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# On Error Performance and Concatenated Coding of Polar Codes in AWGN Channels

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**Abstract.** In additive white Gaussian noise (AWGN) channels, construction of polar codes is needed for every operating signal-to-noise ratio (SNR). Recently, the proposal of the design-SNR reduces the computation effort in constructing polar codes. In this paper, we prove that although the BER performance of the design-SNR construction is not affected, the packet-error-rate (PER) performance is degraded compared with the point-by-point construction. Therefore, a concatenation scheme is proposed to improve the degraded PER performance. Results show the validity of the proposed concatenation scheme when employing the design-SNR construction.

## 1. Introduction

Polar codes, which were proposed by Erdal Arikan in [1], have attracted extensive attention since they are proven to achieve the capacity of binary input discrete memory less symmetric channel (B-DMC) with a low encoding and decoding complexity.

For a given block length  $N$ , a sorting algorithm is needed to select  $K$  best bit channels among  $N$  bit channels. Density evolution (DE) was put forward in [2], and Gaussian approximation (GA) proposed in [3] was designed to choose good bit channels for additive white Gaussian noise (AWGN) channels. The algorithm of selecting good bit channels through degrading or upgrading operations [4] can be used for all types of B-DMCs.

However, in AWGN channels, due to the channel quality change (the operating SNR varies), construction of polar codes needs to be done to select optimal bit channels for each SNR value, which causes high complexity in the construction of polar codes. In this paper, the construction using every operating SNR is called the point-by-point construction. Recently, a method proposed in [5] realized the one-to-one mapping between the code rate and the SNR value from an information theoretical



analysis. According to the study, for a given code rate  $R$ , one can obtain the good bit channels with a fixed SNR and apply these good bit channels to a range of SNR. This fixed SNR is called the design-SNR for the given code rate. By employing a design-SNR for a range of operating SNRs, the construction complexity of polar codes is reduced.

It is shown in [5, 6] that the employment of the design-SNR does not degrade the bit-error-rate (BER) performance. However, as analyzed in this paper, the packet-error-rate (PER) performance employing the design-SNR is degraded, which is undesirable for practical communication systems. The fact that the BER performance is not affected while the PER performance is degraded reveals that the number of errors per error packet is smaller when employing the design-SNR compared with the point-by-point construction. Based on this phenomenon, this paper proposes a simple concatenation scheme to improve the error performance of polar codes employing the efficient design-SNR construction.

The concatenation proposed in this paper is different from the concatenation schemes in [7, 8]: the concatenation in [7, 8] is the direct concatenation while the concatenation in this paper contains a simple interleaving between the inner and outer code. The interleaving scheme in this paper simply divides the coded bits from the outer code into several sections. This is also different from the interleaving scheme in [9] where the interleaving aims to break the possible error correlation among the information bits.

The contribution of the current paper compared with the work in [5] is that: 1) the current paper investigates the PER degradation of polar codes employing the design-SNR in [5, 6]; 2) To combat the PER degradation in [5, 6], polar codes are proposed to be simply interleaved with an outer coder, employing the theoretically calculated design-SNR based on [5]. Compared with the point-by-point construction, the proposed interleaving scheme produces an improved PER performance when employing the design-SNR construction.

The rest of the paper is organized as the following. In Section 2, the basic theories of polar code are introduced. The motivation of the work is set forth in Section 3. In Section 4, to achieve a better error performance employing the design-SNR, a concatenation scheme is proposed to interleave polar codes with BCH codes. Numerical results of the concatenation scheme are shown in Section 5. At the last section, the conclusion is provided.

