7-1基础

① $xy' - y\ln y = 0$;

$x\frac{dy}{dx} = y\ln y$

$\int \frac{1}{y\ln y}dy = \int \frac{1}{x}dx$

$\ln|\ln y| = \ln|x| + \ln|C| \Rightarrow y = e^{cx}$

② $y' - xy' = a(y^2 + y')$;

$(1-x-a)y' = ay^2$

$\int \frac{1}{ay^2}dy = \int \frac{1}{1-x-a}dx$

$-\frac{1}{ay} = -\ln|1-x-a| + C \Rightarrow \frac{1}{y} = a\ln|1-x-a| + C$

$\frac{1}{x} - \frac{1}{4-x}$

③ $y' = e^{x+y}$;

$\frac{dy}{dx} = e^x \cdot e^y$

$\int \frac{1}{e^y}dy = \int e^x dx$

$-e^{-y} = e^x + C$

④ $ydx + (x^2-4x)dy = 0$;

$ydx = (4x-x^2)dy$

$\int \frac{1}{4x-x^2}dx = \int \frac{1}{y}dy$

$\frac{1}{4}\ln\left|\frac{x}{4-x}\right| = \ln|y| + \ln|C|$

$Cy^4 = \frac{x}{4-x}$

⑤ $\cos y dx + (1+e^{-x})\sin y dy = 0, y(0) = \frac{\pi}{4}$;

$\int -\frac{dx}{e^{-x}+1} = \int \tan y\, dy$

$-\ln(e^x+1) = \ln|\sec y| + C$

$\ln\cos y + \ln C = \ln(e^x+1)$

$C\cos y = e^x + 1$

由$y(0) = \frac{\pi}{4} \Rightarrow e^x + 1 = 2\sqrt{2}\cos y$

⑥ $y'\sin x = y\ln y, y\left(\frac{\pi}{2}\right) = e$;

$\int \frac{1}{y\ln y}dy = \int \frac{1}{\sin x}dx$

$\ln(\ln y) = \ln\frac{1}{\cot x + \csc x} + \ln C$

$\ln y = \frac{C}{\cot x + \csc x} \Rightarrow \ln y = \frac{\sin x}{1+\cos x}$

2.求下列微分方程的通解或满足所给初始条件的特解：

① $xy' - y - \sqrt{y^2 - x^2} = 0$;

$x>0$时

$y' - \frac{y}{x} - \sqrt{\left(\frac{y}{x}\right)^2 - 1} = 0$；令$u = \frac{y}{x}$

$u'x + u = u + \sqrt{u^2-1}$

$\int \frac{1}{\sqrt{u^2-1}}du = \int \frac{1}{x}dx \Rightarrow$

$\ln(u + \sqrt{u^2-1}) = \ln Cx$

$Cx = u + \sqrt{u^2-1}$

$Cx^2 = y + \sqrt{y^2-x^2}$

$x<0$时

$y' = \frac{y}{x} - \sqrt{\left(\frac{y}{x}\right)^2 - 1}$

$\Rightarrow$

$y - \sqrt{y^2-x^2} = C$

② $xy' = y\ln\frac{y}{x}$;

$y' = \frac{y}{x}\ln\frac{y}{x}$. 令$u = \frac{y}{x}$

$u'x + u = u\ln u$

$\int \frac{du}{u(\ln u - 1)} = \int \frac{1}{x}dx \Rightarrow \ln y = \ln x + Cx + 1$

③ $y' = \frac{x}{y} + \frac{y}{x}, y(1) = 2$;

$u + u'x = u + \frac{1}{u}$

$\int u\,du = \int \frac{1}{x}dx$

$\frac{1}{2}u^2 = \ln|x| + C$

$y^2 = 2x^2\ln|x| + 4x^2$

④ $(y^2 - 3x^2)dy + 2xydx = 0, y(0) = 1$;

$(u^2-3)dy + 2u\,dx = 0$

$(u^2-3)(u+u'x) + 2u = 0$

$\int \frac{3-u^2}{u(u^2-1)}du = \int \frac{1}{x}dx$

$\ln\left|\frac{u^2-1}{u^3}\right| = \ln x + Cx + 1$

$y^2 - x^2 = y^3$

⑤ $(1+2e^{\frac{x}{y}})dx + 2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$.

$\frac{dy}{dx} = \frac{1+2e^u}{-2e^u(1-\frac{1}{u})}$

$\Rightarrow 2e^u \cdot y + x = C$

3.求下列微分方程的通解或满足所给初始条件的特解：

① $y' + y\cos x = e^{-\sin x}$;

$y = e^{-\int\cos x dx}\left[\int e^{-\sin x}e^{\int\cos x dx} + C\right]$

$= e^{-\sin x}(x + C)$

同一种方法!☆

形如 $y' + p(x)y = q(x)$ 的方程称为一阶线性微分方程.

