1 Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{IZ_2} \tag{1}$$
$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi_0\partial_\mu\varphi_0 + \frac{1}{2}m_0\varphi_0^2 + \lambda_0\varphi_0^4 \tag{2}$$
$$\mathcal{L}_1 = \frac{1}{2}\partial_\mu\varphi_1\partial_\mu\varphi_1 + \frac{1}{2}m_1\varphi_1^2 + \lambda_1\varphi_1^4 \tag{3}$$
$$\mathcal{L}_{IZ_2} = \mu\varphi_0^2\varphi_1^2 \tag{4}$$
$$\mathcal{L}_I = g\varphi_1\varphi_0^3 \tag{5}$$

On the lattice we can discretise the derivative $\partial_\mu\varphi(x) = \frac{1}{a}(\varphi(x+\mu) - \varphi(x))$. On the lattice the above lagrangian can be written in a more convenient way for simulations

$$\mathcal{L}_0 = \sum_x \left[-2\kappa_0 \sum_\mu \phi_0(x)\phi_0(x+\mu) + \lambda_L^0 \left(\phi_0(x)^2 - 1\right)^2 + \phi_0(x)^2\right], \tag{6}$$
$$\mathcal{L}_1 = \sum_x \left[-2\kappa_1 \sum_\mu \phi_1(x)\phi_1(x+\mu) + \lambda_L^1 \left(\phi_1(x)^2 - 1\right)^2 + \phi_1(x)^2\right], \tag{7}$$
$$\mathcal{L}_{IZ_2} = \mu_L \sum_x \phi_0(x)^2\phi_1(x)^2\,, \tag{8}$$
$$\mathcal{L}_I = g_L \sum_x \phi_1(x)\phi_0(x)^3\,. \tag{9}$$

With

$$m_0^2 = \frac{1 - 2\lambda_L^0}{\kappa_0} - 8\,, \quad \lambda_0 = \frac{\lambda_L^0}{4\kappa_0^2}\,, \quad \varphi_0 = \sqrt{2\kappa_0}\phi_0 \tag{10}$$
$$m_1^2 = \frac{1 - 2\lambda_L^1}{\kappa_1} - 8\,, \quad \lambda_1 = \frac{\lambda_L^1}{4\kappa_1^2}\,, \quad \varphi_1 = \sqrt{2\kappa_1}\phi_1\,, \tag{11}$$

and

$$\mu = \frac{\mu_L}{4\kappa_0\kappa_1}\,, \quad g = \frac{g_L}{4\sqrt{\kappa_0}\kappa_1^{3/2}} \tag{12}$$

2 four point function BH arXiv:2012.11488

$$C_4^{BH} = \frac{\langle\phi_0(\frac{T}{2})\phi_1(t)\phi_1(\frac{T}{8})\phi_0(0)\rangle}{\langle\phi_0(\frac{T}{2})\phi_0(0)\rangle\langle\phi_1(t)\phi_1(\frac{T}{8})\rangle} - 1 \tag{13}$$

In the following we will call $|\pi\rangle$ the state interpolated by the field ϕ_0 and $|N\rangle$ the state interpolated by ϕ_1

2.1 Spectral Decomposition for $t_1 < t_2 < t_3 < t_4$

- Numerator

$$\langle\phi_0(t_1)\phi_1(t_2)\phi_1(t_3)\phi_0(t_4)\rangle = \tag{14}$$
$$\sum_{i,j,k} \frac{1}{2m_j 2m_k} e^{-(T-t_1)E_i} e^{-t_1 E_j} \langle i|\phi_0|j\rangle\langle j|\phi_1(t_2)\phi_1(t_3)|k\rangle\langle k|\phi_0|i\rangle e^{-(E_i-E_k)t_4} \tag{15}$$
$$\tag{16}$$

in the limit $T - t_1 \to \infty$ and setting $t_4 = 0$

$$\sum_{j,k} \frac{1}{2m_j 2m_k} \langle 0|\phi_0|j\rangle e^{-t_1 E_j} \langle j|\phi_1(t_2)\phi_1(t_3)|k\rangle\langle k|\phi_0|0\rangle \tag{17}$$
$$= \sum_{j,k} \frac{1}{2m_j 2m_k} \langle 0|\phi_0|j\rangle e^{-(t_1-t_2)E_j} \langle j|\phi_1 e^{-(t_2-t_3)H}\phi_1|k\rangle e^{-t_3 E_k} \langle k|\phi_0|0\rangle \tag{18}$$

