Mixed Integer Programming

Techniques and Applications

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VÉL113F: Design and Optimization

INDUSTRIAL ENGINEERING

Integer Programming (IP)

Exploring the complexities of optimization when decision variables are limited to discrete values. Integer and discrete programming offer solutions when continuity is not feasible.

- IP deals with optimization problems where some or all of the variables are restricted to be integers.
- Common in situations where solutions involving fractions or decimals are not feasible, such as in task scheduling or determining quantities of products to manufacture.
- Generally harder to solve than their LP counterparts due to the discrete nature of solutions.

An IP in which only **some** of the variables are required to be integers is called a **mixed integer programming problem**.

Assumptions of Linear Optimization

To apply traditional linear optimization, the following assumptions are made:

Proportionality
: The contribution of product j to the objective function is proportional to the value of x_j. The same applies to the left-hand side in constraints.

Additivity
: The objective function and the left-hand side of constraints are the sum of contributions from individual products. E.g., $z = 3x_1 + 5x_2 + x_1 x_2$.

Divisibility
: Decision variables can take any real value within the solution area.

Certainty
: We assume that the values of the parameters a_{ij}, b_i and c_j are fully known.

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Example (Wyndor-Glass Company)

Wyndor, a glass product manufacturer, plans to start producing two new types of products which can be made in three different factories.

- Product 1: Produced in factories 1 and 3
- Product 2: Produced in factories 2 and 3

Everything that is produced can be sold.

Factory	Production Time (hrs)		Available Time
	Product 1	Product 2	
1	1	0	4
2	0	2	12
3	3	2	18
Profit	\$3000	\$5000	

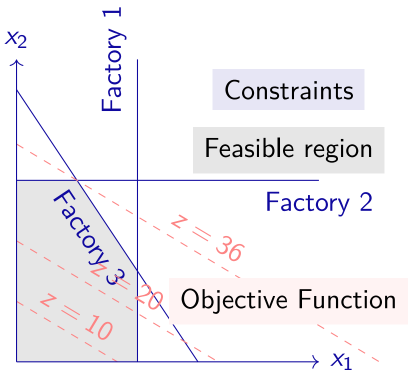

Figure: Graphical solution.

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Example (Wyndor-Glass Company)

Let's explore a simplified scenario for the Wyndor-Glass Company. Suppose we only consider the constraint $x_1 \leq 4$ and disregard all other constraints. With this sole constraint, what is the maximum profit achievable from the production of the products

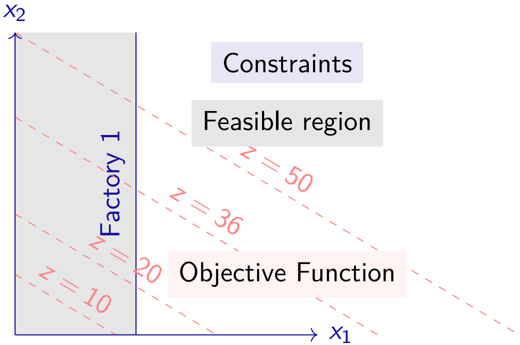

Figure: Graphical solution. The solution space for this problem is unbounded, meaning the profit can potentially be infinite under the given conditions.

Example (Wyndor-Glass Company)

Reintroducing the previous constraints for the Wyndor-Glass Company, and adding a new constraint $3x_1 + 5x_2 \geq 50$. What's the maximum achievable profit given all these conditions?

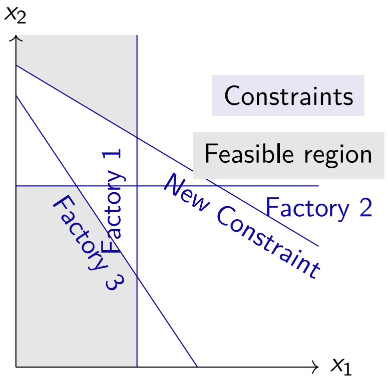

Figure: The feasible region is defined by the area that satisfies **all** constraints simultaneously. Cannot be done, infeasible.

LP Relaxation

- A technique used to find an **approximate** solution to an integer programming problem.
- Involves ignoring the integer constraint and solving the resulting linear program.
- The LP relaxation provides a bound on the value of the optimal solution of the integer program. If the solution to the LP relaxation happens to be integer, it is also a solution to the IP.
- The solution to the LP relaxation can be used as a starting point for other algorithms, like branch and bound, to find the integer solution.

Binary Integer Programming (BIP)

Binary Variables in BIP

Binary Integer Programming is a special case of Integer Programming where decision variables can take only values of 0 or 1.

- Represent binary decisions, e.g., do or do not.
- Often used to activate or deactivate constraints or other decision variables.
- Commonly denoted as y (referred to as **auxiliary variables**), with main decision variables as x.

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Example Usages

1. **Either-Or Decisions:** If a company can open a factory or a retail store, but not both.
2. **Fixed Costs:** If opening a factory incurs a fixed cost, y can indicate whether the factory is opened or not.
3. **Set Values:** Selling products in predefined bundles (e.g. 6-packs).

Big M Method: Either-Or Constraints

Use the **Big M** method with an auxiliary binary variable y to enforce one of two constraints:

- $y = 1$: Constraint A active, B non-binding.
- $y = 0$: Constraint B active, A non-binding.

Given:

$$Ax \leq b \qquad \text{(A)}$$

$$Cx \leq d \qquad \text{(B)}$$

Modify as:

$$\begin{aligned} Ax &\leq b + My \\ Cx &\leq d + M(1 - y) \end{aligned}$$

Choose M sufficiently large.

Elevator Pitch: MIP in Daily Life

Objective

Develop a 30-second pitch for a real-life mixed integer programming problem.

- **Relatability**: Choose a daily life problem that most can resonate with.
- **Hook**: Begin with an engaging statement or question.
- **Problem Description**: Clearly state the problem.
- **Benefits**: Emphasize how solving it would improve daily life.
- **MIP Relevance**:
  - Decision variables: Real, integer, binary.
  - Linear constraints only.

Elevator Pitch: MIP in Daily Life II

Pointers

- Engage the listener quickly.
- Keep it concise and to the point.
- Practice to ensure confidence and timing.

Remember: This is about conveying the importance of the problem, not the solution details.