CRACKING CALCULUS 12

A REVISION BOOK

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Cracking Calculus 12
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MLIS-ZJ CALCULUS 12 PROJECT

This text is based on and includes some exercises and examples from the following calculus textbooks: *MOOCulus Calculus* Jim Fowler and Bart Snapp, *Calculus I with Precalculus: Third Edition* by Ron Larson and Bruce H. Edwards, and *Cracking the AP Calculus BC Exam: 2019 Edition* by David S. Khan.

The contents in this book are organized in a way that is supplemental to the Maple Leaf International School - Zhenjiang 2019 Calculus 12 curriculum and is intended as a revision material for the course.

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Answer Key

Bibliography

Prerequisites

1. Linear and Piecewise Functions

1.1 Introduction to Functions

Many everyday phenomena involve two quantities that are related to each other by some rule of correspondence. The mathematical term for such a rule of correspondence is a **relation**. In mathematics, relations are often represented by mathematical equations and formulas. For instance, the simple interest I earned on \$1000 for 1 year is related to the annual interest rate r by the formula $I = 1000r$. [3]

The formula $I = 1000r$ represents a special kind of relation that matches each item from one set with *exactly one* item from a different set. Such a relation is called a **function**.

> **Definition 1.1.1 — Function.** A **function** f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the **domain** (or set of inputs) of the function f, and the set B contains the **range** (or set of outputs). [3]

Functions are commonlt represented in four ways.

Proposition 1.1.1 — Four ways to represent a function.

1. *Verbally* by a sentence that describes how the input variable is related to the output variable
2. *Numerically* by a table or a list of ordered pairs that matches input values with output values
3. *Graphically* by points on a graph in a coordinate plane in which the input values are represented by the horizontal axis and the output values are represented by the vertical axis
4. *Analytically* by an equation in two variables

[3]

To determine whether or not a relation is a function, you must decide whether each input value is matched with exactly one output value. When any input value is matched with two or more output values, the relation is not a function.

■ Example 1.1 — Testing for Functions.
Determine whether the relation represents y as a function of x. [3]

1. The input value x is the number of representatives from a state, and the output value y is the number of senators.
2. Function f is defined as the following table

Input, x	**Output,** y
2	11
2	10
3	8
4	5
5	1

Solution 1.1

1. This verbal description *does* describe y as a function of x. Regardless of the value of x, the value of y is always 2. Such functions are called *constant functions*.
2. This table *does not* describe y as a function of x. The input value 2 is matched with two different y-values.

■

1.2 Function Notation

When an equation is used to represent a function, it is convenient to name the function so that it can be referenced easily. For example, you know that the equation $y = 3x+4$ describes y as a function of x. Suppose you give this function the name "f." Then you can use the following **function notation**. [3]

Input	**Output**	**Equation**
x	$f(x)$	$f(x) = 3x+4$

The symbol $f(x)$ is read as the value of f at x or simply f of x. The symbol $f(x)$ corresponds to the y-value for a given x. So, you can write $y = f(x)$. Keep in mind that f is the name of the function, whereas $f(x)$ is the value of the function at x. For instance, the function given by

$$f(x) = 3x+4 \tag{1.1}$$

has function values denoted by $f(0)$, $f(1)$, $f(2)$, and so on. To find these values, substitute the specified input values into the given equation.

$$\begin{aligned} f(-1) &= 3(-1)+4 = -3+4 = 1 && x = -1 \\ f(0) &= 3(0)+4 = 0+4 = 4 && x = 0 \\ f(2) &= 3(2)+4 = 6+4 = 10 && x = 2 \end{aligned}$$

Although f is often used as a convenient function name and x is often used as the independent variable, you can use other letters. For instance,

$$f(x) = 3x+4, \quad f(t) = 3t+4, \quad g(s) = 3s+4 \tag{1.2}$$

all define the same function. In fact, the role of the independent variable is that of a "placeholder" that can be replaced by *any real number or algebraic expression*. [3]

■ Example 1.2 — Evaluating a Function.
Let $g(x) = -7x + 20$. Find each function value [3]

1. $g(2)$
2. $g(t)$
3. $g(x+2)$

Solution 1.2

1. Replacing x with 2 in $g(x) = -7x + 20$ yields the following.

$$\begin{aligned} g(2) &= -7(2) + 20 && (1.3)\\ &= -14 + 20 && (1.4)\\ &= 6 && (1.5) \end{aligned}$$

2. Replacing x with t yields the following.

$$\begin{aligned} g(t) &= -7(t) + 20 && (1.6)\\ &= -7t + 20 && (1.7) \end{aligned}$$

3. Replacing x with $x+2$ yields the following.

$$\begin{aligned} g(x+2) &= -7(x+2) + 20 && (1.8)\\ &= -7x - 14 + 20 && (1.9)\\ &= 7x + 20 && (1.10) \end{aligned}$$

■

1.3 Piecewise Function

A function defined by two or more equations over a specified domain is called a **piecewise-defined function**. [3]

■ Example 1.3 — A Piecewise-Defined Function.
Evaluate the function when $x = -1$, 0, and 1. [3]

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases} \tag{1.11}$$

Solution 1.3 Because $x = -1$ is less than 0, use $f(x) = x^2 + 1$ to obtain

$$f(-1) = (-1)^2 + 1 = 2 \tag{1.12}$$

For $x = 0$, use $f(x) = x - 1$ to obtain

$$f(0) = (0) - 1 = -1 \tag{1.13}$$

For $x = 1$, use $f(x) = x - 1$ to obtain

$$f(1) = (1) - 1 = 0 \tag{1.14}$$

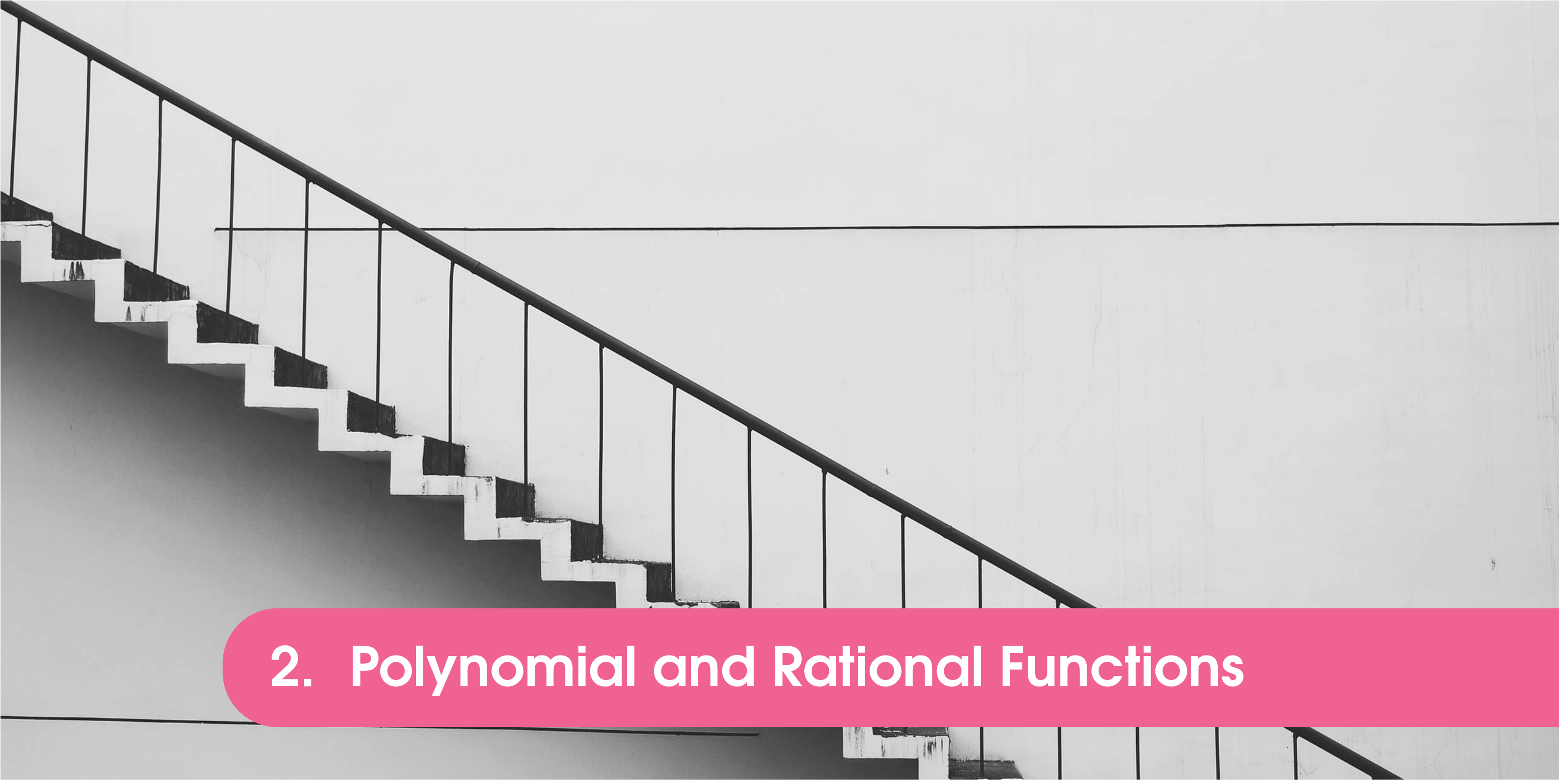
■

Exercises For Chapter 1

1. A relation that assigns to each element x from a set of inputs, or ______, exactly one element y in a set of outputs, or ______, is called a ______. [3]
2. Functions are commonly represented in four different ways, ______, ______, ______, and ______. [3]
3. For an equation that represents y as a function of x, the set of all values taken on by the ______variable x is the domain, and the set of all values taken on by the ______variable y is the range. [3]
4. The function given by
$$f(x) = \begin{cases} 2x-1, & x<0 \\ x^2+4, & x \geq 0 \end{cases}$$
is an example of a ______function.

In Exercises 5–7, evaluate the function at each specified value of the independent variable and simplify.

5. $f(x) = 2x-3$
(a) $f(1)$ (b) $f(-3)$ (c) $f(x-1)$
6. $f(x) = \begin{cases} 3x-1, & x<-1 \\ 4, & -1 \leq x \leq 1 \\ x^2, & x>1 \end{cases}$
(a) $f(-2)$ (b) $f(-\frac{1}{2})$ (c) $f(3)$
7. $f(x) = \begin{cases} 4-5x, & x \leq -2 \\ 0, & -2<x<2 \\ x^2+1, & x \geq 2 \end{cases}$
(a) $f(-3)$ (b) $f(4)$ (c) $f(-1)$



2. Polynomial and Rational Functions

2.1 Quadratic Function

In this and the next section, you will study the graphs of polynomial functions. In Chapter 1, you were introduced to the following basic functions. [3]

$f(x) = ax + b$	Linear function
$f(x) = c$	Constant function
$f(x) = x^2$	Squaring function

These functions are examples of **polynomial functions**.

Definition 2.1.1 — Polynomial Function. Let n be a nonnegative integer and let $a_n, a_{n-1}, \cdots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function given by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function of x with degree n**.
[3]

Polynomial functions are classified by degree. For instance, a constant function $f(x) = c$ with $c \neq 0$ has degree 0, and a linear function $f(x) = ax + b$ with $a \neq 0$ has degree 1. In this section, you will study second-degree polynomial functions, which are called **quadratic functions**. [3]

For instance, each of the following functions is a quadratic function.

$$
\begin{aligned}
f(x) &= x^2+6x+2\\
g(x) &= 2(x+1)^2-3\\
h(x) &= 9+\frac{1}{4}x^2\\
k(x) &= -3x^2+4\\
m(x) &= (x-2)(x+1)
\end{aligned}
$$

Note that the squaring function is a simple quadratic function that has degree 2.

Definition 2.1.2 — Quadratic Function. Let a, b, and c be real numbers with $a \neq 0$. The function given by

$$f(x) = ax^2+bx+c$$

is called a **quadratic function**.
[3]

The graph of a quadratic function is a special type of "U"-shaped curve called a **parabola**. Parabolas occur in many real-life applications—especially those involving reflective properties of satellite dishes and flashlight reflectors.

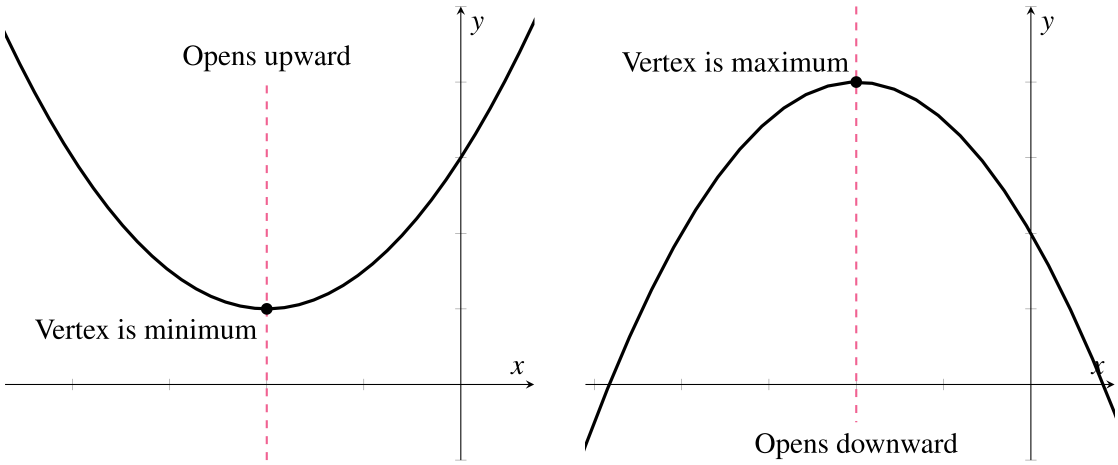
All parabolas are symmetric with respect to a line called the **axis of symmetry**, or simply the **axis** of the parabola. The point where the axis intersects the parabola is the **vertex** of the parabola, as shown in Figure 2.1. When $a > 0$, the graph of

$$f(x) = ax^2+bx+c$$

is a parabola that opens upward. When $a < 0$, the graph of

$$f(x) = ax^2+bx+c$$

is a parabola that opens downward. [3]



$a > 0$ $a < 0$

Figure 2.1: Parabola of ax^2+bx+c

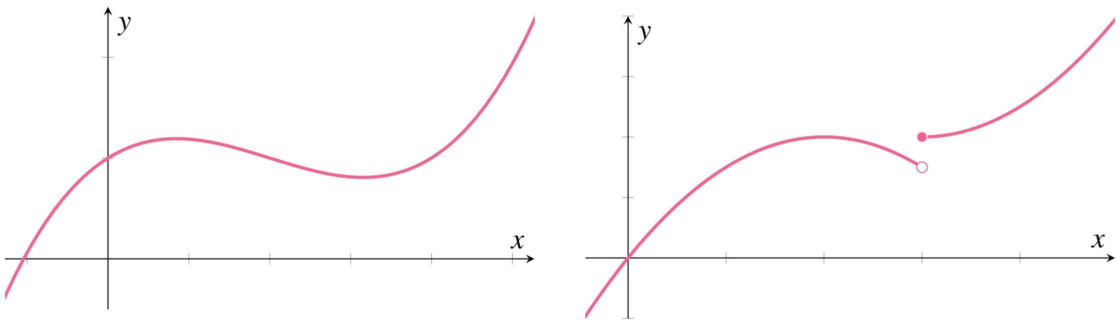
The simplest type of quadratic function is

$$f(x) = ax^2$$

Its graph is a parabola whose vertex is (0, 0). When $a > 0$, the vertex is the point with the *minimum* y-value on the graph, and when $a < 0$, the vertex is the point with the *maximum* y-value on the graph, as shown in Figure 2.1. [3]

2.2 Polynomial Functions of Higher Degree

In this section, you will study basic features of the graphs of polynomial functions. The first feature is that the graph of a polynomial function is *continuous*. Essentially, this means that the graph of a polynomial function has no breaks, holes, or gaps, as shown in Figure 2.2(a). The graph shown in Figure 2.2(b) is an example of a piecewise-defined function that is not continuous.

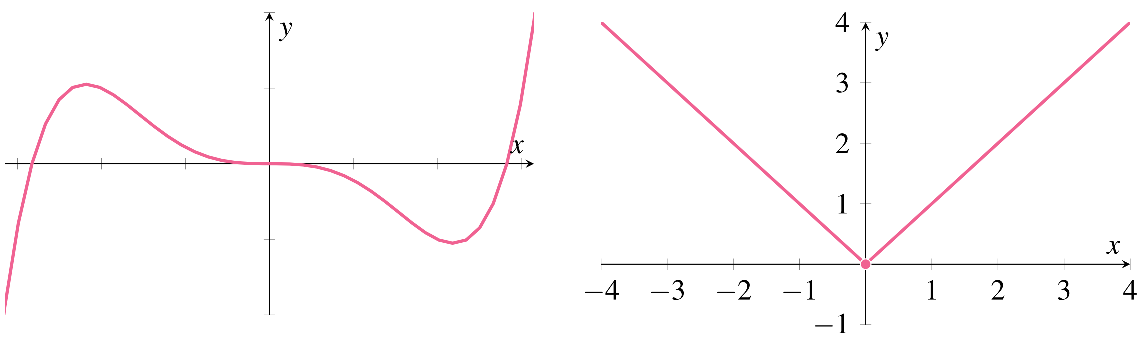


(a) Polynomial functions have continuous graphs.

(b) Functions with graphs that are not continuous are not polynomial functions.

Figure 2.2: Continuity of Polynomial Functions

The second feature is that the graph of a polynomial function has only smooth, rounded turns, as shown in Figure 2.3(a). A polynomial function cannot have a sharp turn. For instance, the function given by $f(x) = |x|$, which has a sharp turn at the point (0, 0), as shown in Figure 2.3(b), is not a polynomial function.



(a) Polynomial functions have graphs with smooth, rounded turns.

(b) Graphs of polynomial functions cannot have sharp turns.

Figure 2.3: Smoothness of Polynomial Functions

2.3 Real Zeros of Polynomial Functions

It can be shown that for a polynomial function f of degree n, the following statements are true. [3]

1. The function f has, at most, n real zeros.
2. The graph of f has, at most, $n-1$ turning points.

Proposition 2.3.1 — Real Zeros of Polynomial Functions . [3]
When f is a polynomial function and a is a real number, the following statements are equivalent.

1. $x = a$ is a *zero* of the function f.
2. $x = a$ is a solution of the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a factor of the polynomial $f(x)$.
4. $(a, 0)$ is an x-intercept of the graph of f.

■ Example 2.1 — Find the Zeros of a Polynomial Function . [3]
Find all real zeros of

$$f(x) = -2x^4 + 2x^2.$$

Then determine the number of turning points of the graph of the function.

Solution 2.1
To find the real zeros of the function, set $f(x)$ equal to zero and solve for x.

$-2x^4 + 2x^2 = 0$	Set $f(x)$ equal to 0.
$-2x^2(x^2 - 1) = 0$	Remove common monomial factor.
$-2x^2(x-1)(x+1) = 0$	Factor completely.

So, the real zeros are $x = 0$, $x = 1$, and $x = -1$. Because the function is a fourth-degree polynomial, the graph of f can have at most $4 - 1 = 3$ turning points.

■

Proposition 2.3.2 — Repeated Zeros . [3]
A factor $(x-a)^k, k > 1$, yields a **repeated zero** $x = a$ of **multiplicity** k.

1. When k is odd, the graph *crosses* the x-axis at $x = a$.
2. When k is even, the graph *touches* the x-axis (but does not cross the x-axis) at $x = a$.

