Linear Optimization

Fundamentals and Applications

Dr. Helga Ingimundardóttir ⟨helgaingim@hi.is⟩

VÉL113F: Design and Optimization

INDUSTRIAL ENGINEERING

Operations Research Overview

- Operations Research (OR) aims to determine the most efficient way to perform tasks within companies or institutions.
- It often involves decision-making where complex issues are presented as optimization problems.
- The solution to these problems usually lies in maximizing or minimizing a specific function, known as the **objective function**.
- OR is an interdisciplinary field combining elements of applied mathematics, statistics, probability theory, computer science, and more.
- A primary focus within OR is on model creation and optimization.
- Its historical roots trace back to World War II, emphasizing its significance in practical applications, especially in resource allocation.

Operations Research Overview II

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Process

1. Problem definition and data collection.
2. Develop a mathematical model that captures the essence of the subject.
3. Develop a computer program to work with the model.
4. Test (verify) the model. Improve the model if necessary.
5. Implement the model – usually in the form of a program.

Operations Research Overview III

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Mathematical Model

Mimics the most crucial aspects of the task.

1. What decisions need to be made?
2. What are the options?
3. What is the outcome (the gain)?
4. What are the conditions for good decision making?
5. Which factors influence the decision?
6. How can we ensure that we've made the right decision?

Linear Optimization / Linear Programming

It's a technique used to find the best possible outcome in a model where the conditions are expressed through **linear** equations and inequalities.

Beer Substitute Optimization Formulation

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Example (Old Exam Question)

Before beer was legalized in Iceland, a beer substitute was in demand. Now, imagine a brewery's challenge to recreate it. The mix must contain 3-5% malt, a minimum of 2% brandy, no more than 7% vodka, and a hard liquor cap of 10% to avoid an overpowering spirit taste.

Ingredient	**Alcohol %**	**Cost (kr. per liter)**
Pilsner	2.25%	100
Vodka	40%	2000
Brandy	40%	3000
Malt Extract	1.5%	120

1. Develop a linear optimization for the strongest (tasty) beer substitute.
2. Design one for a cost-effective 4% beer substitute.

Key Components

- Decision Variables
- Objective Function
- Constraints

Extensive data collection is needed to estimate the model's parameters. **Sensitivity analysis** evaluates the effects of changes in individual parameters. If the model is significantly sensitive to certain parameters, extra care is needed in their estimation. **Stochastic programming** formally handles uncertainties in the parameters.

Types of Models

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- Decision variables can be continuous, discrete, or both.
- Objective function can have one or more min/max values.
- Constraints can be linear or nonlinear.

For now we will focus on models with continuous variables, linear objectives, and linear constraints (i.e. **linear optimization**). We will also look at so-called integer optimization (decision variables take values $0, 1, 2, \ldots$).

Classification of Optimization Problems

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- Continuous Decision Variables
  - Unconstrained
    - Nonlinear equations
    - Least squares method
    - Global optimization
    - Non-differentiable optimization
  - Constrained
    - **Linear Programming**
    - Semidefinite programming
    - Nonlinearly constrained
    - Bound constraints
    - Network Optimization
- Discrete Decision Variables
  - **Integer Programming**
  - Stochastic programming

Solving LP: The Simplex Method

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Introduction:

- Developed by George Dantzig in 1947.
- A popular algorithm for numerical solution of linear programming problems.
- Works by moving along edges of the feasible region defined by the constraints to find the optimal solution.
- Utilizes the fact that the optimal solution lies at a vertex of the feasible region.
- Efficient for large-scale linear problems.

Solving LP: The Simplex Method II

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Advantages:

- Highly systematic and provides a lot of information about the problem structure.
- Can handle additional constraints easily.

Challenges:

- May face issues like cycling (though rare with anti-cycling rules).
- May not be as efficient for some nonlinear problems.

Resources

- For guidance on using the Simplex method, refer to the video How to use the simplex method by StudyForce.
- For the algorithm details, check Wikipedia's page on the Simplex Algorithm.

Solving LP: The Simplex Method III

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Layman's Explanation

Imagine you're a drone over a land with marked paths. Your task is to find the highest point. Instead of searching randomly:

- You start at a corner (vertex) of a path.
- You 'fly' from corner to corner, always choosing the path leading upwards.
- You know the highest point is at a corner due to the problem's structure.
- Once all paths around you lead downwards, you've found the highest spot!

In essence, Simplex is like a savvy drone finding the best peak on a constrained landscape.

Beer Substitute Optimization with Simplex

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Example (Continued from Optimization Formulation)

Now, use the Simplex method to solve:

1. Find the linear optimization solution for the strongest (yet tasty) beer substitute.
2. Determine the solution for a cost-effective 4% beer substitute.

Note

Remember to set up the problem with the appropriate decision variables, constraints, and objective function before applying the Simplex algorithm.

Solving LP: A Computational Approach

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- What is **Gurobi**?
  - Gurobi is a state-of-the-art optimization solver.
  - It efficiently handles linear, quadratic, and mixed-integer optimization problems.
- **Language Integration:**
  - Supports various languages, notably C++ and Python.
  - For our purposes, we'll utilize the **Python** implementation (pip package `gurobipy`).
- **Licensing:**
  - Gurobi offers a free academic license.
  - Important: To register for the license, ensure you are connected to the university's network.
- **Resources:**
  - Gurobi provides extensive resources for learners.
  - Tutorials range from introductory to advanced models.
  - All tutorials are hosted in Jupyter Notebooks, facilitating hands-on learning.

Beer Substitute Optimization with Gurobi

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Example (Continued from Optimization Formulation)

Now, use the Gurobi method to solve:

1. Find the linear optimization solution for the strongest (yet tasty) beer substitute.
2. Determine the solution for a cost-effective 4% beer substitute.

Note

Compare the results from the Simplex method and Gurobi. Are they the same? If not, why?