

An Algorithm for Evaluating the Validity of Singly-Quantified Monadic Predicate Logic Arguments

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1 Introduction

One way in which the validity of a predicate logic argument may be demonstrated is by writing a formal proof. Consider the following monadic predicate logic argument:

$$\frac{(\forall x)(Px \supset Rx) \quad (\exists x)(\neg Rx \wedge Qx)}{\therefore (\exists x)(Qx \wedge \neg Px)}$$

The formal proof would be similar to the following:

Steps	Reasons	<u>$\therefore (\exists x)(Qx \wedge \neg Px)$</u>
1. $(\forall x)(Px \supset Rx)$	1. premise	
2. $(\exists x)(\neg Rx \wedge Qx)$	2. premise	
3. $\neg Ry \wedge Qy$	3. 2 E.I.	
4. $Py \supset Ry$	4. 1 U.I.	
5. $\neg Ry$	5. 3 Simp.	
6. Qy	6. 3 Simp.	
7. $\neg Py$	7. 4,5 M.T.	
8. $Qy \wedge \neg Py$	8. 6,7 Conj.	
9. $(\exists x)(Qx \wedge \neg Px)$	9. 8 E.G.	
Q.E.D.		

Formal proofs such as the one given above may use the four laws of predicate logic (two laws of instantiation and two laws of generalization), the ten laws of inference, and the ten laws of replacement.

In this article, we present an algorithm that can be used for evaluating monadic predicate logic arguments as to validity/invalidity. An example of how to use the algorithm is discussed in Section 2 and an explanation of the algorithm is given in Section 3.

In Section 4, we discuss the possibility of extending the use of the algorithm to relational predicate logic arguments.

2 An Example

The algorithm will be described first through an example. The summary of the steps and explanation will be given in the next section. Consider the predicate logic argument given in the previous section.

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- Step 1. *Write down the symbolic logic argument underlying each instantiation of the given predicate logic argument.*

Consider the example given in Section 1 at the instance when $x = c$, we then have

$$\frac{Pc \supset Rc \quad \neg Rc \wedge Qc}{\therefore Qc \wedge \neg Pc}$$

The symbolic logic argument underlying this one would be

$$\frac{P \supset R \quad \neg R \wedge Q}{\therefore Q \wedge \neg P}$$

- Step 2. *Construct the truth table of the symbolic logic argument obtained in Step 1 and put at the topmost part of each column the quantifier of the proposition for that column. Each simple proposition that is not a premise of the argument is considered to be existentially quantified. (Observe that the truth table has columns for the simple propositions, premises, and conclusion.)*

Object Types	\exists P	\exists Q	\exists R	\forall $P \supset R$	\exists $\neg R \wedge Q$	\exists $Q \wedge \neg P$
1	1	1	1	1	0	0
2	1	1	0	0	1	0
3	1	0	1	1	0	0
4	1	0	0	0	0	0
5	0	1	1	1	0	1
6	0	1	0	1	1	1
7	0	0	1	1	0	0
8	0	0	0	1	0	0

- Step 3. *Delete the rows of object types that have entries of 0 in a column of a universally quantified premise (i.e., rows with a 0 entry in a \forall premise column).*

For this particular example, Rows 2 and 4 are deleted (they are marked with asterisks below).

Object Types	\exists P	\exists Q	\exists R	\forall $P \supset R$	\exists $\neg R \wedge Q$	\exists $Q \wedge \neg P$
1	1	1	1	1	0	0
2*	1	1	0	0*	1	0
3	1	0	1	1	0	0
4*	1	0	0	0*	0	0
5	0	1	1	1	0	1
6	0	1	0	1	1	1
7	0	0	1	1	0	0
8	0	0	0	1	0	0

- Step 4. *Delete the rows of object types that have entries of 1 in a column of an existentially quantified conclusion (i.e., rows with a 1 directly below a \exists conclusion column).*

For this particular example, Rows 5 and 6 are further deleted.

Object Types	\exists P	\exists Q	\exists R	\forall $P \supset R$	\exists $\neg R \wedge Q$	\exists $Q \wedge \neg P$
1	1	1	1	1	0	0
3	1	0	1	1	0	0
5*	0	1	1	1	0	1*
6*	0	1	0	1	1	1*
7	0	0	1	1	0	0
8	0	0	0	1	0	0

Step 5. For each \forall column, conjoin the entries of the remaining rows and indicate the combined truth value for the column at the bottom of the table.