2.4 Rational Functions

A rational function is a quotient of polynomial functions. It can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.[3]

In general, the *domain* of a rational function of x includes all real numbers except x-values that make the denominator zero. Much of the discussion of rational functions will focus on their graphical behavior near these x-values excluded from the domain. [3]

2.5 Vertical and Horizontal Asymptotes

Consider this function:

$$f(x) = \frac{1}{x}$$

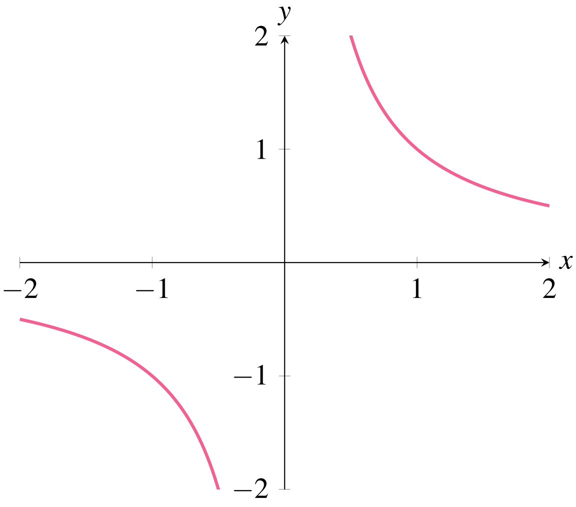


Figure 2.4: A plot of $\frac{1}{x}$.

the behaviour of f near $x = 0$ is denoted as follows.

$$f(x) \to -\infty \text{ as } x \to 0^- \qquad f(x) \to \infty \text{ as } x \to 0^+$$

The line $x = 0$ is a **vertical asymptote** of the graph of f, as shown in Figure 2.4. From this figure, you can see that the graph of f also has a **horizontal asymptote** — the line $y = 0$. This means that the values of $f(x) = \frac{1}{x}$ approach zero as x increases or decreases without bound.

$$f(x) \to 0 \text{ as } x \to -\infty \qquad f(x) \to 0 \text{ as } x \to \infty$$

Definition 2.5.1 — Vertical and Horizontal Asymptotes.

1. The line $x = a$ is a **vertical asymptote** of the graph of f when

$$f(x) \to -\infty \text{ or } f(x) \to \infty$$

as $x \to a$, either from the right or from the left.

2. The line $y = b$ is a **horizontal asymptote** of the graph of f when

$$f(x) \to b$$

as $x \to \infty$ or $x \to -\infty$.

[3]

Proposition 2.5.1 — Vertical and Horizontal Asymptotes of a Rational Function. [3]
Let f be the rational function given by

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of f has *vertical* asymptotes at the zeros of $D(x)$.
2. The graph of f has one or no *horizontal* asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.
   (a) When $n < m$,the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.
   (b) When $n = m$,the graph of f has the line $y = \frac{a_n}{b_m}$ (ratio of the leading coefficients) as a horizontal asymptote.
   (c) When $n > m$,the graph of f has no horizontal asymptote.

■ **Example 2.2 — Finding Vertical and Horizontal Asymptotes.** [3]
Find all vertical and horizontal asymptotes of the graph of each rational function.

1. $f(x) = \dfrac{2x^2}{x^2 - 1}$
2. $f(x) = \dfrac{x^2 + x - 2}{x^2 - x - 6}$

Solution 2.2

1. For this rational function, the degree of the numerator is equal to the degree of the denominator. The leading coefficient of the numerator is 2 and the leading coefficient of the denominator is 1, so the graph has the line $y = 2$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$$\begin{aligned} x^2 - 1 &= 0 && \text{Set denominator equal to zero.} \\ (x+1)(x-1) &= 0 && \text{Factor.} \\ x + 1 &= 0, \quad x = -1 && \text{Set 1st factor equal to 0.} \\ x - 1 &= 0, \quad x = 1 && \text{Set 2nd factor equal to 0.} \end{aligned}$$

   This equation has two real solutions, $x = 1$ and $x = -1$, so the graph has the lines $x = 1$ and $x = -1$ as vertical asymptotes.
2. For this rational function, the degree of the numerator is equal to the degree of the denominator. The leading coefficient of both the numerator and denominator is 1, so the graph has the line $y = 1$ as a horizontal asymptote. To find any vertical asymptotes, first factor the numerator and denominator as follows.

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6} = \frac{(x-1)(x+2)}{(x+2)(x-3)} = \frac{x-1}{x-3}, \quad x \neq -2$$

   By setting the denominator $x - 3$ (of the simplified function) equal to zero, you can determine that the graph has the line $x = 3$ as a vertical asymptote.

■

Exercises For Chapter 2

In Exercises 1–4, (a) find all the real zeros of the polynomial function, (b) determine the multiplicity of each zero and the number of turning points of the graph of the function. [3]

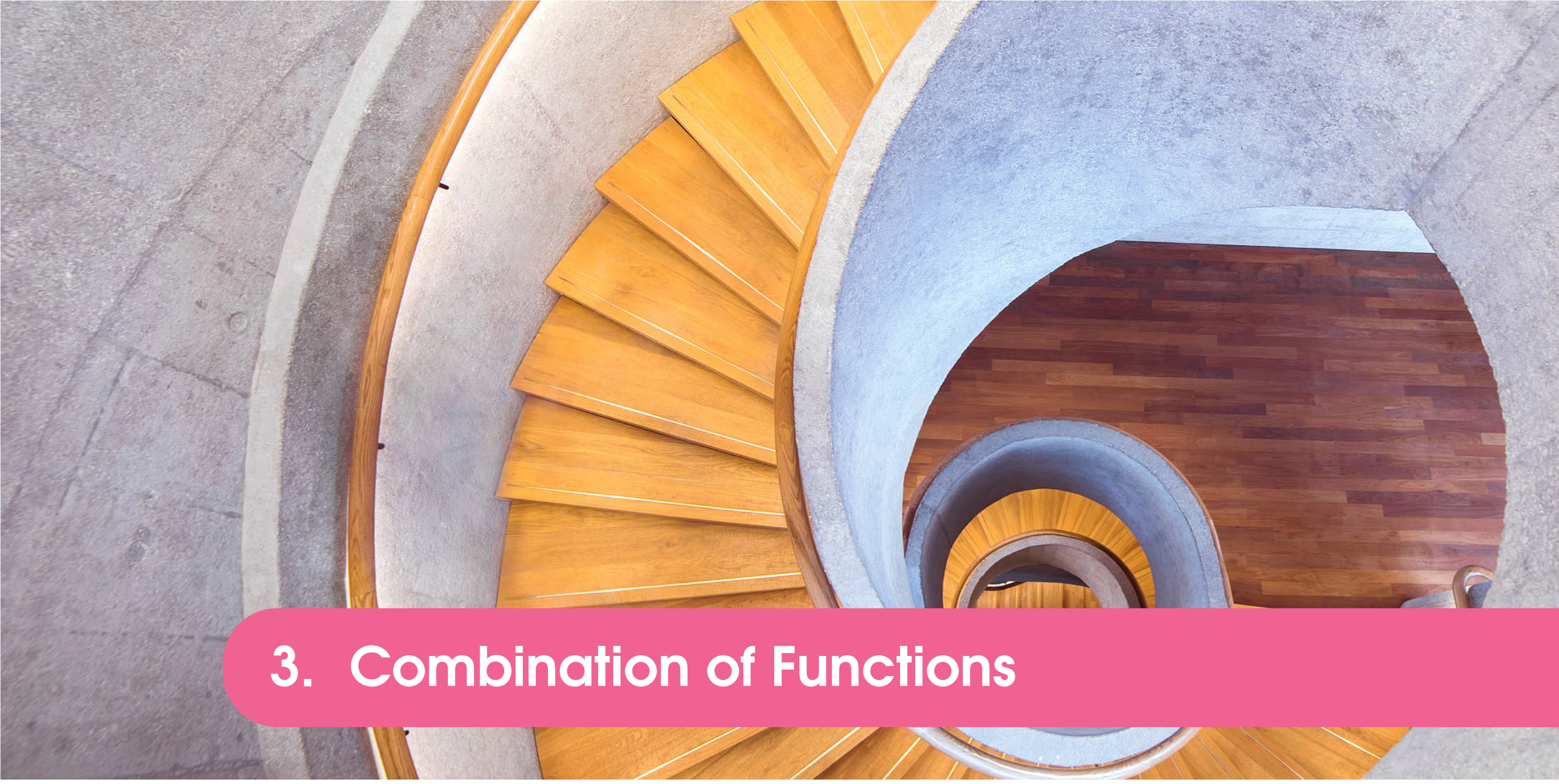
1. $f(x) = x^2 - 36$
2. $g(x) = 3x^3 - 12x^2 + 3x$
3. $f(c) = 3x^3 + 3x^2 - 4x - 12$
4. $f(t) = t^5 - 6t^3 + 9t$

In Exercises 5–8, find a polynomial function that has the given zeros. (There are many correct answers.) [3]

5. $0, 8$
6. $4, -3, 3, 0$
7. $1+\sqrt{3}, 1-\sqrt{3}$
8. $0, -4, -5$

In Exercises 9–12, find any vertical and horizontal Asymptotes. [3]

9. $f(x) = -\dfrac{1}{(x-2)^2}$
10. $f(x) = \dfrac{2x^2 - 5x - 3}{x^3 - 2x^2 - 5x + 6}$
11. $f(x) = \dfrac{x^2 + 3x)}{x^2 + x - 6}$
12. $f(t) = \dfrac{t^2 - 1}{t - 1}$



3. Combination of Functions

3.1 Arithmetic Combinations of Functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two functions can be combined to create new functions. For example, the functions given by $f(x) = 2x - 3$ and $g(x) = x^2 - 1$ can be combined to form the sum, difference, product, and quotient of f and g. [3]

$$\begin{aligned}
f(x) + g(x) &= (2x-3) + (x^2-1) & \\
&= x^2 + 2x + 4 & \text{Sum} \\
f(x) - g(x) &= (2x-3) - (x^2-1) & \\
&= -x^2 + 2x - 2 & \text{Difference} \\
f(x)g(x) &= (2x-3)(x^2-1) & \\
&= 2x^3 - 3x^2 - 2x + 3 & \text{Product} \\
\frac{f(x)}{g(x)} &= \frac{2x-3}{x^2-1}, \qquad x \neq \pm 1 & \text{Quotient}
\end{aligned}$$

The domain of an **arithmetic combination** of functions f and g consists of all real numbers that are common to the domains of f and g. In the case of the quotient $\frac{f(x)}{g(x)}$, there is the further restriction that $g(x) \neq 0$. [3]

■ **Example 3.1 — Finding the Sum of Two Functions.** [3]
Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f+g)(x)$. Then evaluate the sum when $x = 3$.

Solution 3.1

$$(f+g)(x) = f(x) + g(x) = (2x+1) + (x^2+2x-1) = x^2 + 4x$$

When $x = 3$, the value of this sum is

$$(f+g)(3) = (3)^2 + 4(3) = 21.$$

■

■ **Example 3.2 — Finding the Difference of Two Functions.** [3]
Given $f(x) = 2x+1$ and $g(x) = x^2+2x-1$, find $(f-g)(x)$. Then evaluate the sum when $x = 2$.

Solution 3.2

$$(f-g)(x) = f(x) - g(x) = (2x+1) - (x^2+2x-1) = -x^2+2$$

When $x = 2$, the value of this sum is

$$(f-g)(3) = -(2)^2+2 = -1.$$

■

■ **Example 3.3 — Finding the Product of Two Functions.** [3]
Given $f(x) = x^2$ and $g(x) = x-3$, find $(fg)(x)$. Then evaluate the sum when $x = 4$.

Solution 3.3

$$(fg)(x) = f(x)g(x) = (x^2)(x-3) = x^3-3x^2$$

When $x = 3$, the value of this sum is

$$(fg)(4) = (4)^3-3(4)^2 = 16.$$

■

■ **Example 3.4 — Finding the Quotients of Two Functions.** [3]
Given $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4-x^2}$, find $(f/g)(x)$ and $(g/f)(x)$. Then find the domains of f/g and g/f

Solution 3.4 The quotient of f and g is

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4-x^2}}$$

and the quotient of g and f is

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4-x^2}}{\sqrt{x}}$$

The domain of f is $[0,\infty)$ and the domain of g is $[2,-2]$. The intersection of these domains is $[0,2]$. So, the domains of f/g and g/f are as follows.

$$\text{Domain of } \frac{f}{g} : [0,2) \qquad \text{Domain of } \frac{g}{f} : (0,2]$$

■

3.2 Composition of Functions

Definition 3.2.1 — Composition of Two Functions. The **composition** of the function f with the function g is

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 3.1.)
[3]

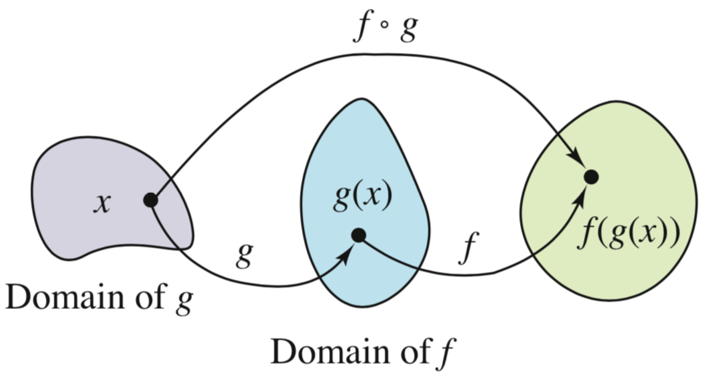


Figure 3.1: Domain of a Composition Function [3]

■ Example 3.5 — Composition of Functions. [3]
Given $f(x) = x+2$ and $g(x) = 4-x^2$, find the following.

1. $(f \circ g)(x)$
2. $(g \circ f)(x)$
3. $(g \circ f)(-2)$

Solution 3.5

1. The composition of f with g is as follows.

$$(f \circ g)(x) = f(g(x))$$
$$= f(4-x^2) \qquad = (4-x^2)+2 = -x^2+6$$

2. The composition of g with f is as follows.

$$(g \circ f)(x) = g(f(x))$$
$$= g(x+2) \qquad = 4-(x+2)^2 = -x^2-4x$$

Note that, in this case, $(f \circ g)(x) \neq (g \circ f)(x)$.

3. Using the result of part 2, you can write the following.

$$(g \circ f)(-2) = -(-2)^2 - 4(-2) \qquad = -4+8 = 4$$

■

3.3 Application

■ Example 3.6 — Bacteria Count. [3]
The number N of bacteria in a refrigerated food is given by

$$N(T) = 20T^2 - 80T + 500, \qquad 2 \leq T \leq 14$$

where T is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature of the food is given by

$$T(t) = 4t + 2, \qquad 0 \leq t \leq 3$$

where t is the time in hours.

1. Find the composition $N(T(t))$ and interpret its meaning in context.
2. Find the time when the bacteria count reaches 2000.

Solution 3.6

1. $$\begin{aligned} N(T(t)) &= 20(4t+2)^2 - 80(4t+2) + 500 \\ &= 20(16t^2 + 16t + 4) - 320t - 160 + 500 \\ &= 320t^2 + 320t + 80 - 320t - 160 + 500 \\ &= 320t^2 + 420 \end{aligned}$$

   The composite function $N(T(t))$ represents the number of bacteria in the food as a function of the amount of time the food has been out of refrigeration.
2. The bacteria count will reach 2000 when $320t^2 + 420 = 2000$. Solve this equation for t as shown.

   $$\begin{aligned} 320t^2 + 420 &= 2000 \\ 320t^2 &= 1580 \\ t^2 &= \frac{79}{16} \\ t &= \frac{\sqrt{79}}{4} \\ t &\approx 2.2 \end{aligned}$$

   So, the count will reach 2000 when $t \approx 2.2$ hours. When you solve this equation, note that the negative value is rejected because it is not in the domain of the composite function.

■

Exercises For Chapter 3

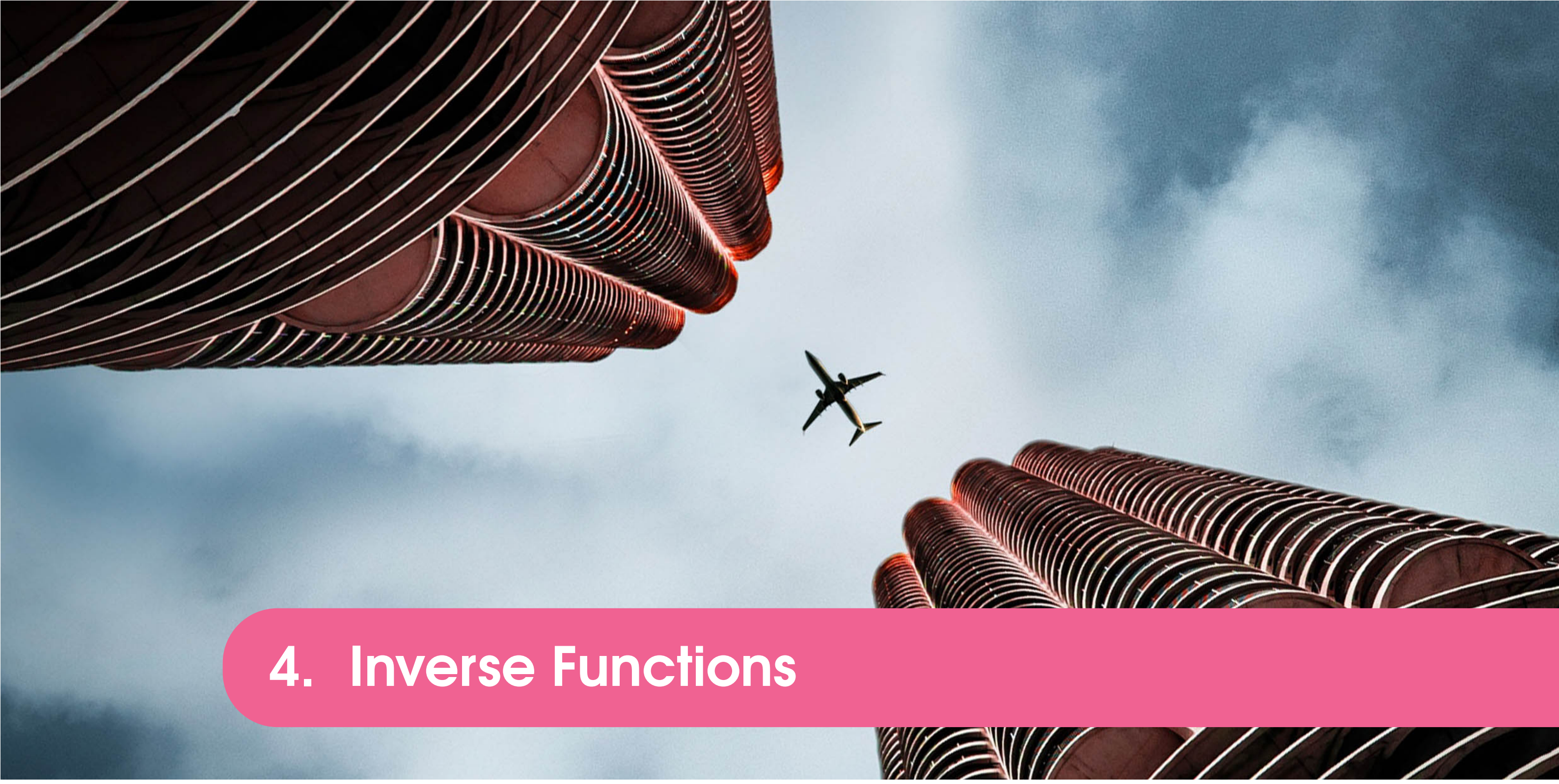
In Exercises 1–2, find (a) $(f+g)(x)$, (b) $(f-g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. [3]

1. $f(x) = x^2, \quad g(x) = 4x - 5$
2. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x^2}$

In Exercises 3–5, evaluate the indicated function for $f(x) = x^2 + 1$ and $g(x) = x - 4$. [3]

3. $(f+g)(2)$
4. $(fg)(6)$
5. $(f/g)(-1) - g(3)$

■



4. Inverse Functions

4.1 Inverse Functions

Recall that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x+4$ from the set $A = 1,2,3,4$ to the set $B = 5,6,7,8$ can be written as follows. [3]

$$f(x) = x+4 : (1,5),(2,6),(3,7),(4,8)$$

In this case, by interchanging the first and second coordinates of each of these ordered pairs, you can form the **inverse function** of f, which is denoted by f^{-1}. It is a function from the set B to the set A, and can be written as follows. [3]

$$f^{-1}(x) = x-4 : (5,1),(6,2),(7,3),(8,4)$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 4.1. Also note that the functions f and f^{-1} have the effect of "undoing" each other. In other words, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function. [3]

$$f(f^{-1}(x)) = f(x-4) = (x-4)+4 = x$$
$$f^{-1}(f(x)) = f^{-1}(x+4) = (x+4)-4 = x$$

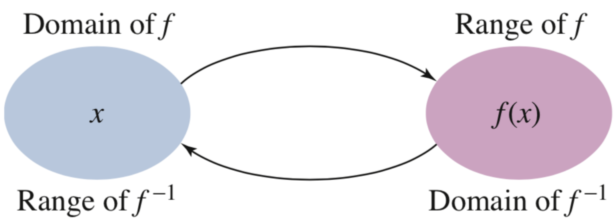


Figure 4.1: Domain of an Inverse Function [3]

Definition 4.1.1 Let f and g be two functions such that

$$f(g(x)) = x \qquad \text{for every } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \qquad \text{for every } x \text{ in the domain of } f$$

Under these conditions, the function g is the **inverse function** of the function f. The function g is denoted by f^{-1} (read "f-inverse").
[3]

If the function g is the inverse function of the function f, it must also be true that the function f is the inverse function of the function g. For this reason, you can say that the functions f and g are *inverse functions of each other*.

