Computational Physics - Project 5

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Abstract

We use Monte Carlo simulation to model a simple transaction-based market and calculate the economic distribution of agents. We add a fixed saving fraction and reproduce the Gibbs- and Gamma distributions in the work of Patriarca, Chakarborti and Kaski (2004). Furthermore, we add agent transaction preferences in the form of near economic neighbours and former transactions. Our results are comparable to those of Goswami and Sen (2014). We calculate the distribution drop off in the different models for comparison to the Pareto law, but lack a good criteria for defining the tails of the distributions.

1 Introduction

It has been shown that the upper parts of money distribution in populations follows the Pareto law [1],

$$w_m \propto m^{-1-\nu} \qquad \nu \in [1,2] \tag{1}$$

Many groups have tried to reproduce this macroscopic behavior by simulating a market built on rule-based transactions between microscopic agents.

In their work, Patriarca, Chakraborti and Kaski show that including a saving criterion to the model gives something close to a gamma distribution of wealth [2]. Goswami and Sen demonstrate that adding interaction preference criteria, such as economic proximity and number of previous interactions, significantly changes the distribution [3].

Here, we attempt to reproduce the work of these two groups, using Monte Carlo simulations. We start out only considering straightforward exchanges of money between two random agents, before we gradually add layers of complexity to the model. In particular, we implement the three criteria described above; savings, economic proximity and previous interactions. Finally, we compare the higher end of all the distributions to the Pareto law.

2 Theory

In its simplest form, the model consists of N agents starting out with an equal fortune m_0. Two agents i and j are then drawn randomly among the population and perform a transaction so that

$$\begin{aligned} m_i' &= \epsilon(m_i + m_j) \\ m_j' &= (1-\epsilon)(m_i + m_j), \end{aligned} \tag{2}$$

where m_i' and m_j' are their respective fortunes after the transaction and $\epsilon \in (0,1]$ is a random number drawn from a uniform distribution. In other words, the total amount of money is conserved,

$$m_i + m_j = m_i' + m_j', \tag{3}$$

and there is no possibility for debt. After a sufficient number of transactions, this simple system should reach an equilibrium state where the money follows a Gibbs distribution,

$$w_m = \beta \exp(-\beta m) \qquad \beta = \frac{1}{\langle m \rangle}. \tag{4}$$

2.1 Saving criterion

If each agent save a fixed fraction $\lambda \in (0, 1]$ of their money in each transaction, a smaller part of their fortune is available for exchange. This gives a revised version of Equation 2,

$$\begin{aligned} m_i' &= \lambda m_i + \epsilon(1-\lambda)(m_i + m_j) \\ m_j' &= \lambda m_j + (1-\epsilon)(1-\lambda)(m_i + m_j) \end{aligned} \tag{5}$$

where the conservation law from Equation 3 still holds. Patriarca et. al. has shown that the equilibrium state closely follows the gamma distribution,

$$\gamma_m = \frac{x_n^{n-1} \exp(-x_n)}{\Gamma(\lambda)}, \qquad n \equiv 1 + \frac{3\lambda}{1-\lambda}, \quad x_n \equiv \frac{mn}{\langle m \rangle}. \tag{6}$$

2.2 Neighbour criterion

In the real world, it is unrealistic to expect every pair of agents to interact with each other with equal probability. As a rule of thumb, we expect agents that are economically closer to interact more frequently. A way of accounting for this, is to only accept a transactions with the probability

$$p_{ij} \propto |m_i - m_j|^{-\alpha} \qquad \alpha > 0. \tag{7}$$

Thus "nearer neighbours" are more likely to interact. The bigger α is, the more sensitive the model is to neighbours. If $\alpha = 0$, the neighbour criterion disappears.

2.3 Former transaction criterion

Additionally, we consider agents that has interacted before more likely to interact again. We can implement this by modifying the probability from Equation 7,

$$p_{ij} \propto |m_i - m_j|^{-\alpha}(c_{ij} + 1)^{\gamma} \qquad \alpha, \gamma > 0, \tag{8}$$

where c_{ij} is the number of previous interactions between agents i and j. For $\gamma = 0$, we are back to the pure neighbour criterion.

