ASSIGNMENT 8: The Digital Fourier Transform

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1 Brief Up:

The tasks of this assignment is the following:

- Implementation of DFT in python.
- Using FFT which is an implementation of the DFT.
- Plot graphs to analyze them by altering window sizes.

2 Spectrum of $sin(5t)$:

The solution is already presented in the assignment. To fix the non-zero phase near the peaks for some values we sample the input signal at an appropriate frequency.

```
N1 = 128
t1 = linspace(0,2*pi,N1+1); t1 = t1[:-1]
y1 = sin(5*t1)
Y1 = fftshift(fft(y1))/N1
w1 = linspace(-64,63,N1)

# Magnitude and phase plot for the DFT of sin(5t)
figure(0)
plot(w1,abs(Y1))
title(r"Spectrum of $\sin(5t)$")
ylabel(r"$|Y(\omega)|\rightarrow$")
xlabel(r"$\omega\rightarrow$")
xlim([-10,10])
grid(True)

figure(1)
plot(w1,angle(Y1),'ro')
ii = where(abs(Y1)>1e-3)
plot(w1[ii],angle(Y1[ii]),'go')
```

```
title(r"Phase of $\sin(5t)$")
ylabel(r"$\angle Y(\omega)\rightarrow$")
xlabel(r"$\omega\rightarrow$")
xlim([-10,10])
grid(True)
```

As expected we get 2 peaks at +5 and -5 with height 0.5. The phases of the peaks at $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ are also expected based on the expansion of a sine wave ie:

$$sin(5t) = 0.5(\frac{e^{5t}}{j} - \frac{e^{-5t}}{j}) \tag{1}$$

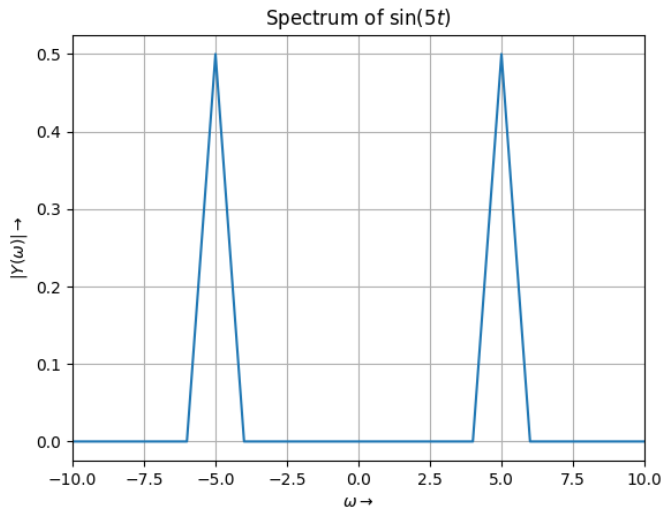

Figure 1: Magnitude plot of DFT of sin(5t)

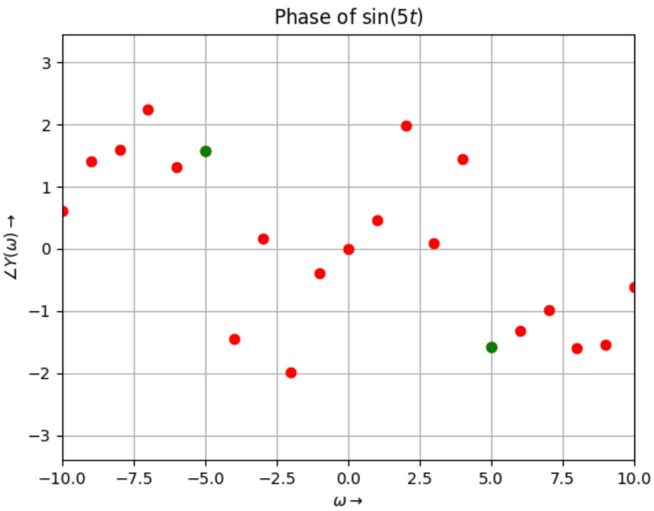

Figure 2: Phase plot of DFT of sin(5t)