4.2 The Graph of an Inverse Function

The graphs of a function f and its inverse function f^{-1} are related to each other in the following way. If the point (a,b) lies on the graph of f, then the point (b,a) must lie on the graph of f^{-1}, and vice versa. This means that the graph of f^{-1} is a reflection of the graph of f in the line $y = x$, as shown in Figure 4.2.

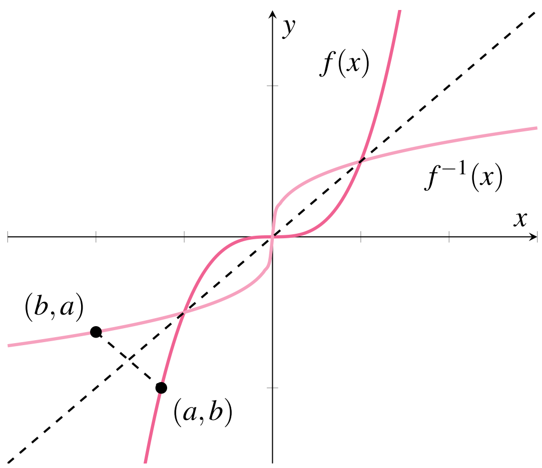


Figure 4.2: A plot of a pair of Inverse Functions.

4.3 Finding Inverse Functions Analytically

For simple functions, you can find inverse functions by inspection. For more complicated functions, however, it is best to use the following guidelines. The key step in these guidelines is Step 3—interchanging the roles of x and y. This step corresponds to the fact that inverse functions have ordered pairs with the coordinates reversed. [3]

Proposition 4.3.1 — Guidelines For Finding Inverse Functions. [3]

1. Use the Horizontal Line Test to decide whether f has an inverse function.
2. In the equation for $f(x)$, replace $f(x)$ by y.
3. Interchange the roles of x and y, and solve for y.
4. Replace y by $f^{-1}(x)$ in the new equation.

■ Example 4.1 — Finding an Inverse Function Analytically. [3]
Find the inverse function of $f(x) = \dfrac{5-3x}{2}$.

Solution 4.1
Since $f(x)$ is a linear function, the graph of f is a line which passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function.

$$\begin{aligned}
f(x) &= \frac{5-3x}{2} && \text{Write original function.}\\
y &= \frac{5-3x}{2} && \text{Replace } f(x) \text{ by } y.\\
x &= \frac{5-3y}{2} && \text{Interchange } x \text{ and } y.\\
2x &= 5-3y && \text{Multiply each side by 2.}\\
3y &= 5-2x && \text{Isolate the } y\text{-term.}\\
y &= \frac{5-2x}{3} && \text{Solve for } y.\\
f^{-1}(x) &= \frac{5-2x}{3} && \text{Replace } y \text{ by } f^{-1}(x).
\end{aligned}$$

Note that both f and f^{-1} have domains and ranges that consist of the entire set of real numbers.

■

■ Example 4.2 — Finding an Inverse Function Analytically. [3]
Find the inverse function of $f(x) = \sqrt{2x-3}$.

Solution 4.2
The graph of f is a curve, as shown in Figure 4.3. Because this graph passes the Horizontal Line Test, you know that f is one-to-one and has an inverse function.

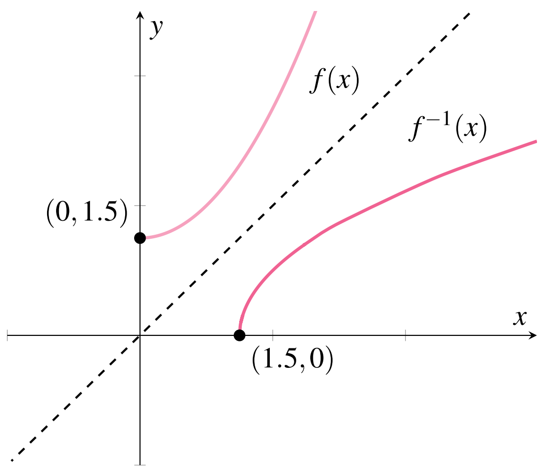


Figure 4.3: A plot of $\sqrt{2x-3}$.

$$f(x) = \sqrt{2x-3}$$ Write original function.

$$y = \sqrt{2x-3}$$ Replace $f(x)$ by y.

$$x = \sqrt{2y-3}$$ Interchange x and y.

$$x^2 = 2y-3$$ Square each side.

$$2y = x^2+3$$ Isolate y.

$$y = \frac{x^2+3}{2}$$ Solve for y.

$$f^{-1}(x) = \frac{x^2+3}{2}$$ Replace y by $f^{-1}(x)$.

The graph of f^{-1} in Figure 4.3 is the reflection of the graph of f in the line $y = x$. Note that the range of f is the interval $[0,\infty)$, which implies that the domain of f^{-1} is the interval $[0,\infty)$. Moreover, the domain of f is the interval $\left[\frac{2}{3},\infty\right)$, which implies that the range of f^{-1} is the interval $\left[\frac{2}{3},\infty\right)$.

■

Exercises For Chapter 4

In Exercises 1–3, show that f and g are inverse functions analytically. [3]

1. $f(x) = 2x, \quad g(x) = \frac{x}{2}$
2. $f(x) = \frac{1}{x}, \quad g(x) = \frac{1}{x}$

In Exercises 3–6, determine whether the function has an inverse function. If it does, find the inverse function. [3]

3. $f(x) = \frac{x}{8}$
4. $f(x) = -4$
5. $f(x) = \sqrt{2x+3}$
6. $h(x) = -\frac{4}{x^2}$

■

II

Limits

5. Introduction to Limits

5.1 What is a Limit?

In orther to understand calculus, you need to know what a "limit" is. A limit is the value a function approaches as the variable within the function (usually "x") gets nearer and nearer to a particular value. In other words, when x is very close to a certain number, what is $f(x)$ very close to?

Let's look at an example of a limit: What is the limit of the function $f(x) = x^2$ as x approaches 2? In limit notation, the expression of "the limit of $f(x)$ as x approaches 2" is written like this: $\lim x \to 2 f(x)$. In order to evaluate the limit, let's check out some values of $\lim x \to 2 f(x)$ as x increases and gets close to 2 (without ever exactly getting there).

When $x = 1.9, f(x) = 3.61$.
When $x = 1.99, f(x) = 3.9601$.
When $x = 1.999, f(x) = 3.996001$.
When $x = 1.9999, f(x) = 3.99960001$.

As x increases and approaches 2, $f(x)$ gets closer and closer to 4. This is called the **left-hand limit** and is written: $\lim x \to 2^- f(x)$. Notice the littleminus sign!

What about when x is bigger than 2?

When $x = 2.1, f(x) = 4.41$.
When $x = 2.01, f(x) = 4.0401$.
When $x = 2.001, f(x) = 4.004001$.
When $x = 2.0001, f(x) = 4.00040001$.

As x increases and approaches 2, $f(x)$ still approaches 4. This is called the **right-hand limit** and is written: $\lim x \to 2^+ f(x)$. Notice the little plus sign!

We got the same answer when evaluating both lift- and right-hand limits, because when x is 2, $f(x)$ is 4. You should always check both sides of the independent variable because, as you'll see shortly, sometimes you don't get the same answser. Therefore, we write that $\lim x \to 2x^2 = 4$.

Let's consider the function [1]:

$$f(x) = \frac{x^2 - 3x + 2}{x - 2}$$

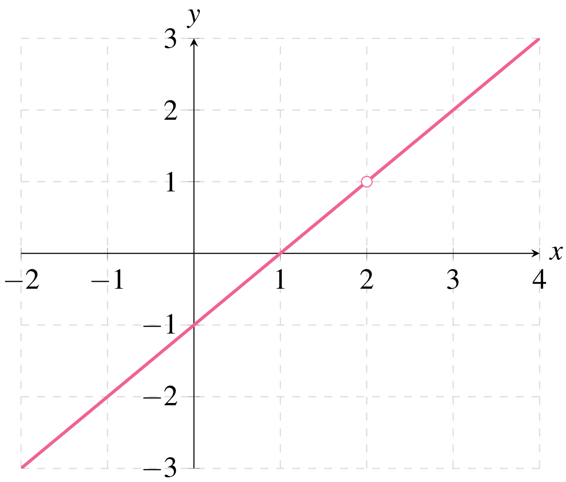


Figure 5.1: A plot of $f(x) = \frac{x^2-3x+2}{x-2}$.

x	$f(x)$
1.7	0.7
1.9	0.9
1.99	0.99
1.999	0.999
2	undefined
2.001	1.001
2.01	1.01
2.1	1.1
2.3	1.3

Table 5.1: Values of $f(x) = \frac{x^2-3x+2}{x-2}$.

While $f(x)$ is undefined at $x = 2$, we can still plot $f(x)$ at other values, see Figure 1.1. Examining Table 1.1, we see that as x approaches 2, $f(x)$ approaches 1. We write this:

$$\text{As } x \to 2,\ f(x) \to 1 \qquad \text{or} \qquad \lim_{x\to 2} f(x) = 1$$

Intuitively, $\lim_{x\to a} f(x) = L$ when the value of $f(x)$ can be made arbitrarily close to L by making x sufficiently close, but not equal to, a. This leads us to the formal definition of a limit.

Definition 5.1.1 — Limit. The **limit** of $f(x)$ as x approaches a is L,

$$\lim_{x\to a} f(x) = L,$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever

$$0 < |x - a| < \delta, \qquad \text{we have} \qquad |f(x) - L| < \varepsilon$$

If not such value of L can be found, the the $\lim_{x\to a} f(x)$ **does not exist**.
[1]

The geometric interpretation of this definition can be seen in Figure 1.2.

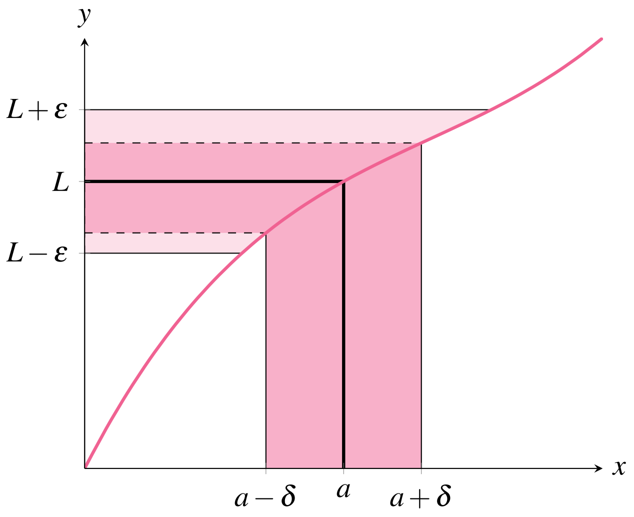


Figure 5.2: A geometric interpretation of the (ε, δ)-criterion for limits. If $0 < |x-a| < \delta$, then we have that $a-\delta < x < a+\delta$. In our diagram, we see that for all such x we are sure to have $L-\varepsilon < f(x) < L+\varepsilon$, and hence $|f(x)-L| < \varepsilon$. [1]

And as we've seen, sometimes the limit of a function exists from one side or the other (or both) even though the limit does not exist. Since it is useful to be able to talk about this situation, we introduce the concept of a *one-sided limit*:

Definition 5.1.2 — One-Sided Limit. We say that the **limit** of $f(x)$ as x goes to a from the **left** is L,

$$\lim_{x\to a^-} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x < a$ and

$$a-\delta < x \qquad \text{we have} \qquad |f(x)-L| < \varepsilon.$$

We say that the **limit** of $f(x)$ as x goes to a from the **right** is L,

$$\lim_{x\to a^+} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x > a$ and

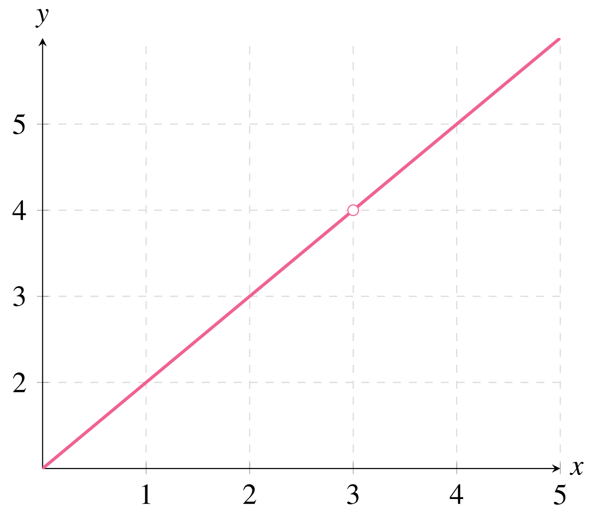
$$x < a+\delta \qquad \text{we have} \qquad |f(x)-L| < \varepsilon.$$

[1]

5.2 Finding Limits with Graphs and Tables

We can sometimes determine the limit of a function simply through its graph or a table of values. Let's do a few examples.

■ **Example 5.1** Find $\lim_{x\to 3} \dfrac{x^2-2x-3}{x-3}$



Solution 5.1
Even though $f(x)$ is undefined at $x = 3$, the limit still exists. We can see from the graph that $f(x)$ goes closer and closer to 4 as $x \to 3$, so the answer is

$$\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} = 4$$

■

■ **Example 5.2** Find $\lim_{x \to 0} f(x)$

x	$f(x)$
1	54.9989164415
0.1	56.2373785384
0.01	56.2498737743
0.001	56.2499987377
0	undefined
-0.001	56.2499987377
-0.01	56.2498737743
-0.1	56.2373785384
-1	54.9989164415

Solution 5.2
Again, a limit can exist even if the original function is undefined at a certain value, as long as the one-handed limits from both sides equals. We can see from the table that $f(x)$ goes closer to 56.25 as from both left and right, that is,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 56.25$$

therefore,

$$\lim_{x \to 0} f(x) = 56.25$$

■

5.3 Limits That Failed to Exist

In the next two examples you will examine some limits that fail to exist.

■ **Example 5.3 — Behavior That Differs from the Right and from the Left .** [3]
Show that the limit

$$\lim_{x \to 0} \frac{|x|}{x}$$

does not exist.

Solution 5.3

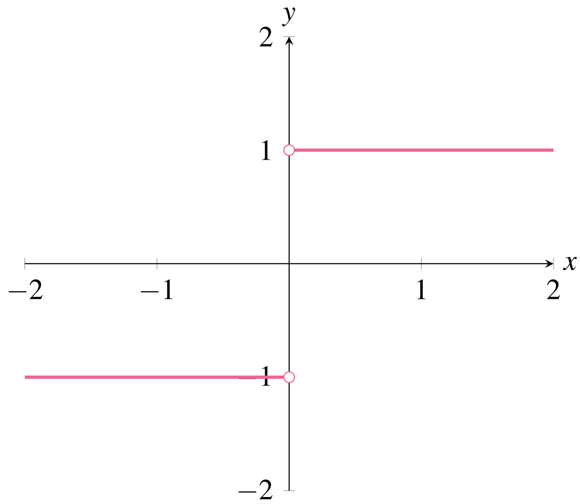


Figure 5.3: A plot of $\frac{|x|}{x}$.

Consider the graph of the function $\frac{|x|}{x}$. From Figure 5.3 and the definition of absolute value

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

you can see that

$$\frac{|x|}{x} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

This means that no matter how close x gets to 0, there will be both positive and negative x-values that yield $f(x) = 1$ or $f(x) = -1$. Specifically, if δ is a positive number, then for x-values satisfying the inequality $0 < |x| < \delta$, you can classify the values of $\frac{|x|}{x}$ as follows.

within $(-\delta, 0)$, $\qquad \frac{|x|}{x} = -1$

within $(0, \delta)$, $\qquad \frac{|x|}{x} = 1$

Because $\frac{|x|}{x}$ approaches a different number from the right side of 0 than it approaches from the left side, the limit $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

■ Example 5.4 — Unbounded Behaviour . [3]
Discuss the existance of

$$\lim_{x\to -1} \frac{1}{x+1}$$

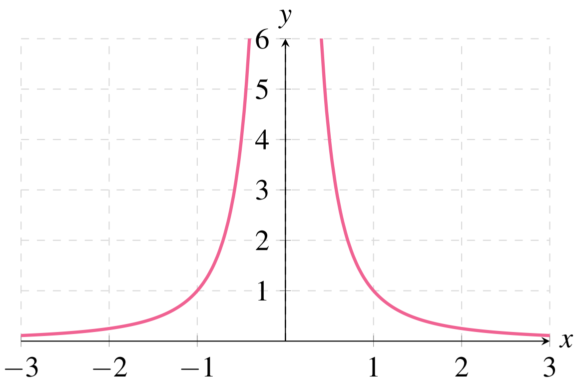


Figure 5.4: A plot of $\frac{1}{x^2}$.

x	$\frac{1}{x^2}$
1	1
0.1	100
0.01	10,000
0.001	1,000,000
0	undefined
-0.001	1,000,000
-0.01	10,000
-0.1	100
-1	1

Table 5.2: Values of $f(x) = \frac{1}{x^2}$.

