1 Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{IZ_2} \tag{1}$$
$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi_0\partial_\mu\varphi_0 + \frac{1}{2}m_0\varphi_0^2 + \lambda_0\varphi_0^4 \tag{2}$$
$$\mathcal{L}_1 = \frac{1}{2}\partial_\mu\varphi_1\partial_\mu\varphi_1 + \frac{1}{2}m_1\varphi_1^2 + \lambda_1\varphi_1^4 \tag{3}$$
$$\mathcal{L}_{IZ_2} = \mu\varphi_0^2\varphi_1^2 \tag{4}$$
$$\mathcal{L}_I = g\varphi_1\varphi_0^3 \tag{5}$$

On the lattice we can discretise the derivative $\partial_\mu\varphi(x) = \frac{1}{a}(\varphi(x+\mu) - \varphi(x))$. On the lattice the above lagrangian can be written in a more convenient way for simulations

$$\mathcal{L}_0 = \sum_x \left[-2\kappa_0 \sum_\mu \phi_0(x)\phi_0(x+\mu) + \lambda_L^0 \left(\phi_0(x)^2 - 1\right)^2 + \phi_0(x)^2 \right], \tag{6}$$
$$\mathcal{L}_1 = \sum_x \left[-2\kappa_1 \sum_\mu \phi_1(x)\phi_1(x+\mu) + \lambda_L^1 \left(\phi_1(x)^2 - 1\right)^2 + \phi_1(x)^2 \right], \tag{7}$$
$$\mathcal{L}_{IZ_2} = \mu_L \sum_x \phi_0(x)^2\phi_1(x)^2\,, \tag{8}$$
$$\mathcal{L}_I = g_L \sum_x \phi_1(x)\phi_0(x)^3\,. \tag{9}$$

With

$$m_0^2 = \frac{1-2\lambda_L^0}{\kappa_0} - 8\,, \quad \lambda_0 = \frac{\lambda_L^0}{4\kappa_0^2}\,, \quad \varphi_0 = \sqrt{2\kappa_0}\phi_0 \tag{10}$$
$$m_1^2 = \frac{1-2\lambda_L^1}{\kappa_1} - 8\,, \quad \lambda_1 = \frac{\lambda_L^1}{4\kappa_1^2}\,, \quad \varphi_1 = \sqrt{2\kappa_1}\phi_1\,, \tag{11}$$

and

$$\mu = \frac{\mu_L}{4\kappa_0\kappa_1}\,, \quad g = \frac{g_L}{4\sqrt{\kappa_0}\kappa_1^{3/2}} \tag{12}$$

2 BH four point function

$$C_4^{BH} = \frac{\langle\phi_0(\frac{T}{2})\phi_1(t)\phi_1(\frac{T}{8})\phi_0(0)\rangle}{\langle\phi_0(\frac{T}{2})\phi_0(0)\rangle\langle\phi_1(t)\phi_1(\frac{T}{8})\rangle} - 1$$

2.1 Spectral Decomposition for $t_1 < t_2 < t_3 < t_4$

- Numerator

$$\langle\phi_0(t_1)\phi_1(t_2)\phi_1(t_3)\phi_0(t_4)\rangle = \tag{13}$$
$$\sum_{i,j,k} \frac{1}{2m_j 2m_k} e^{-(T-t_1)E_i} e^{-t_1 E_j} \langle i|\phi_0|j\rangle\langle j|\phi_1(t_2)\phi_1(t_3)|k\rangle\langle k|\phi_0|i\rangle e^{-(E_i-E_k)t_4} \tag{14}$$
$$\tag{15}$$

in the limit $T - t_1 \to \infty$ and setting $t_4 = 0$

$$\sum_{j,k} \frac{1}{2m_j 2m_k} \langle 0|\phi_0|j\rangle e^{-t_1 E_j} \langle j|\phi_1(t_2)\phi_1(t_3)|k\rangle\langle k|\phi_0|0\rangle \tag{16}$$
$$= \sum_{j,k} \frac{1}{2m_j 2m_k} \langle 0|\phi_0|j\rangle e^{-(t_1-t_2)E_j} \langle j|\phi_1 e^{-(t_2-t_3)H}\phi_1|k\rangle e^{-t_3 E_k}\langle k|\phi_0|0\rangle \tag{17}$$

