation, multiplication in the time domain corresponds to convolution in the frequency domain. In addition, amplitude modulation provides an excellent example of how the elusive *negative* frequencies enter into everyday science and engineering problems.

Audio signals are great for short distance communication; when you speak, someone across the room hears you. On the other hand, radio frequencies are very good at propagating long distances. For instance, if a 100 volt, 1 MHz sine wave is fed into an antenna, the resulting radio wave can be detected in the next *room*, the next *country*, and even on the next *planet*. **Modulation** is the process of merging two signals to form a third signal with desirable characteristics of both. This always involves nonlinear processes such as multiplication; you can't just add the two signals together. In radio communication, modulation results in radio signals that can propagate long distances *and* carry along audio or other information.

Radio communication is an extremely well developed discipline, and many modulation schemes have been developed. One of the simplest is called **amplitude modulation**. Figure 10-14 shows an example of how amplitude modulation appears in both the time and frequency domains. Continuous signals will be used in this example, since modulation is usually carried out in analog electronics. However, the whole procedure could be carried out in discrete form if needed (the shape of the future!).

Figure (a) shows an audio signal with a DC bias such that the signal always has a positive value. Figure (b) shows that its frequency spectrum is composed of frequencies from 300 Hz to 3 kHz, the range needed for voice communication, plus a spike for the DC component. All other frequencies have been removed by analog filtering. Figures (c) and (d) show the **carrier wave**, a pure sinusoid of much higher frequency than the audio signal. In the time domain, amplitude modulation consists of *multiplying* the audio signal by the carrier wave. As shown in (e), this results in an oscillatory waveform that has an instantaneous amplitude proportional to the original audio signal. In the jargon of the field, the *envelope* of the carrier wave is equal to the modulating signal. This signal can be routed to an antenna, converted into a radio wave, and then detected by a receiving antenna. This results in a signal identical to (e) being generated in the radio receiver's electronics. A *detector* or *demodulator* circuit is then used to convert the waveform in (e) back into the waveform in (a).

Since the time domain signals are multiplied, the corresponding frequency spectra are convolved. That is, (f) is found by convolving (b) & (d). Since the spectrum of the carrier is a shifted delta function, the spectrum of the

信号相乘（调幅）

傅里叶变换的一个重要特性是：一个域中的*卷积运算*对应另一个域中的*乘法运算*。这一特性在上一章已有讨论——时域信号可以通过频谱相乘实现卷积运算。而调幅技术则呈现了相反的情况：时域中的乘法运算对应频域中的卷积运算。此外，调幅技术还生动展示了那些难以捉摸的*负频分量*如何渗透到日常科学与工程问题中的实际应用。

音频信号特别适合短距离通信，比如你说话时，房间另一头的人就能听到。而无线电波则擅长远距离传播。举个例子，当100伏特、1兆赫的正弦波输入天线时，产生的无线电波不仅能传到隔壁*房间*，还能传到*邻国*，甚至*到外太空*。调制技术就是把两个信号融合成第三个信号，让新信号兼具两者的优点。这个过程总涉及乘法等非线性运算，不能简单地把两个信号相加。在无线电通信中，调制后的信号既能远距离传播，*又能*传输音频或其他信息。

无线电通信是一门高度成熟的学科，目前已发展出多种调制方案。其中最简单的当属**幅度调制**。图10-14展示了幅度调制在时域和频域中的典型表现。本例采用连续信号，因为调制操作通常在模拟电子领域进行。不过如果需要的话，整个过程也可以用离散信号来实现（这可是未来的技术趋势！）。

图(a)展示了一个带有直流偏置的音频信号，该信号始终为正值。图(b)显示其频谱由300赫兹至3千赫兹的频率组成——这是语音通信所需的范围，外加直流分量的尖峰。所有其他频率均通过模拟滤波去除。图(c)和(d)展示了**载波**，即频率远高于音频信号的纯正弦波。在时域中，幅度调制是通过将音频信号与载波*相乘*实现的。如图(e)所示，这会产生一个瞬时振幅与原始音频信号成正比的振荡波形。用行业术语来说，载波的*包络线*等于调制信号。该信号可传输至天线，转换为无线电波后由接收天线检测。这会在无线电接收器的电子设备中生成与图(e)完全相同的信号。随后使用*检波器*或*解调器*电路，将图(e)中的波形还原为图(a)中的波形。