Solution 5.4 Let $f(x) = \frac{1}{x^2}$. In Figure 5.4, you can see that as x approaches 0 from either the right or the left, $f(x)$ increases without bound. This means that by choosing x close enough to 0, you can force $f(x)$ to be as large as you want. For instance, $f(x)$ will be larger than 100 if you choose x that is within $\frac{1}{10}$ of 0. That is,

$$0 < |x| < \frac{1}{10} \qquad \rightarrow \qquad f(x) = \frac{1}{x^2} > 100 \tag{5.1}$$

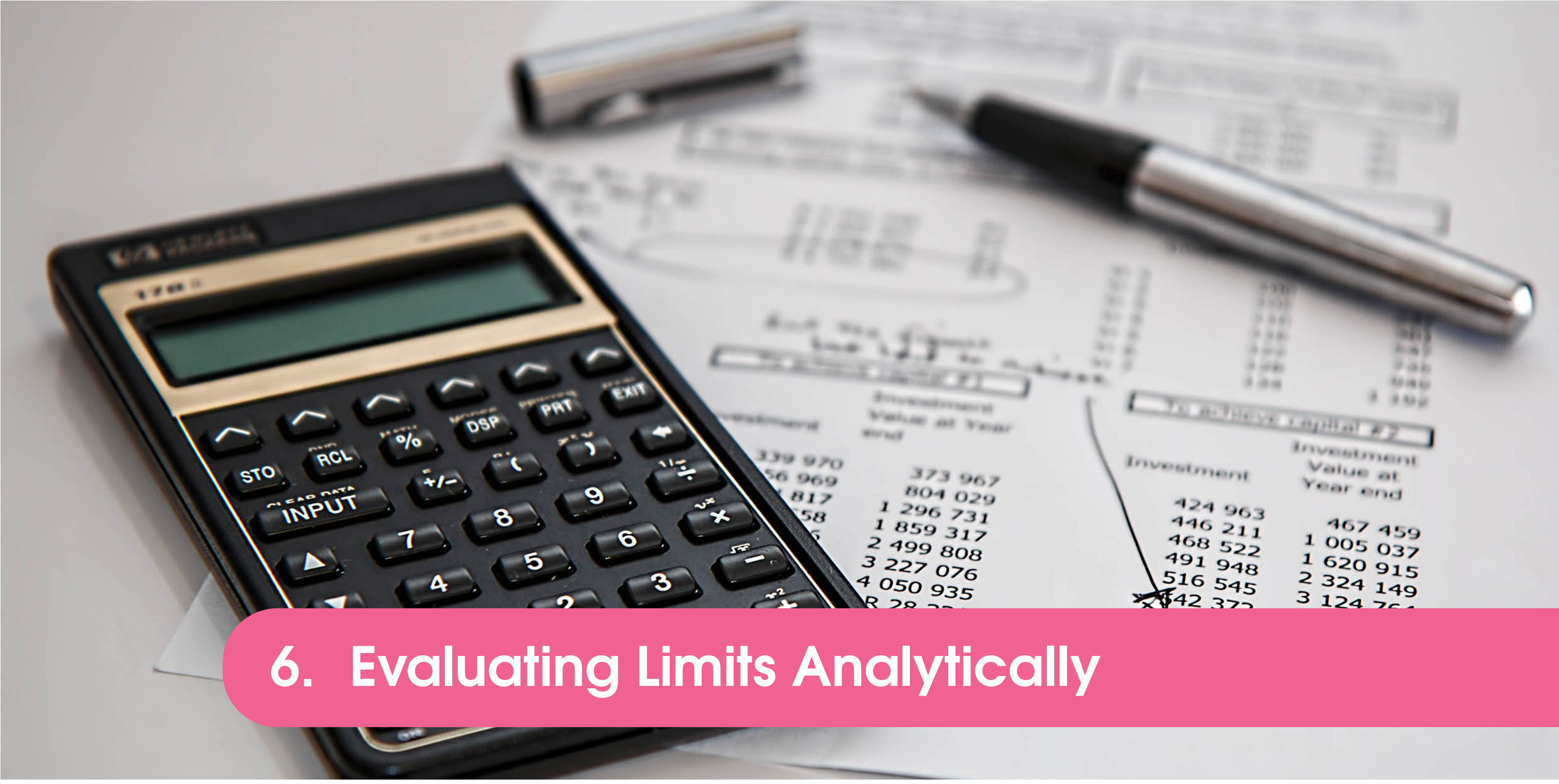
Similarly, you can force $f(x)$ to be larger than $1,000,000$, as follows.

$$0 < |x| < \frac{1}{1000} \qquad \rightarrow \qquad f(x) = \frac{1}{x^2} > 1,000,000 \tag{5.2}$$

Because $f(x)$ is not approaching a real number L as x approaches 0, you can conclude that the limit does not exist.

Exercises For Chapter 5

1. Use the definition of limits to explain why $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$. Hint: Use the fact that $|\sin(a)| \leq 1$ for any real number a. [1]
2. Use the definition of limits to explain why $\lim_{x\to -2} \pi = \pi$. [1]
3. Use the definition of limits to explain why $\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} = 6$. [1]
4. Sketch a plot of $f(x) = \frac{x}{|x|}$ and explain why $\lim_{x\to 0} \frac{x}{|x|}$ does not exist. [1]



6. Evaluating Limits Analytically

6.1 Properties of Limits

We didn't really need to look at all of the graphs or decimal values to know what was going to appen when x get really close to some number. But it's important to go through the exercise because, typically, the answers get a lot more complicated.

Keep in mind that $\lim_{x\to c} f(x)$ does not depend on the value of $f(x)$. It may happen, however, that the limit is precisely $f(c)$. In such cases, the limit can be evaluated by **direct substitution**. That is,

$$\lim_{x\to c} f(x) = f(c)$$

There are also some simple algebraic rules of limits that you should know.

Theorem 6.1.1 — Properties of Limits. Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x\to c} f(x) = L \qquad \text{and} \qquad \lim_{x\to c} g(x) = K$$

1. $\lim_{x\to c}[bf(x)] = bL$
2. $\lim_{x\to c}[f(x) \pm g(x)] = L \pm K$
3. $\lim_{x\to c}[f(x)g(x)] = LK$
4. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}, \qquad K \neq 0$
5. $\lim_{x\to c}[f(x)]^n = L^n$
6. $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}, \qquad c \in \mathbb{R}$ if n is odd or $c > 0$ if n is even
7. $\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x)) = f(K)$

[3]

Let's do a few examples.

■ **Example 6.1** Find $\lim\limits_{x \to 5} x^2$.

Solution 6.1 The approach is simple: Plug 5 for x, and you get 25.

■

■ **Example 6.2** Find $\lim\limits_{x \to 3} x^3$.

Solution 6.2 Here the answer is 27.

■

■ **Example 6.3** Find $\lim\limits_{x \to 5} [x^2 + x^3]$.

Solution 6.3

$$\lim_{x \to 5} [x^2 + x^3] = \lim_{x \to 5} x^2 + \lim_{x \to 5} x^3 \tag{6.1}$$

$$= 25 + 125 \tag{6.2}$$

$$= 150 \tag{6.3}$$

■

■ **Example 6.4** Find $\lim\limits_{x \to 5} [(x^2 + 1)\sqrt{x-1}]$.

Solution 6.4

$$\lim_{x \to 5} [(x^2 + 1)\sqrt{x-1}] = \lim_{x \to 5} (x^2 + 1) \lim_{x \to 5} \sqrt{x-1} \tag{6.4}$$

$$= 52 \tag{6.5}$$

■

So far, so good. All so to find the limit of a simple polynomial is plug in the number that the variable is approaching and you get the answer. Natually, this may not be the case.

6.2 A Strategy for Finding Limits

Theorem 6.2.1 — Functions That Agree At All But One Point.
Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

[3]

■ **Example 6.5** Find $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}$. [3]

Solution 6.5 Let $f(x) = \dfrac{x^3 - 1}{x - 1}$. By factoring and dividing out like factors, you can rewrite f as

$$f(x) = \frac{(x-1)(x^2 + x + 1)}{(x-1)} = x^2 + x + 1 = g(x), \qquad x \neq 1$$

So, for all x-values other than $x = 1$, the functions f and g agree. Because $\lim\limits_{x \to 1} g(x)$ exists, you can apply Theorem 6.2.1 to conclude that f and g have the same limit at $x = 1$.

$$\begin{aligned}
\lim_{x\to 1} \frac{x^3-1}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)} \\
&= \lim_{x\to 1}(x^2+x+1) \\
&= 1^2+1+1 \\
&= 3
\end{aligned}$$

■

6.3 Dividing Out and Rationalizing Techniques

■ **Example 6.6** Find $\displaystyle\lim_{x\to -3} \frac{x^2+x-6}{x+3}$. [3]

Solution 6.6

$$\begin{aligned}
\lim_{x\to -3} \frac{x^2+x-6}{x+3} &= \lim_{x\to -3} \frac{(x+3)(x-2)}{(x+3)} \\
&= \lim_{x\to -3}(x-2) \\
&= -5
\end{aligned}$$

■

In Example 6.6, direct substitution produced the meaningless fractional form $\frac{0}{0}$. An expression such as $\frac{0}{0}$ is called an **indeterminate form** because you cannot (from the form alone) determine the limit. When you try to evaluate a limit and encounter this form, remember that you must rewrite the fraction so that the new denominator does not have 0 as its limit. One way to do this is to *divide out like factors*, as shown in Example 6.6.

A second way is to *rationalize the numerator*, as shown in Example 6.7.

■ **Example 6.7** Find $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$. [3]

Solution 6.7 By direst substitution, you obtain $\frac{0}{0}$. In this case, you can rewrite the fraction by rationalizing the numerator.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0} \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) \\
&= \lim_{x\to 0} \frac{(x+1)-1}{x(\sqrt{x+1}+1)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} \\
&= \frac{1}{1+1} \\
&= \frac{1}{2}
\end{aligned}$$

■

6.4 The Squeeze Theorem

The last thing you need to know for this chapter is the **Squeeze Theorem**. It concerns the limit of a function that is squeezed between two other functions, each of which has the same limit at a given x-value, as shown in Figure 6.1.

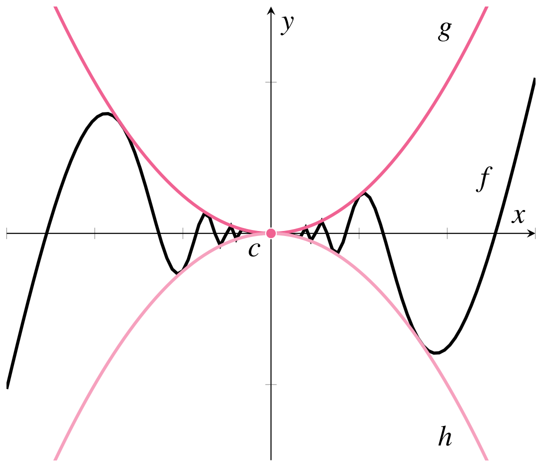


Figure 6.1: The Squeeze Theorem

Theorem 6.4.1 — The Squeeze Theorem.
If $h(x) \leq f(x) \leq g(x)$ for all x in n an open interval containing c, except possibly at c itself, and if $\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$, then $\lim_{x \to c} f(x)$ exists and is equal to L.
[3]

Exercises For Chapter 6

In Exercises 1–5, find the limit. [3]

1. $\lim_{x \to 2} x^3$
2. $\lim_{x \to 1} \dfrac{x}{x^2+4}$
3. $\lim_{x \to 0} \dfrac{\frac{1}{3+x} - \frac{1}{3}}{x}$
4. $\lim_{x \to 4} \dfrac{\sqrt{x+5}-3}{x-4}$
5. $\lim_{\Delta x \to 0} \dfrac{(x+\Delta x)^3 - x^3}{\Delta x}$

In Exercises 6–7, use the Squeeze Theorem to find $\lim_{x \to c} f(x)$ [3]

6. $c = 0; \quad 4 - x^2 \leq f(x) \leq 4 + x^2$
7. $c = a; \quad b - |x-a| \leq f(x) \leq b + |x-a|$

7. Continuity

7.1 Continuity at a Point and on an Interval

Informally, a function is continuous if you can "draw it" without "lifting your pencil." We need a formal definition. [1]

Definition 7.1.1 — Continuity at a Point. In order for a functiuon $f(x)$ to be continuous at a point $x = c$, it must fulfill *all three* of the following conditions:

1. $f(x)$ exists.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(x)$.

[2]

■ **Example 7.1** Find the discontinuities (the values for x where a function is not continuous) for the function given in Figure 7.1. [1]

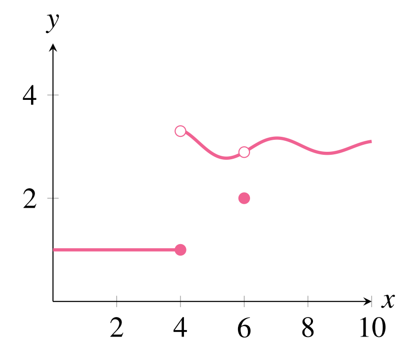


Figure 7.1: A plot of a function with discontinuities at $x = 4$ and $x = 6$. [1]

Solution 7.1 From Figure 7.1 we see that $\lim_{x\to 4} f(x)$ does not exist as

$$\lim_{x\to 4^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to 4^+} f(x) \approx 3.5$$

Hence $\lim_{x\to 4} f(x) \neq f(4)$, and so $f(x)$ is not continuous at $x = 4$.

We also see that $\lim_{x\to 6} f(x) \approx 3$ while $f(6) = 2$. Hence $\lim_{x\to 6} f(x) \neq f(6)$, and so $f(x)$ is not continuous at $x = 6$.

■

Building from the definition of *continuous at a point*, we can now define what it means for a function to be *continuous* on an interval. [1]

Definition 7.1.2 — Continuity at an Open Interval. A function f is **continuous on open interval** (a,b) if it is continuous at every point in the interval. A function that is continuous on the entire real line $(-\infty,\infty)$ is **everywhere continuous**.
[1]

Definition 7.1.3 — Continuity at Closed Interval. A function f is **continuous on an closed interval** $[a,b]$ if it is continuous on the open interval (a,b), and

$$\lim_{x\to a^+} f(x) = f(a) \qquad \text{and} \qquad \lim_{x\to b^-} f(x) = f(b)$$

The function f is **continuous from the right** at a and **continuous from the left** at b.
[3]

In particular, we should note that if a function is not defined on an interval, then it **cannot** be continuous on that interval. [1]

■ **Example 7.2** Consider the function

$$f(x) = \begin{cases} \sqrt[5]{x}\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

see Figure 7.2. Is this function continuous? [1]

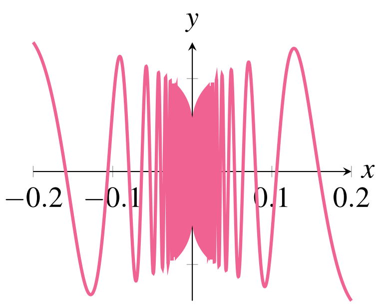


Figure 7.2: A plot of $f(x) = \begin{cases} \sqrt[5]{x}\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

Solution 7.2 Considering $f(x)$, the only issue is when $x=0$. We must show that $\lim_{x\to 0} f(x)=0$. Note

$$-|\sqrt[5]{x}| \leq f(x) \leq |\sqrt[5]{x}|.$$

Since

$$\lim_{x\to 0} -|\sqrt[5]{x}| = 0 = \lim_{x\to 0} |\sqrt[5]{x}|,$$

we see by the Squeeze Theorem, that $\lim_{x\to 0} f(x)=0$. Hence $f(x)$ is continuous.

Here we see how the informal definition of continuity being that you can "draw it" without "lifting your pencil" differs from the formal definition.

■

7.2 Types of Discontinuity

There are three typoes of discontinuity you have to know: jump, essential, removable.

Definition 7.2.1 — Jump Discontinuity. A **jump** discontinuity occurs when the curve "breaks" at a particular place and starts somewhere else. The limits from the left and the right both exist, but they will not match.
[2]

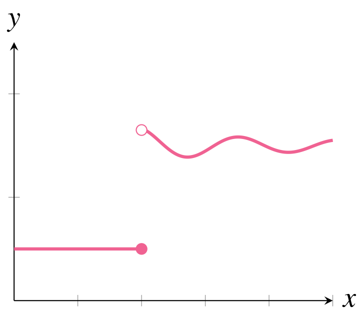


Figure 7.3: Jump Discontinuity [1]

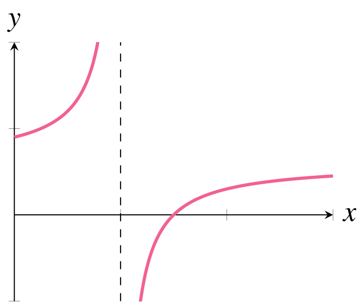


Figure 7.4: Essential Discontinuity [1]

Definition 7.2.2 — Essential Discontinuity. An **essential** discontinuity occurs when the curve has a vertical asymptote.
[2]

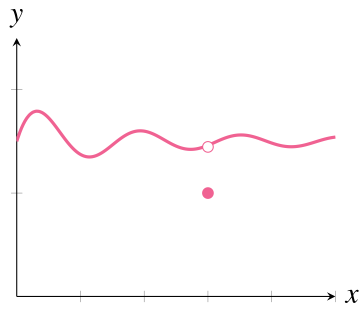


Figure 7.5: Removable Discontinuity [1]

Definition 7.2.3 — Removable Discontinuity. An **removable** discontinuity occurs when the curve has a "hole" in it. It is "removable" because one can remove the discontinuity by properly defining the value (filling the hole).
[2]

7.3 The Intermediate Value Theorem

We close with a useful theorem about continuous functions:

Theorem 7.3.1 — Intermediate Value Theorem.
If $f(x)$ is a continuous function for all x in the closed interval $[a,b]$ and d is between $f(a)$ and $f(b)$, then there is a number c in $[a,b]$ such that $f(c) = d$.
[1]

In Figure 7.6, we see a geometric interpretation of this theorem.

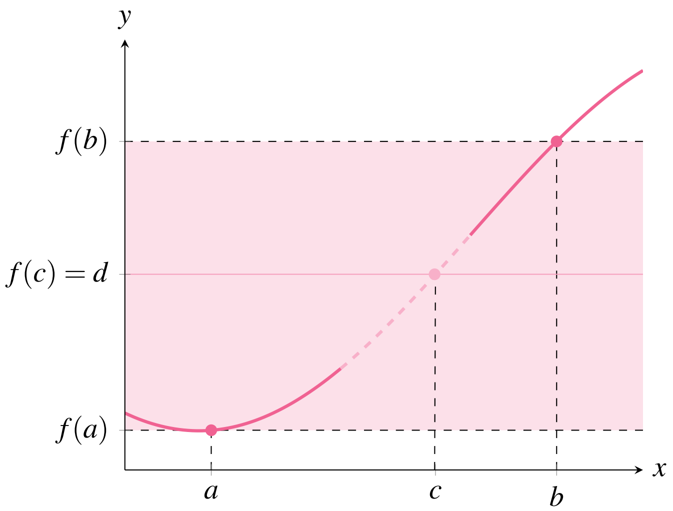


Figure 7.6: A geometric interpretation of the Intermediate Value Theorem. The function $f(x)$ is continuous on the interval $[a,b]$. Since d is in the interval $[f(a), f(b)]$, there exists a value c in $[a,b]$ such that $f(c) = d$. [1]

■ **Example 7.3** Explain why the function $f(x) = x^3 + 3x^2 + x - 2$ has a root between 0 and 1. [1]

Solution 7.3

By Theorem 6.1.1, $\lim_{x\to a} f(x) = f(a)$, for all real values of a, and hence f is continuous. Since $f(0) = -2$ and $f(1) = 3$, and 0 is between -2 and 3, by the Intermediate Value Theorem, Theorem 7.3.1, there is a $c \in [0,1]$ such that $f(c) = 0$.

■

Exercises For Chapter 7

In Exercises 1–4, find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable? [3]

1. $x^2 - 2x + 1$
2. $\dfrac{x}{x^2 - x}$
3. $\dfrac{x+2}{x^2 - 3x - 10}$
4. $\dfrac{|x+7|}{x+7}$

In Exercises 5–6, find the constant a, or the constants a and b, such that the function is continuous on the entire real line. [3]

5. $f(x) = \begin{cases} 3x^2, & x \geq 1 \\ ax - 4, & x < 1 \end{cases}$
6. $f(x) = \begin{cases} 2, & x \leq -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \geq 3 \end{cases}$

■

8. Limits and Asymptotes

The last thing you need to know about limits is its relationship with the asymptotes on the graph of a function, which is usually tested upon (advise: memorize the propositions for quizzes).

8.1 Essential Discontinuities and Vertical Asymptotes

As aforementioned in the previous chapter, the essential discontinuity, by Definition 7.2.2, occurs when the curve has a vertical asymptote.

Consider the function

$$f(x) = \frac{1}{(x+1)^2}$$

While the $\lim_{x\to -1} f(x)$ does not exist, see Figure 8.1, something can still be said. [1]

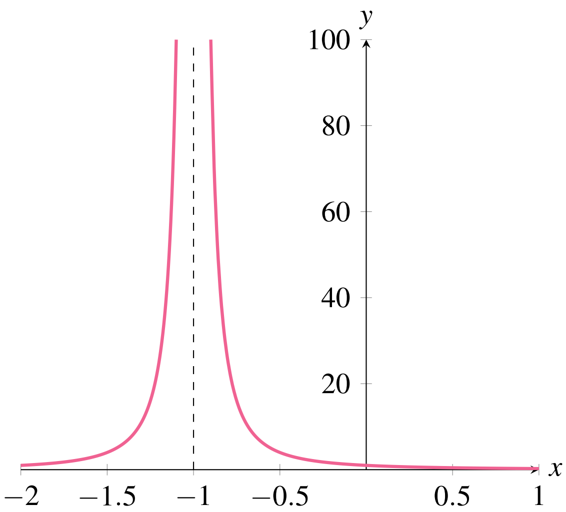


Figure 8.1: A plot of $f(x) = \frac{1}{(x+1)^2}$. [1]

Definition 8.1.1 — Infinite Limit. If $f(x)$ grows arbitrarily large as x approaches a, we write

$$\lim_{x\to a} f(x) = \infty$$

and say that the limit of $f(x)$ **approaches infinity** as x goes to a.