Assuming $t_1 - t_2 >> 0$ and $t_3 >> 0$ the states $|j\rangle = |\pi\rangle$ and $|k\rangle = |\pi\rangle$ so we get

$$= \frac{1}{2m_\pi 2m_\pi} |\langle 0|\phi_0|\pi\rangle|^2 e^{-(t_1-t_2)E_\pi} e^{-t_3 E_\pi} \langle \pi|\phi_1 e^{-(t_2-t_3)H}\phi_1|\pi\rangle \tag{19}$$

Setting $t_1 = T/2$, $t_2 = t$ and $t_3 = T/8$

$$\langle \phi_0(\frac{T}{2})\phi_1(t)\phi_1(\frac{T}{8})\phi_0(0)\rangle = \frac{1}{2m_\pi 2m_\pi} |\langle 0|\phi_0|\pi\rangle|^2 e^{-\frac{T}{2}E_\pi} e^{-(\frac{T}{8}-t)E_\pi} \langle \pi|\phi_1 e^{-(t-\frac{T}{8})H}\phi_1|\pi\rangle \tag{20}$$

- Denominator 0

$$\langle \phi_0(\frac{T}{2})\phi_0(0)\rangle = \frac{1}{m_\pi} |\langle 0|\phi_0|\pi\rangle|^2 e^{-\frac{T}{2}E_\pi}$$

  Where we have summed both the forward and backward signal

- Denominator 1

$$\langle \phi_1(t)\phi_1(\frac{T}{8})\rangle = \frac{1}{2m_N} |\langle 0|\phi_1|N\rangle|^2 \left(e^{-(t-\frac{T}{8})E_N} + e^{(T-t+\frac{T}{8})E_N}\right) \approx \frac{1}{2m_N} |\langle 0|\phi_1|N\rangle|^2 e^{-(t-\frac{T}{8})E_N}$$

  We can ignore the second term since $T/8 < t < T/2$

Putting the various pieces toghether we get

$$C_4^{BH} = \frac{m_N}{2m_\pi} \frac{e^{(t-\frac{T}{8})(E_N+E\pi)} \langle \pi|\phi_1 e^{-(t-\frac{T}{8})H}\phi_1|\pi\rangle}{|\langle 0|\phi_1|N\rangle|^2} - 1 \tag{21}$$

The above equation need to be compared with the expression on the paper BH arXiv:2012.11488:

$$c^{\Theta,N\pi}_{\vec{q}_1'\vec{q}_2'\vec{q}_1\vec{q}_2}(t',t|\,M_0) \equiv \frac{2\omega_{\vec{q}_2'} e^{\omega_{\vec{q}_2'}t'} 2\omega_{\vec{q}_2} e^{-\omega_{\vec{q}_2}t}}{Z_N} C_{a'b'}C_{ab}$$
$$\times \langle \pi^{a'}, \vec{q}_1'|\widetilde{N}^{b'}_{\vec{q}_2'}(0)\,\Theta(\hat{M}-M_0,\Delta)\,e^{-\hat{H}(t'-t)} e^{\omega_{\vec{q}_1'}t'} e^{-\omega_{\vec{q}_1}t}\,\widetilde{N}^{\dagger b}_{-\vec{q}_2}(0)|\pi^a,\vec{q}_1\rangle\,.$$

where $\omega_{\vec{q}} = \sqrt{\vec{q}^2 + m_\pi^2}$ and $Z_\pi = \langle \pi, \vec{p}|\pi(0)|0\rangle^2$. Here the single particle state has the usual relativistic normalization, $\langle \pi, \vec{q}|\pi, \vec{q}'\rangle = 2\omega_{\vec{q}}(2\pi)^3\delta^3(\vec{q}-\vec{q}')$.

