8-2基础过关

1.计算下列偏导数

（1）设 $z=u^2\ln v$，而 $u=\dfrac{x}{y}, v=3x-2y$，求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

$\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}=2u\ln v\cdot\dfrac{1}{y}+\dfrac{u^2}{v}\cdot(3)=\dfrac{2x}{y^2}\ln(3x-2y)+\dfrac{3x^2}{y^2(3x-2y)}$

$\dfrac{\partial z}{\partial y}=2u\ln v\cdot\left(-\dfrac{x}{y^2}\right)+\dfrac{u^2}{v}\cdot(-2)=-\dfrac{2x^2}{y^3}\ln(3x-2y)-\dfrac{2x^2}{y^2(3x-2y)}$

（2）设 $u=\dfrac{e^{ax}(y-z)}{a^2+1}$，而 $y=a\sin x, z=\cos x$，求 $\dfrac{du}{dx}$.

$\dfrac{du}{dx}=\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial x}+\dfrac{\partial u}{\partial z}\cdot\dfrac{\partial z}{\partial x}$

$=\dfrac{ae^{ax}(y-z)}{a^2+1}+\dfrac{e^{ax}}{a^2+1}\cdot a\cos x+\dfrac{e^{ax}}{a^2+1}\cdot\sin x=e^{ax}\sin x$

2.设 $z=\arctan\dfrac{x}{y}$，而 $x=u+v, y=u-v$，求 $\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}=$___.

$\dfrac{\partial z}{\partial u}=\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial u}+\dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial u}=\dfrac{\frac{1}{y}}{1+(\frac{x}{y})^2}+\dfrac{-\frac{x}{y^2}}{1+(\frac{x}{y})^2}$

$\dfrac{\partial z}{\partial v}=\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial v}+\dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial v}=\dfrac{\frac{1}{y}}{1+(\frac{x}{y})^2}+\dfrac{-\frac{x}{y^2}}{1+(\frac{x}{y})^2}\cdot(-1)$

相加：$\dfrac{2y}{x^2+y^2}=\dfrac{u-v}{u^2+v^2}$

3.求下列函数的一阶偏导数（其中 f 具有一阶连续偏导数）：

① $u=f(x^2-y^2, e^{xy})$；

$\dfrac{\partial u}{\partial x}=2xf_1'+ye^{xy}f_2'$

$\dfrac{\partial u}{\partial y}=-2yf_1'+xe^{xy}f_2'$

② $u=f\left(\dfrac{x}{y},\dfrac{y}{z}\right)$；

$\dfrac{\partial u}{\partial x}=\dfrac{1}{y}f_1'$

$\dfrac{\partial u}{\partial y}=-\dfrac{x}{y^2}f_1'+\dfrac{1}{z}f_2'$

$\dfrac{\partial u}{\partial z}=-\dfrac{y}{z^2}f_2'$

③ $u=f(x,xy,xyz)$.

$\dfrac{\partial u}{\partial x}=f_1'+yf_2'+yzf_3'$；$\dfrac{\partial u}{\partial z}=xyf_3'$

$\dfrac{\partial u}{\partial y}=xf_2'+xzf_3'$

4.设 $z=xy+xF(u)$，而 $u=\dfrac{y}{x}$，$F(u)$ 为可导函数，则 $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=$ $2xy+xF(u)$.

4. $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}=y+F(u)+xF'(u)\cdot-\dfrac{y}{x^2}$

$\dfrac{\partial z}{\partial y}=x+xF'(u)\cdot\dfrac{1}{x}=x+F'(u)$

5. $\frac{\partial z}{\partial x}=\frac{-y}{f^2(x^2-y^2)}\cdot(2x)$

$\frac{\partial z}{\partial y}=\frac{1}{f(x^2-y^2)}+\frac{-y}{f^2(x^2-y^2)}\cdot(-2y)$

$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=$

5.设 $z=\frac{y}{f(x^2-y^2)}$，其中 $f(u)$ 为可导函数，则 $\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=$ $\frac{1}{y\cdot f(x^2-y^2)}$

$f_{12}''=f_{21}''$

6.求下列函数的二阶偏导数（其中 f 具有二阶连续偏导数）：

①$z=f(x,\frac{x}{y})$；

6①
$\frac{\partial z}{\partial x}=f_1'+f_2'\cdot\frac{1}{y}$
$\frac{\partial^2 z}{\partial x^2}=f_{11}''+\frac{2}{y}f_{12}''+f_{22}''\cdot\frac{1}{y^2}$
$\frac{\partial z}{\partial y}=-f_2'\cdot\frac{x}{y^2}$；$\frac{\partial^2 z}{\partial y^2}=-\frac{-2yx}{y^4}f_2'-\frac{x}{y^2}\cdot f_{22}''\cdot-\frac{x}{y^2}=\frac{2x}{y^3}f_2'+\frac{x^2}{y^4}f_{22}''$

②$z=f(xy^2,x^2y)$.