Assuming $t_1 - t_2 >> 0$ and $t_3 >> 0$ the states $|j\rangle = |\pi\rangle$ and $|k\rangle = |\pi\rangle$ so we get

$$= \frac{1}{2m_\pi 2m_\pi} |\langle 0|\phi_0|\pi\rangle|^2 e^{-(t_1-t_2)E_\pi} e^{-t_3 E_\pi} \langle\pi|\phi_1 e^{-(t_2-t_3)H}\phi_1|\pi\rangle \tag{18}$$

Setting $t_1 = T/2$, $t_2 = t$ and $t_3 = T/8$

$$\langle \phi_0(\frac{T}{2})\phi_1(t)\phi_1(\frac{T}{8})\phi_0(0)\rangle = \frac{1}{2m_\pi 2m_\pi}|\langle 0|\phi_0|\pi\rangle|^2 e^{-\frac{T}{2}E_\pi} e^{-(\frac{T}{8}-t)E_\pi}\langle \pi|\phi_1 e^{-(t-\frac{T}{8})H}\phi_1|\pi\rangle \tag{19}$$

- Denominator 0

$$\langle \phi_0(\frac{T}{2})\phi_0(0)\rangle = \frac{1}{m_\pi}|\langle 0|\phi_0|\pi\rangle|^2 e^{-\frac{T}{2}E_\pi}$$

  Where we have summed both the forward and backward signal

- Denominator 1

$$\langle \phi_1(t)\phi_1(\frac{T}{8})\rangle = \frac{1}{2m_N}|\langle 0|N|\pi\rangle|^2 \left(e^{-(t-\frac{T}{8})E_N} + e^{(T-t+\frac{T}{8})E_N}\right) = \frac{1}{2m_N}|\langle 0|N|\pi\rangle|^2 e^{-(t-\frac{T}{8})E_N}$$

  We can ignore the second term since $T/8 < t < T/2$

Putting the various pieces toghether we get

$$C_4^{BH} = \frac{m_N}{2m_\pi}\frac{e^{(t-\frac{T}{8})(E_N+E\pi)}\langle \pi|\phi_1 e^{-(t-\frac{T}{8})H}\phi_1|\pi\rangle}{|\langle 0|N|\pi\rangle|^2} - 1 \tag{20}$$

To be compared with the expression on the paper BH:

$$c^{\Theta,N\pi}_{\vec{q}_1'\vec{q}_2'\vec{q}_1\vec{q}_2}(t',t|\,M_0) \equiv \frac{2\omega_{\vec{q}_2'}e^{\omega_{\vec{q}_2'}t'}2\omega_{\vec{q}_2}e^{-\omega_{\vec{q}_2}t}}{Z_N}C_{a'b'}C_{ab}$$
$$\times\,\langle \pi^{a'},\vec{q}_1'|\widetilde{N}^{b'}_{\vec{q}_2'}(0)\,\Theta(\hat{M}-M_0,\Delta)\,e^{-\hat{H}(t'-t)}e^{\omega_{\vec{q}_1'}t'}e^{-\omega_{\vec{q}_1}t}\,\widetilde{N}^{\dagger b}_{-\vec{q}_2}(0)|\pi^a,\vec{q}_1\rangle\,.$$

where $\omega_{\vec{q}} = \sqrt{\vec{q}^2 + m_\pi^2}$ and $Z_\pi = \langle \pi,\vec{p}|\pi(0)|0\rangle^2$. Here the single particle state has the usual relativistic normalization, $\langle \pi,\vec{q}|\pi,\vec{q}'\rangle = 2\omega_{\vec{q}}(2\pi)^3\delta^3(\vec{q}-\vec{q}')$.

Comparing the two equation above we get

$$8m_\pi m_N C_4^{BH} = c^{\Theta,N\pi}_{\vec{q}_1'\vec{q}_2'\vec{q}_1\vec{q}_2}(t',t|\,M_0) \tag{21}$$

To be compared with the expression on the paper BH:

$$c^{\Theta,N\pi}_{\vec{0}}(t',t|\,M_0)_{\mathsf{c}} = 8\pi(m_N+m_\pi)\,a_{N\pi}\,(t'-t)$$
$$-\,16\,a_{N\pi}^2\sqrt{2\pi(m_N+m_\pi)m_N m_\pi(t'-t)} + \mathcal{O}\big((t'-t)^0\big)\,.$$

Finally our formula become

$$C_4^{BH} \quad = \quad \frac{1}{8m_\pi m_N}\left[8\pi(m_N+m_\pi)\,a_{N\pi}\,(t'-t) - 16\,a_{N\pi}^2\sqrt{2\pi(m_N+m_\pi)m_N m_\pi(t'-t)} + \mathcal{O}\big((t'-t)^0\big)\right]\,.$$