Comparing the two equation above we get

$$8m_\pi m_N C_4^{BH} = c^{\Theta,N\pi}_{\vec{q}_1'\vec{q}_2'\vec{q}_1\vec{q}_2}(t',t|\,M_0) \tag{22}$$

The result of the paper paper BH is

$$c^{\Theta,N\pi}_{\vec{0}}(t',t|\,M_0)_{\mathsf{c}} = 8\pi(m_N+m_\pi)\,a_{N\pi}\,(t'-t)$$
$$-\,16\,a_{N\pi}^2\sqrt{2\pi(m_N+m_\pi)m_N m_\pi(t'-t)} + \mathcal{O}\big((t'-t)^0\big)\,.$$

thus we get (we add a factor L^3 to match the dimension, we need to understand better the factor in front!)

$$C_4^{BH} = \frac{1}{8m_\pi m_N L^3}\left[8\pi(m_N+m_\pi)\,a_{N\pi}\,(t'-t) - 16\,a_{N\pi}^2\sqrt{2\pi(m_N+m_\pi)m_N m_\pi(t'-t)} + \mathcal{O}\big((t'-t)^0\big)\right]\,. \tag{23}$$

3 Numerical Test

For simplicity we set the couplings $\lambda_0 = \lambda_1 = \mu/2 = 2.5$, the masses $m_0^2 = -4.925$, $m_1^2 = -4.85$ and $g = 0$. First we measured the mass of the two particles from the two point functions

$$\langle \phi_0(t)\phi_0(0)\rangle \quad \text{and} \quad \langle \phi_1(t)\phi_1(0)\rangle \tag{24}$$

obtaining $m_\pi = 0.14653(11)$ and $m_N = 0.274853(80)$, the corresponding plateaux are in Fig. 1

In Fig. 2 we show our result for in lattice $L20T32$ for the correlator in eq.(13), the value we got from the fit is $a_{N\pi}^{BH} = -0.980(72)$ while with the Lusher method we get $a_{N\pi}^{Luscher} = -0.401(35)$