3 Spectrum of Amplitude Modulated Wave:

Consider the signal:

$$f(t) = (1 + 0.1\cos(t))\cos(10t) \tag{2}$$

```
N2 = 512
t2 = linspace(-4*pi,4*pi,N2+1); t2 = t2[:-1]
y2 = (1 + 0.1*cos(t2))*cos(10*t2)
Y2 = fftshift(fft(y2))/N2
w2 = linspace(-64,64,N2+1); w2 = w2[:-1]

# Magnitude and phase plot for the DFT of (1 + 0.1cos(t))cos(10t)
figure(2)
plot(w2,abs(Y2))
title(r"Spectrum of $(1 + 0.1*cos(t))*cos(10*t)$")
ylabel(r"$|Y(\omega)|\rightarrow$")
xlabel(r"$\omega\rightarrow$")
xlim([-15,15])
grid(True)
figure(3)
plot(w2,angle(Y2),'ro')
title(r"Phase of $(1 + 0.1*cos(t))*cos(10*t)$")
ylabel(r"$\angle Y(\omega)\rightarrow$")
xlabel(r"$\omega\rightarrow$")
```

```
xlim([-15,15])
grid(True)
```

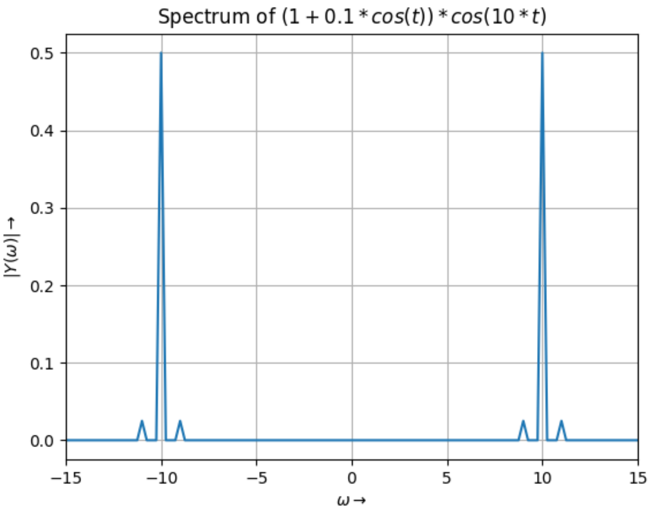

Figure 3: Spectrum of mag of $f(t) = (1 + 0.1\cos(t))\cos(10t)$

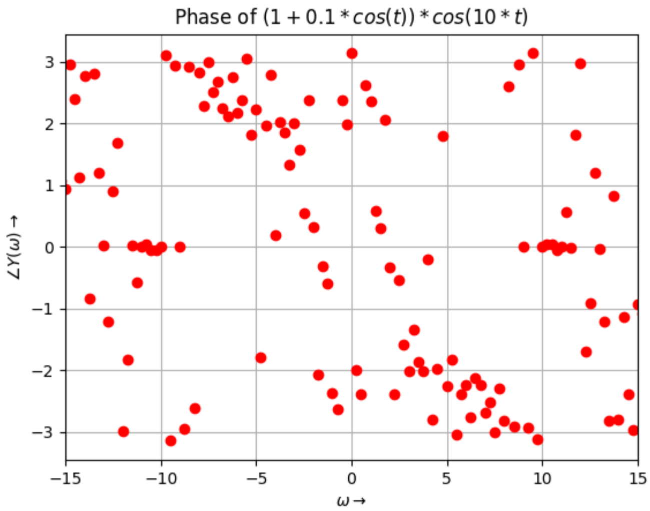

Figure 4: Spectrum of phase of $f(t) = (1 + 0.1\cos(t))\cos(10t)$

At all 3 peaks, the phase is 0 as expected for a cosine wave.

4 Spectrum of $sin^3(t)$:

This signal can be expressed as a sum of sine waves using this identity:
$\sin^3(t) = \frac{3}{4}\sin(t) - \frac{1}{4}\sin(3t)$
There has to be 2 peaks at frequencies 1 and 3, and phases similar to that expected from a sum of sinusoids.

```
N3 = 512
t3 = linspace(-4*pi,4*pi,N3+1); t3 = t3[:-1]
y3 = (3*sin(t3) - sin(3*t3))/4
Y3 = fftshift(fft(y3))/N3
w3 = linspace(-64,64,N3+1); w3 = w3[:-1]

# Magnitude and phase plot for the DFT of sin^3(t)
figure(4)
plot(w3,abs(Y3))
title(r"Spectrum of $sin^{3}(t)$")
ylabel(r"$|Y(\omega)|\rightarrow$")
xlabel(r"$\omega\rightarrow$")
xlim([-15,15])
grid(True)
figure(5)
plot(w3,angle(Y3),'ro')
title(r"Phase of $sin^{3}(t)$")
ylabel(r"$\angle Y(\omega)\rightarrow$")
xlabel(r"$\omega\rightarrow$")
xlim([-15,15])
grid(True)
```