由于时域信号被相乘，相应的频谱则进行卷积运算。即通过卷积(b)与(d)可得到(f)。由于载波的频谱为移位δ函数，因此其频谱

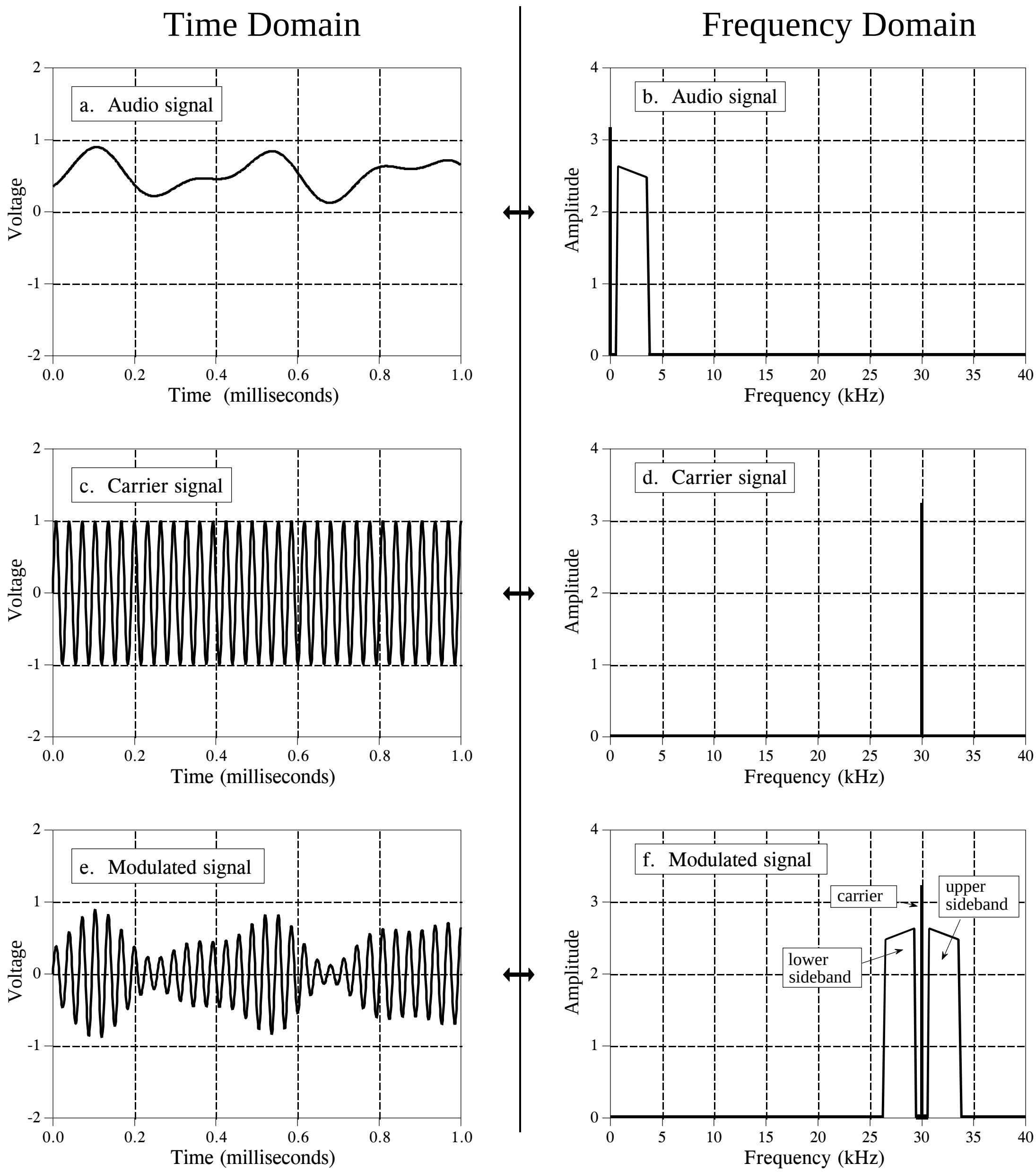


FIGURE 10-14
Amplitude modulation. In the time domain, amplitude modulation is achieved by multiplying the audio signal, (a), by the carrier signal, (c), to produce the modulated signal, (e). Since multiplication in the time domain corresponds to convolution in the frequency domain, the spectrum of the modulated signal is the spectrum of the audio signal shifted to the frequency of the carrier.