Object Types	\exists P	\exists Q	\exists R	\forall $P \supset R$	\exists $\neg R \wedge Q$	\exists $Q \wedge \neg P$
1	1	1	1	1	0	0
3	1	0	1	1	0	0
7	0	0	1	1	0	0
8	0	0	0	1	0	0
				1		

Step 6. For each \exists column, disjoin the entries of the remaining rows and indicate the combined truth value for the column at the bottom of the table.

Object Types	\exists P	\exists Q	\exists R	\forall $P \supset R$	\exists $\neg R \wedge Q$	\exists $Q \wedge \neg P$
1	1	1	1	1	0	0
3	1	0	1	1	0	0
7	0	0	1	1	0	0
8	0	0	0	1	0	0
	1	1	1	1	0	0

Step 7. Determine the validity of the argument according to the following rules:

Rule I.	If all object types are deleted then the argument is valid.
Rule II.	If in the universe that contains these remaining object types and only these, not all the premises are true and the conclusion false, then the argument is valid.
Rule III.	A) If in the universe that contains these remaining object types and only these, all the existentially quantified simple propositions are true, the premises are true, and the conclusion is false, then the argument is invalid.
	B) However, if in the universe just described, not all of the existentially quantified simple propositions are true but the premises are all true and the conclusion false, then the argument is valid in universes in which it has existential import (EIm) and invalid otherwise.

For this particular example, Rule II is applicable and we conclude that the argument is valid (of course, the formal proof presented in Section 1 already shows this).

3 Explanation of the Algorithm

Observe that if a monadic predicate logic argument has p predicates, then, in so far as this argument is concerned, each universe can contain only up to 2^p object types or categories (each type or category arising from a possible combination of the presence or absence of each of the p predicates on an object). That is to say, the p predicates can potentially partition a universe into 2^p classes.

An argument is valid if it is valid in every universe, i.e., there does not exist a universe in which it is invalid. Hence, to prove that an argument is invalid we simply need to construct a universe in which it is invalid.

To prove that an argument is invalid in a universe, we have to show that it is possible for each of the premises to be true and the conclusion false simultaneously. That is to say, that we can find a subset of the universe for which the premises are all true and the conclusion false simultaneously. This is the primary basis of the algorithm.

The algorithm discussed in Section 2, attempts to construct a universe in which the given argument is invalid in the following manner. First, a list of all of the 2^p object types are obtained. Then the truth value of the instantiation of each premise and conclusion for each object type is determined. All object types which render a premise false or a conclusion true are then discarded (this requires that an object type that renders a \forall premise false and a \exists conclusion true to be discarded).

In applying the algorithm, there are two possibilities for the number of remaining object types. Either: a) all the 2^p object types are deleted (Case A); or, b) not all of the 2^p object types are deleted (Case B). If Case A ensues, then it is not possible to construct a universe in which the argument is invalid since each object type either makes at least one premise false or the conclusion true. Rule I deals with this eventuality. An illustration is given in Table 1 below using the argument:

Argument:

$$\frac{(\forall x)(Px \supset Qx) \quad (\exists x)(\neg Qx)}{\therefore (\exists x)(\neg Px)}$$

Object Types	\exists	\exists	\forall	\forall	\exists
	P	Q	$P \supset Q$	$\neg Q$	$\neg P$
1	1	1	1	0*	0
2	1	0	0*	1	0
3	0	1	1	0*	1*
4	0	0	1	1	1*

The algorithm applied to the argument given leads to the deleting of all object types (Case A). Hence, the given argument is valid by Rule I (see Step 7 in Section 2).

Table 1: An example for Rule I.

Observe that in Table 1, the argument involves two ($p = 2$) predicates giving a total of $2^2 = 4$ object types. All object types are eventually deleted and hence the argument is valid. Another example in which Rule I applies is given in Table 2. In this table, object types 2 to 8 are deleted because of false truth values under a \forall -quantified premise (see column 5, 6, and 7 of Table 2). Object type 1 is deleted because of a true truth value under a \exists -quantified conclusion (see column 8 of Table 2).