求解一阶线性微分方程的一般方法:常数变易法,或直接利用以下通解公式

$$y = e^{-\int p(x)dx}\left[\int q(x)e^{\int p(x)dx}dx + C\right].$$

② $(x^2-1)y' + 2xy - \cos x = 0$;

$y' + \frac{2x}{x^2-1}y = \frac{\cos x}{x^2-1}$

$y = \frac{1}{x^2-1}(\sin x + C)$

$(y^2-6x)y'+2y$

$y^2\cdot y'-6xy'+2y=0$

反过来更方便

③ $y\ln y\,dx+(x-\ln y)dy=0$;

$x'+\frac{1}{y\ln y}x=\frac{1}{y}$

$x=\frac{\ln y}{2}+\frac{c}{\ln y}$

④ $(y^2-6x)\frac{dy}{dx}+2y=0$;

$x'=\frac{6}{y}x-\frac{1}{2}y$

$x=\frac{1}{2}y^2+cy^3$

⑤ $\frac{dy}{dx}-y\tan x=\sec x, y(0)=0$;

$y'-\tan x\, y=\sec x$

$y=e^{\int\tan x dx}[\int\sec x\, e^{-\int\tan x dx}dx+C]=\frac{x}{\cos x}$ (代入 $y(0)=0$)

⑥ $\frac{dy}{dx}+\frac{y}{x}=\frac{\sin x}{x}, y(\pi)=1$.

$y'+\frac{1}{x}y=\frac{\sin x}{x}$

$y=e^{-\int\frac{1}{x}dx}[\int\frac{\sin x}{x}e^{\int\frac{1}{x}dx}dx+C]$

$=\frac{1}{x}[\pi-1-\cos x]$

4.用适当的变量代换化简下列微分方程化，然后求出通解：

① $\frac{dy}{dx}=(x+y)^2$;

令 $u=(x+y)$

$\frac{d(u-x)}{dx}=u'-1=u^2$

$\int\frac{1}{u^2+1}du=\int dx$

$\Rightarrow \arctan u=x+C$

$u=\tan(x+C)$

$\Rightarrow y=\tan(x+C)-x$

② $xy'+y=y(\ln x+\ln y)$.

令 $u=xy$

$\frac{dy}{dx}=\frac{d\frac{u}{x}}{dx}=\frac{u'x-u}{x^2}$

$\therefore \frac{1}{x}(u'-u)+\frac{u}{x}=\frac{u\ln u}{x}$

$\Rightarrow u=e^{cx}$

$xy=e^{cx}$

5.（数一、二）求下列微分方程的通解或满足所给初始条件的特解：

可降阶的

① $y''=y'+x$;

令 $y'=p\Rightarrow p'=p+x$

$p=e^{\int dx}[\int xe^{-\int dx}dx+C_1]$

$y=\int p\,dx=\int(C_1e^x-x-1)dx=C_1e^x-\frac{1}{2}x^2-x+C_2$

② $xy''+y'=0$;

$xp'+p=0\Rightarrow |p|=C_1\frac{1}{x}$

$y=C_1\ln|x|+C_2$

③ $yy''+2y'^2=0$;

令 $y'=p$. $y''=\frac{dp}{dy}p$

$y\frac{dp}{dy}p=-2p^2\Rightarrow y^3=C_1x+C_2$

④ $y''=1+y'^2$;

$p'=1+p^2$

$\int\frac{1}{1+p^2}dp=\int dx$

$\arctan p=x+C_1$

$y=\ln|\cos(x+C_1)|+C_2$

⑤ $y''-ay'^2=0, y(0)=0, y'(0)=-1$;

$\frac{dp}{dx}=ap^2$

$\int\frac{1}{p^2}dp=\int a\,dx$

$-\frac{1}{p}=ax+C_1\Rightarrow C_1=1$

$y=\ln|ax+1|\cdot\frac{-1}{a}$

⑥ $y''+(y')^2=1, y(0)=y'(0)=0$.

$p'+p^2=1$

$\int\frac{1}{1-p^2}dp=\int dx$

$\frac{1}{2}\ln|\frac{1+p}{1-p}|=x+C_1$

由 $y'(0)=0$, $C_1=0$

$y=\ln(\frac{e^x+e^{-x}}{2})$

6.线性微分方程解的结构：

（1）验证 $y_1=e^{x^2}$ 及 $y_2=xe^{x^2}$ 都是方程 $y''-4xy'+(4x^2-2)y=0$ 的解，并写出该方程的通解.