If $|f(x)|$ grows arbitrarily large as x approaches a and $f(x)$ is negative, we write

$$\lim_{x\to a} f(x) = -\infty$$

and say that the limit of $f(x)$ **approaches negative infinity** as x goes to a.
[1]

On the other hand, if we consider the function

$$f(x) = \frac{1}{(x-1)}$$

While we have $\lim_{x\to 1} f(x) \neq \pm\infty$, we do have one-sided limits, $\lim_{x\to 1+} f(x) = \infty$ and $\lim_{x\to 1-} f(x) = -\infty$, see Figure 8.2. [1]

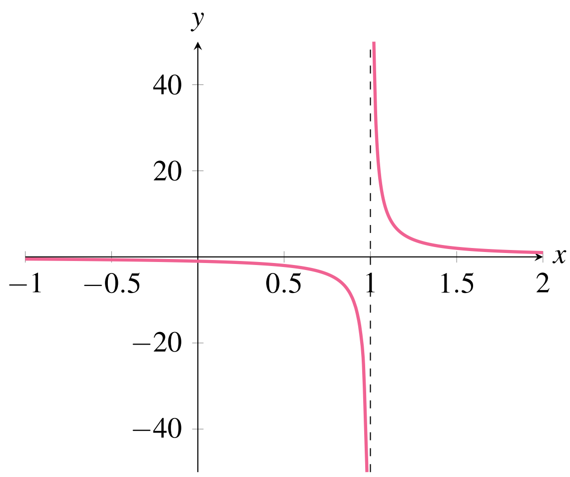


Figure 8.2: A plot of $f(x) = \frac{1}{x-1}$. [1]

Definition 8.1.2 — Vertical Asymptote. If

$$\lim_{x\to a} f(x) = \pm\infty, \qquad \lim_{x\to a+} f(x) = \pm\infty, \qquad \text{or} \qquad \lim_{x\to a-} f(x) = \pm\infty,$$

then the line $x = a$ is a **vertical asymptote** of $f(x)$.
[1]

■ **Example 8.1** Find the vertical asymptotes of

$$f(x) = \frac{x^2 - 9x + 14}{x^2 - 5x + 6}.$$

Solution 8.1 Start by factoring both the numerator and the denominator:

$$\frac{x^2-9x+14}{x^2-5x+6} = \frac{(x-2)(x-7)}{(x-2)(x-3)}$$

Using limits, we must investigate when $x \to 2$ and $x \to 3$. Write

$$\begin{aligned}\lim_{x\to 2} \frac{(x-2)(x-7)}{(x-2)(x-3)} &= \lim_{x\to 2} \frac{(x-7)}{(x-3)} \\ &= \frac{-5}{-1} \\ &= 5.\end{aligned}$$

Now write

$$\begin{aligned}\lim_{x\to 3} \frac{(x-2)(x-7)}{(x-2)(x-3)} &= \lim_{x\to 3} \frac{(x-7)}{(x-3)} \\ &= \lim_{x\to 3} \frac{-4}{x-3}.\end{aligned}$$

Since $\lim\limits_{x\to 3+} x-3$ approaches 0 from the right and the numerator is negative, $\lim\limits_{x\to 3+} f(x) = -\infty$. Since $\lim\limits_{x\to 3-} x-3$ approaches 0 from the left and the numerator is negative, $\lim\limits_{x\to 3-} f(x) = \infty$. Hence we have a vertical asymptote at $x = 3$, see Figure 8.3.

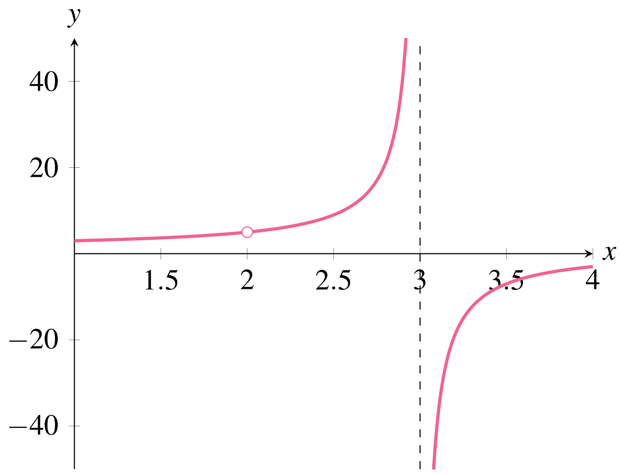


Figure 8.3: A plot of $f(x) = \frac{x^2-9x+14}{x^2-5+6}$. [1]

■

8.2 Limits at Infinity and Horizontal Asymptotes

Consider the function:

$$f(x) = \frac{6x-9}{x-1}$$

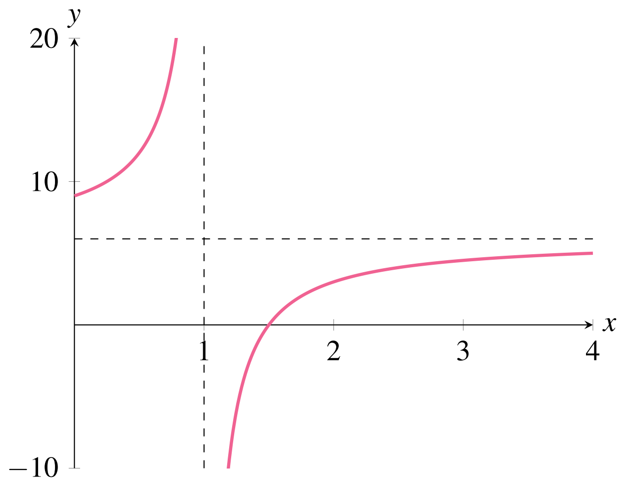


Figure 8.4: A plot of $f(x) = \frac{6x-9}{x-1}$. [1]

As x approaches infinity, it seems like $f(x)$ approaches a specific value. This is a *limit at infinity*. [1]

Definition 8.2.1 — Limit At Infinity. If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large, we write

$$\lim_{x\to\infty} f(x) = L$$

and we say, the **limit at infinity** of $f(x)$ is L.

If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large and negative, we write

$$\lim_{x\to-\infty} f(x) = L$$

and we say, the **limit at negative infinity** of $f(x)$ is L.
[1]

You might have guessed it, this results in a **horizontal asymptote**.

Definition 8.2.2 — Horizontal Asymptote. If

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L,$$

then the line $y = L$ is a **horizontal asymptote** of $f(x)$.
[1]

■ **Example 8.2** Give the horizontal asymptotes of

$$f(x) = \frac{6x-9}{x-1}$$

[1]

Solution 8.2 From our previous work, we see that $\lim_{x\to\infty} f(x) = 6$, and upon further inspection, we see that $\lim_{x\to-\infty} f(x) = 6$. Hence the horizontal asymptote of $f(x)$ is the line $y = 6$.

■

Exercises For Chapter 8

In Exercises 1–2, find the vertical asymptote(s). [1]

1. $f(x) = \dfrac{x-3}{x^2+2x-3}$.
2. $f(x) = \dfrac{x^2-x-6}{x+4}$.

In Exercises 3–4, find the horizontal asymptote(s). [1]

3. $f(x) = \dfrac{\sin\left(x^7\right)}{\sqrt{x}}$
4. $f(x) = \left(17 + \dfrac{32}{x} - \dfrac{(\sin(x/2))^2}{x^3}\right)$
5. Suppose a population of feral cats on a certain college campus t years from now is approximated by
$$p(t) = \frac{1000}{5+2e^{-0.1t}}.$$
Approximately how many feral cats are on campus 10 years from now? 50 years from now? 100 years from now? 1000 years from now? What do you notice about the prediction—is this realistic? [1]
6. The amplitude of an oscillating spring is given by
$$a(t) = \frac{\sin(t)}{t}.$$
What happens to the amplitude of the oscillation over a long period of time? [1]

■

III

Differentiation

9. Definition of the Derivative

The main tool that you'll use in differential calculus is called the **derivative**. All of the problems that you'll encounter in differential calculus make use of the derivative, so your goal should be to become an expert at finding, or "taking", derivatives by the end of the next chapter. However, before you learn a simple waya to take the derivative, your teacher will probably make you learn how derivatives are calculated by teaching you something called the "Definition of the Derivative".

9.1 Deriving the Formula

The best way to understand the definition of the derivative is to start by looking at the simplest continuous function: a line. As you should recall, you can determine the slope of a line by taking two points on that line and plugging then into the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad m \text{ stands for slope.}$$

Now for a few change in notation. Instead of calling the x-coordinates x_1 and x_2, we're going to call them x and $x+h$, where h is the difference between the two x-coordinates. Second, instead of using y_1 and y_2, we use $f(x)$ and $f(x+h)$.

9.2 The Slope of a Curve

Suppose that instead of finding the slope of a line, we wanted to find the slope of a curve. Here the slope formula no longer works because the distance from one point to the other is along a curve, not a strait line. But we could find the approximate slope if we took the slope of the line between the two points. This is called a **secant line**.

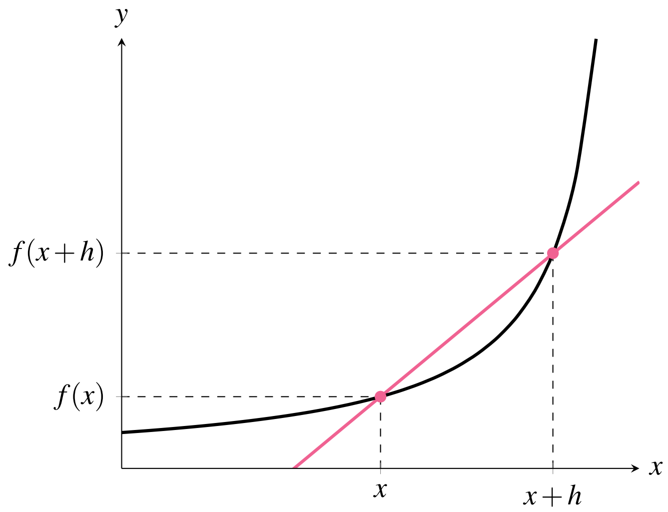


Figure 9.1: Secant lines can be found by connecting two points ont he curve.

The slope of the secant line is sometimes called a **difference quotient**.

Definition 9.2.1 — Difference Quotient. The difference quotient of a function f with respect to x and $x+h$ in its domain is

$$\frac{f(x+h)-f(x)}{h}.$$

[2]

9.3 The Secant and the Tangent

As you can see in Figure 9.2, the farther apart the two points are, the less the slope corresponds to the slope of the curve. Conversely, the close the two points are, the more accurate the approximation is.

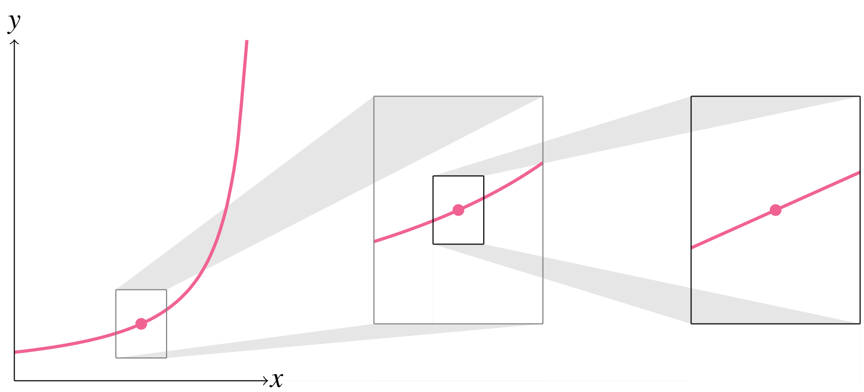


Figure 9.2: Given a function $f(x)$, if one can "zoom in" on $f(x)$ sufficiently so that $f(x)$ seems to be a straight line, then that line is the **tangent line** to $f(x)$ at the point determined by x. [1]

In fact, there is one line, called the **tangent line**, that touches the curve at exactly one point.

The slope of the tangentline is equal to the slope of the curve at exactly this point. The object of using the above formula, therefore, is to shrink h down to an infinitestimally small amount. If we do that, then the difference between $(x+h)$ and x would be a point.

Graphically, it looks like the following:

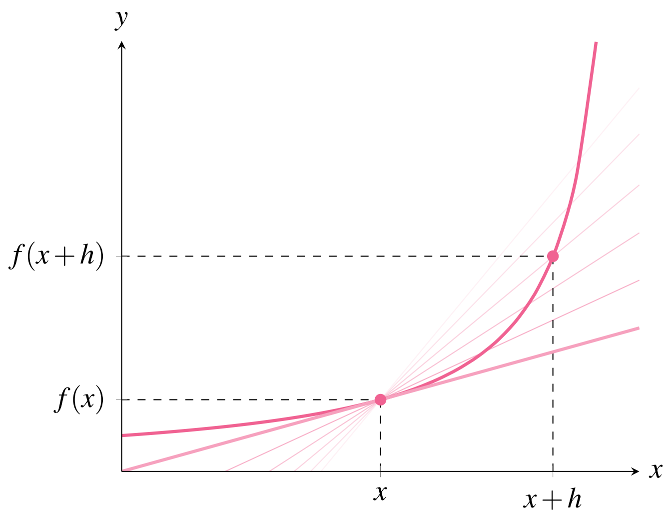


Figure 9.3: Tangent lines can be found as the limit of secant lines. [1]

How do we perform this shrinking act? By using the limits we just discussed. We set up a limit during which h approaches zero, which is the definition of the derivative.

Definition 9.3.1 — Derivative. The **derivative** of $f(x)$ is the function

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

If this limit does not exist for a given value of x, then $f(x)$ is not **differentiable** at x. [1]

The derivative may also appear in other forms, but all means the same thing.

Definition 9.3.2 — Derivative Notations. There are several different notations for the derivative, we'll mainly use

$$\frac{d}{dx}f(x) = f'(x).$$

If one is working with a function of a variable other than x, say t we write

$$\frac{d}{dt}f(t) = f'(t).$$

However, if $y = f(x)$, $\frac{dy}{dx}$, $\dot{y}$, and $D_x f(x)$ are also used. [1]

■ **Example 9.1** Compute $\frac{d}{dx}(x^3+1)$.

Solution 9.1 Using the definition of the derivative,

$$\begin{aligned}\frac{d}{dx}f(x) &= \lim_{h\to 0}\frac{(x+h)^3+1-(x^3+1)}{h}\\ &= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3+1-x^3-1}{h}\\ &= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}\\ &= \lim_{h\to 0}(3x^2+3xh+h^2)\\ &= 3x^2.\end{aligned}$$

■

Next we will consider the derivative a function that is not continuous on $\mathbb{R}$.

■ **Example 9.2** Compute $\frac{d}{dt}\frac{1}{t}$.

Solution 9.2 Using the definition of the derivative,

$$\begin{aligned}\frac{d}{dt}\frac{1}{t} &= \lim_{h\to 0}\frac{\frac{1}{t+h}-\frac{1}{t}}{h}\\ &= \lim_{h\to 0}\frac{\frac{t}{t(t+h)}-\frac{t+h}{t(t+h)}}{h}\\ &= \lim_{h\to 0}\frac{\frac{t-(t+h)}{t(t+h)}}{h}\\ &= \lim_{h\to 0}\frac{t-t-h}{t(t+h)h}\\ &= \lim_{h\to 0}\frac{-h}{t(t+h)h}\\ &= \lim_{h\to 0}\frac{-1}{t(t+h)}\\ &= \frac{-1}{t^2}.\end{aligned}$$

This function is differentiable at all real numbers except for $t=0$.

■

9.4 Differentiability

One of the important requirements for differentiability of a function is that the **function is continuous**. But, even if a function is continuous at a point, the functiuon is not necessarilly differentiable there. Check out the graph below.

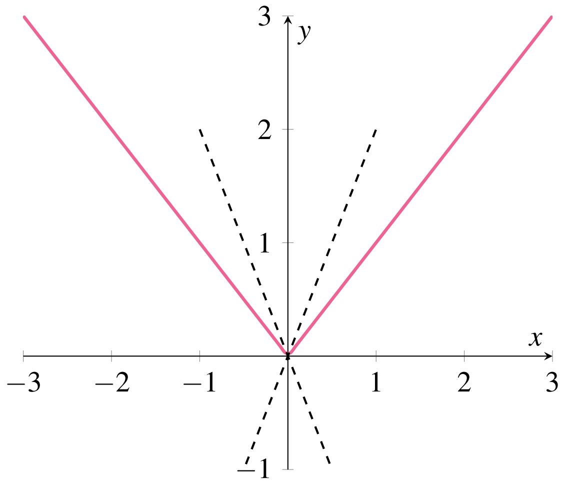


Figure 9.4: A plot of $|x|$

If a function has a **"sharp turn"**, you can draw more than one tangent line at that point, and because the slopes of these tangent lines are not equal, the function is not differentiable there.

Another possible problem occurs when the tangent line is **vertical** because a vertical line has an infinite slope.

Fortunately the reverse is true: if a function is differentiable at a point, it is continuous there.

Theorem 9.4.1 — Differentiability Implies Continuity. If $f(x)$ is a differentiable function at $x=a$, then $f(x)$ is continuous at $x=a$.

Proof. We want to show that $f(x)$ is continuous at $x=a$, hence we must show that

$$\lim_{x\to a} f(x) = f(a).$$

Consider

$$\begin{aligned}
\lim_{x\to a}(f(x)-f(a)) &= \lim_{x\to a}\left((x-a)\frac{f(x)-f(a)}{x-a}\right) && \text{Multiply and divide by } (x-a).\\
&= \lim_{h\to 0} h\cdot\frac{f(a+h)-f(a)}{h} && \text{Set } x=a+h.\\
&= \left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right) && \text{Limit Law.}\\
&= 0\cdot f'(a) = 0.
\end{aligned}$$

Since $\lim_{x\to a}(f(x)-f(a)) = 0$ we see that $\lim_{x\to a} f(x) = f(a)$, and so $f(x)$ is continuous. ■

[1]

Exercises For Chapter 9

1. If the line $y = 7x - 4$ is tangent to $f(x)$ at $x = 2$, find $f(2)$ and $f'(2)$. [1]
2. If $f(-2) = 4$ and $f(-2+h) = (h+2)^2$, compute $f'(-2)$. [1]
3. If $f'(x) = x^3$ and $f(1) = 2$, approximate $f(1.2)$ [1]

In Exercises 4–7, consider the plot in Figure 9.5. [1]

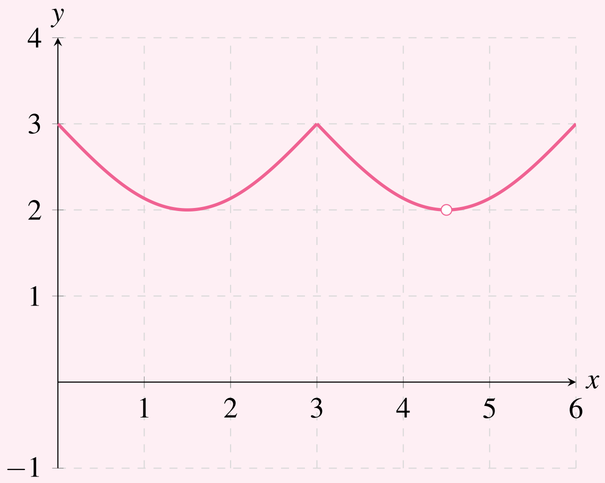


Figure 9.5: A plot of $f(x)$. [1]

4. On which subinterval(s) of $[0,6]$ is $f(x)$ continuous?
5. On which subinterval(s) of $[0,6]$ is $f(x)$ differentiable?
6. Sketch a plot of $f'(x)$.

10. Basic Differentiation

In calculus, you'll be asked to do two things: differentiate and integrate. In this chapter, you're going to learn differentiation.

10.1 Notation

As we've talked about, there are several different notations for derivatives in calculus. We'll use two different types interchangably throughout this book.