## 2. Polar Code Basics

Given a B-DMC  $W$  whose input alphabet  $X$  takes values in  $\{0,1\}$  and the output alphabet is  $Y$ , the transition probability of  $W$  is written as  $W(y|x)$  with  $x \in X$  and  $y \in Y$ . Assuming that there are  $N$  independent copies of  $W$ , where  $N = 2^n, n \geq 1$ . The channel combination will produce a vector channel  $W_N : X_1^N \rightarrow Y_1^N$ ,

$$W_N(y_1^N | u_1^N) = W^N(y_1^N | u_1^N G_N) \quad (1)$$

Where  $W^N$  denotes the transition probability of the  $N$  independent channels  $W$ ,  $u_1^N$  is the binary source vector, and  $G_N$  is the generator matrix. Let  $B_N$  be the bit-reversal permutation matrix and  $F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ . The generator matrix is  $G_N = B_N F^{\otimes n}$ . For the channel splitting,  $W_N$  is split into  $N$  bit channels. The transition probability of the  $i$ th bit channel is defined as

$$W_N^{(i)}(y_1^N, u_1^{i-1} | u_i) = \sum_{u_{i+1}^N \in \mathcal{X}^{N-i}} \frac{1}{2^{N-i}} w_N(y_1^N | u_1^N) \quad (2)$$

Where  $u_i^j$  is the subvector taking entries from  $u_i$  to  $u_j$  of the vector  $u_1^N$ .

The codeword of polar codes can be obtained with the parameter vector  $(N, K, A, u_{A^c})$ , where  $A$  contains the location of the information bits,  $K = |A|$ ,  $A^c$  contains the location of the frozen bits, and  $u_{A^c}$  contains the frozen bits. In this paper, the subvector  $u_{A^c}$  takes entries from  $u_1^N$  defined by the set  $A^c$ . Once these parameters are defined, the codeword can be produced with

$$x_1^N = u_A G_A + u_{A^c} G_{A^c} \quad (3)$$

Where  $G_A$  is the sub-matrix of  $G_N$  formed by the rows with indices in  $A$  and  $G_{A^c}$  is the sub-matrix of  $G_N$  formed by the rows whose indices belong to  $A^c$ .

SC decoding proposed by Arikan in [1] is the first decoding algorithm of polar codes. For a code with the length of  $N$ , SC decoder consists of  $N$  decision elements, each calculating the likelihood ratio (LR) of a source bit. The LR value for each source bit can be calculated recursively as in [1].

### 3. Motivation of the work

From [5, 6], it is shown that for a range of SNR, there exists a design-SNR that works for the whole range in the construction stage. Here ‘works’ means that the resultant BER using the design-SNR is almost the same as that using the operating SNR itself.

#### 3.1. Calculation of the design-SNR

Theoretically, a new good set of bit channels is needed with the change of SNR. If a system operates at several SNR points, then a channel sorting is needed for each SNR point. Different from the algorithm of locality search mentioned in [6], the work in [5] points out that for the underlying AWGN channels, there is a mapping between the code rate  $R$  and the SNR value  $\gamma$ :

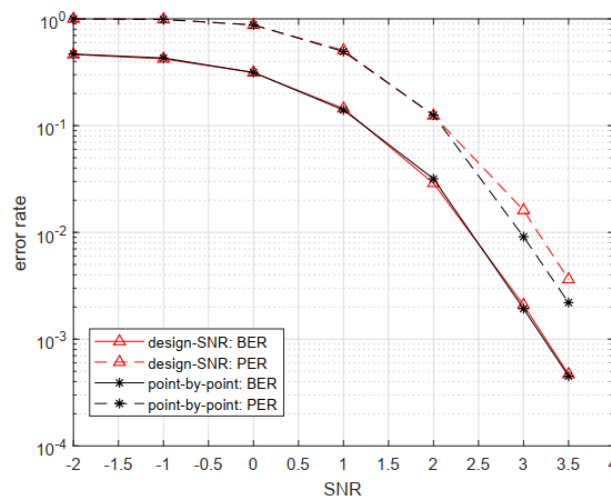
$$\gamma = C^{-1}(R) \quad (4)$$

Where  $C^{-1}(\cdot)$  is the inverse capacity function of the binary input AWGN channels. This calculated SNR is called the design-SNR.