代入验证即可 ✓ $y_1'=2xe^{x^2}$. $y_1''=2e^{x^2}+2x\cdot e^{x^2}\cdot 2x$

通解：由于 $y_1/y_2\neq$ 常. y_1 和 y_2 线性无关. $y=C_1y_1+C_2y_2$

（2）已知 $y_1=x+\cos x$，$y_2=x+\sin x$，$y_3=x$ 是二阶非齐次线性微分方程的三个解，求该微分方程的通解.

$y_a=y_1-y_3=\cos x$. $y_b=y_2-y_3=\sin x$.

通解：$y=C_1\cos x+C_2\sin x+x$

（3）已知 $y = xe^{-x}$ 是二阶常系数齐次线性微分方程 $y'' + py' + qy = 0$ 的一个特解，求 p, q.

$y' = e^{-x} - xe^{-x}$ $\quad y'' = -e^{-x} - (e^{-x} - xe^{-x}) = e^{-x}(x-2)$

代入 $p=2,\ q=1$

7.求下列微分方程的通解或满足所给初始条件的特解：

① $y'' - 4y' = 0$； 令 $y' = p$

$p' - 4p = 0$

$y = c_1 + c_2 e^{4x}$

② $y'' - 4y' + 4y = 0$；

$r^2 - 4r + 4 = 0 \Rightarrow r_1 = r_2 = 2$

$y = (c_1 + c_2 x)e^{2x}$

③ $y'' + 6y' + 13y = 0$；

$r^2 + 6r + 13 = 0 \Rightarrow r_{1,2} = -3 \pm 2i$

$y = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x)$

④ $y^{(4)} - 2y''' + y'' = 0$；

$r^4 - 2r^3 + r^2 = 0$

$\Rightarrow r_1 = r_2 = 1 \quad r_3 = r_4 = 0$

$y = (c_1 + c_2 x)e^{x} + c_3 + c_4 x$

⑤ $4y'' + 4y' + y = 0,\ y(0) = 2,\ y'(0) = 0$；

$4r^2 + 4r + 1 = 0 \Rightarrow r_1 = r_2 = -\frac{1}{2}$

$y = (c_1 + c_2 x)e^{-\frac{1}{2}x}$

由初始条件，$y = (2 + x)e^{-\frac{1}{2}x}$

⑥ $y'' - 4y' + 13y = 0,\ y(0) = 0,\ y'(0) = 3$.

$r^2 - 4r + 13 = 0 \Rightarrow r_{1,2} = 2 \pm 3i$

$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$

$\Rightarrow y = e^{2x} \sin 3x$

8.求下列微分方程的通解或满足所给初始条件的特解：

① $2y'' + y' - y = 2e^x$；

$r^2 + r - 1 = 0 \Rightarrow r_1 = -1,\ r_2 = \frac{1}{2}$

$y = c_1 e^{-x} + c_2 e^{\frac{1}{2}x} + e^x$

② $y'' + 3y' + 2y = 3xe^{-x}$；

$r^2 + 3r + 2 \Rightarrow r_1 = -1,\ r_2 = -2$

$y = c_1 e^{-x} + c_2 e^{-2x} + (\frac{3}{2}x^2 - 3x)e^{-x}$

③ $y'' - 2y' + 5y = e^x \sin 2x$；

$r^2 - 2r + 5 = 0.\ r_{1,2} = 1 \pm 2i$

$y^* = xe^x(A\cos 2x + B\sin 2x)$

$\Rightarrow y = e^x(c_1 \cos 2x + c_2 \sin 2x) - \frac{1}{4}xe^x \cos 2x$

④ $y'' + 4y = x\cos x$；

$r^2 + 4r = 0 \Rightarrow r_{1,2} = \pm 2i$

$y^* = (Ax + B)\cos x + (Cx + D)\sin x$

$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3}x\cos x + \frac{2}{9}\sin x$

⑤ $y'' + y = e^x + \cos x$；

$r^2 + r = 0 \Rightarrow r_{1,2} = \pm i$

$y^* = ae^x + x(A\cos x + B\sin x)$

$y = c_1 \cos x + c_2 \sin x + \frac{1}{2}e^x + \frac{1}{2}x\sin x$

⑥ $y'' - y = 4xe^x,\ y(0) = 0,\ y'(0) = 1$.

$r^2 - 1 = 0 \Rightarrow r_1 = -1 \quad r_2 = 1$

$y^* = x(Ax + B)e^x$

$\Rightarrow A = 1.\ \ b = -1$

$\Rightarrow y = c_1 e^{-x} + c_2 e^x + (x^2 - x)e^x$

$\Rightarrow c_1 = 1.\ c_2 = -1$

$y = e^x - e^{-x} + (x^2 - x)e^x$