The derivatives of the functions will use notation that depends on the function, as shown in the following table:

Function	First Derivative	Second Derivative
$f(x)$	$f'(x)$	$f''(x)$
$g(x)$	$g'(x)$	$g''(x)$
y	y' or $\frac{dy}{dx}$	y'' or $\frac{d^2y}{dx^2}$

It is tedious to compute a limit every time we need to know the derivative of a function. Fortunately, we can develop a small collection of examples and rules that allow us to compute the derivative of almost any function we are likely to encounter. [1] We will start simply and build-up to more complicated examples.

10.2 The Constant Rule

The simplest function is a constant function. Recall that derivatives measure the rate of change of a function at a given point. Hence, the derivative of a constant function is zero. For example, the constant function plots a horizontal line—so the slope of the tangent line is 0. [1] This lead us to our next theorem.

Theorem 10.2.1 — The Constant Rule. Given a constant c,

$$\frac{d}{dx}c = 0.$$

■ **Example 10.1** Find $\dfrac{d}{dx}114514$.

Solution 10.1 Guess what, it's 0. (Differentiation is easy!)

■

10.3 The Power Rule

The basic technique for taking a derivative of non-constants is called the **Power Rule**.

Theorem 10.3.1 — The Power Rule. For any real number n,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

■ **Example 10.2** Compute $\dfrac{d}{dx}x^{13}$. [1]

Solution 10.2 Applying the power rule, we write $\dfrac{d}{dx}x^{13} = 13x^{12}$.

■

Sometimes, it is not as obvious that one should apply the power rule.

■ **Example 10.3** Compute $\dfrac{d}{dx}\dfrac{1}{x^4}$. [1]

Solution 10.3 Applying the power rule, we write $\dfrac{d}{dx}\dfrac{1}{x^4} = \dfrac{d}{dx}x^{-4} = -4x^{-5}$.

■

The power rule also applies to radicals once we rewrite them as exponents.

■ **Example 10.4** Compute $\dfrac{d}{dx}\sqrt[5]{x}$. [1]

Solution 10.4 Applying the power rule, we write $\dfrac{d}{dx}\sqrt[5]{x} = \dfrac{d}{dx}x^{1/5} = \dfrac{x^{-4/5}}{5}$.

■

10.4 The Sum Rule

The *sum rule* allows us to take derivatives of functions "one piece at a time".

Theorem 10.4.1 — The Sum Rule. If $f(x)$ and $g(x)$ are differentiable and c is a constant, then

1. $\dfrac{d}{dx}\big(f(x)+g(x)\big) = f'(x)+g'(x)$,
2. $\dfrac{d}{dx}\big(f(x)-g(x)\big) = f'(x)-g'(x)$,
3. $\dfrac{d}{dx}\big(c\cdot f(x)\big) = c\cdot f'(x)$.

Example 10.5 Compute $\frac{d}{dx}\left(x^5+\frac{1}{x}\right)$. [1]

Solution 10.5

$$\begin{aligned}\frac{d}{dx}\left(x^5+\frac{1}{x}\right) &= \frac{d}{dx}x^5+\frac{d}{dx}x^{-1} \\ &= 5x^4-x^{-2}.\end{aligned}$$

■

Example 10.6 Compute $\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}}-2\sqrt{x}+\frac{1}{x^7}\right)$. [1]

Solution 10.6

$$\begin{aligned}\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}}-2\sqrt{x}+\frac{1}{x^7}\right) &= 3\frac{d}{dx}x^{-1/3}-2\frac{d}{dx}x^{1/2}+\frac{d}{dx}x^{-7} \\ &= -x^{-4/3}-x^{-1/2}-7x^{-8}.\end{aligned}$$

■

10.5 The Product Rule

Consider the product of two simple functions, say $f(x)\cdot g(x)$, where $f(x)=x^2+1$ and $g(x)=x^3-3x$. An obvious guess for the derivative of $f(x)g(x)$ is the product of the derivatives:

$$\begin{aligned}f'(x)g'(x) &= (2x)(3x^2-3) \\ &= 6x^3-6x.\end{aligned}$$

Is this guess correct? We can check by rewriting $f(x)$ and $g(x)$ and doing the calculation in a way that is known to work. Write

$$\begin{aligned}f(x)g(x) &= (x^2+1)(x^3-3x) \\ &= x^5-3x^3+x^3-3x \\ &= x^5-2x^3-3x.\end{aligned}$$

Hence

$$\frac{d}{dx}f(x)g(x) = 5x^4-6x^2-3,$$

so we see that

$$\frac{d}{dx}f(x)g(x) \neq f'(x)g'(x).$$

So the derivative of $f(x)g(x)$ is **not** as simple as $f'(x)g'(x)$. Never fear, we have a rule for exactly this situation. [1]

Theorem 10.5.1 — The Product Rule. If $f(x)$ and $g(x)$ are differentiable, then

$$\frac{d}{dx}f(x)g(x) = f(x)g'(x)+f'(x)g(x).$$

■ **Example 10.7** Let $f(x) = (x^2+1)$ and $g(x) = (x^3-3x)$. Compute: $\frac{d}{dx}f(x)g(x)$. [1]

Solution 10.7

$$\begin{aligned}\frac{d}{dx}f(x)g(x) &= f(x)g'(x) + f'(x)g(x)\\ &= (x^2+1)(3x^2-3) + 2x(x^3-3x).\end{aligned}$$

We could stop here—or expand it if you're asked to

$$\begin{aligned}(x^2+1)(3x^2-3) + 2x(x^3-3x) &= 3x^4 - 3x^2 + 3x^2 - 3 + 2x^4 - 6x^2\\ &= 5x^4 - 6x^2 - 3,\end{aligned}$$

■

10.6 The Quotient Rule

We'd like to have a formula to compute

$$\frac{d}{dx}\frac{f(x)}{g(x)}$$

if we already know $f'(x)$ and $g'(x)$. Instead of attacking this problem head-on, let's notice that we've already done part of the problem: $f(x)/g(x) = f(x)\cdot(1/g(x))$, that is, this is really a product, and we can compute the derivative if we know $f'(x)$ and $(1/g(x))'$. This brings us to our next derivative rule. [1]

Theorem 10.6.1 — The Quotient Rule. If $f(x)$ and $g(x)$ are differentiable, then

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

A great way to remember this (how I memorized this) is to say:

$$\frac{\text{“}LoDeHi - HiDeLo\text{”}}{(Lo)^2}$$

■ **Example 10.8** Compute: $\frac{d}{dx}\frac{x^2+1}{x^3-3x}$. [1]

Solution 10.8

$$\begin{aligned}\frac{d}{dx}\frac{x^2+1}{x^3-3x} &= \frac{2x(x^3-3x)-(x^2+1)(3x^2-3)}{(x^3-3x)^2}\\ &= \frac{-x^4-6x^2+3}{(x^3-3x)^2}.\end{aligned}$$

■

It is often possible to calculate derivatives in more than one way, as we have already seen. Since every quotient can be written as a product, it is always possible to use the product rule to compute the derivative, though it is not always simpler. [1]

■ **Example 10.9** Compute

$$\frac{d}{dx}\frac{625-x^2}{\sqrt{x}}$$

in two ways. First using the quotient rule and then using the product rule. [1]

Solution 10.9 First, we'll compute the derivative using the quotient rule.

$$\frac{d}{dx}\frac{625-x^2}{\sqrt{x}} = \frac{(-2x)\left(\sqrt{x}\right)-\left(625-x^2\right)\left(\frac{1}{2}x^{-1/2}\right)}{x}.$$

Second, we'll compute the derivative using the product rule:

$$\begin{aligned}\frac{d}{dx}\frac{625-x^2}{\sqrt{x}} &= \frac{d}{dx}\left(625-x^2\right)x^{-1/2} \\ &= \left(625-x^2\right)\left(\frac{-x^{-3/2}}{2}\right)+(-2x)\left(x^{-1/2}\right).\end{aligned}$$

With a bit of algebra, both of these simplify to

$$-\frac{3x^2+625}{2x^{3/2}}.$$

■

10.7 The Chain Rule

Consider

$$h(x) = (1+2x)^5.$$

While there are several different ways to differentiate this function, if we let $f(x) = x^5$ and $g(x) = 1+2x$, then we can express $h(x) = f(g(x))$. The question is, can we compute the derivative of a composition of functions using the derivatives of the constituents $f(x)$ and $g(x)$? To do so, we need the *chain rule*. [1]

Theorem 10.7.1 — Chain Rule. If $f(x)$ and $g(x)$ are differentiable, then

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

And the last bits of examples.

■ **Example 10.10** Compute: $\dfrac{d}{dx}(1+2x)^5$. [1]

Solution 10.10 Set $f(x) = x^5$ and $g(x) = 1+2x$, now

$$f'(x) = 5x^4 \qquad \text{and} \qquad g'(x) = 2.$$

Hence

$$\begin{aligned}\frac{d}{dx}(1+2x)^5 &= \frac{d}{dx}f(g(x)) \\ &= f'(g(x))g'(x) \\ &= 5(1+2x)^4\cdot 2 \\ &= 10(1+2x)^4.\end{aligned}$$

■

■ **Example 10.11** Compute: $\dfrac{d}{dx}\sqrt{1+\sqrt{x}}$. [1]

Solution 10.11 Set $f(x)=\sqrt{x}$ and $g(x)=1+x$. Hence,

$$\sqrt{1+\sqrt{x}} = f(g(f(x))) \qquad \text{and} \qquad \frac{d}{dx}f(g(f(x))) = f'(g(f(x)))g'(f(x))f'(x).$$

Since

$$f'(x)=\frac{1}{2\sqrt{x}} \qquad \text{and} \qquad g'(x)=1$$

We have that

$$\frac{d}{dx}\sqrt{1+\sqrt{x}} = \frac{1}{2\sqrt{1+\sqrt{x}}}\cdot 1\cdot \frac{1}{2\sqrt{x}}.$$

■

Exercises For Chapter 10

In Exercises 1–18, find the *derivative*. [1]

1. $\dfrac{d}{dx}2147483647$
2. $\dfrac{d}{dx}\dfrac{1}{\sqrt{2}}$
3. $\dfrac{d}{dx}x^{\pi}$
4. $\dfrac{d}{dx}\dfrac{1}{(\sqrt[7]{x})^9}$
5. $\dfrac{d}{dx}\left(5x^3+12x^2-15\right)$
6. $\dfrac{d}{dx}\left(\dfrac{x^2}{x^7}+\dfrac{\sqrt{x}}{x}\right)$
7. $\dfrac{d}{dx}x^3(x^3-5x+10)$
8. $\dfrac{d}{dx}(x^2+5x-3)(x^5-6x^3+3x^2-7x+1)$
9. $\dfrac{d}{dx}\dfrac{x^3}{x^3-5x+10}$
10. $\dfrac{d}{dx}\dfrac{x^2+5x-3}{x^5-6x^3+3x^2-7x+1}$
11. $\dfrac{d}{dx}(1+3x)^2$
12. $\dfrac{d}{dx}\sqrt{\dfrac{169}{x}-x}$
13. $\dfrac{d}{dx}100/(100-x^2)^{3/2}$
14. $\dfrac{d}{dx}\sqrt{(x^2+1)^2+\sqrt{1+(x^2+1)^2}}$
15. $\dfrac{d}{dx}(3x^2+1)(2x-4)^3$
16. $\dfrac{d}{dx}\dfrac{2x^{-1}-x^{-2}}{3x^{-1}-4x^{-2}}$
17. $\dfrac{d}{dx}(2x+1)^3(x^2+1)^2$
18. $\dfrac{d}{dx}x^4-3x^3+(1/2)x^2+7x-\pi$

■

11. Implicit Differentiation

11.1 How To Do It

The functions we've been dealing with so far have been *explicit functions*, meaning that the dependent variable is written in terms of the independent variable. [1] For example:

$$y = 3x^2 - 2x + 1, \qquad y = e^{3x}, \qquad y = \frac{x-2}{x^2 - 3x + 2}.$$

However, there are another type of functions, called *implicit functions*. In this case, the dependent variable is not stated explicitly in terms of the independent variable. [1] For example:

$$x^2 + y^2 = 4, \qquad x^3 + y^3 = 9xy, \qquad x^4 + 3x^2 = x^{2/3} + y^{2/3} = 1.$$

Your inclination might be simply to solve each of these for y and go merrily on your way. However this can be difficult and it may require two *branches*, for example to explicitly plot $x^2 + y^2 = 4$, one needs both $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$. Moreover, it may not even be possible to solve for y. To deal with such situations, we use *implicit differentiation*. [1] Let's see an illustrative example:

■ **Example 11.1** Consider the curve defined by

$$x^3 + y^3 = 9xy.$$

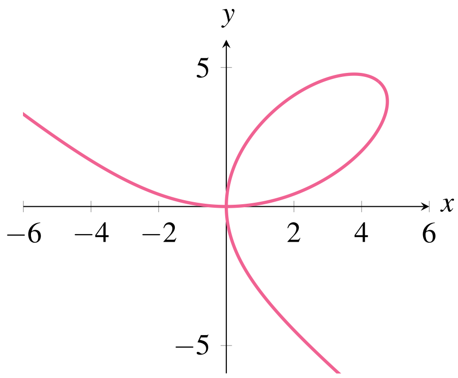


Figure 11.1: A plot of $x^3 + y^3 = 9xy$. [1]

1. Compute $\frac{dy}{dx}$. [1]
2. Find the slope of the tangent line at $(4,2)$. [1]

Solution 11.1 Starting with $x^3 + y^3 = 9xy$, we apply the differential operator $\frac{d}{dx}$ to both sides of the equation to obtain

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx} 9xy.$$

Applying the sum rule we see

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}9xy.$$

Let's examine each of these terms in turn. To start

$$\frac{d}{dx}x^3 = 3x^2.$$

On the other hand $\frac{d}{dx}y^3$ is somewhat different. Here you imagine that $y = y(x)$, and hence by the chain rule

$$\begin{aligned}\frac{d}{dx}y^3 &= \frac{d}{dx}(y(x))^3 \\ &= 3(y(x))^2 \cdot y'(x) \\ &= 3y^2\frac{dy}{dx}.\end{aligned}$$

Considering the final term $\frac{d}{dx}9xy$, we again imagine that $y = y(x)$. Hence

$$\begin{aligned}\frac{d}{dx}9xy &= 9\frac{d}{dx}x \cdot y(x) \\ &= 9\left(x \cdot y'(x) + y(x)\right) \\ &= 9x\frac{dy}{dx} + 9y.\end{aligned}$$

Putting this all together we are left with the equation

$$3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y.$$

At this point, we solve for $\frac{dy}{dx}$. Write

$$\begin{aligned}3x^2+3y^2\frac{dy}{dx} &= 9x\frac{dy}{dx}+9y\\ 3y^2\frac{dy}{dx}-9x\frac{dy}{dx} &= 9y-3x^2\\ \frac{dy}{dx}\left(3y^2-9x\right) &= 9y-3x^2\\ \frac{dy}{dx} &= \frac{9y-3x^2}{3y^2-9x}=\frac{3y-x^2}{y^2-3x}.\end{aligned}$$

For the second part of the problem, we simply plug $x=4$ and $y=2$ into the formula above, hence the slope of the tangent line at $(4,2)$ is $\frac{5}{4}$, see Figure 11.1.

■

11.2 Second Derivatives

Sometimes, you'll be asked to find a second derivative implicitly.

■ **Example 11.2** Find $\frac{d^2y}{dx^2}$ if $y^2+2y=4x^2+2x$. [2]

Solution 11.2 Differentiating implicitly, you get

$$2y\frac{dy}{dx}+2\frac{dy}{dx}=8x+2$$

Now, simplify and solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx}=\frac{4x+1}{y+1}$$

Noew, it's time to take the derivative again.

$$\frac{d^2y}{dx^2}=\frac{4(y+1)-(4x+1)\left(\frac{dy}{dx}\right)}{(y+1)^2}$$

What's $\frac{dy}{dx}$? Well, you just defined it yourself.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{4(y+1)-(4x+1)\left(\frac{4x+1}{y+1}\right)}{(y+1)^2}\\ &= \frac{4(y+1)^2-(4x+1)^2}{(y+1)^3}\end{aligned}$$

■

That's how you do implicit differntiation. Give yourself a rest before starting these exercises.

Exercises For Chapter 11

In Exercises 1–6, find $\dfrac{dy}{dx}$**.** [1]

1. $x^2 + y^2 = 4$
2. $x^2 + xy + y^2 = 7$
3. $x^3 + xy^2 = y^3 + yx^2$
4. $\sqrt{x} + \sqrt{y} = 9$
5. $xy^{3/2} + 4 = 2x + y$
6. $\frac{1}{x} + \frac{1}{y} = 7$

In Exercises 7–8, find $\dfrac{d^2y}{dx^2}$**.** [3]

7. $x^2 + y^2 = 4$
8. $y^2 = x^3$

IV

Differential Calculus Applications

12. Basic Applications of the Derivative

Consider this problem: Suppose you drive a car from toll booth on a toll road to another toll booth at an average speed of 70 miles per hour. What can be concluded about your actual speed during the trip? In particular, did you exceed the 65 mile per hour speed limit? To solve this, you need to use the **Mean Value Theorem**.

12.1 Mean Value Theorem for Derivatives

Theorem 12.1.1 — Mean Value Theorem. Suppose that $f(x)$ has a derivative on the interval (a,b) and is continuous on the interval $[a,b]$. Then at some value $c \in (a,b)$,

$$f'(c) = \frac{f(b)-f(a)}{b-a}. \text{ [1]}$$

In other words, there's come point in the interval where the slope of the tangent line equals the slope of the secant line that connects the endpoints of the interval. You can see this in Figure 12.1:

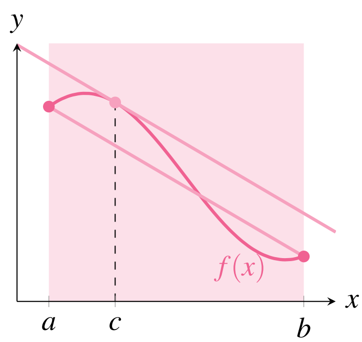


Figure 12.1: A geometric interpretation of the Mean Value Theorem

■ **Example 12.1** Suppose you drive a car from toll booth on a toll road to another toll booth 30 miles away in half of an hour. Must you have been driving at 60 miles per hour at some point? [1]

Solution 12.1 If $p(t)$ is the position of the car at time t, and 0 hours is the starting time with $1/2$ hours being the final time, the Mean Value Theorem states there is a time c

$$p'(c) = \frac{30-0}{1/2} = 60 \qquad \text{where } 0 < c < 1/2.$$

Since the derivative of position is velocity, this says that the car must have been driving at 60 miles per hour at some point.

■

12.2 Rolle's Theorem

Now let's learn Rolle's Theorem, which is a special case of the MVTD.

Theorem 12.2.1 — Rolle's Theorem. Suppose that $f(x)$ is differentiable on the interval (a,b), is continuous on the interval $[a,b]$, and $f(a) = f(b)$. Then

$$f'(c) = 0$$

for some $a < c < b$.

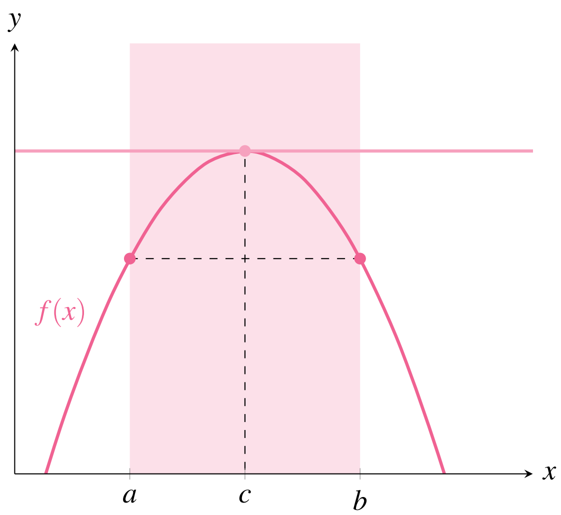


Figure 12.2: A geometric interpretation of Rolle's Theorem.

■ **Example 12.2** Suppose you toss a ball into the air and then catch it. Must the ball's vertical velocity have been zero at some point? [1]

Solution 12.2 If $p(t)$ is the position of the ball at time t, then we may apply Rolle's Theorem to see at some time c, $p'(c) = 0$. Hence the velocity must be zero at some point.

