1. Introduction

The Finite Element (FE) Method is a very powerful mathematical technique used to solve complex systems of partial differential equations. When applied with respect to structural analysis of trusses and frames, the method discretizes the structure into a limited and countable number of elements, reformulates the governing differential euqations – equilibrium (forces), constitutive (the material law) and kinematic equations (strain displacement relations), and finally assembles the elements to obtain the full structure again. This allows us to build up a matrix of simultaneous algebraic equations that can now be solved. Using this concept, these procedures are applied to the codes contained in the classes 'TRUSS' and 'FRAMES' under the functions 'assemble' and 'solve' and will be used throughout the coursework

2. Structural Stability

Being assigned a truss of 5 stories and 3 bays, this gives the structure below:

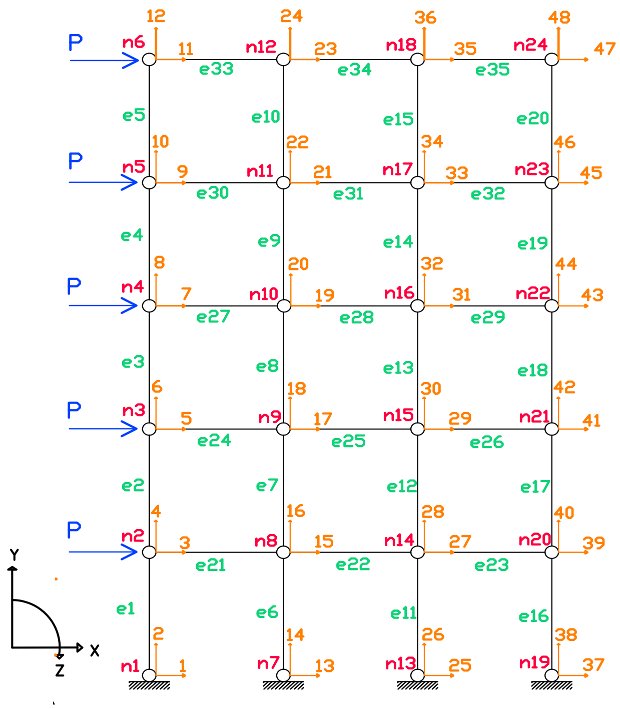

Figure 2.1: Truss with its element, node and degrees of freedom designation.

Looking at the 5x3 truss from a structural point of view, we can test the stability of this truss by assessing its redundancy, r, using:

$$r = m + s - 2j$$

whereby:

- m is the no. of individual members = 35
- s is the no. of support reactions = 8
- j is the no. of hinge joints, including supports. = 24

This gives us r = -5, meaning that the structure is statically deficient and is a mechanism. Thus, 5 additional cross members are required to ensure that the structure is determinate (r = 0). However, the method of redundancy is limited as it does not take into due account of the arrangement of the cross members. Despite so, purely by assessing the structure visually, it is evident that cross members are required at each floor. This is because floors without cross members will collapse due to its inability to resist horizontal loads and thus will remain a mechanism regardless of its redundancy.

This can be further proven by assessing the structure computationally. To do this, we can carry out the checks on the truss by assessing either the determinant, condition or rcondition value of the reduced stiffness matrix, K_{FF}. The K_{FF} matrix is assessed as it is the only matrix that must be inverted to solve for the reaction forces and displacements. Hence, the matrix should be checked to ensure that it is singular, invertible and thus able to provide us with a unique solution (well-conditioned).

However, as the determinant is computationally expensive to calculate, it makes a poor means of testing the conditioning of the K_{FF} matrix and hence we will only look at the ‘rcond’ value.

In the MATLAB file ‘Qn1_rcond’, the code was ran multiple times, each time adding a cross member on a new floor. The corresponding ‘rcond’ values were given as:

No. of Cross Members, n	rcond value
1	2.387e-34
2	2.0775e-34
3	2.0775e-34
4	2.0775e-34
5	0.000163204

Table 2.1: rcond values for a truss with single cross members on n floors.