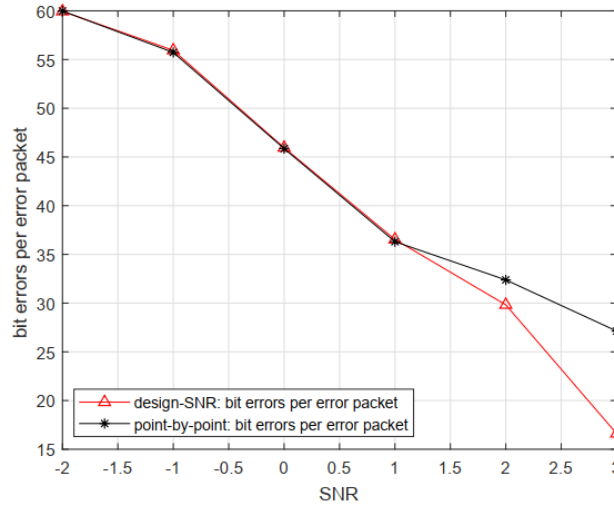
### 3.2. The error performance

The BER performance loss employing a design-SNR is shown in [5, 6] to be almost negligible compared with the point-by-point construction of polar codes. Here is an example:  $N = 256$ ,  $R = 0.5$ , and the design-SNR is calculated to be 0.188 dB by (4). The difference of simulations between the system employing the design-SNR and that with the point-by-point construction is the information set. For the point-by-point construction, the information set changes with the operating SNR. For the design-SNR construction, the information set is fixed with the design-SNR.

Figure. 1 and Figure. 2 are produced with the SC decoding algorithm, in which the BER, PER and bit errors per error packet curves are reported. As in [5, 6], the BER performance between the design-SNR construction and the point-by-point construction is almost the same. However, the PER of the design-SNR construction is worse than the point-by-point construction. This phenomenon is theoretically stated below.



**Figure 1.** The code length is 256 with the code rate 0.5. At SNR = 3.5 dB, the PER employing the design-SNR is 1.6455 times bigger than that of the point-by-point construction.



**Figure 2.** The bit errors per error packet of polar codes constructed using the design-SNR and every operating SNR (the point-by-point construction). The code length is 256 with the code rate 0.5.

**Lemma 1.** For a range of SNR and a given code rate  $R$ , the union bound of PER employing a design-SNR cannot be better than that of the point-by-point construction.

Proof: Let the interested SNR range being  $\text{SNR} = \{\gamma_1, \gamma_2, \dots, \gamma_J\}$ , where  $J$  is the number of operating SNR values. For the SNR  $\gamma_j (j \leq J)$ , the point-by-point construction selects an information set  $A$ .

Define the union bound conditioned on this information set as  $P_B(A)$  [4]. Then

$$P_B(A) \leq \sum_{i \in A} P(W_N^{(i)}) \quad (5)$$

Where  $P(W_N^{(i)})$  is the error probability of the  $i$ th information bit. This packet error rate is optimized for the operating SNR  $\gamma_j$ . For the design-SNR construction, a design-SNR  $\gamma_d$  is calculated with (4) for the whole range of SNR with the given code rate  $R$ . Let  $A'$  be the information set optimized with the SNR  $\gamma_d$ . The corresponding union bound is defined as  $P_B(A')$ , and we have

$$P_B(A') \leq \sum_{i \in A'} P(W_N^{(i)}) \quad (6)$$

Considering  $P_B(A)$  and  $P_B(A')$  for the whole range of the  $\text{SNR} = \{\gamma_1, \gamma_2, \dots, \gamma_J\}$ , it can be immediately concluded that  $P_B(A') \geq P_B(A)$  since  $P_B(A)$  is optimized (with a minimum value) for

each SNR values while  $P_B(A')$  is optimized only for the design-SNR value  $\gamma_d$ . Note that  $P_B(A') = P_B(A)$  when  $\lambda_d = \gamma_j$ .

From Lemma 1, it can be concluded that the PER of a polar code employing the design-SNR construction is generally larger than that of the system employing the point-by-point construction. The following reasoning reveals that the average number of bit errors per error packet of the construction using the design-SNR is generally smaller than that of the point-by-point construction. Define the following variables:

- $N_B$ : the number of packets considered;
- $EB_p$ : the number of packets in error when employing the point-by-point construction;
- $EB_d$ : the number of packets in error when employing the design-SNR construction;
- $E_{bp}$ : the average number of bits in error per error packet from the point-by-point construction;
- $E_{bd}$ : the average number of bits in error per error packet from the design-SNR construction.