■

Exercises For Chapter 12

In Exercises 1–3, determine whether Rolle's Theorem can be applied to f on the closed interval $[a,b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a,b) such that $f'(c)=0$. If Rolle's Theorem cannot be applied, explain why not. [3]

1. $f(x)=(x-1)(x-2)(x-3), \quad [1,3]$
2. $f(x)=x^{2/3}-1, \quad [-8,8]$
3. $f(x)=\dfrac{x^2-2x-3}{x+2}, \quad [-1,3]$

In Exercises 4–6, determine whether Mean Value Theorem can be applied to f on the closed interval $[a,b]$. If Mean Value Theorem can be applied, find all values of c in the open interval (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$. If Mean Value Theorem cannot be applied, explain why not. [3]

4. $f(x)=x^2, \quad [-2,1]$
5. $f(x)=x^3+2x, \quad [-1,1]$
6. $f(x)=|2x+1|, \quad [-1,3]$

13. Maxima and Minima

One of the most interesting question to ask when looking at a function's graph is, what's that highest/lowest point? Many of these problems can be solved by finding the appropriate function and then using techniques of calculus to find the maximum or the minimum value required.

13.1 Absolute Extrema

First let's start with the most concerned ones, finding the *absolute extrema*.

Definition 13.1.1

1. A point $(x, f(x))$ is an **absolute maximum** on an interval if $f(x) \geq f(z)$ for every z in that interval.
2. A point $(x, f(x))$ is an **absolute minimum** on an interval if $f(x) \leq f(z)$ for every z in that interval.

An **absolute extremum** is either an absolute maximum or an absolute minimum.
[1]

If we are working on an finite closed interval, then we have the following theorem. [1]

Theorem 13.1.1 — Extreme Value Theorem. If $f(x)$ is a continuous function for all x in the closed interval $[a,b]$, then there are points c and d in $[a,b]$, such that $(c, f(c))$ is an absolute maximum and $(d, f(d))$ is an absolute minimum on $[a,b]$.
[1]

In Figure 13.1, we see a geometric interpretation of this theorem.

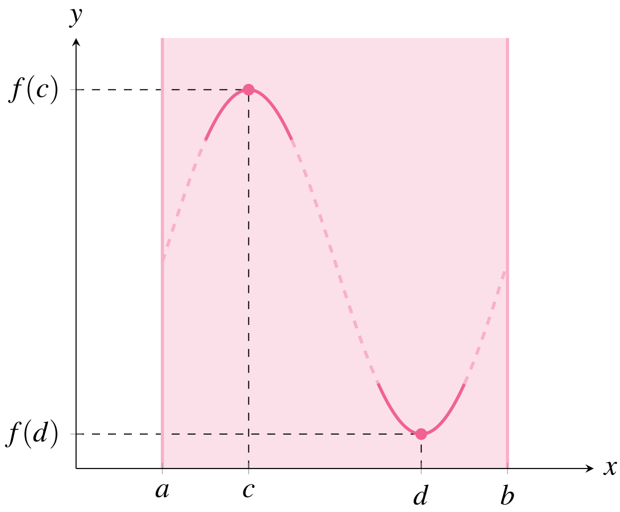


Figure 13.1: A geometric interpretation of the Extreme Value Theorem. A continuous function $f(x)$ attains both an absolute maximum and an absolute minimum on an interval $[a,b]$. Note, it may be the case that $a = c$, $b = d$, or that $d < c$. [1]

13.2 Relative Extrema

Apart from the top and bottom, for a continuous function, you can think of a relative maximum as occurring on a "hill" on the graph, and a relative minimum as occurring in a "valley" on the graph. [3]

Definition 13.2.1 — Relative Extrema.

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f, or you can say that f has a relative maximum at $(c, f(c))$.
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f, or you can say that f has a relative minimum at $(c, f(c))$.

An **relative extremum** is either an relative maximum or an relative minimum.
[3]

Continuing with the analogy before, such a hill and valley can occur in two ways. If the hill (or valley) is smooth and rounded, the graph has a horizontal tangent line at the high point (or low point). If the hill (or valley) is sharp and peaked, the graph represents a function that is not differentiable at the high point (or low point). [3] That said, the slope of the curve can be used to find these hills and valleys. The x-values indicating the existance possible extrema are called the **critical numbers**.

Definition 13.2.2 — Critical Number. Let f be defined at c. If $f(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f. [3]

At a point where the first derivative is zero, the curve has a horizontal tangent line, at which point it could be reaching a "hill" (maximum) or a "valley" (minimum). Then, how do you know it's a maximum or a minimum? You can use either the first or second derivative test, which we will explore in the next chapter. For now, just look at this example:

■ **Example 13.1** Find all local maximum and minimum points for the function $f(x) = x^3 - x$.
Solution 13.1 First things first, the first derivative,

$$\frac{d}{dx} f(x) = 3x^2 - 1.$$

This is defined everywhere and is zero at $x = \pm\sqrt{3}/3$. Looking first at $x = \sqrt{3}/3$, we see that

$$f(\sqrt{3}/3) = -2\sqrt{3}/9.$$

Now we test two points on either side of $x = \sqrt{3}/3$, making sure that neither is farther away than the nearest critical point; since $\sqrt{3} < 3$, $\sqrt{3}/3 < 1$ and we can use $x = 0$ and $x = 1$. Since

$$f(0) = 0 > -2\sqrt{3}/9 \qquad \text{and} \qquad f(1) = 0 > -2\sqrt{3}/9,$$

there must be a local minimum at $x = \sqrt{3}/3$.

For $x = -\sqrt{3}/3$, we see that $f(-\sqrt{3}/3) = 2\sqrt{3}/9$. This time we can use $x = 0$ and $x = -1$, and we find that $f(-1) = f(0) = 0 < 2\sqrt{3}/9$, so there must be a local maximum at $x = -\sqrt{3}/3$.

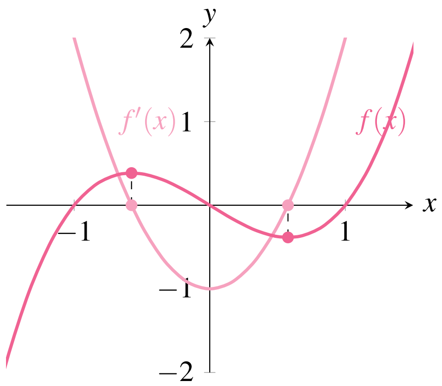


Figure 13.2: A plot of $f(x) = x^3 - x$ and $f'(x) = 3x^2 - 1$.

■

Exercises For Chapter 13

Find the x values for relative maximum and minimum points. [1]

1. $y = x^2 - x$
2. $y = 2 + 3x - x^3$
3. $y = x^3 - 9x^2 + 24x$
4. $y = x^4 - 2x^2 + 3$
5. $y = -\frac{x^4}{4} + x^3 + x^2$
6. $f(x) = \begin{cases} x - 1 & x < 2 \\ x^2 & x \geq 2 \end{cases}$
7. $f(x) = \begin{cases} -2 & x = 0 \\ 1/x^2 & x \neq 0 \end{cases}$

■

14. Curve Sketching

14.1 The First Derivative Test

The method of the previous section for deciding whether there is a relative maximum or minimum at a critical point by testing "near-by" points is not always convenient. Instead, since we have already had to compute the derivative to find the critical points, we can use information about the derivative to decide. [1] Recall that

- If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.
- If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.

So how exactly does the derivative tell us whether there is a maximum, minimum, or neither at a point? Use the *first derivative test*. [1]

Theorem 14.1.1 — First Derivative Test. Suppose that $f(x)$ is continuous on an interval, and that $f'(a) = 0$ for some value of a in that interval.

- If $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a, then $f(a)$ is a relative maximum.
- If $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a, then $f(a)$ is a relative minimum.
- If $f'(x)$ has the same sign to the left and right of a, then $f(a)$ is not a relative extremum.

[1]

■ **Example 14.1** Consider the function

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$$

Find the intervals on which $f(x)$ is increasing and decreasing and identify the relative extrema of $f(x)$. [1]

Solution 14.1 Start by computing

$$\frac{d}{dx}f(x) = x^3 + x^2 - 2x.$$

Now we need to find when this function is positive and when it is negative. To do this, solve

$$f'(x) = x^3 + x^2 - 2x = 0.$$

Factor $f'(x)$

$$\begin{aligned} f'(x) &= x^3 + x^2 - 2x \\ &= x(x^2 + x - 2) \\ &= x(x+2)(x-1). \end{aligned}$$

So the critical points (when $f'(x) = 0$) are when $x = -2$, $x = 0$, and $x = 1$. Now we can check points **between** the critical points to find when $f'(x)$ is increasing and decreasing:

$$f'(-3) = -12 \qquad f'(.5) = -0.625 \qquad f'(-1) = 2 \qquad f'(2) = 8$$

From this we can make a sign table:

$f'(x) < 0$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
-2	0	1	
Decreasing	Increasing	Decreasing	Increasing

Hence $f(x)$ is increasing on $(-2,0) \cup (1,\infty)$ and $f(x)$ is decreasing on $(-\infty,-2) \cup (0,1)$. Moreover, from the first derivative test, Theorem 14.1.1, the relative maximum is at $x = 0$ while the relative minima are at $x = -2$ and $x = 1$, see Figure 14.1.

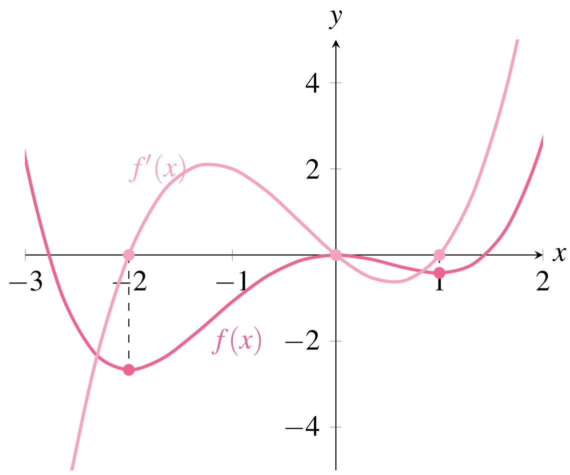


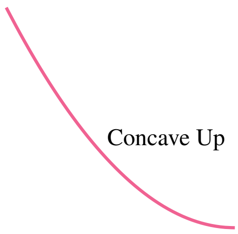
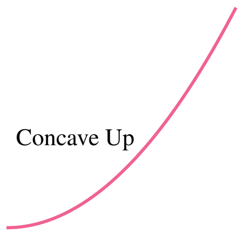
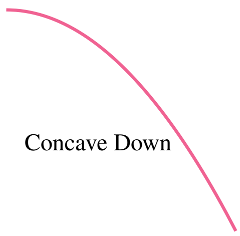
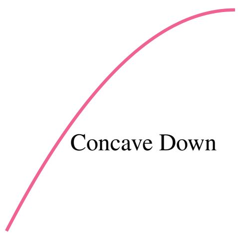
Figure 14.1: A plot of $f(x) = x^4/4 + x^3/3 - x^2$ and $f'(x) = x^3 + x^2 - 2x$. [1]

■

Hence we have seen that if $f'(x)$ is zero and increasing at a point, then $f(x)$ has a relative minimum at the point. If $f'(x)$ is zero and decreasing at a point then $f(x)$ has a relative maximum at the point. Thus, we see that we can gain information about $f(x)$ by studying how $f'(x)$ changes. This leads us to our next section. [1]

14.2 Concavity and Inflection Points

We know that the sign of the derivative tells us whether a function is increasing or decreasing. Likewise, the sign of the second derivative $f''(x)$ tells us whether $f'(x)$ is increasing or decreasing. We summarize this in the table below: [1]

	$f'(x) < 0$	$f'(x) > 0$
$f''(x) > 0$	 Here $f'(x) < 0$ and $f''(x) > 0$. This means that $f(x)$ slopes down and is getting *less steep*. In this case the curve is **concave up**.	 Here $f'(x) > 0$ and $f''(x) > 0$. This means that $f(x)$ slopes up and is getting *steeper*. In this case the curve is **concave up**.
$f''(x) < 0$	 Here $f'(x) < 0$ and $f''(x) < 0$. This means that $f(x)$ slopes down and is getting *steeper*. In this case the curve is **concave down**.	 Here $f'(x) > 0$ and $f''(x) < 0$. This means that $f(x)$ slopes up and is getting less *steep*. In this case the curve is **concave down**.

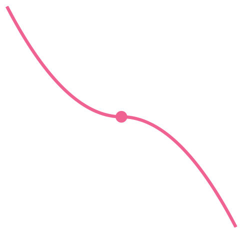
If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. It is worth summarizing what we have seen already in to a single theorem.

Theorem 14.2.1 — Test for Concavity. Suppose that $f''(x)$ exists on an interval.
1. If $f''(x) > 0$ on an interval, then $f(x)$ is concave up on that interval.
2. If $f''(x) < 0$ on an interval, then $f(x)$ is concave down on that interval.

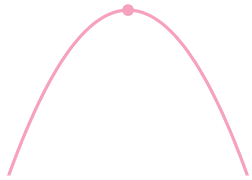
Of particular interest are points at which the concavity changes from up to down or down to up.

Definition 14.2.1 If $f(x)$ is continuous and its concavity changes either from up to down or down to up at $x = a$, then $f(x)$ has an **inflection point** at $x = a$.

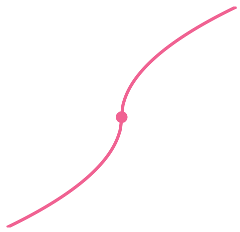
It is instructive to see some examples and nonexamples of inflection points.



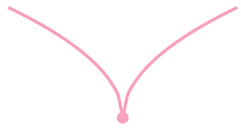
This is an inflection point. The concavity changes from concave up to concave down.



This is **not** an inflection point. The curve is concave down on either side of the point.



This is an inflection point. The concavity changes from concave up to concave down.



This is **not** an inflection point. The curve is concave down on either side of the point.

We identify inflection points by first finding where $f''(x)$ is zero or undefined and then checking to see whether $f''(x)$ does in fact go from positive to negative or negative to positive at these points.

14.3 The Second Derivative Test

Recall the first derivative test, Theorem 14.1.1:

- If $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a, then $f(a)$ is a relative maximum.
- If $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a, then $f(a)$ is a relative minimum.

If $f'(x)$ changes from positive to negative it is decreasing. In this case, $f''(x)$ might be negative, and if in fact $f''(x)$ is negative then $f'(x)$ is definitely decreasing, so there is a relative maximum at the point in question. On the other hand, if $f'(x)$ changes from negative to positive it is increasing. Again, this means that $f''(x)$ might be positive, and if in fact $f''(x)$ is positive then $f'(x)$ is definitely increasing, so there is a relative minimum at the point in question. We summarize this as the *second derivative test*. [1]

Theorem 14.3.1 — Second Derivative Test. Suppose that $f''(x)$ is continuous on an open interval and that $f'(a) = 0$ for some value of a in that interval.
- If $f''(a) < 0$, then $f(x)$ has a relative maximum at a.
- If $f''(a) > 0$, then $f(x)$ has a relative minimum at a.
- If $f''(a) = 0$, then the test is inconclusive. In this case, $f(x)$ may or may not have a relative extremum at $x = a$.

The second derivative test is often the easiest way to identify relative maximum and minimum points. Sometimes the test fails and sometimes the second derivative is quite difficult to evaluate. In such cases we must fall back on one of the previous tests. [1]

■ **Example 14.2** Once again, consider the function

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$$

Use the second derivative test, Theorem 14.3.1, to locate the relative extrema of $f(x)$. [1] ■

Solution 14.2 Start by computing

$$f'(x) = x^3 + x^2 - 2x \qquad \text{and} \qquad f''(x) = 3x^2 + 2x - 2.$$

Using the same technique as used in the solution of Example 14.1, we find that

$$f'(-2) = 0, \qquad f'(0) = 0, \qquad f'(1) = 0.$$

Now we'll attempt to use the second derivative test, Theorem 14.3.1,

$$f''(-2) = 6, \qquad f''(0) = -2, \qquad f''(1) = 3.$$

Hence we see that $f(x)$ has a relative minimum at $x = -2$, a relative maximum at $x = 0$, and a relative minimum at $x = 1$, see Figure 14.2.

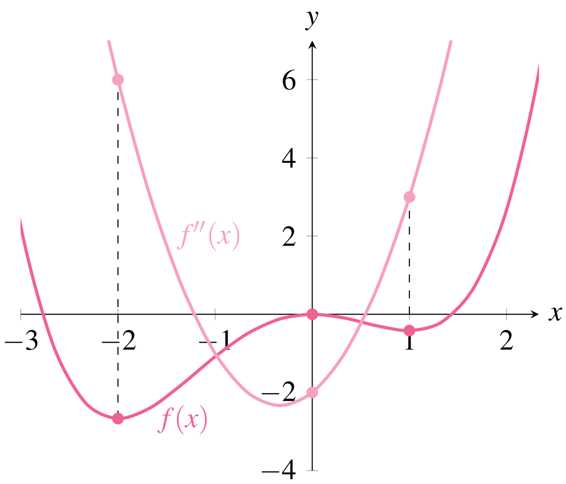


Figure 14.2: A plot of $f(x) = x^4/4 + x^3/3 - x^2$ and $f''(x) = 3x^2 + 2x - 2$. [1]

14.4 Sketching the Plot of a Function

In this section, we will give some general guidelines for sketching the plot of a function.

Proposition 14.4.1 — Procedure for Sketching the Plots of Functions.

- Find the y-intercept, this is the point $(0, f(0))$. Place this point on your graph.
- Find candidates for vertical asymptotes, these are points where $f(x)$ is undefined.
- Compute $f'(x)$ and $f''(x)$.
- Find the critical points, the points where $f'(x) = 0$ or $f'(x)$ is undefined.
- Use the second derivative test to identify relative extrema and/or find the intervals where your function is increasing/decreasing.

- Find the candidates for inflection points, the points where $f''(x) = 0$ or $f''(x)$ is undefined.
- Identify inflection points and concavity.
- If possible find the x-intercepts, the points where $f(x) = 0$. Place these points on your graph.
- Find horizontal asymptotes.
- Determine an interval that shows all relevant behavior.

[1] At this point you should be able to sketch the plot of your function.