To further illustrate the need for a cross-member to be present on each floor, the rcond value was calculated again for a truss containing 5 cross members, but with 1 floor containing no cross members (i.e.: there was one floor that contained 2 cross members). This yielded the following results:

No. of Cross Members	rcond value
5 (but not 1 in each floor)	2.0933e-34

Table 2.2: rcond values for a truss with 5 cross members but not 1 on each floor.

Computationally, when an rcond value is smaller than the machine precison, $\varepsilon = 1 \times 10^{-15}$, the matrix is considered badly conditioned. As observed from Tables 2.1 and 2.2, for a truss without a cross-member on each floor, the rcond value is very much smaller than ε and is thus badly conditioned. This means that the structure in non-invertible and does not have a unique relationship between its applied loads and structure responses and hence, is statically unstable.

In conclusion, not only are 5 cross-members needed, but one is required on each floor for the truss to be characterized as a stable structure.

3. Trusses

When considering trusses, as its bar elements are unable to carry bending, shear or twisting due to its pinned ends, only axial deformations are taken into account. Thus, at each node, only 2 degrees of freedom will be present (i.e.: no bending dof). This gives the element, node and dof designation as shown in Figure 2.1.

By permuting through all combinations of possible cross-member configurations, the results below were obtained:

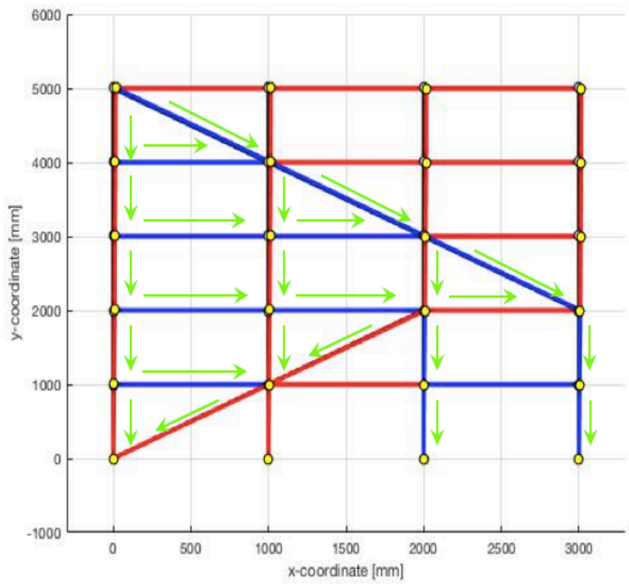

Figure 3.1: Cross-member configuration and key force path for minimum lateral displacement (Truss A)

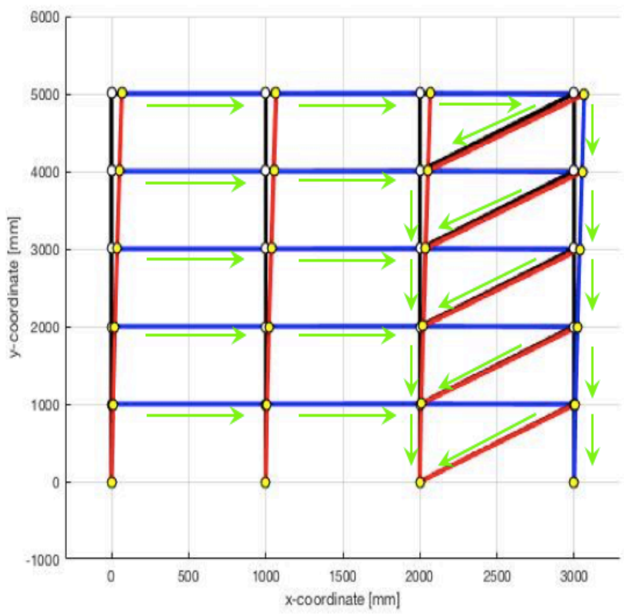

Figure 3.2: Cross-member configuration and key force path for maximum lateral displacement (Truss B)

By tracing the key force paths from the application of the loads to the base supports, it is evident that Truss A would provide a smaller overall lateral displacement compared to Truss B due to the following reasons:

1. Distribution of loads across supports
   For A, as the applied loads are dispersed across more base supports than B, Truss A will be able to provide more axial resistance against the force applied. Contrast this to Truss B, less load will be able to be resisted as only 2 supports are used. This is proven true by Table 3.1 which showed that supports n1 and n7 provided no reaction forces for Truss B, whereas only n7 had no reaction forces for Truss A.