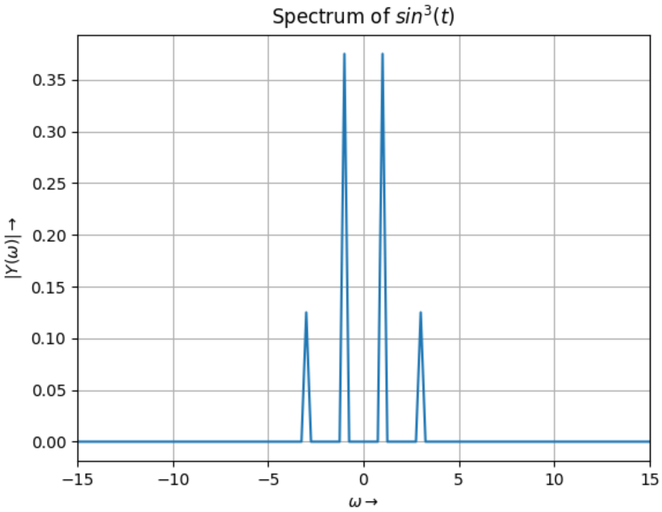

Figure 5: Spectrum of mag of $f(t) = sin^3(t)$

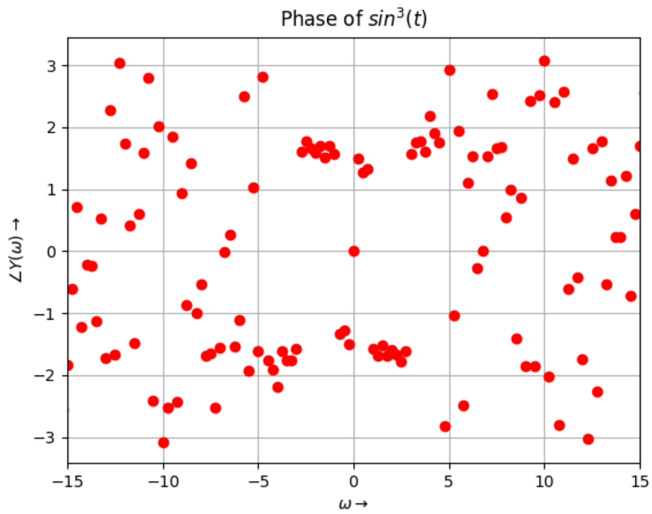

Figure 6: Spectrum of phase of $f(t) = sin^3(t)$

5 Spectrum of $cos^3(t)$:

This signal can be expressed as a sum of cosine waves using this identity:
$\sin^3(t) = \frac{3}{4}\cos(t) + \frac{1}{4}\cos(3t)$
There has to be 2 peaks at frequencies 1 and 3, and phase=0 at the peaks.

```
y4 = (3*cos(t3) + cos(3*t3))/4
Y4 = fftshift(fft(y4))/N3
```

```
# Magnitude and phase plot for the DFT of cos^3(t)
figure(6)
plot(w3,abs(Y4))
title(r"Spectrum of $cos^{3}(t)$")
ylabel(r"$|Y(\omega)|\rightarrow$")
xlabel(r"$\omega\rightarrow$")
xlim([-15,15])
grid(True)
figure(7)
plot(w3,angle(Y4),'ro')
title(r"Phase of $cos^{3}(t)$")
ylabel(r"$\angle Y(\omega)\rightarrow$")
xlabel(r"$\omega\rightarrow$")
xlim([-15,15])
grid(True)
```