modulated signal is equal to the audio spectrum *shifted* to the frequency of the carrier. This results in a modulated spectrum composed of three components: a **carrier wave**, an **upper sideband**, and a **lower sideband**.

These correspond to the three parts of the original audio signal: the DC component, the positive frequencies between 0.3 and 3 kHz, and the negative

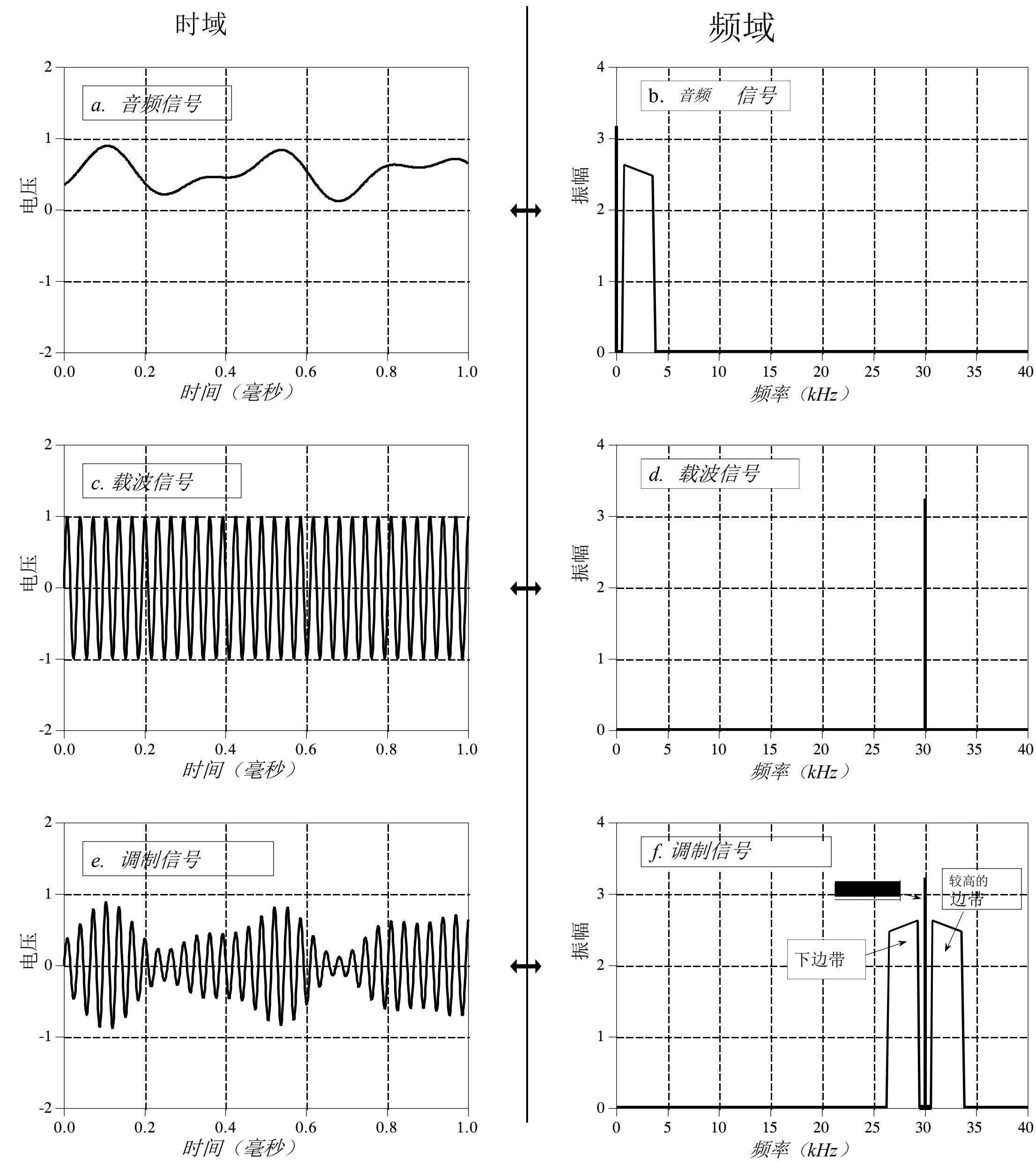


图10-14

振幅调制。在时域中，振幅调制是通过将音频信号(a)与载波信号(c)相乘，从而产生调制信号(e)来实现的。由于时域中的乘法运算在频域中对应于卷积运算，因此调制信号的频谱即为音频信号频谱向载波频率方向平移后的频谱。

调制信号等于音频频谱*移位*到载波频率。这导致调制频谱由三个部分组成：**载波**、**上边带**和**下边带**。

这些对应于原始音频信号的三个部分：直流分量、0.3至3 kHz之间的正频分量以及负频分量

frequencies between -0.3 and -3 kHz, respectively. Even though the negative frequencies in the original audio signal are somewhat elusive and abstract, the resulting frequencies in the lower sideband are as real as you could want them to be. The ghosts have taken human form!

Communication engineers live and die by this type of frequency domain analysis. For example, consider the frequency spectrum for television transmission. A standard TV signal has a frequency spectrum from DC to 6 MHz. By using these frequency shifting techniques, 82 of these 6 MHz wide channels are stacked end-to-end. For instance, channel 3 is from 60 to 66 MHz, channel 4 is from 66 to 72 MHz, channel 83 is from 884 to 890 MHz, etc. The television receiver moves the desired channel back to the DC to 6 MHz band for display on the screen. This scheme is called **frequency domain multiplexing**.

The Discrete Time Fourier Transform