2.4 Putting everything together

The final model is then completely described by Equation 5 and 8. For a chosen set of parameters λ, α and γ, a given transaction between two agents i and j is accepted with the probability in Equation 8. If it is accepted, the agents exchange money according to Equation 5, where $\epsilon \in (0, 1]$ is a randomly drawn number.

3 Method

All of the implementation of the model, along with the obtained datafiles can be found in the Github repository https://github.com/HHaughom/FYS3150 in the StockMarket folder.

We simulate the financial transactions with the different combination of parameters shown below.

Agents, N	**Saving, λ**	**Neighbour, α**	**Former transaction, γ**
500	0.0, 0.25, 0.5, 0.9	0.0	0.0
500	0.0	0.5, 1.0, 1.5, 2.0	0.0
500	0.5	0.5, 1.0, 1.5, 2.0	0.0
1000	0.0	0.5, 1.0, 1.5, 2.0	0.0
1000	0.5	0.5, 1.0, 1.5, 2.0	0.0
1000	0.0	1.0	0.0, 1.0, 2.0, 3.0, 4.0
1000	0.5	1.0	0.0, 1.0, 2.0, 3.0, 4.0
1000	0.0	2.0	0.0, 1.0, 2.0, 3.0, 4.0
1000	0.5	2.0	0.0, 1.0, 2.0, 3.0, 4.0

Table 1: The different combination of parameters used in the simulations.

We let every agent start out with $m_0 = 1.0$. We perform 10^7 transactions, where two random agents are drawn and the transaction accepted and performed according to Equation 8 and 5. After every 10^5 transaction, we find the normalised distribution of money in the population by counting the number of agents in each bin $\Delta m = 0.1$ and dividing by the total number of agents, N. We then check how much the distribution has changed over the previous 10^5 transactions.

$$\Delta P = \frac{\sum_{bin} n_{bin}(t) - n_{bin}(t - 100\,000)}{N}$$

If $\Delta P < 10^{-3}$, we assume the distribution has reached an equilibrium and stop the transactions. We repeat the steps above for 10^4 Monte Carlo cycles, and take the average. This is done for all the combination of parameters in Table 1.

Finally, we fit the tails of the distributions to power laws. We define the start of the tails as the point on the right hand side of the peak that is 1% of the peak value. We then use the curve_fit function from SciPy to estimate the exponent of the power law fit[4].

4 Results

All graphical tail power law fits are collected in Appendix A.

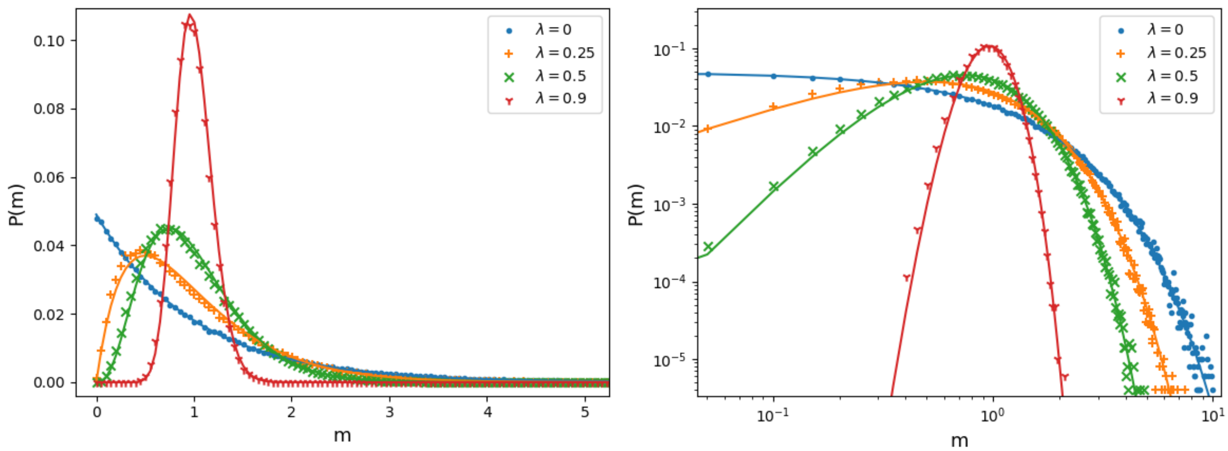
4.1 Saving criterion

Agents, N	**Saving, λ**	**Neighbour, α**	**Former transaction, γ**	**Tail exponent**
500	0.0, 0.25, 0.5, 0.9	0.0	0.0	-5.3, -9.3, -9.3, -21.5

Table 2: The different combinations of parameters for the agent-independent model.