Argument:

$$\frac{(\forall x)(Px \wedge Qx) \quad (\forall x)(Px \supset Rx) \quad (\forall x)(Qx \supset Rx)}{\therefore (\exists x)(Px \wedge Rx)}$$

Object Types	\exists	\exists	\exists	\forall	\forall	\forall	\exists
	P	Q	R	$P \wedge Q$	$P \supset R$	$Q \supset R$	$P \wedge R$
1	1	1	1	1	1	1	1*
2	1	1	0	1	1	0*	0
3	1	0	1	0*	0*	1	1*
4	1	0	0	0*	0*	1	0
5	0	1	1	0*	1	1	0
6	0	1	0	0*	1	0*	0
7	0	0	1	0*	1	1	0
8	0	0	0	0*	1	1	0

An example of an argument for which all the object types are discarded based on the algorithm and hence is valid by Rule I (see Step 7 in Section 2).

Table 2: Another Rule I example.

If not all of the 2^p object types are discarded (Case B), then there are two possibilities. Either: i) the remaining object types cannot simultaneously make all the premises true and the conclusion false (Case B.i), in which case the argument is valid (see the example in Section 2); or, ii) the remaining object types together can make

all the premises true and the conclusion false simultaneously (Case B.ii)), in which case the argument is either valid or invalid depending on whether or not it has existential import (**EIm**) in the subuniverse containing only the remaining object types (see the discussion in the paragraph after next).

If Case B.i) ensues, then the argument is valid since it is not possible to construct a universe in which all the premises are true and the conclusion false simultaneously even after discarding those object types that either make the truth value of a premise false or the truth value of the conclusion true. The example in Section 2 falls under this case. Rule II deals with this situation (the corresponding full truth table is in Step 2 of Section 2 and the reduced truth table is in Step 6 of Section 2). Most valid arguments are covered by Rule II.

If Case B.ii) ensues (i.e., all premises true and conclusion false simultaneously), then there are two further possibilities. Either all the existentially quantified simple propositions are true or at least one existentially quantified simple proposition is false. If the former occurs, then the argument is invalid. If the latter occurs, then the argument is valid in universes (subuniverses of the original universe with 2^p object types) wherein it has existential import and invalid otherwise. This is so since for the given argument to have existential import, the subuniverse must include at least one more object type that was discarded in the forward application of the algorithm (see Steps 3 and 4). When this is done, then either at least one premise becomes false or the conclusion becomes true. Consequently, the argument becomes valid in the “augmented” subuniverse. Rules III.A) and III.B) deal with these two eventualities.

An example covered by Rule III.A) is given in Table 3 with the reduced truth table given in Table 4. Object types 3 and 4 are deleted because of either false truth values under a \forall -quantified premise (see column 5 of Table 3) or true truth values under a \exists -quantified conclusion (see column 8 of Table 3). Object types 5 to 8 are deleted because of true truth values under a \exists -quantified conclusion (see column 8 of Table 3). The given argument in Table 3 has existential import in the subuniverse that contains the remaining two object types (see cells 2 to 4 of the last row of Table 4). Furthermore, all the premises are true and the conclusion is false simultaneously. Hence, the argument is invalid by Rule III.A).

Argument:	Object Types	\exists	\exists	\exists	\forall	\exists	\exists	\exists
		P	Q	R	$P \supset Q$	$Q \wedge R$	$\neg Q \vee \neg R$	$P \supset \neg Q$
	1	1	1	1	1	1	0	0
	2	1	1	0	1	0	1	0
	3	1	0	1	0*	0	1	1*
	4	1	0	0	0*	0	1	1*
	5	0	1	1	1	1	0	1*
	6	0	1	0	1	0	1	1*
	7	0	0	1	1	0	1	1*
	8	0	0	0	1	0	1	1*

This is an example of an invalid argument in which the two remaining object types together make the premises false and the conclusion true simultaneously. The combined truth values are given in Table 4.

Table 3: An example in which Rule III.A) applies.