The Discrete Time Fourier Transform (DTFT) is the member of the Fourier transform family that operates on *aperiodic*, *discrete* signals. The best way to understand the DTFT is how it relates to the DFT. To start, imagine that you acquire an N sample signal, and want to find its frequency spectrum. By using the DFT, the signal can be decomposed into $N/2+1$ sine and cosine waves, with frequencies equally spaced between zero and one-half of the sampling rate. As discussed in the last chapter, padding the time domain signal with zeros makes the period of the time domain *longer*, as well as making the spacing between samples in the frequency domain *narrower*. As N approaches infinity, the time domain becomes *aperiodic*, and the frequency domain becomes a *continuous* signal. This is the DTFT, the Fourier transform that relates an *aperiodic*, *discrete* signal, with a *periodic*, *continuous* frequency spectrum.

The mathematics of the DTFT can be understood by starting with the synthesis and analysis equations for the DFT (Eqs. 8-2, 8-3 and 8-4), and taking N to infinity:

EQUATION 10-1
The DTFT analysis equation. In this relation, $x[n]$ is the time domain signal with n running from 0 to $N-1$. The frequency spectrum is held in: $Re\,X(\omega)$ and $Im\,X(\omega)$, with ω taking on values

$$Re\,X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]\cos(\omega n)$$

$$Im\,X(\omega) = -\sum_{n=-\infty}^{+\infty} x[n]\sin(\omega n)$$

EQUATION 10-2
The DTFT synthesis equation.

$$x[n] = \frac{1}{\pi}\int_{0}^{\pi} Re\,X(\omega)\cos(\omega n) - Im\,X(\omega)\sin(\omega n)\,d\omega$$

频率范围分别在-0.3至-3千赫兹之间。虽然原始音频信号中的负频分量较为难以捉摸且抽象，但生成的低频边带频率却真实得令人惊叹——这些幽灵竟已化身为人类形态！

通信工程师的生存之道就在于这种频域分析技术。以电视传输的频谱为例，标准电视信号的频谱范围从直流到6兆赫。通过运用频移技术，这6兆赫的宽频带被82个信道首尾相连地堆叠起来。比如第3频道覆盖60-66兆赫，第4频道覆盖66-72兆赫，第83频道覆盖884-890兆赫，依此类推。电视接收器会将目标频道重新调回直流到6兆赫的频段，在屏幕上显示出来。这种技术方案被称为**频域复用**。