In the simplest case with no savings and equal transactions probability between all agents, the population follows the Gibbs distribution. This, of course, gives a linear graph when the y-axis is put on a logarithmic scale, as is the case in Figure 2.

When saving is included in the model, the distributions closely follow the γ_m given in Equation 6. Increasing the saving fraction gives a more strongly peaked distribution and a higher average. This is also manifested in the increasing exponent in the power law fits of the tails in Table 2.



(a) Linear (b) Logarithmic

Figure 1: The normalised distribution of the $N = 500$ agents with different saving fractions. For the $\lambda = 0$ case, the solid line is given by the Gibbs distribution from Equation 4, while the rest are given by the gamma distribution from Equation 6.

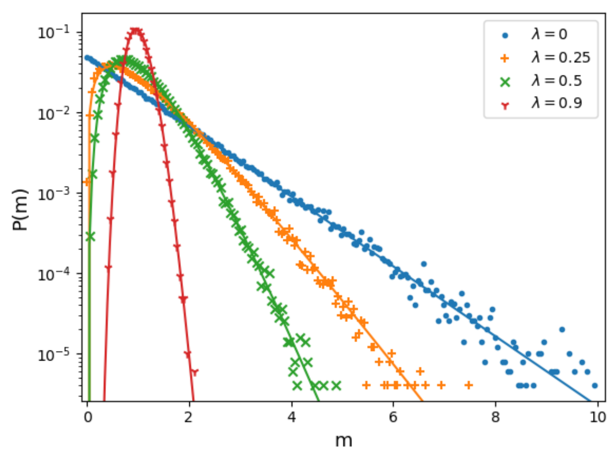


Figure 2: The distribution for the different saving fractions plotted in a log scale to illustrate the linearity of the $\lambda = 0$ case.