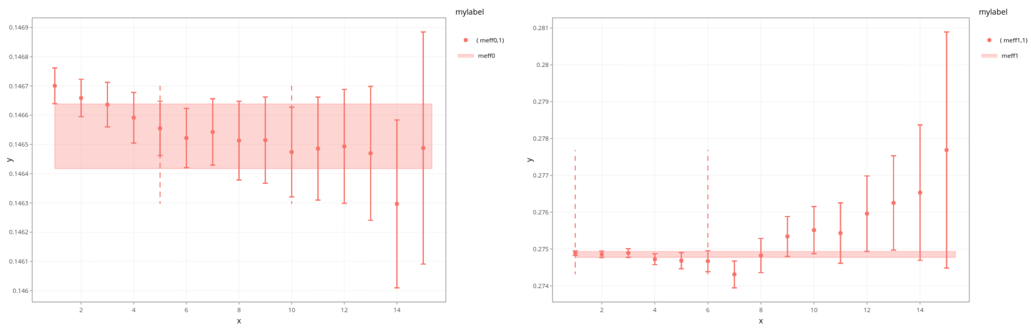

Figure 1: Effective mass plot of the m_π left and m_N right

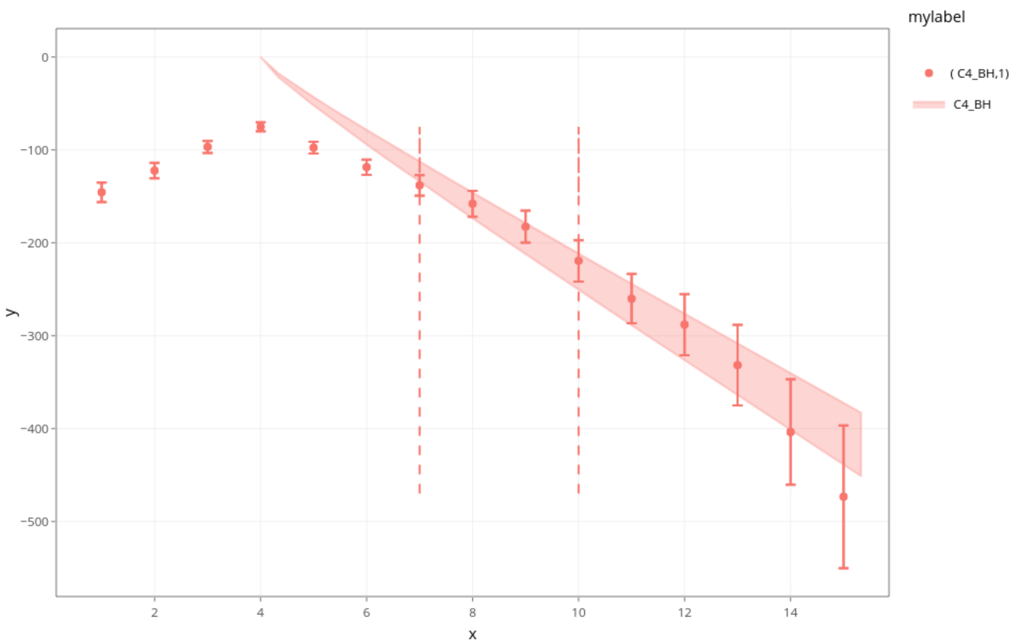

Figure 2: Correlator of eq.(13) fitted with eq.(23) in a L20T32 lattice

4 Lüscher method

The Lüscher method requires to measure the energy shift of the two particle energy, then the scattering length can be computed from the formula

$$\Delta E_{N\pi} = -\frac{2\pi a_{N\pi}}{\mu L^3}\left[1 + c_1\frac{a_{N\pi}}{L} + c_2\left(\frac{a_{N\pi}}{L}\right)^2\right] + O(\frac{1}{L^6}), \tag{25}$$

where μ is the reduced mass, $c_1 = -2.837297$, $c_2 = 6.375183$. The measurement for the masses and the two particle energy are in Table 1. The values of Table 1 can be fitted with eq.(25) as shown in Fig. 3 obtaining

$$\chi^2/d.o.f. = 2.82759 \pm 2.1 \tag{26}$$
$$a_{N\pi} = -0.401 \pm 0.035 \tag{27}$$
$$\tag{28}$$

L	m_π	m_N	$E_{N\pi}$	$\Delta E_{N\pi}$
20	0.14653(11)	0.274850(84)	0.42407(45)	0.00269(47)
24	0.145783(77)	0.274484(89)	0.42288(30)	0.00261(31)
26	0.145593(96)	0.274699(80)	0.42190(37)	0.00161(35)
32	0.145261(76)	0.274538(86)	0.42095(36)	0.00116(35)

Table 1: Spectrum for $\lambda_0 = \lambda_1 = \mu/2 = 2.5$, $m_0^2 = -4.925$, $m_1^2 = -4.85$ and $g = 0$

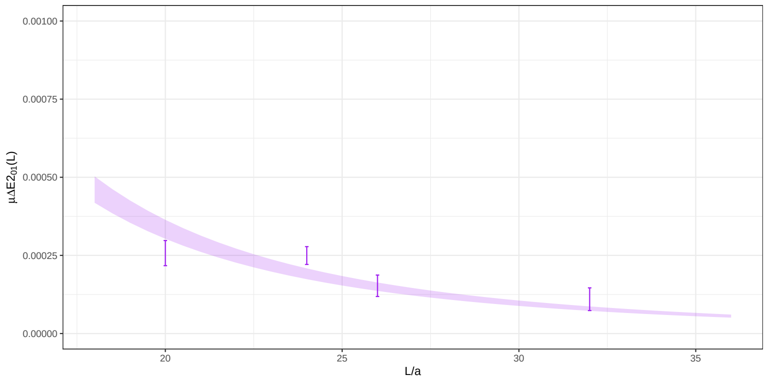

Figure 3: Fit of eq.(25)