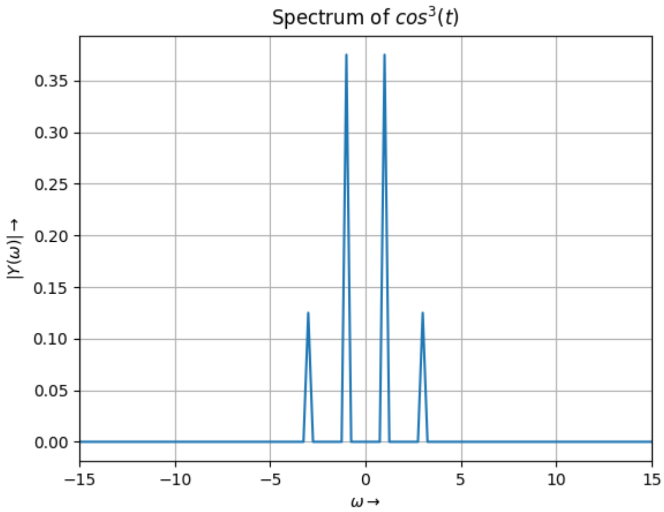

Figure 7: Spectrum of mag of $f(t) = cos^3(t)$

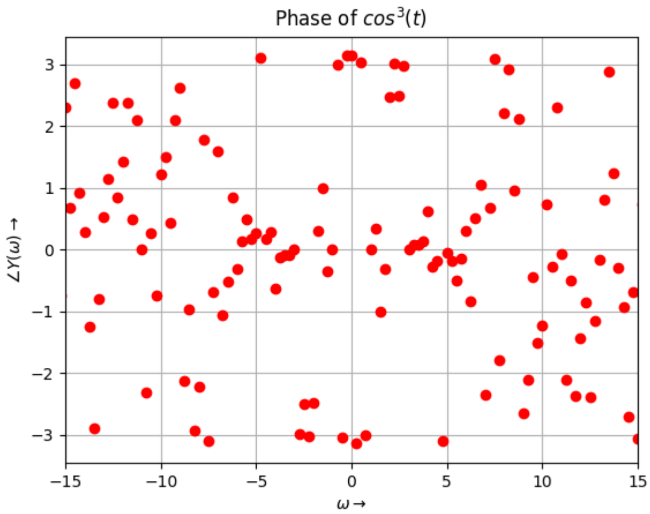

Figure 8: Spectrum of phase of $f(t) = cos^3(t)$

6 Spectrum of Frequency Modulated Wave:

Consider the signal:

$$f(t) = \cos(20t + 5\cos(t)) \tag{3}$$

```
y5 = cos(20*t3 + 5*cos(t3))
Y5 = fftshift(fft(y5))/N3

# Magnitude plot for the DFT of cos(20t + 5cos(t))
figure(8)
plot(w3,abs(Y5))
title(r"Spectrum of $cos(20t + 5cos(t))$")
ylabel(r"$|Y(\omega)|\rightarrow$")
xlabel(r"$\omega\rightarrow$")
xlim([-40,40])
grid(True)

# Phase plot for the DFT of cos(20t + 5cos(t)) ( magnitude is greater tha
figure(9)
ii = where(abs(Y5)>=1e-3)
plot(w3[ii],angle(Y5[ii]),'go')
title(r"Phase of $cos(20t + 5cos(t))$")
ylabel(r"$\angle Y(\omega)\rightarrow$")
xlabel(r"$\omega\rightarrow$")
xlim([-40,40])
grid(True)
```

The number of peaks has increased. The energy in the side bands is comparable to that of the main signal.

- We see that the plot is Phase modulated since phase of the signal, varying proportionally to amplitude of the message signal, is $= 20$ and infinite side band frequencies which are produced by 5 cost, because cos(t) is infinitely long signal.
- The strength of the side band frequencies either decays or is very small as they are away from the center frequency or carrier frequency component .
- Phase spectra is a mix of different phases from [,] because the phase is changed continuously with respect to time, the carrier signal can represent either a sine or cosine depending on the phase contribution from cos(t).