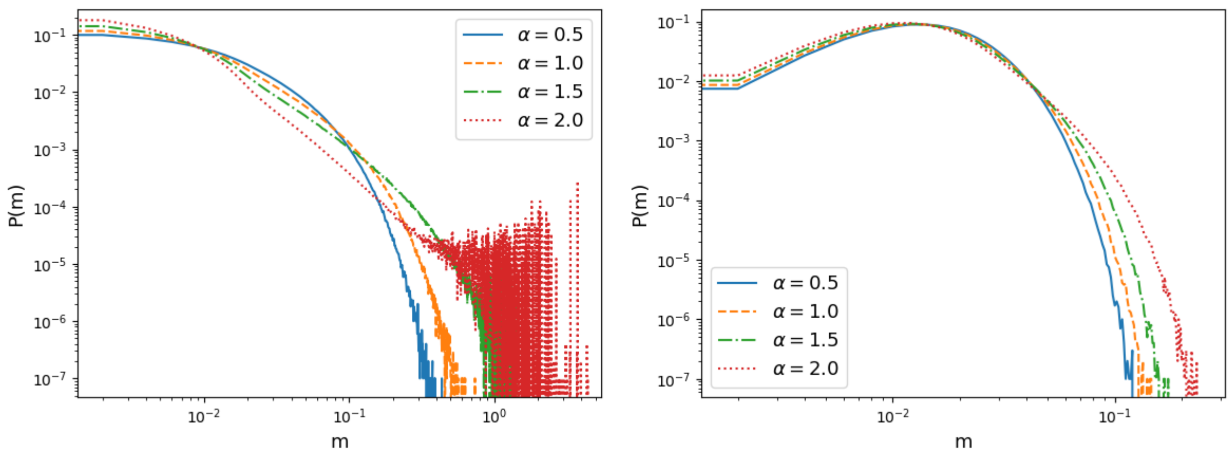
4.2 Neighbour criterion

Agents, N	**Saving, λ**	**Neighbour, α**	**Former transaction, γ**	**Tail exponent**
500	0.0	0.5, 1.0, 1.5, 2.0	0.0	-4.5, -3.2, -2.2, -3.2
500	0.5	0.5, 1.0, 1.5, 2.0	0.0	-9.1, -8.0, -6.9, -5.2
1000	0.0	0.5, 1.0, 1.5, 2.0	0.0	-4.5, -3.2, -2.2, -2.1
1000	0.5	0.5, 1.0, 1.5, 2.0	0.0	-9.2, -8.0, -6.9, -5.3

Table 3: The combinations of parameters for the neighbour criterion model.

The distributions change significantly when you add a nearest neighbour preference to the transactions. The shape of the distribution is similar for the different α, with maxima at the lowest value $m \sim 10^{-3}$. Higher α gives a slightly higher maximum and a less steep tail. For the case without saving, the $\alpha = 2$ distribution becomes unstable at the high end.

As in the previous model, adding the saving fraction $\lambda = 0.5$ shifts the maxima to the right at $m \sim 10^{-2}$ and gives a much steeper tail. Here, the $\alpha = 2$ case is more stable. In Figure 3 below, we have plotted the distributions for $N = 1000$. As the tail exponents in Table 3 indicate, the distributions are very similar for $N = 500$. For this reason, these plots are not included.



(a) No saving, $\lambda = 0.0$.

(b) Saving, $\lambda = 0.5$.

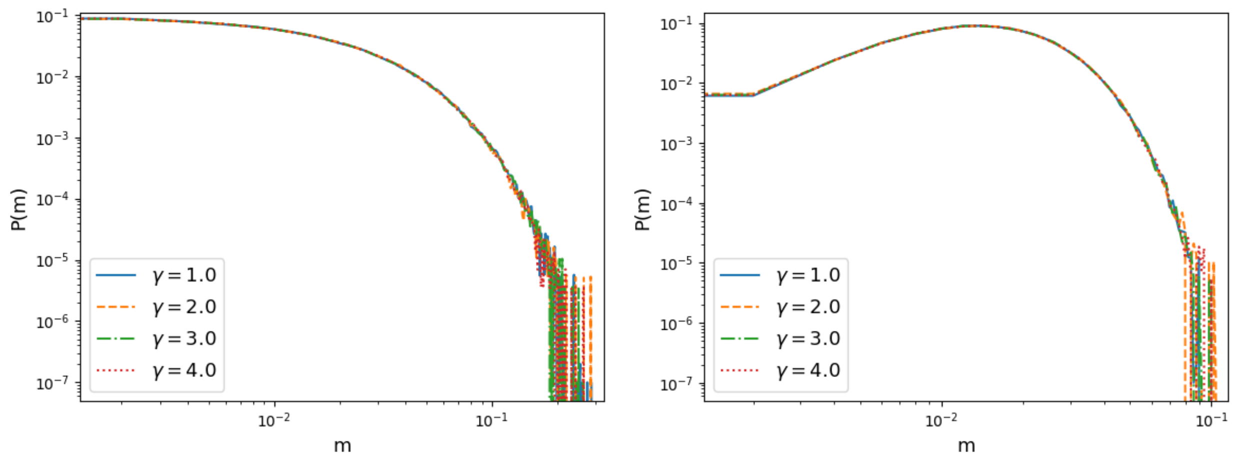
Figure 3: The distribution of $N = 1\,000$ agents with and without saving and for different nearest neighbour parameters α.

4.3 Former transaction criterion

Agents, N	**Saving, λ**	**Neighbour, α**	**Former transaction, γ**	**Tail exponent**
1000	0.0	1.0	1.0, 2.0, 3.0, 4.0	-5.5, -5.5, -5.3, -5.9
1000	0.5	1.0	1.0, 2.0, 3.0, 4.0	-10.7, -11.9, -10.4, -8.8
1000	0.0	2.0	1.0, 2.0, 3.0, 4.0	-5.5, -5.7, -5.6, -5.3
1000	0.5	2.0	1.0, 2.0, 3.0, 4.0	-11.5, -11.0, -10.2, -11.1

Table 4: The different combinations of parameters for the complete model with both the neighbour and former transactions criteria.