离散时间傅里叶变换

离散时间傅里叶变换（DTFT）是傅里叶变换家族中用于处理*非周期性*、*离散*信号的成员。理解 DTFT 的最佳方式是考察其与DFT的关系。首先，假设你获取了一个N个采样点的信号，并希望找到其频谱。通过使用DFT，该信号可以分解为$N/2+1$个正弦和余弦波，其频率均匀分布在零点到采样率一半之间。如上一章所述，用零值填充时域信号会使时域周期*延长*，同时使频域中的采样间隔*变窄*。当N趋近于无穷大时，时域变为*非周期性*，而频域则成为*连续*信号。这就是 DTFT ——将*非周期性*、*离散*信号与*周期性*、*连续*频谱联系起来的傅里叶变换。

DTFT 的数学原理可以通过从DFT的合成与分析方程（方程8-2、8-3和8-4）入手，并将N取为无穷大来理解：

方程10-1
DTFT 分析方程。在此关系中，$x[n]$是时域信号，其中n从 0 到N-1运行。频谱表示为：ReX（T）和$Im\ X$（T），其中T取值

$$Re X(\omega) = \sum_{n=-\infty}^{+\infty} x[n] \cos(\omega n)$$

$$Im X(\omega) = -\sum_{n=-\infty}^{+\infty} x[n] \sin(\omega n)$$

方程10-2 DTFT 合成方程。

$$x[n] = \frac{1}{\pi} \int_0^{\pi} Re X(\omega) \cos(\omega n) - Im X(\omega) \sin(\omega n) \, d\omega$$

There are many subtle details in these relations. First, the time domain signal, $x[n]$, is still discrete, and therefore is represented by *brackets*. In comparison, the frequency domain signals, $Re\,X(\omega)$ & $Im\,X(\omega)$, are continuous, and are thus written with *parentheses*. Since the frequency domain is continuous, the synthesis equation must be written as an integral, rather than a summation.

As discussed in Chapter 8, frequency is represented in the DFT's frequency domain by one of three variables: k, an index that runs from 0 to $N/2$; f, the fraction of the sampling rate, running from 0 to 0.5; or ω, the fraction of the sampling rate expressed as a natural frequency, running from 0 to π. The spectrum of the DTFT is continuous, so either f or ω can be used. The common choice is ω, because it makes the equations shorter by eliminating the always present factor of 2π. Remember, when ω is used, the frequency spectrum extends from 0 to π, which corresponds to DC to one-half of the sampling rate. To make things even more complicated, many authors use Ω (an upper case omega) to represent this frequency in the DTFT, rather than ω (a lower case omega).

When calculating the inverse DFT, samples 0 and $N/2$ must be divided by two (Eq. 8-3) before the synthesis can be carried out (Eq. 8-2). This is not necessary with the DTFT. As you recall, this action in the DFT is related to the frequency spectrum being defined as a *spectral density*, i.e., amplitude per unit of bandwidth. When the spectrum becomes continuous, the special treatment of the end points disappear. However, there is still a normalization factor that must be included, the $2/N$ in the DFT (Eq. 8-3) becomes $1/\pi$ in the DTFT (Eq. 10-2). Some authors place these terms in front of the *synthesis* equation, while others place them in front of the *analysis* equation. Suppose you start with some time domain signal. After taking the Fourier transform, and then the Inverse Fourier transform, you want to end up with what you started. That is, the $1/\pi$ term (or the $2/N$ term) must be encountered somewhere along the way, either in the forward or in the inverse transform. Some authors even split the term between the two transforms by placing $1/\sqrt{\pi}$ in front of both.