From Lemma 1, the following holds (the subscript 'p' indicates the point-by-point construction and 'd' indicates the design-SNR construction):

$$PER_p = \frac{E_{Bp}}{N_B} \leq PER_d = \frac{E_{Bd}}{N_B} \quad (7)$$

Form [5, 6], the following holds:

$$BER_p = \frac{E_{Bp} E_{bp}}{N_B K} \approx BER_d = \frac{E_{Bd} E_{bd}}{N_B K} \quad (8)$$

With (7), it is seen that  $E_{Bp} \leq E_{Bd}$ . Applying this to (8), it can be concluded that  $E_{bp} \geq E_{bd}$ .

This fact can be employed to improve the system performance while maintaining an efficient construction by using the design-SNR.

#### 4. The concatenation scheme

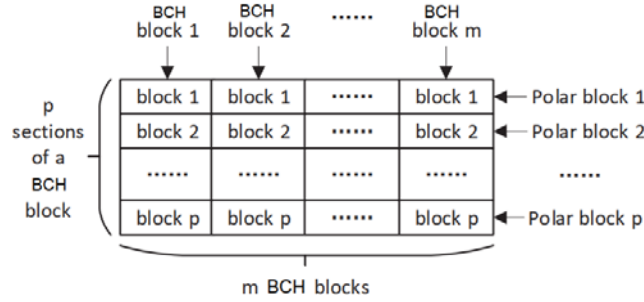
Suppose the outer code is a BCH code with a code length  $N_l$ . There are  $K_l$  information bits in each coded block. The concatenation scheme is to divide the encoded  $N_l$  BCH bits into  $p$  sections. Let

$(K_1, K_2, K_3, \dots, K_p)$  be the number of bits in each section, and we have  $\sum_{i=1}^p K_i = N_l$ . These

$p$  sections of coded BCH bits are treated as the information bits for  $p$  polar blocks. The required number of BCH blocks to form the input information bits of the  $p$  polar blocks is approximately

$$m \approx \frac{K}{\max\{K_1, K_2, \dots, K_p\}}, \quad m \in \mathbb{Z} \quad (9)$$

This interleaving scheme is illustrated in Figure. 3.



**Figure 3.**The general scheme of interleaving

To determine the value of  $p$ , assume that  $A_e$  is the average number of bit error per error block of polar codes and  $d$  indicates the number of errors that the BCH code can correct in each BCH block. The value of  $p$  can be determined from

$$\frac{A_e}{N \cdot R} \cdot \frac{N_l}{p} \leq d, \quad p, d \in \mathbb{Z} \quad (10)$$

Note that the value of  $A_e$  can be estimated first for a given polar code in a practical environment.

Or  $A_e$  can be assumed to be half of the information bits in a polar code block. This is the worst case assumption for independent errors. A little over-design is also encouraged in practice to account for the non-independent errors from the SC decoder.

After the polar codes encoding and the SC decoding, the BER performance of the design-SNR construction is close to that of the point-by-point construction, while the PER performance of the design-SNR construction is degraded with respect to the point-by-point construction from Lemma 1. Based on this performance, the design-SNR construction has less errors in each error packet than the point-by-point construction and these errors will be further scattered to BCH blocks. This results in less errors in each BCH block when employing the design-SNR construction. With this simple interleaving, it can be predicted that the error performance of the concatenation system using the design-SNR construction will have a bigger improvement than the point-by-point construction.

## 5. Numerical results

The outer code is the BCH code (127,57). The polar code has the block length  $N = 256$  and the code rate  $R = 1/2$ . For polar codes, the design-SNR is calculated from (4) to be 0.188 dB. Then Tal-Vardy's algorithm [4] is used to construct polar codes using this design-SNR. For the point-by-point construction, the information set is selected by feeding each simulated SNR to Tal-Vardy's algorithm. The SC decoding is employed to decode polar codes. In our simulation,  $A_e$  is half of the number of information bits in a polar code block. With BCH code (127, 57), the number of