Finally, including former transactions, we see that the distributions change little for different γ as well as for different α. The results for $\alpha = 1$ and $\alpha = 2$ is visually indistinguishable, and we have only included the first below in Figure 4. As in the previous cases, both the maxima and the tail exponent increase when saving is included.



(a) No saving, $\lambda = 0.0$.

(b) Saving, $\lambda = 0.5$.

Figure 4: The distribution of $N = 1000$ agents with and without saving, and for different former transaction parameters γ. The nearest neighbour parameter is $\alpha = 1.0$.

5 Discussion

At its core, the market model is a simple, random interaction based model. For the most primitive case without saving and agent preference $\lambda = \alpha = \gamma = 0$, we get the Gibbs distribution that peaks at the smallest m and exponentially drops off from there.

5.1 Saving criterion

When the agents then are forced to save a given fraction of their money in each transaction, the average agent keeps a bigger part of their fortune. The higher the saving fraction, the smaller variance in the distribution, giving a more equal population. The shrinking variance also gives a steeper tail for higher saving fractions, giving an exponent far away from the $[-2, -3]$ range in Equation 1.

The results are very similar to those of Patriarca et.al. (see Fig. 1 and 2 in Patriarca et.al.). This is of course expected, as we are following the same method, but it is nonetheless a good validation of our implementation.

5.2 Neighbour criterion

When agent preference is added to the models, the distributions get more strongly peaked at low values of m. First, in the model with nearest neighbour preference only, increasing the value of α gives a less steep tail. The no-saving case is very similar to the results of Goswami et. al. (see Fig. 1, Goswami et.al.). In other words, the higher the preference of the nearest neighbour, the higher the variance in the distribution. A possible interpretation is that a strong neighbour preference gives less chance for the entire population to mix and become more homogeneous.

We tested this model for $N = 500$ and $N = 1000$ and reached equilibria that were more or less indistinguishable. Adding saving again decreased the variance and gave a steeper tail drop-off.

5.3 Former transactions criterion

Finally, when accounting for previous transactions, while also including the nearest neighbour element, the distributions are nearly indistinguishable for the different γ's (Figure 4). In other words, the biggest effect come from merely adding the criteria, while the additional change from strengthening it, is negligible in comparison. Furthermore, changing the α parameter has almost no effect. While in the previous case, the exponent dropped from 8.0 to 5.3 when α went from 1.0 to 2.0, the exponent now remains within the same order of magnitude. It is almost like the former transaction factor overpowers the nearest neighbour effect.

While Goswami et.al. does not include saving in their work, our no-saving case in Figure 4 is qualitatively similar to their results (see Fig. 5 in Goswami et.al.). However, we note that their results for change for different α. They do not explicitly give all their simulation parameters in their article, and the discrepancy in our work might for example be addressed by using different values for the bin size Δm.

Perhaps most interesting, is the cases that combine the saving element of Patriarca et.al. and the agent preferences of Goswami et.al. The two criteria are not coupled, and we get a sense of their relative effect in Figure 3b and 4b.

5.4 Tails of the distributions in terms of power laws

As a final comment on the tail drop-offs, we should note that none of the calculated exponents in the final model (see Table 4) come close to the $-1-\nu$, $\nu \in [1,2]$, value in the Pareto law, nor the values of Goswami et.al. (see Table 1, Goswami et.al.). While some articles give higher values for ν (see for example Goswami et.al.), our exponents ~ 5.5 are still far off.

Now, the model with the neighbour criterion does a lot better for $\alpha \in [1,2]$ (Table 3). However, our greatest concern is the arbitrary beginning of the tail. As mentioned in the method, we chose to define the beginning of the tail at 1% of the peak value. This choice remains very unmotivated, and changing it dramatically changes the exponent values. For this reason, the exponent values in our work are most useful in comparison to each other. In further work, we should use a more robust tail definition.

