

1 Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{IZ_2} \quad (1)$$

$$\mathcal{L}_0 = \frac{1}{2} \partial_\mu \varphi_0 \partial_\mu \varphi_0 + \frac{1}{2} m_0 \varphi_0^2 + \lambda_0 \varphi_0^4 \quad (2)$$

$$\mathcal{L}_1 = \frac{1}{2} \partial_\mu \varphi_1 \partial_\mu \varphi_1 + \frac{1}{2} m_1 \varphi_1^2 + \lambda_1 \varphi_1^4 \quad (3)$$

$$\mathcal{L}_{IZ_2} = \mu \varphi_0^2 \varphi_1^2 \quad (4)$$

$$\mathcal{L}_I = g \varphi_1 \varphi_0^3 \quad (5)$$

On the lattice we can discretise the derivative $\partial_\mu \varphi(x) = \frac{1}{a}(\varphi(x+\mu) - \varphi(x))$. On the lattice the above lagrangian can be written in a more convenient way for simulations

$$\mathcal{L}_0 = \sum_x \left[-2\kappa_0 \sum_\mu \phi_0(x) \phi_0(x+\mu) + \lambda_L^0 (\phi_0(x)^2 - 1)^2 + \phi_0(x)^2 \right], \quad (6)$$

$$\mathcal{L}_1 = \sum_x \left[-2\kappa_1 \sum_\mu \phi_1(x) \phi_1(x+\mu) + \lambda_L^1 (\phi_1(x)^2 - 1)^2 + \phi_1(x)^2 \right], \quad (7)$$

$$\mathcal{L}_{IZ_2} = \mu_L \sum_x \phi_0(x)^2 \phi_1(x)^2, \quad (8)$$

$$\mathcal{L}_I = g_L \sum_x \phi_1(x) \phi_0(x)^3. \quad (9)$$

With

$$m_0^2 = \frac{1 - 2\lambda_L^0}{\kappa_0} - 8, \quad \lambda_0 = \frac{\lambda_L^0}{4\kappa_0^2}, \quad \varphi_0 = \sqrt{2\kappa_0} \phi_0 \quad (10)$$

$$m_1^2 = \frac{1 - 2\lambda_L^1}{\kappa_1} - 8, \quad \lambda_1 = \frac{\lambda_L^1}{4\kappa_1^2}, \quad \varphi_1 = \sqrt{2\kappa_1} \phi_1, \quad (11)$$

and

$$\mu = \frac{\mu_L}{4\kappa_0\kappa_1}, \quad g = \frac{g_L}{4\sqrt{\kappa_0\kappa_1}^{3/2}} \quad (12)$$

2 BH four point function

$$C_4^{BH} = \frac{\langle \phi_0(\frac{T}{2}) \phi_1(t) \phi_1(\frac{T}{8}) \phi_0(0) \rangle}{\langle \phi_0(\frac{T}{2}) \phi_0(0) \rangle \langle \phi_1(t) \phi_1(\frac{T}{8}) \rangle} - 1$$

2.1 Spectral Decomposition for $t_1 < t_2 < t_3 < t_4$

- Numerator

$$\langle \phi_0(t_1) \phi_1(t_2) \phi_1(t_3) \phi_0(t_4) \rangle = \quad (13)$$

$$\sum_{i,j,k} \frac{1}{2m_j 2m_k} e^{-(T-t_1)E_i} e^{-t_1 E_j} \langle i | \phi_0 | j \rangle \langle j | \phi_1(t_2) \phi_1(t_3) | k \rangle \langle k | \phi_0 | i \rangle e^{-(E_i - E_k)t_4} \quad (14)$$

$$(15)$$

in the limit $T - t_1 \rightarrow \infty$ and setting $t_4 = 0$

$$\sum_{j,k} \frac{1}{2m_j 2m_k} \langle 0 | \phi_0 | j \rangle e^{-t_1 E_j} \langle j | \phi_1(t_2) \phi_1(t_3) | k \rangle \langle k | \phi_0 | 0 \rangle \quad (16)$$

$$= \sum_{j,k} \frac{1}{2m_j 2m_k} \langle 0 | \phi_0 | j \rangle e^{-(t_1-t_2)E_j} \langle j | \phi_1 e^{-(t_2-t_3)H} \phi_1 | k \rangle e^{-t_3 E_k} \langle k | \phi_0 | 0 \rangle \quad (17)$$