   Furthermore, this means that less load must be taken up by each support for A, as seen from the resultant forces listed in Table 3.1. For Truss B, its supports contain reaction forces approximately 5 times of those of Truss A and will thus experience greater axial deformations at its supports.

REACTIONS			
Support	**Component**	**Min Displacement Truss - A**	**Max Displacement Truss - B**
1 (n1)	Hori	-500000	~ 0
	Vert	-600000	~ 0
	Resultant	**781025**	**0**
2 (n7)	Hori	~ 0	~ 0
	Vert	~ 0	~ 0
	Resultant	**0**	**0**
3 (n13)	Hori	~ 0	-500000
	Vert	300000	-1500000
	Resultant	**300000**	**1581139**
4 (n19)	Hori	~ 0	~ 0
	Vert	300000	1500000
	Resultant	**300000**	**1500000**

Table 3.1: Reaction forces at each support for Truss A and B.

2. Distribution of loads across elements

   There is a better distribution of load across bars for A compared to B. Upon the point of application of force at node 6, the load is immediately transferred to both the vertical supports and the cross members for Truss A, whereas for Truss B, loads were concentrated on the horizontal bars until the third bay was reached. Thus, the better distribution of forces in Truss A helped maximize the structure's ability to resist the loading with a smaller sway.

3. Efficiency of load path

   In relation to point 2, this also indicates that there is a better transfer of load for Truss A than Truss B. By considering only the load transfer of the horizontal force applied on node 6, it is evident that Truss A uses the shortest pathway, requiring only 5 bars. Truss B, on the other hand, required 8 bars. Thus, A will be more effective in its response in resisting the loading.

4. Frames

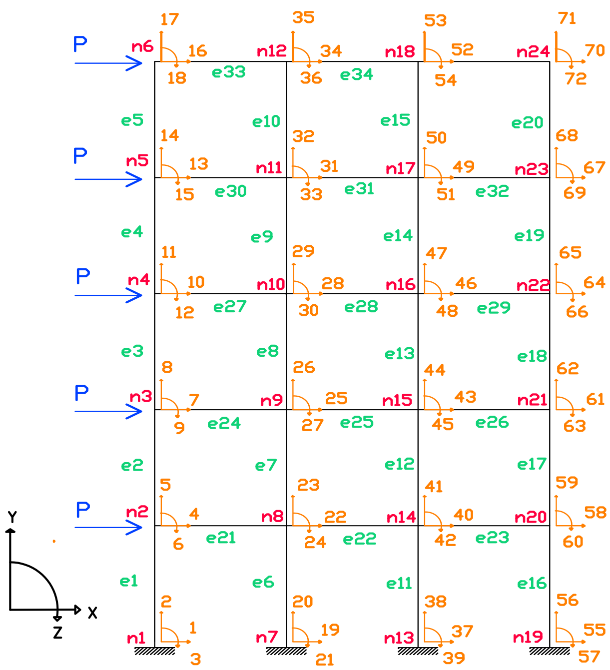

Figure 4.1: Frames with element, node and degrees of freedom designation.

Unlike trusses, elements of frames are considered as beams, whereby the ends are rigidly joined. This allows the elements to carry all transverse loads, axial loads and bending moments through a combination of axial deformations, transverse deformations and cross-sectional rotations. Thus, at each node, 3 dofs will be present, giving the element, node and dof designation as shown in Figure 4.1.

Cross-members were not needed for frames because, as mentioned previously, the nodes contained a third bending dof and thus will be able to resist the applied loads through rotation.