Since the DTFT involves infinite summations and integrals, it cannot be calculated with a digital computer. Its main use is in theoretical problems as an alternative to the DFT. For instance, suppose you want to find the frequency response of a system from its impulse response. If the impulse response is known as an *array of numbers*, such as might be obtained from an experimental measurement or computer simulation, a DFT program is run on a computer. This provides the frequency spectrum as another *array of numbers*, equally spaced between 0 and 0.5 of the sampling rate.

In other cases, the impulse response might be know as an *equation*, such as a sinc function (described in the next chapter) or an exponentially decaying sinusoid. The DTFT is used here to mathematically calculate the frequency domain as another *equation*, specifying the entire continuous curve between 0 and 0.5. While the DFT could also be used for this calculation, it would only provide an equation for *samples* of the frequency response, not the entire curve.

这些关系中存在许多微妙的细节。首先，时域信号$x[n]$仍然是离散的，因此用*括号*表示。相比之下，频域信号*ReX*（T）和*ImX*（T）是连续的，因此用*括号*表示。由于频域是连续的，合成方程必须写成积分形式，而非求和形式。

如第8章所述，频率在DFT的频域中由三个变量之一表示：*k*（一个从0到*N*/2的索引）；*f*（采样率的分数，范围从0到0.5）；或T（以固有频率表示的采样率分数，范围从0到**π**）。DTFT 的频谱是连续的，因此可以使用*f*或T。常用的选择是T，因为它通过消除始终存在的2**π**因子使方程更简短。记住，当使用T时，频率谱从0延伸到**π**，这对应于直流到采样率的一半。为了使事情更加复杂，许多作者在 DTFT 中使用**Ω**（大写**Ω**）来表示这个频率，而不是T（小写**Ω**）。

在计算逆DFT时，样本0和*N*/2必须除以二（*等式8-3）才能进行合成（等式8-2）。这在 DTFT 中并非必要。如你所忆，DFT中的这一操作与将频谱定义为谱密度*有关，即单位带宽内的振幅。当频谱变为连续时，端点的特殊处理消失。然而仍需包含一个归一化因子，DFT中的2/*N*（*等式8-3）在 DTFT 中变为1/π*（*等式10-2）。部分作者将这些项置于合成*方程前，而另一些则置于*分析*方程前。假设你从某个时域信号开始，经过傅里叶变换和逆傅里叶变换后，希望最终得到初始信号。也就是说，1/**π**项（或2/*N*项）必须在正向或逆向变换的某个环节出现。部分作者甚至通过放置$1\sqrt{\pi}$ 在两者面前

由于傅里叶变换（DTFT）涉及无限级数求和与积分运算，无法通过数字计算机直接计算。其主要应用领域是作为离散傅里叶变换（DFT）的替代方案，用于解决理论问题。例如，若需根据系统的脉冲响应求得其频率响应，当已知脉冲响应为*数值序列*（如实验测量或计算机模拟所得）时，可运行DFT程序。该程序将生成频率谱，即在采样率0到0.5区间内等间距分布的*数值序列*。

在其他情况下，脉冲响应可能被描述为一个*方程*，例如sinc函数（将在下一章详细说明）或指数衰减的正弦波。此处采用 DTFT 来数学计算频域，将其表示为另一个*方程*，完整呈现0到0.5之间的连续曲线。虽然也可以使用DFT进行此类计算，但它只能提供频率响应*样本*的方程，而无法呈现完整的曲线。

Parseval's Relation

Since the time and frequency domains are equivalent representations of the same signal, they must have the same *energy*. This is called Parseval's relation, and holds for all members of the Fourier transform family. For the DFT, Parseval's relation is expressed:

EQUATION 10-3
Parseval's relation. In this equation, $x[i]$ is a time domain signal with i running from 0 to N-1, and $X[k]$ is its *modified* frequency spectrum, with k running from 0 to N/2. The *modified* frequency spectrum is found by taking the DFT of the signal, and dividing the first and last frequencies (sample 0 and N/2) by the square-root of two.

$$\sum_{i=0}^{N-1} x[i]^2 = \frac{2}{N} \sum_{k=0}^{N/2} Mag\, X[k]^2$$

The left side of this equation is the total energy contained in the time domain signal, found by summing the energies of the N individual samples. Likewise, the right side is the energy contained in the frequency domain, found by summing the energies of the $N/2+1$ sinusoids. Remember from physics that *energy* is proportional to the *amplitude squared*. For example, the energy in a spring is proportional to the displacement squared, and the energy stored in a capacitor is proportional to the voltage squared. In Eq. 10-3, $X[f]$ is the frequency spectrum of $x[n]$, with one slight modification: the first and last frequency components, $X[0]$ & $X[N/2]$, have been divided by $\sqrt{2}$. This modification, along with the $2/N$ factor on the right side of the equation, accounts for several subtle details of calculating and summing energies.