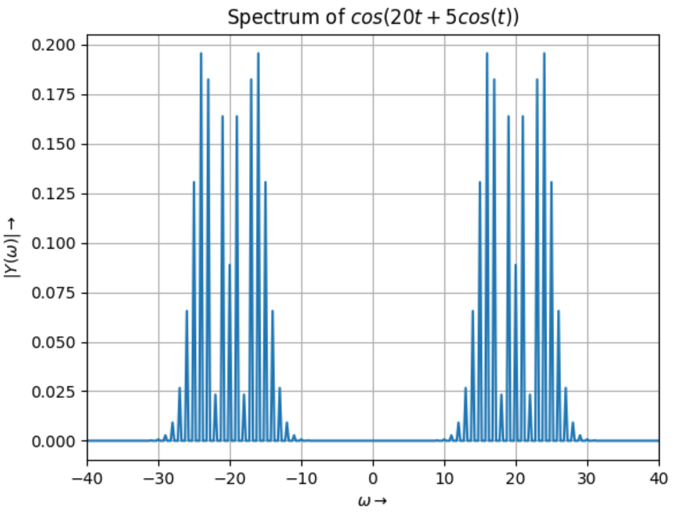

Figure 9: Spectrum of mag of $f(t) = (\cos(20t) + 5\cos(t)$

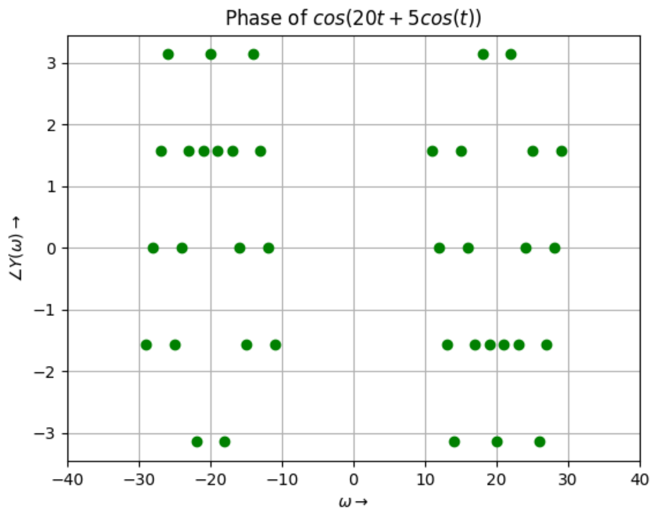

Figure 10: Spectrum of phase of $f(t) = (\cos(20t) + 5\cos(t)$

7 CTFT and DFT of a Gaussian:

The Fourier transform of a signal $x(t)$ is given by:

$$X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{4}$$

- We can approximate(as we have integration) this by the Fourier transform of the windowed version of the signal $x(t)$, with a sufficiently large window as Gaussian curves tend to 0 for large values of t . Let the window be of size T. We get:

$$X(\omega) \approx \frac{1}{2\pi} \int_{-T/2}^{T/2} x(t) e^{-j\omega t} dt \tag{5}$$

- We shall write this integral as a Riemann sum with a small time step $\Delta t = \frac{T}{N}$, which gives:

$$X(\omega) \approx \frac{\Delta t}{2\pi} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} x(n\Delta t) e^{-j\omega n\Delta t} \tag{6}$$

- We sample our spectrum with a sampling period in the frequency domain of $\Delta\omega = \frac{2\pi}{T}$, which makes our continuous time signal periodic with period equal to the window size T. Our transform then becomes:

$$X(k\Delta\omega) \approx \frac{\Delta t}{2\pi} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} x(n\Delta t) e^{-j\frac{2\pi}{N}kn} \tag{7}$$

- We now obtain a form which is similar to DFT(for a finite window size). Thus:

$$X(k\Delta\omega) \approx \frac{\Delta t}{2\pi} DFT\{x(n\Delta t)\} \tag{8}$$

- We have assumed few approximations as we use a finite window size the Riemann approximation.

- It is possible to improve these approximations by making the window size T larger, and by decreasing the time domain sampling period or increasing the number of samples N. We compute the appropriate values for these using iterative loop , by keeping the sampling frequency constant and setting a tolerance.

  The expression for the Gaussian equation given is :

$$x(t) = e^{\frac{-t^2}{2}} \tag{9}$$

  The CTFT is given by:

$$X(j\omega) = \frac{1}{\sqrt{2\pi}} e^{\frac{-\omega^2}{2}} \tag{10}$$

```
#The gaussian and it's precision
T = 2*pi
N = 128
Y_previous=0
tolerance=1e-6
err_t=tolerance+1

count = 0
#finding the correct window size considering tolerance using iterative lo
while err_t>tolerance:
    x=linspace(-T/2,T/2,N+1)[:-1]
    w = linspace(-N*pi/T,N*pi/T,N+1)[:-1]
    y = exp(-0.5*x**2)
    Y=fftshift(fft(ifftshift(y)))*T/(2*pi*N)
    err_t = sum(abs(Y[::2] - Y_previous))
    Y_previous = Y
    count+=1
    T*=2
    N*=2

Y_expec=1/sqrt(2*pi) * exp(-w**2/2)
#computing error
error = sum(abs(abs(Y)-Y_expec))
print("True error: ", error)
print("samples = "+str(N)+" time period = "+str(T/pi) + "*pi")

figure(10)
subplot(2,1,1)
plot(w,abs(Y),lw=2)
xlim([-5,5])
ylabel('Magnitude',size=16)
title("Estimate fft of gaussian")
grid(True)
subplot(2,1,2)
plot(w,angle(Y),'ro',lw=2)
ii=where(abs(Y)>1e-3)
plot(w[ii],angle(Y[ii]),'go',lw=2)
xlim([-5,5])
ylabel("Phase",size=16)
xlabel("w",size=16)
grid(True)
show()

figure(11)
```

```
subplot(2,1,1)
plot(w,abs(Y_expec),lw=2)
xlim([-5,5])
ylabel('Magnitude',size=16)
title("True fft of gaussian")
grid(True)
subplot(2,1,2)
plot(w,angle(Y_expec),'ro',lw=2)
ii=where(abs(Y_expec)>1e-3)
plot(w[ii],angle(Y_expec[ii]),'go',lw=2)
xlim([-5,5])
ylabel("Phase",size=16)
xlabel("w",size=16)
grid(True)
show()
```

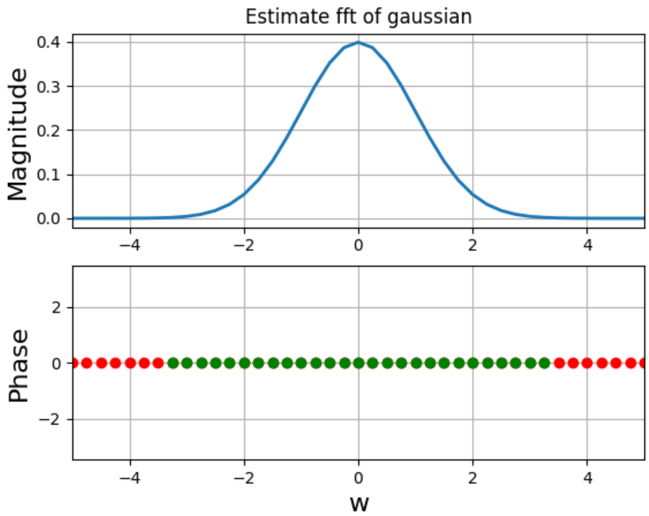

Figure 11: Estimated FFT of Gaussian

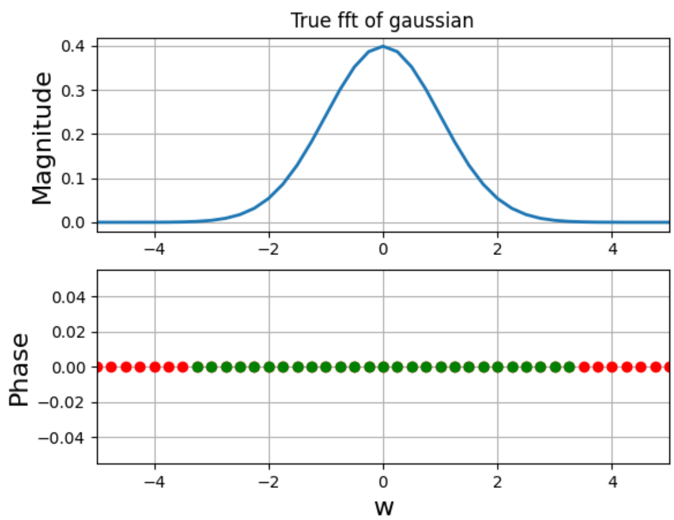

Figure 12: True FFT of Gaussian

8 Results:

The summation of the absolute values of the error, number of samples and window size are as follows:
True error: 6.846904845668299e-15
samples =1024 time period = 16.0*pi

9 Conclusion

- We used fft on a random signal to check the error in in reconstructed signal,which was low.
- We analysed on how to find DFT and normalising factors for Gaussian functions and hence recover the analog Fourier transform using DFT ,also to find parameters like window size and sampling rate by minimizing the error with tolerance upto 10-6.
- We need to use ifftshift and fftshift to fix distorted phase responses.
- FFT works well for signals with samples in powers of 2 , as it divides the samples into even and odd and goes dividing further to compute the DFT.