While solving for the frames, we assume that the members were slender. By doing so, we can ignore shear strains as its contribution to the strain energy balance becomes insignificant. Thus, the Euler-Bernoulli beam theory can be applied in place of Timoshenko's beam theory, which simplifies calculations significantly as only bending strain energy and not shear strain energy needs to be considered for the kinematic relationships. The application of this theory thus indicates that the following assumptions are applied:

1. Plane sections remain plane.
2. Plane sections remain perpendicular to the neutral axis.

Using this to derive our FE code under 'Frames', the following deformed structure was obtained:

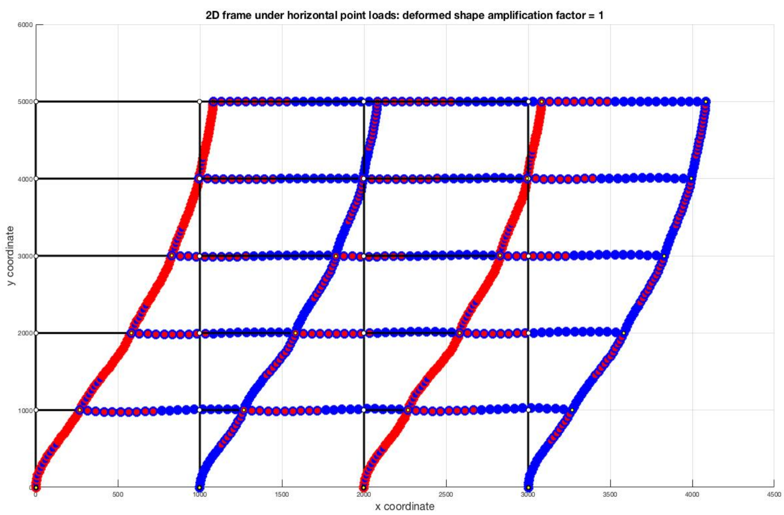

Figure 4.2: Frame deformation.

Comparing the displacements of the structures we have obtained thus far:

Structure	Displacement [mm]
Min Displacement Truss - A	15.9
Max Displacement Truss- B	67.34
Frame	1082.04

Table 4.1: Displacement of corresponding structures.

As seen from Table 4.1, displacements in trusses are significantly smaller than when in frames. This lies in the efficiency of trusses in transferring all applied loads through the members as axial loads. However, for frames, applied loads are transferred through members as not only axial loads, but also shear stresses (which in this case are neglected) and bending moments. As frames uses bending, the deformation of the structure at its nodes as a result causes the displacements to be significantly larger. This efficiency can be illustrated through assessing the reactions at the supports of the frames:

REACTIONS (FRAME)				
Component	**Support 1 (n1)**	**Support 2 (n7)**	**Support 3 (n13)**	**Support 4 (n19)**
Hori [N]	-110298	-139848	-139765	-110089
Vert [N]	-424740	71631	-71561	424670
Resultant [N]	438828	157126	157020	438708
Bending [Nmm]	69494999	79324895	79281518	69378380

Table 4.2: Support reactions for frames.

From Table 4.2, it is evident that the horizontal and vertical reaction components are not utilized as efficiently in resisting the applied loads as compared to the trusses. Despite the load being distributed across all supports, each support is exerted a reaction force greater than those experienced by trusses. The fixed ends cause the frames to resist the loading primarily through bending moments and thus, resulted in its reduced efficiency in resisting lateral sway.

In conclusion, the minimum displacement truss, A, is the most efficient structure.

5. Constraint: Lateral Sway of 25mm

Under this constraint, the results of the required radius of each structure were obtained:

Structure	Radius [mm]	Volume [m^3]
Min Displacement Truss – Truss A	19.9353668	0.052526912
Max Displacement Truss – Truss B	41.02914561	0.222494026
Frame	64.63487566	0.459358495

Table 5.1: Minimum radius and corresponding volume for 25mm lateral sway at point of application of top-most load.

From Table 5.1, it is evident that Truss A is the most economical as not only does it use the least volume of material, it is the most efficient structure. Thus, the price per performance value is the greatest. The frame on the other hand is the least economical due to its poor efficiency and large volume of material required.