To understand these corrections, start by finding the frequency domain representation of the signal by using the DFT. Next, convert the frequency domain into the amplitudes of the sinusoids needed to reconstruct the signal, as previously defined in Eq. 8-3. This is done by dividing the first and last points (sample 0 and N/2) by 2, and then dividing all of the points by N/2. While this provides the amplitudes of the sinusoids, they are expressed as a *peak* amplitude, not the *root-mean-square* (rms) amplitude needed for energy calculations. In a sinusoid, the peak amplitude is converted to rms by dividing by $\sqrt{2}$. This correction must be made to all of the frequency domain values, *except* sample 0 and N/2. This is because these two sinusoids are unique; one is a constant value, while the other alternates between two constant values. For these two special cases, the *peak* amplitude is already equal to the *rms* value. All of the values in the frequency domain are squared and then summed. The last step is to divide the summed value by N, to account for each sample in the frequency domain being converted into a sinusoid that covers N values in the time domain. Working through all of these details produces Eq. 10-3.

While Parseval's relation is interesting from the physics it describes (conservation of energy), it has few practical uses in DSP.

帕斯卡关系

由于时域和频域是同一信号的等效表示，它们必须具有相同的*能量*。这被称为Parseval关系，适用于傅里叶变换家族的所有成员。对于DFT，Parseval关系表示为：

方程10-3
帕斯卡关系。在此方程中，$x[i]$ 是一个时域信号，其中i从 0 到N-1，而$X[k]$ 是其*改进* 频率谱，其中k从 0 到N/2。该*改进*频率谱通过对信号进行 DFT 并将首尾频率（样本 0 和N/2）除以根号二获得。

$$\sum_{i=0}^{N-1} x[i]^2 = \frac{2}{N} \sum_{k=0}^{N/2} Mag\, X[k]^2$$

这个方程的左侧是时域信号中包含的总能量，通过求N个独立样本的能量之和得到。同样地，右侧是频域中包含的能量，通过求N/2+1个正弦波的能量之和得到。记得物理学中*能量* 与*振幅的平方* 成正比。例如，弹簧中的能量与位移的平方成正比，电容器中储存的能量与电压的平方成正比。在等式10-3中，$X[f]$ 是$x[n]$的频谱，有一个细微的修改：第一个和最后一个频率分量，$X[0]$ &$X[N/2]$，被$\sqrt{2}$除。这个修改，加上方程右侧的2/N因子，解释了计算和求和能量时的几个微妙细节。

要理解这些校正，首先通过使用DFT找到信号的频域表示。接着，将频域转换为重建信号所需的正弦波振幅，如等式8-3中先前所定义的那样。这是通过将第一个和最后一个点（采样点0和N/2）除以2，然后将所有点除以N/2来实现的。虽然这提供了正弦波的振幅，但它们以*峰值* 振幅表示，而非能量计算所需的*均方根* （rms）振幅。在正弦波中，峰值振幅通过除以转换为均方根。$\sqrt{2}$. 这一修正必须应用于所有频域值，*除了*样本0和N/2。这是因为这两个正弦波是独特的：一个是恒定值，另一个则在两个恒定值之间交替变化。对于这两个特殊情况，*峰值* 幅度已经等于*rms*值。频域中的所有值都先平方再求和。最后一步是将求和值除以N，以解释频域中的每个样本被转换为覆盖时域N个值的正弦波。通过处理所有这些细节，最终得到等式10-3。

尽管帕尔塞瓦尔关系式从其所描述的物理学角度（能量守恒）具有研究价值，但在数字信号处理（DSP）领域实际应用价值有限。