Table 5 gives an example of an argument the determination of whose validity is covered by Rule III.B). The reduced truth table for Table 5 is given in Table 6. Observe that although the argument is apparently invalid since all of the premises are true and the conclusion false simultaneously, it is also true that the argument doesn't have existential import in this subuniverse in that the simple proposition $(\exists x) Px$ is always false. For the given argument to have existential import, we have to additionally include at least one of object types 1, 2, 3, or 4 that have previously been deleted. In so doing, the argument becomes valid. Hence, in a subuniverse wherein the quantified statement $(\exists x) Px$ is sometimes true (i.e., existential import holds), it is not the case that all the premises are true and the conclusion false simultaneously. Observe that if we additionally include only object type 5 to the subuniverse already containing object types 6, 7, and 8, then existential import would still be

Object Types	\exists P	\exists Q	\exists R	\forall $P \supset Q$	\exists $Q \wedge R$	\exists $\neg Q \vee \neg R$	\exists $P \supset \neg Q$
1	1	1	1	1	1	0	0
2	1	1	0	1	0	1	0
	1	1	1	1	1	1	0

This table is Table 3 with the unnecessary object types deleted (object types 3 to 8). Observe that all the premises are true and the conclusion is false and that there is existential import (see Columns 2 to 4; all existentially quantified simple propositions Px , Qx , and Rx are true in this subuniverse consisting of the two remaining object types).

Table 4: Reduced Table 3.

absent since in such an augmented subuniverse, the quantified statement $(\exists x) Px$ would still be always false!

Argument:

$$\frac{(\forall x)(Px \supset Qx) \quad (\forall x)(Px \supset Rx)}{\therefore (\exists x)(Qx \wedge Rx)}$$

Object Types	\exists P	\exists Q	\exists R	\forall $P \supset Q$	\forall $P \supset R$	\exists $Q \wedge R$
1	1	1	1	1	1	1*
2	1	1	0	1	0*	0
3	1	0	1	0*	1	0
4	1	0	0	0*	0*	0
5	0	1	1	1	1	1*
6	0	1	0	1	1	0
7	0	0	1	1	1	0
8	0	0	0	1	1	0

The argument given is invalid if existential import is not required. It is valid if existential import is required. Object types 1 to 5 are removed (object types 2, 3, and 4 because of false truth values under a \forall -quantified premise; object types 1 and 5 because of true truth values under a \exists -quantified conclusion).

Table 5: An example covered by Rule III.B).

The summary of Cases, corresponding Conditions and Rules to be applied, and consequent determination of Validity in Step 7 of the algorithm are given in Table 7. Applying the algorithm to a given singly-quantified monadic predicate logic argument can result in one of among the following four cases: A; B.i); B.ii) with **EIm**; or, B.ii) without **EIm**. The determination that a given argument is invalid is arrived at for the two cases B.ii) with **EIm** and B.ii) without **EIm**. All the other cases lead to a determination that the given argument is valid.

4 Avenues for Future Study

The determination of a similar algorithm for relational predicate logic arguments is apparently an open problem. One can possibly study simple arguments first, such as:

$$\frac{(\forall x)(\exists y)Fxy}{\therefore (\exists x)(\forall y)Fxy}.$$

This argument is invalid in that one can find a particular universe in which the premise is true and the conclusion false. For example, take the universe to be the set of integers (**Z**) and the relationship F to be the “is equal to” ($=$)

Object Types	\exists P	\exists Q	\exists R	\forall $P \supset Q$	\forall $P \supset R$	\exists $Q \wedge R$
6	0	1	0	1	1	0
7	0	0	1	1	1	0
8	0	0	0	1	1	0
	0	1	1	1	1	0

The argument given in Table 5 is invalid in the subuniverse containing only subject types 6, 7 and 8, but becomes valid when existential import (**EIm**) is required. The additional inclusion of at least one of the previously deleted object types (1, 2, 3, or 4 but not 5) from Table 5 brings about existential import.

Table 6: Reduced version of Table 5.

Case	Condition		Rule	Validity
A	all object types are deleted		I	Valid
B.i)	it is <u>not</u> the case that all the premises are true and the conclusion false simultaneously		II	Valid
B.ii)	all the premises are true and the conclusion false simultaneously	w/ EIm	III.A	Invalid
		w/o EIm	III.B	Invalid (subuniverse as is; no EIm)
				Valid (subuniverse augmented for EIm)

Table 7: Summary of Cases, corresponding applicable Rules, and consequent determination of Validity in Step 7 of the algorithm.

relationship, i.e., $Fxy \iff x = y$. This particular argument in words would then be: “If for each integer x , there exists an integer y such that x is equal to y , then there exists an integer x that is equal to every integer y .” The premise: “For each integer x , there exists an integer y such that x is equal to y .” is true. However, the conclusion: “There exists an integer x that is equal to every integer y .” is false and thus the argument is invalid.

References

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