Assuming $t_1 - t_2 \gg 0$ and $t_3 \gg 0$ the states $|j\rangle = |\pi\rangle$ and $|k\rangle = |\pi\rangle$ so we get

$$= \frac{1}{2m_\pi 2m_\pi} |\langle 0 | \phi_0 | \pi \rangle|^2 e^{-(t_1-t_2)E_\pi} e^{-t_3 E_\pi} \langle \pi | \phi_1 e^{-(t_2-t_3)H} \phi_1 | \pi \rangle \quad (18)$$

Setting $t_1 = T/2$, $t_2 = t$ and $t_3 = T/8$

$$\langle \phi_0(\frac{T}{2}) \phi_1(t) \phi_1(\frac{T}{8}) \phi_0(0) \rangle = \frac{1}{2m_\pi 2m_\pi} |\langle 0 | \phi_0 | \pi \rangle|^2 e^{-\frac{T}{2} E_\pi} e^{-(\frac{T}{8}-t) E_\pi} \langle \pi | \phi_1 e^{-(t-\frac{T}{8}) H} \phi_1 | \pi \rangle \quad (19)$$

- Denominator 0

$$\langle \phi_0(\frac{T}{2}) \phi_0(0) \rangle = \frac{1}{m_\pi} |\langle 0 | \phi_0 | \pi \rangle|^2 e^{-\frac{T}{2} E_\pi}$$

Where we have summed both the forward and backward signal

- Denominator 1

$$\langle \phi_1(t) \phi_1(\frac{T}{8}) \rangle = \frac{1}{2m_N} |\langle 0 | N | \pi \rangle|^2 \left(e^{-(t-\frac{T}{8}) E_N} + e^{(T-t+\frac{T}{8}) E_N} \right) = \frac{1}{2m_N} |\langle 0 | N | \pi \rangle|^2 e^{-(t-\frac{T}{8}) E_N}$$

We can ignore the second term since $T/8 < t < T/2$

Putting the various pieces together we get

$$C_4^{BH} = \frac{m_N}{2m_\pi} \frac{e^{(t-\frac{T}{8})(E_N-E_\pi)} \langle \pi | \phi_1 e^{-(t-\frac{T}{8}) H} \phi_1 | \pi \rangle}{|\langle 0 | N | \pi \rangle|^2} - 1 \quad (20)$$

To be compared with the expression on the paper BH:

$$c_{\vec{q}_1' \vec{q}_2' \vec{q}_1 \vec{q}_2}^{\Theta, N\pi}(t', t | M_0) \equiv \frac{2\omega_{\vec{q}_2'} e^{\omega_{\vec{q}_2'} t'} 2\omega_{\vec{q}_2} e^{-\omega_{\vec{q}_2} t}}{Z_N} C_{a'b'} C_{ab} \\ \times \langle \pi^{a'}, \vec{q}_1' | \tilde{N}_{\vec{q}_2'}^{b'}(0) \Theta(\hat{M} - M_0, \Delta) e^{-\hat{H}(t'-t)} e^{\omega_{\vec{q}_1'} t'} e^{-\omega_{\vec{q}_1} t} \tilde{N}_{-\vec{q}_2}^{\dagger b}(0) | \pi^a, \vec{q}_1 \rangle.$$

where $\omega_{\vec{q}} = \sqrt{\vec{q}^2 + m_\pi^2}$ and $Z_\pi = \langle \pi, \vec{p} | \pi(0) | 0 \rangle^2$. Here the single particle state has the usual relativistic normalization, $\langle \pi, \vec{q} | \pi, \vec{q}' \rangle = 2\omega_{\vec{q}} (2\pi)^3 \delta^3(\vec{q} - \vec{q}')$.

Comparing the two equation above we get

$$8m_\pi m_N C_4^{BH} = c_{\vec{q}_1' \vec{q}_2' \vec{q}_1 \vec{q}_2}^{\Theta, N\pi}(t', t | M_0) \quad (21)$$

To be compared with the expression on the paper BH:

$$c_0^{\Theta, N\pi}(t', t | M_0)_c = 8\pi(m_N + m_\pi) a_{N\pi}(t' - t) \\ - 16 a_{N\pi}^2 \sqrt{2\pi(m_N + m_\pi) m_N m_\pi (t' - t)} + \mathcal{O}((t' - t)^0).$$

Finally our formula become

$$C_4^{BH} = \frac{1}{8m_\pi m_N} \left[8\pi(m_N + m_\pi) a_{N\pi}(t' - t) - 16 a_{N\pi}^2 \sqrt{2\pi(m_N + m_\pi) m_N m_\pi (t' - t)} + \mathcal{O}((t' - t)^0) \right].$$