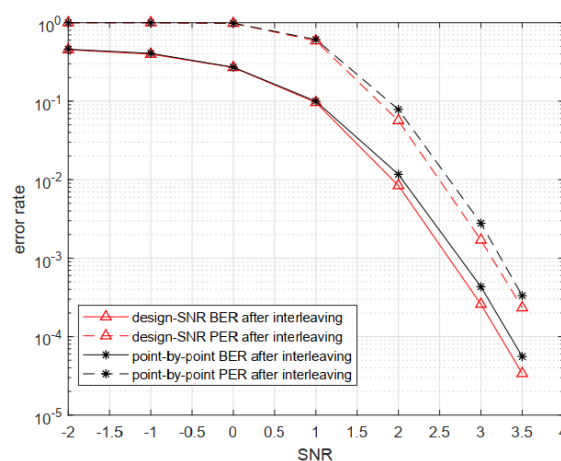
errors that it can correct is  $d = 11$ . Then from (10), it can be calculated that  $p \geq 5.77$ . In the simulations,  $p = 6$  is selected. By employing (9),  $m$  is calculated to be 6.

The BER and PER performance of the concatenation scheme is shown in Figure. 4, where the lines with triangle are the performance of the construction using the design-SNR and the starred lines are the performance of the point-by-point construction. Compared with Figure. 1, it can be found that the BER performance of the construction of polar codes using the design-SNR is still the same as that of the point-by-point construction in the proposed concatenation scheme.

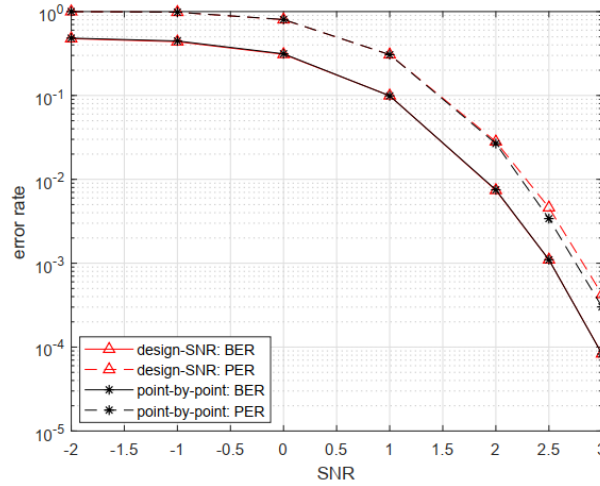
**Table 1.** The Comparison Of Per Between The Design-SNR and Point-by-point Construction Under SC Decoding.

	Before interleaving	After interleaving
Point-by-Point	0.0022	0.000333
Design-SNR	0.00362	0.000233
Design-SNR/Point-by-point	1.6455	0.6997
Improvement	1.6455/0.6997=2.3517	

The improvement of the concatenated PER performance of the construction using the design-SNR is obvious from Figure. 4. With the concatenation scheme, the efficient construction using the design-SNR matches the point-by-point construction in terms of the PER performance. Take SNR = 3.5 dB as an example. The numerical comparison of PER at this SNR is shown in Table 1. Before interleaving, the PER of the design-SNR construction is 1.65 times of that of the point-by-point construction. With interleaving, the PER of the design-SNR construction is only 0.70 times of the point-by-point construction, resulting in a 2.35 times improvement.

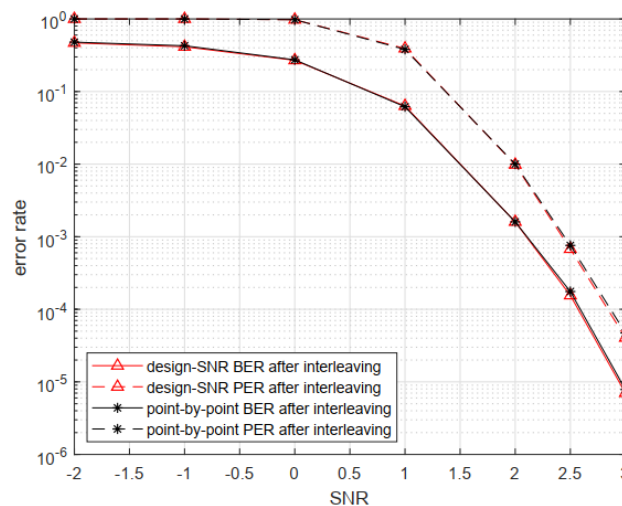


**Figure 4.** The BER and PER performance of the concatenated code using the point-by-point construction and the design-SNR Construction under SC decoding.