The difference in radius and volume for the frame lies in its poorer resistance to load. As a result, a larger radius is required to increase the axial and flexural rigidity of the frame to enhance its resistance. With this similar concept, as Truss A is efficient in resisting loading, a smaller axial rigidity and thus a smaller radius and volume is required, allowing it to be more economical.

6. Buckling and Margin of Safety

As we have assumed that our members are slender, this means that the structures are susceptible to buckling. In order to determine the buckling of each individual element for each structure, the axial load, P, and the critical axial load, P_{crit}, were obtained.

The axial load, P, was obtained by utilizing the axial global dofs computed and then transforming it to obtain the local axial dofs. This allowed us to calculate for the strain, ε, in each member and obtain the axial load exerted on the beam using:

$$P = EA\varepsilon$$

while the critical axial load, P_{crit}, was determined by:

$$P_{crit} = EI\frac{\pi^2}{l^2}$$

whereby l, the critical length was set as:

- L for pin-ended members (truss), and
- $\frac{L}{2}$ for clamped members (frames)

If an element experienced an axial load of $P > P_{crit}$, the member would buckle. This only occurred for the Minimum Displacement Truss, A, for the elements shown in red:

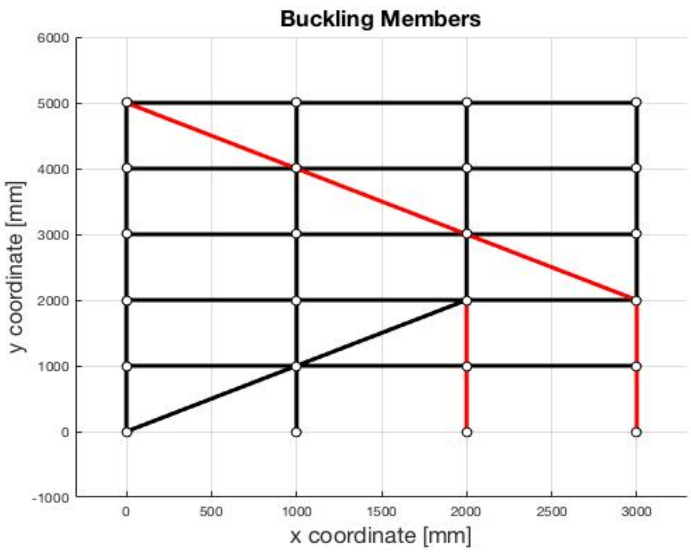

Figure 6.1: Buckling Members.

The margin of safety of the structures were also calculated. As we assume that our bars are inextensible, tension on the bars are not of concern. Thus, only bars in compression were considered as they could fail through buckling. This was obtained using:

$$\text{MoS} = \frac{Pcrit}{P} - 1$$

The smallest margin of safety across the entire structure is then taken as the margin of safety for the entire structure, giving:

Structure	Margin of Safety
Min Displacement Truss - A	-0.711431
Max Displacement Truss- B	1.92886
Frame	261.833

Table 6.1: Margin of safety for each structure.

From this, it is evident that Truss A is considerably less safe when compared against Truss B and especially against the frame. This could be due to the slenderness of the elements of Truss A. Elements of Truss A had the smallest radius and thus had the slenderest members, causing it to be more prone to buckling and as a result, a reduced margin of safety. Conversely, frames had members with the largest radius and thus is less slender in comparison, causing it to be less prone to buckling and in turn attributing to its larger margin of safety.

7. Conclusion

It is important to note that there are many aspects that needs to be taken into consideration to determine what makes a 'good' structure. While Truss A may appear to be the obvious choice given that it is the most economical and effective structure, this was proven wrong after taking into account of other aspects such as buckling and factor of safety.

As engineers, though it is important that we derive an efficient structure, it is crucial that we ensure the safety of our structures. Hence with this, it may mean that a trade-off between efficiency and safety is necessary in order to achieve the most optimal structure.