6 Conclusion

Following the work of Patriarca, Chakraborti and Kaski, as well as Goswami and Sen, we have studied the equilibrium distributions resulting from simple, rule-based transactions between agents. While the saving criterion gives a more equal population with a smaller variance, a stronger neighbour preference in the transaction, broadens the distribution. Finally, we found that the former transaction preference overpowered the neighbour criteria. Furthermore, the resulting distribution was not sensitive to the γ parameter.

The tails of the distributions are all more or less fittable to power laws. However, here the quantitative results are uncertain due to our vague choice of tail definition.

References

[1] Pareto, V. *Cours d'économie politique* (1896) Lausanne

[2] Patriarca, M., Chakraborti, A., Kaski, K. *Gibbs versus non-Gibbs distributions in money dynamics* (2004) Physica A, 340:334–339

[3] Goswami, S., Sen, P. *Agent based models for wealth distribution with preference in interaction* (2014) Physica A, 415:514–524

[4] Virtanen, P., et.al. *SciPy 1.0–Fundamental Algorithms for Scientific Computing in Python* (2019) arXiv:1907.10121

A Power law fits

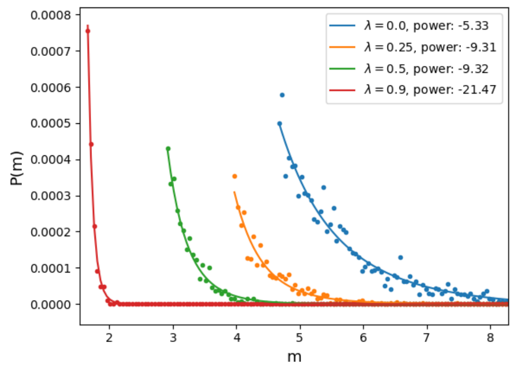
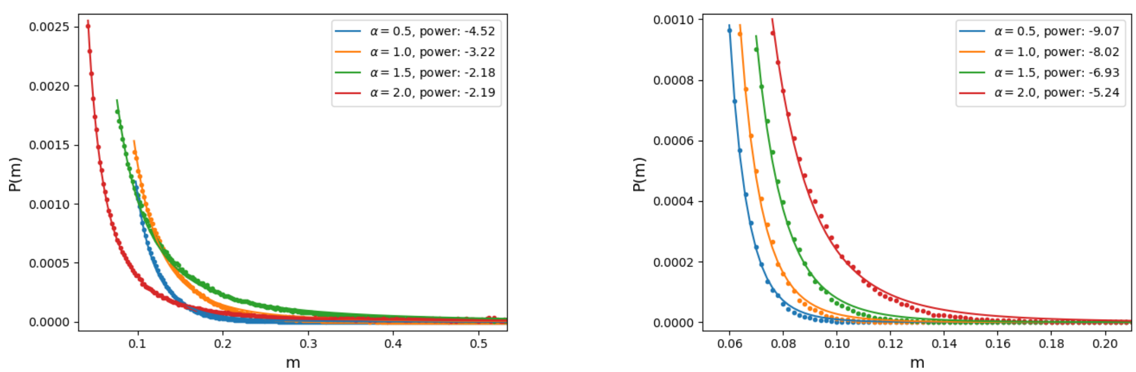
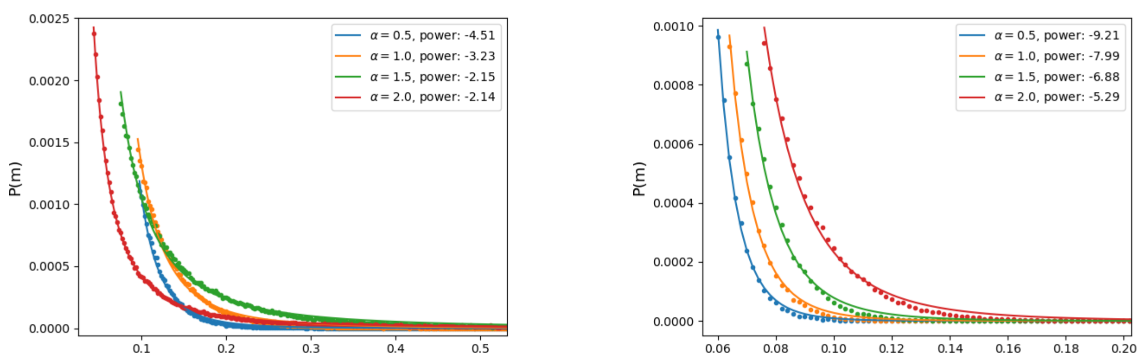