**Figure 5.** The code length is 256 and the number of information bits is 128 with 8 CRC check bits. The list size is 8.

The effect of the design-SNR construction is also simulated for successive cancellation list (SCL) decoding [10, 11] of polar codes in Figure. 5 and Figure. 6. There are eight CRC check bits and the list size is 8. As in the SC decoding case, for the SCL decoding, the design-SNR construction shows almost the same BER performance as the point-by-point construction. However, the design-SNR construction for the SCL decoding also degrades the PER performance. Similar to Table 1, the PER performance before and after the proposed interleaving is calculated with 3 dB as an example. It is found to be a 1.65 times improvement for the SCL decoding at this SNR. It is of interest to note that if adaptive list decoding [12] is employed, the average number of list is found to be 2 when  $E_b/N_0 \geq 1.6$  dB for the reported cases there. Similar to the calculations in Table 1, there is a 2.7 times improvement when the list size 2 is employed in the proposed scheme.



**Figure 6.** The same parameters as in Figure. 5.

## 6. Conclusion

In this paper, the design-SNR construction is shown theoretically to have less bit errors per error packet than the point-by-point construction. Based on this phenomenon, a concatenation scheme is proposed to improve PER performance of polar codes. The numerical results validate the proposed concatenation scheme. This study shows that the application of the design-SNR can simplify the construction of polar codes compared with the point-by-point construction while maintaining a good system performance through the proposed concatenation scheme.

## 7. Acknowledgments

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## References

- [1] E. Arikan, "Channel polarization: A method for constructing capacity achieving codes for symmetric binary-input memoryless channels," *IEEE Trans. Inf. Theory*, vol. 55, no. 7, 2009, pp. 3051–3073.
- [2] R. Mori and T. Tanaka, "Performance of polar codes with the construction using density evolution," *IEEE Commun. Lett.*, vol. 13, no. 7, 2019, pp. 519–521.
- [3] D. Wu, Y. Li, and Y. Sun, "Construction and block error rate analysis of polar codes over awgn channel based on gaussian approximation," *IEEE Commun. Lett.*, vol. 18, no. 7, 2014, pp. 1099–1102.
- [4] I. Tal and A. Vardy, "How to construct polar codes," *IEEE Trans. Inf. Theory*, vol. 59, no. 10, 2013, pp. 6562–6582.
- [5] R. Deng, L. Li, and Y. Hu, "On the polar code encoding in fading channels," in *Proc. 2016 IEEE 84th Veh. Technol. Conf. (VTC-Fall)*, Montreal, QC, Canada, 2016, pp. 1–5.
- [6] H. Vangala, E. Viterbo, and Y. Hong, "A comparative study of polar code constructions for the AWGN channel," 2015, [Online]. Available: <http://arxiv.org/abs/1501.02473>.
- [7] A. Eslami and H. Pishro-Nik, "A practical approach to polar codes," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, 2011, pp. 16–20.
- [8] J. Guo, M. Qin, A. G. i Fabregas, and P. H. Siegel, "Enhanced belief propagation decoding of polar codes through concatenation," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, 2014, pp. 2987 – 2991.
- [9] Y. Meng, L. Li, and Y. Hu, "A novel interleaving scheme for polar codes," in *Proc. 2016 IEEE 84th Veh. Technol. Conf. (VTC-Fall)*, Montreal, QC, Canada, 2016, pp. 1–5.
- [10] K. Chen, K. Niu, and J. Lin, "Improved Successive Cancellation Decoding of Polar Codes," *IEEE Trans. Commun.*, vol. 61, no. 8, 2013, pp. 3100–3107.
- [11] I. Tal and A. Vardy, "List Decoding of Polar Codes," *IEEE Trans. Inf. Theory*, vol. 61, no. 5, 2015, pp. 2213–2226.
- [12] B. Li, H. Shen, and D. Tse, "An adaptive successive cancellation list decoder for polar codes with cyclic redundancy check," *IEEE Communications Letters*, vol. 16, no. 12, 2013, pp. 2044–2047.