6(2)
$\frac{\partial z}{\partial x}=f_1'\cdot y^2+f_2'\cdot 2yx$
$\frac{\partial^2 z}{\partial x\partial y}=y^2(f_{11}''\cdot y^2+f_{12}''2yx)+2yf_2'+2yx(f_{21}''y^2+f_{22}''\cdot 2yx)$
$=y^4f_{11}''+4y^3xf_{12}''+2yf_2'+f_{22}''\cdot 4y^2x^2$
同理：$\frac{\partial^2 z}{\partial y^2}=2xf_1'+4x^3yf_{12}''+4x^2y^2f_{11}''+x^4f_{22}''$

7.设 $u=f(x,y)$ 的所有二阶偏导数连续，而 $x=\frac{s-\sqrt{3}t}{2}$，$y=\frac{\sqrt{3}s+t}{2}$，证明：

①$(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2=(\frac{\partial u}{\partial s})^2+(\frac{\partial u}{\partial t})^2$；

7① $(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2=(f_1')^2+(f_2')^2$
$(\frac{\partial u}{\partial s})^2=(f_1'\cdot\frac{1}{2}+f_2'\cdot\frac{\sqrt{3}}{2})^2$，$(\frac{\partial u}{\partial t})^2=(f_1'\cdot(-\frac{\sqrt{3}}{2})+f_2'\cdot\frac{1}{2})^2$
证得✓

②$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}$.

7② $\frac{\partial^2 u}{\partial x^2}=f_{11}''+f_{22}''$；$\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=\frac{1}{2}(\frac{1}{2}f_{11}''+\frac{\sqrt{3}}{2}f_{12}'')+\frac{\sqrt{3}}{2}(\frac{1}{2}f_{21}''+\frac{\sqrt{3}}{2}f_{22}'')$
$=f_{11}''+f_{22}''$
证得✓

8.计算下列偏导数

（1）设 $x+2y+z-2\sqrt{xyz}=0$，求 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$.

8(1) $1+\frac{\partial z}{\partial x}-\frac{(yz+xy\frac{\partial z}{\partial x})}{\sqrt{xyz}}=0$ $\Rightarrow$ $\frac{\partial z}{\partial x}=\frac{yz-\sqrt{xyz}}{\sqrt{xyz}-xy}$

$2+\frac{\partial z}{\partial y}-\frac{(xz+xy\frac{\partial z}{\partial y})}{\sqrt{xyz}}=0$ $\Rightarrow$ $\frac{\partial z}{\partial y}=\frac{xz-2\sqrt{xyz}}{\sqrt{xyz}-xy}$

（2）设 $\frac{x}{z}=\ln\frac{z}{y}$，求 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$.

(2) 设 $F(x,y,z)=\frac{x}{z}-\ln\frac{z}{y}$. $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{-\frac{1}{z}}{(-\frac{x+z}{z^2})}=\frac{z}{x+z}$. $\frac{\partial z}{\partial y}=-\frac{F_y}{F_x}=\frac{-\frac{1}{y}}{(-\frac{x+z}{z^2})}=\frac{z^2}{y(x+z)}$

9.设 $2\sin(x+2y-3z)=x+2y-3z$，则 $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=$____.

9. 设 $F(x,y,z)=2\sin(x+2y-3z)-x-2y+3z$

$F_x=2\cos(x+2y-3z)-1$; $F_z=2\cos(x+2y-3z)\cdot(-3)+3=-3F_x$

$F_y=2\cos(x+2y-3z)\cdot 2-2=2F_x$; 所以 $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\frac{1}{3}+\frac{2}{3}=1$

10.设 $x=x(y,z),y=y(x,z),z=z(x,y)$ 都是由方程 $F(x,y,z)=0$ 所确定的具有连续偏导数的函数，则 $\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial x}=$ -1.

10. $\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial x}=(-\frac{F_y}{F_x})\cdot(-\frac{F_z}{F_y})\cdot(-\frac{F_x}{F_z})$
$=-1$

11.设 $\Phi(u,v)$ 具有连续偏导数，$z=f(x,y)$ 由方程 $\Phi(cx-az,cy-bz)=0$ 所确定，则 $a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=$ c.

11. 设 $u=cx-az$, $v=cy-bz$

$\Phi_x=\Phi_u\cdot\frac{\partial u}{\partial x}=c\Phi_u$, $\Phi_y=\Phi_v\cdot\frac{\partial v}{\partial y}=c\Phi_v$

$\Phi_z=\Phi_u\cdot\frac{\partial u}{\partial z}\cdots=-a\Phi_u-b\Phi_v$

所以 $\frac{\partial z}{\partial x}=\frac{c\Phi_u}{a\Phi_u+b\Phi_v}$, $\frac{\partial z}{\partial y}=\frac{c\Phi_v}{a\Phi_u+b\Phi_v}$

⇒ 答案：c

12.设 $e^z-xyz=0$，则 $\frac{\partial^2 z}{\partial x^2}=\frac{2y^2ze^z-2xy^3z-y^2z^2e^z}{(e^z-xy)^3}$.

13.设 $z^3-3xyz=a^3$，则 $\frac{\partial^2 z}{\partial x\partial y}=\frac{z^5-2xyz^3-x^2y^2z}{(z^2-xy)^3}$

14.设 $y=f(x,t)$，而 $t=t(x,y)$ 是由方程 $F(x,y,t)=0$ 所确定的函数，其中 f,F 都具有一阶连续偏导数，试证明：$\frac{dy}{dx}=\frac{\frac{\partial f}{\partial x}\frac{\partial F}{\partial t}-\frac{\partial f}{\partial t}\frac{\partial F}{\partial x}}{\frac{\partial f}{\partial t}\frac{\partial F}{\partial y}+\frac{\partial F}{\partial t}}$.

$\frac{dy}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial t}\cdot\left(\frac{\partial t}{\partial x}+\frac{\partial t}{\partial y}\cdot\frac{dy}{dx}\right)$

$\frac{\partial t}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial t}}$, $\frac{\partial t}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial t}}$. 代入原式. $\frac{dy}{dx}=\frac{\frac{\partial f}{\partial x}\frac{\partial F}{\partial t}-\frac{\partial f}{\partial t}\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial t}+\frac{\partial F}{\partial y}\frac{\partial f}{\partial t}}$