Figure 5: Tail power fit on the saving model for $N = 500$ agents.



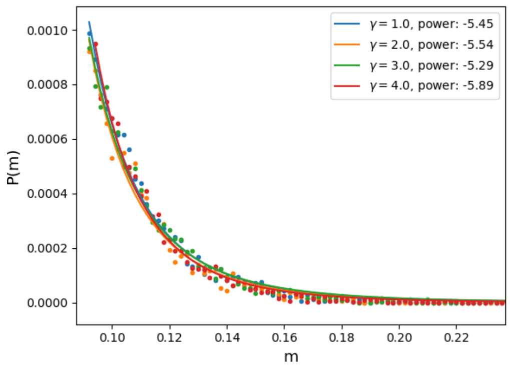
(a) No saving, $\lambda = 0.0$. (b) Saving, $\lambda = 0.5$.

Figure 6: Tail power fit on the neighbour model for $N = 500$ agents.

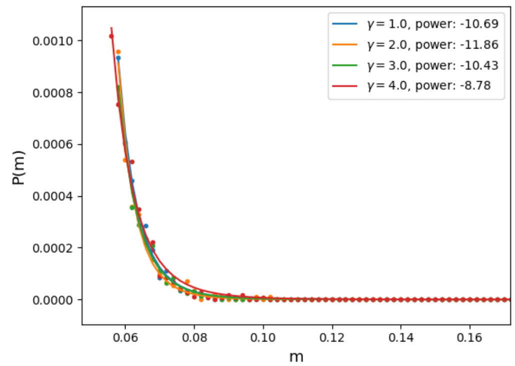


(a) No saving, $\lambda = 0.0$. (b) Saving, $\lambda = 0.5$.

Figure 7: Tail power fit on the neighbour model for $N = 1000$ agents.

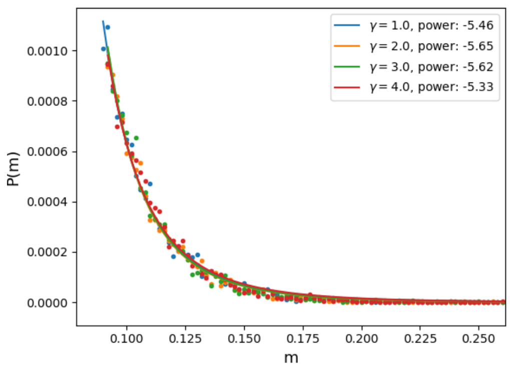


(a) No saving, $\lambda = 0.0$.

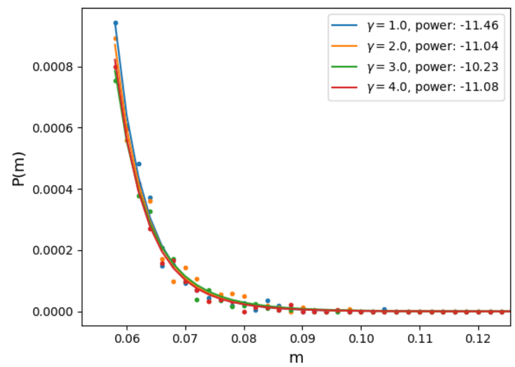


(b) Saving, $\lambda = 0.5$.

Figure 8: Tail power fit on the complete model for $\alpha = 1.0$ agents.



(a) No saving, $\lambda = 0.0$.



(b) Saving, $\lambda = 0.5$.

Figure 9: Tail power fit on the complete model for $\alpha = 2.0$ agents.