Derivations

1 Heat Transfer Equation

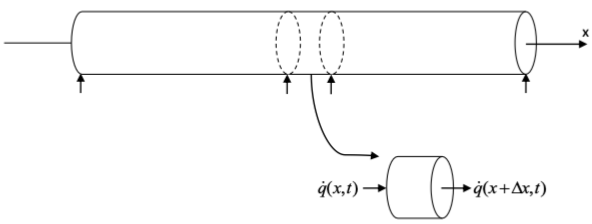

Figure 1: Diagram of a one-dimensional heat rod with its axis along the x-axis

Start with the conservation of energy expressed as:

$$\dot{Q}_{in}(x,t) = \dot{Q}_{out}(x+\triangle x,t) + \dot{Q}_{stored} \tag{1}$$

where

$$\dot{Q}_{in}(x,t) = \dot{q}_{in}(x,t)A \quad [W] \tag{2}$$

$$\dot{Q}_{out}(x,t) = \dot{q}_{out}(x,t)A \quad [W] \tag{3}$$

The rate of energy that is stored in a unit cross-section of the material can be defined as the product of the specific heat, mass, and change in temperature with respect to time.

$$Q = mc_p\triangle T \quad [J] \tag{4}$$

The mass of the material can be written as the product of the density and volume.

$$m = \rho V = \rho A\triangle x \quad [kg] \tag{5}$$

Plugging the mass equation into the rate of heat energy equation, $\dot{Q}_{stored}$ can be expressed as:

$$\dot{Q}_{stored}(x,t) = \rho c_p A\triangle x\frac{\partial u(x,t)}{\partial t} \quad [W] \tag{6}$$

where u(x,t) is a unique solution to the heat equation.
From here, apply all the equations of the rate of heat energy to the conservation of energy, which yields:

$$\dot{q}_{in}(x,t) = \dot{q}_{out}(x+\triangle x,t) + \rho c_p\triangle x\frac{\partial u(x,t)}{\partial t} \quad [W] \tag{7}$$

Rearrange the above equation:

$$\frac{\dot{q}_{in}(x,t) - \dot{q}_{out}(x+\triangle x,t)}{\triangle x} = \rho c_p\triangle x\frac{\partial u(x,t)}{\partial t} \tag{8}$$

Simplify the left-hand term as the length of the element goes to zero:

$$\frac{\dot{q}_{in}(x,t) - \dot{q}_{out}(x+\triangle x,t)}{\triangle x} = \rho c_p\triangle x\frac{\partial u(x,t)}{\partial t} \tag{9}$$

$$\lim_{\triangle x\to 0}\frac{\dot{q}_{in}(x,t) - \dot{q}_{out}(x+\triangle x,t)}{\triangle x} = -\frac{\partial \dot{q}(x,t)}{\partial t} \tag{10}$$

Since only one dimension is considered, it turns out that the conservation of energy is simplified as below:

$$-\frac{\partial \dot{q}(x,t)}{\partial t} = \rho c_p\frac{\partial u(x,t)}{\partial t} \tag{11}$$

Additionally, Fourier's Law of Heat Conduction defines heat flux as the negative of the product of the thermal conductivity, k,and the change in

2 Steady State Solution

- An initial condition for the temperature distribution $f(x)$

$$u(x,0) = f(x) \quad at \quad 0 < x < L, \quad t = 0 \tag{16}$$

- A fixed temperature boundary condition at the beginning of the rod

$$u(0,t) = T_0 = A \quad at \quad x = 0 \tag{17}$$

- A steady state heat flow rate at the end of the rod

$$\frac{\partial u(L,t)}{\partial x} = H = \frac{\dot{Q}}{kA} = B \quad at \quad x = L \tag{18}$$

- A steady state solution

$$\frac{d^2 v(x)}{dx^2} = 0 \quad when \quad 0 \leq x \leq L \tag{19}$$

Considering these initial and boundary conditions, the steady-state solution can be described as:

$$v(x) = A + Bx \tag{20}$$

$$v(x) = T_0 + Hx \quad [^\circ C] \tag{21}$$

where

$$H = \frac{\dot{Q}}{kA} \quad [^\circ C/m] \tag{22}$$

*In this lab, the system is treated as the steady-state as the time goes to infinity. Thus, the heat transfer equation is only a function of the location of the rod.

3 Transient Solution

The steady-state solution does not describe the entire equation of the temperature change. Therefore, the transient solution w(x,t) needs to be taken into account to describe the complete temperature change u(x,t).

$$u(x,t) = w(x,t) + v(x) \tag{23}$$

To start with, initial and boundary conditions have to be set. For the initial condition, it can be said:

$$u(x,0) = w(x,0) + v(x) = f(x) \tag{24}$$

For the boundary conditions,

- As one of the assumptions, the temperature at the beginning of the rod is constant over time. Therefore,

$$u(0,t) = w(0,t) + v(0) = T_0 \tag{25}$$

- The derivative of $w(L,t)$ must be zero

$$\frac{\partial u(L,t)}{\partial x} = \frac{\partial w(L,t)}{\partial x} + \frac{\partial v(L)}{\partial x} = H \tag{26}$$

Using the separation of variables and Fourier Series based on these initial and boundary conditions, the temperature at a given point in time and along the rod can be defined as follows:

$$u(x,t) = T_0 + Hx + \sum_{n=1}^{\infty} b_n sin(\lambda_n x) e^{-\lambda_n^2 \alpha t} \quad [^\circ C] \tag{27}$$

where

$$\lambda_n = \frac{(2n-1)\pi}{2L} \quad where \quad n = 1, 2, 3, ... \tag{28}$$

$$b_n = -\frac{2H}{L} \int_0^L x sin(\lambda_n x) \, dx \tag{29}$$