Let's see this procedure in action. We'll sketch the plot of $2x^3 - 3x^2 - 12x$. Following our guidelines above, we start by computing $f(0) = 0$. Hence we see that the y-intercept is $(0,0)$. Place this point on your plot, see Figure 14.3. [1]

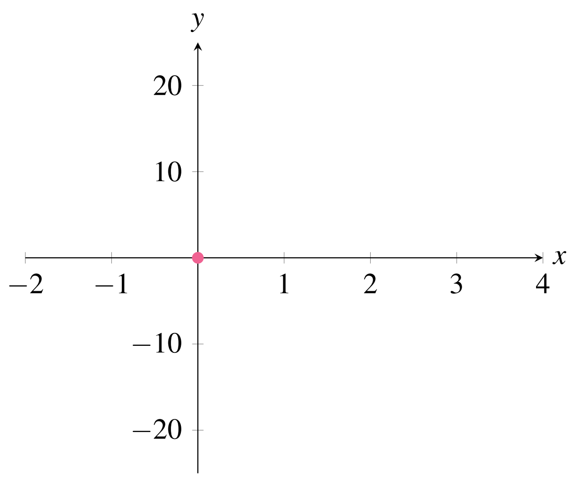


Figure 14.3: We start by placing the point $(0,0)$. [1]

Note that there are no vertical asymptotes as our function is defined for all real numbers. Now compute $f'(x)$ and $f''(x)$,

$$f'(x) = 6x^2 - 6x - 12 \qquad \text{and} \qquad f''(x) = 12x - 6.$$

The critical points are where $f'(x) = 0$, thus we need to solve $6x^2 - 6x - 12 = 0$ for x. Write

$$\begin{aligned} 6x^2 - 6x - 12 &= 0 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0. \end{aligned}$$

Thus

$$f'(2) = 0 \qquad \text{and} \qquad f'(-1) = 0.$$

Mark the critical points $x = 2$ and $x = -1$ on your plot, see Figure 14.4. [1]

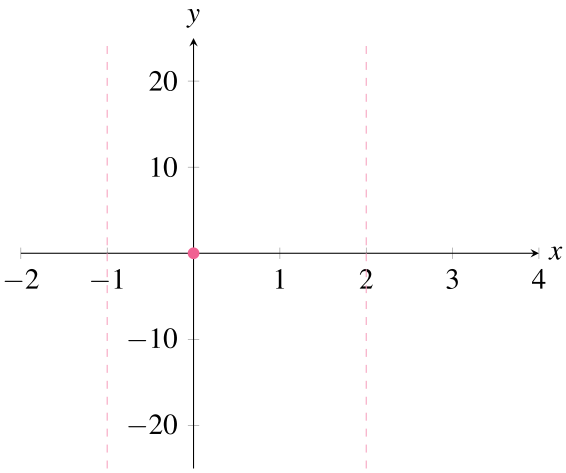


Figure 14.4: Now we add the critical points $x = -1$ and $x = 2$. [1]

Check the second derivative evaluated at the critical points. In this case,

$$f''(-1) = -18 \qquad \text{and} \qquad f''(2) = 18,$$

hence $x = -1$, corresponding to the point $(-1, 7)$ is a relative maximum and $x = 2$, corresponding to the point $(2, -20)$ is relative minimum of $f(x)$. Moreover, this tells us that our function is increasing on $[-2, -1)$, decreasing on $(-1, 2)$, and increasing on $(2, 4]$. Identify this on your plot, see Figure 14.5. [1]

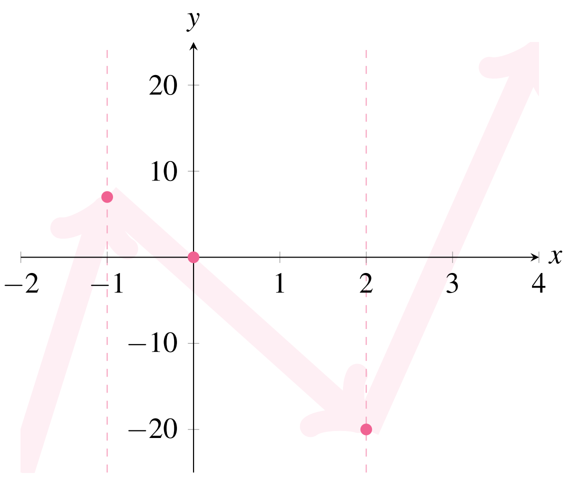


Figure 14.5: We have identified the relative extrema of $f(x)$ and where this function is increasing and decreasing. [1]

The candidates for the inflection points are where $f''(x) = 0$, thus we need to solve $12x - 6 = 0$ for x. Write

$$\begin{aligned} 12x - 6 &= 0 \\ x - 1/2 &= 0 \\ x &= 1/2. \end{aligned}$$

Thus $f''(1/2) = 0$. Checking points, $f''(0) = -6$ and $f''(1) = 6$. Hence $x = 1/2$ is an inflection point, with $f(x)$ concave down to the left of $x = 1/2$ and $f(x)$ concave up to the right of $x = 1/2$. We can add this information to our plot, see Figure 14.6. [1]

Finally, in this case, $f(x) = 2x^3 - 3x^2 - 12x$, we can find the x-intercepts. Write

$$2x^3 - 3x^2 - 12x = 0$$
$$x(2x^2 - 3x - 12) = 0.$$

Using the quadratic formula, we see that the x-intercepts of $f(x)$ are

$$x = 0, \qquad x = \frac{3 - \sqrt{105}}{4}, \qquad x = \frac{3 + \sqrt{105}}{4}.$$

Since all of this behavior as described above occurs on the interval $[-2, 4]$, we now have a complete sketch of $f(x)$ on this interval, see the figure below. [1]

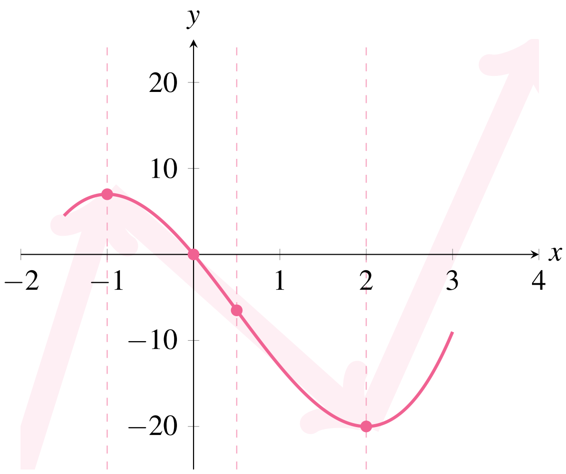
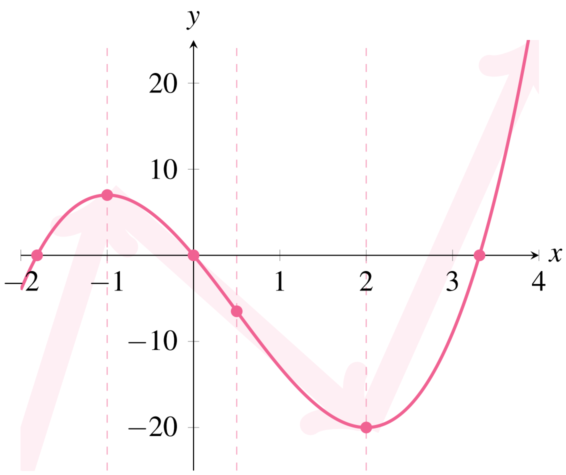


Figure 14.6: We identify the inflection point and note that the curve is concave down when $x < 1/2$ and concave up when $x > 1/2$. [1]



Exercises For Chapter 14

In Exercises 1-4, find all critical points and identify them as relative maximum points, relative minimum points, or neither. [1]

1. $y = x^2 - x$
2. $y = 2 + 3x - x^3$
3. $y = x^3 - 9x^2 + 24x$
4. $f(x) = |x^2 - 121|$
5. Let $f(x) = ax^2 + bx + c$ with $a \neq 0$. Show that $f(x)$ has exactly one critical point using the first derivative test. Give conditions on a and b which guarantee that the critical point will be a maximum.

In Exercises 6-9, Sketch the curves via the procedure outlined in this chapter. Clearly identify any interesting features, including relative maximum and minimum points, inflection points, asymptotes, and intercepts. [1]

6. $y = x^5 - x$
7. $y = 2\sqrt{x} - x$
8. $y = x^5 - 5x^4 + 5x^3$
9. $y = x^2 + 1/x$

15. Motion and Related Rates

This chapter deals with two different types of word problems that involve motion: related rates and the relationship between velocity and acceleration of a particle. The subject matter might seem acrane, but once you get the hang of them, you'll see that these aren't so hard, either.

15.1 Related Rates

The idea behind these problems is very simple. In a typical problem, you'll be given an equation relating two or more variables. These variables will change with respect to time, and you'll use derivatives to determine how the rates of change are related. Sounds easy, soesn't it?

Proposition 15.1.1 — Guidelines for Related Rates Problems.

- **Draw a picture.** If possible, draw a schematic picture with all the relevant information.
- **Find an equation.** We want an equation that relates all relevant functions.
- **Differentiate the equation.** Here we will often use implicit differentiation.
- **Evaluate the equation at the desired values.** The known values should let you solve for the relevant rate.

[1]

Let's see some examples. [1]

■ **Example 15.1** A circular pool of water is expanding at the rate of $16\pi m^2/s$. At what rate is the radius expanding when the radius is 4 meters?

Hint: What equation relates the area of a circle to its radius? $A = \pi r^2$. [2]

Solution 15.1

Find an equation and **differentiate the equation** with respect to t (time).

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

In this equation, $\frac{dA}{dt}$ represents the rate at which the area is changing, and $\frac{dr}{dt}$ is the rate at which the radius is changing. The simplest way to explain this is that whenever you have a variable in an equation (r, for example), the derivative with respect to time $\left(\frac{dr}{dt}\right)$ represents the rate at which that variable is increasing or decreasing.

Now we can **Evaluate the equation at the desired values**, that is, the rate of change of the area and for the radius.

$$16\pi = 2\pi(4)\frac{dr}{dt}$$

Solving for $\frac{dr}{dt}$, we get

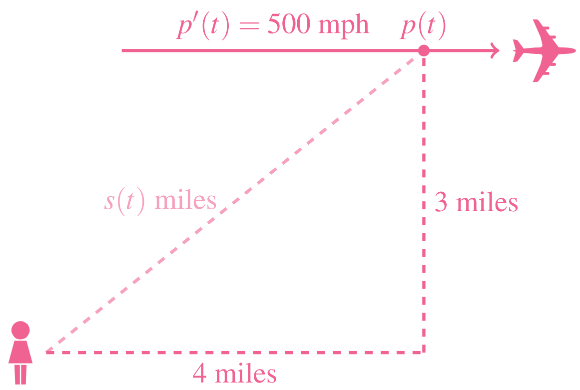
$$16\pi = 8\pi\frac{dr}{dt} \text{ and } \frac{dr}{dt} = 2$$

The radius is changing at a rate of $2m/s$. It's important ot note that this is the rate only when the radius is 4 meters. As the circle gets bigger and bigger, the radius will expand at a slower and sloer rate.

■

■ **Example 15.2** A plane is flying directly away from you at 500 mph at an altitude of 3 miles. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you? [1]

Solution 15.2 To start, **draw a picture**.



Next we need to **find an equation**. By the Pythagorean Theorem we know that

$$p^2 + 3^2 = s^2.$$

Now we **differentiate the equation**. Write

$$2p(t)p'(t) = 2s(t)s'(t).$$

Now we'll **evaluate the equation at the desired values**. We are interested in the time at which $p(t) = 4$ and $p'(t) = 500$. Additionally, at this time we know that $4^2 + 9 = s^2$, so $s(t) = 5$. Putting together all the information we get

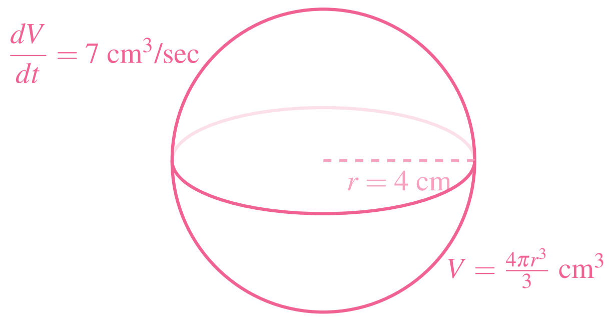
$$2(4)(500) = 2(5)s'(t),$$

thus $s'(t) = 400$ mph.

■

■ **Example 15.3** You are inflating a spherical balloon at the rate of 7 cm^3/sec. How fast is its radius increasing when the radius is 4 cm? [1]

Solution 15.3 To start, **draw a picture**.



Next we need to **find an equation**. Thinking of the variables r and V as functions of time, they are related by the equation

$$V(t) = \frac{4\pi(r(t))^3}{3}.$$

Now we need to **differentiate the equation**. Taking the derivative of both sides gives

$$\frac{dV}{dt} = 4\pi(r(t))^2 \cdot r'(t).$$

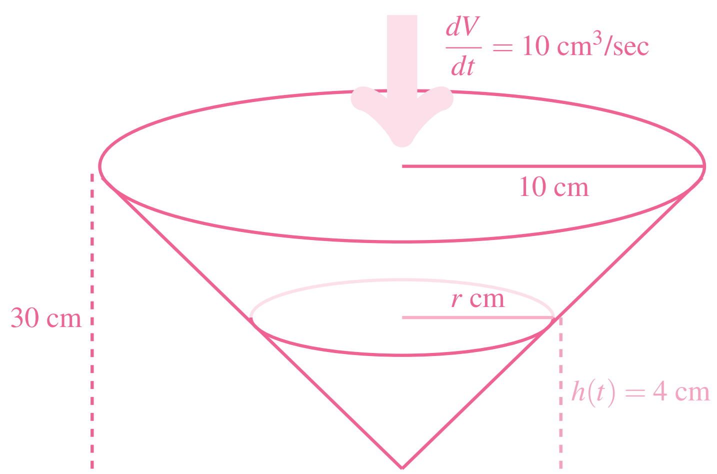
Finally we **evaluate the equation at the desired values**. Set $r(t) = 4$ cm and $\frac{dV}{dt} = 7$ cm^3/sec. Write

$$7 = 4\pi 4^2 r'(t),$$
$$r'(t) = 7/(64\pi) \text{ cm/sec}.$$

■

■ **Example 15.4** Water is poured into a conical container at the rate of 10 cm^3/sec. The cone points directly down, and it has a height of 30 cm and a base radius of 10 cm. How fast is the water level rising when the water is 4 cm deep? [1]

Solution 15.4 To start, **draw a picture**.



Note, no attempt was made to draw this picture to scale, rather we want all of the relevant information to be available to the mathematician.

Now we need to **find an equation**. The formula for the volume of a cone tells us that

$$V = \frac{\pi}{3} r^2 h.$$

Now we must **differentiate the equation**. We should use implicit differentiation, and treat each of the variables as functions of t. Write

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right). \tag{15.1}$$

At this point we **evaluate the equation at the desired values**. At first something seems to be wrong, we do not know $\frac{dr}{dt}$. However, the dimensions of the cone of water must have the same proportions as those of the container. That is, because of similar triangles,

$$\frac{r}{h} = \frac{10}{30} \qquad \text{so} \qquad r = h/3.$$

In particular, we see that when $h = 4$, $r = 4/3$ and

$$\frac{dr}{dt} = \frac{1}{3} \cdot \frac{dh}{dt}.$$

Now we can **evaluate the equation at the desired values**. Starting with Equation 15.1, we plug in $\frac{dV}{dt} = 10$, $r = 4/3$, $\frac{dr}{dt} = \frac{1}{3} \cdot \frac{dh}{dt}$ and $h = 4$. Write

$$\begin{aligned}
10 &= \frac{\pi}{3}\left(2 \cdot \frac{4}{3} \cdot 4 \cdot \frac{1}{3} \cdot \frac{dh}{dt} + \left(\frac{4}{3}\right)^2 \frac{dh}{dt}\right) \\
10 &= \frac{\pi}{3}\left(\frac{32}{9}\frac{dh}{dt} + \frac{16}{9}\frac{dh}{dt}\right) \\
10 &= \frac{16\pi}{9}\frac{dh}{dt} \\
\frac{90}{16\pi} &= \frac{dh}{dt}.
\end{aligned}$$

Thus, $\frac{dh}{dt} = \frac{90}{16\pi}$ cm/sec.

■

15.2 Position, Velocity, and Acceleration

If you have a function that gives you the position of an object (usually a "particle") at a specified time, then the derivative of that function with respect to time is the velocity of the object, and the second derivative is the acceleration. These are usually represented by the following: [2]

$p(t) =$ position with respect to time.
$v(t) = p'(t) =$ velocity with respect to time.
$s(t) = |v(t)| =$ speed, the absolute value of velocity.
$a(t) = v'(t) =$ acceleration with respect to time.

Let's see an example.

■ **Example 15.5** The Mostar bridge in Bosnia is 25 meters above the river Neretva. For fun, you decided to dive off the bridge. Your position t seconds after jumping off is

$$p(t) = -4.9t^2 + 25.$$

When do you hit the water? What is your instantaneous velocity as you enter the water? What is your average velocity during your dive? [1]

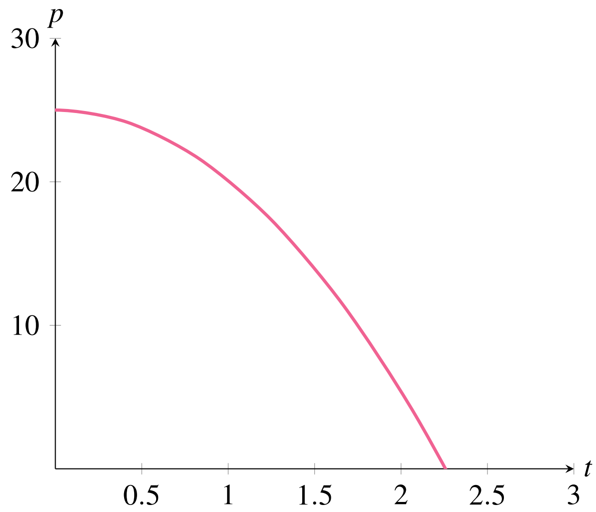


Figure 15.1: Here we see a plot of $p(t) = -4.9t^2 + 25$. Note, time is on the t-axis and vertical height is on the p-axis. [1]

Solution 15.5 To find when you hit the water, you must solve

$$-4.9t^2 + 25 = 0$$

Write

$$\begin{aligned} -4.9t^2 &= -25 \\ t^2 &\approx 5.1 \\ t &\approx 2.26. \end{aligned}$$

Hence after approximately 2.26 seconds, you gracefully enter the river.

Your instantaneous velocity is given by $p'(t)$. Write

$$p'(t) = -9.8t,$$

so your instantaneous velocity when you enter the water is approximately $-9.8 \cdot 2.26 \approx -22$ meters per second.

Finally, your average velocity during your dive is given by

$$\frac{p(2.26) - p(0)}{2.26} \approx \frac{0 - 25}{2.26} = -11.06 \text{ meters per second.}$$

■

Exercises For Chapter 15

1. Sand is poured onto a surface at 15 cm^3/sec, forming a conical pile whose base diameter is always equal to its altitude. How fast is the altitude of the pile increasing when the pile is 3 cm high? [1]
2. A balloon is at a height of 50 meters, and is rising at the constant rate of 5 m/sec. A bicyclist passes beneath it, traveling in a straight line at the constant speed of 10 m/sec. How fast is the distance between the bicyclist and the balloon increasing 2 seconds later? [1]
3. A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.6 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 5 m from the wall? [1]
4. A woman 5 ft tall walks at the rate of 3.5 ft/sec away from a streetlight that is 12 ft above the ground. At what rate is the tip of her shadow moving? At what rate is her shadow lengthening? [1]
5. The position of a particle in meters is given by $1/t^3$ where is t is measured in seconds. What is the acceleration of the particle after 4 seconds? [1]
6. On the Earth, the position of a ball dropped from a height of 100 meters is given by

$$-4.9t^2 + 100, \qquad \text{(ignoring air resistance)}$$

where time is in seconds. On the Moon, the position of a ball dropped from a height of 100 meters is given by

$$-0.8t^2 + 100,$$

where time is in seconds. How long does it take the ball to hit the ground on the Earth? What is the speed immediately before it hits the ground? How long does it take the ball to hit the ground on the Moon? What is the speed immediately before it hits the ground? [1]

16. Optimization

16.1 Applied Maxima and Mimina Problems

Many important applied problems involve finding the best way to accomplish some task. Often this involves finding the maximum or minimum value of some function: The minimum time to make a certain journey, the minimum cost for doing a task, the maximum power that can be generated by a device, and so on. These problems can be solved by finding the **absolute maxima** and **absolute minima** of a function as we had previously discussed.

Proposition 16.1.1 — General Strategies for Optimization Problems.

1. Identify all given quantities and all quantities to be determined. If possible, make a sketch and label it with any relevant measurements.
2. Write a primary equation for the quantity that is to be maximized or minimized.
3. Reduce the primary equation to one having a single independent variable. This may involve the use of secondary equations relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques.

Since you are aware of the concept, let's get straight into the examples.

■ **Example 16.1** Of all rectangles of area 100 cm^2, which has the smallest perimeter? [1]

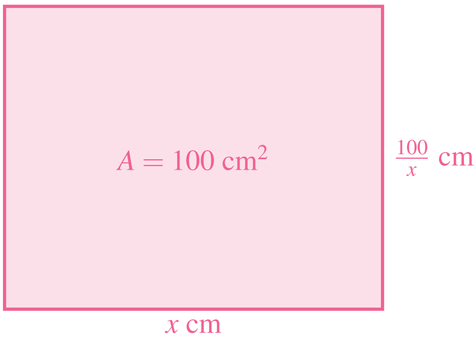


Figure 16.1: A rectangle with an area of 100 cm^2.

Solution 16.1 First we draw a picture, see Figure 16.1. If x denotes one of the sides of the rectangle, then the adjacent side must be $100/x$.

The perimeter of this rectangle is given by

$$p(x) = 2x + 2\frac{100}{x}.$$

We wish to minimize $p(x)$. Note, not all values of x make sense in this problem: lengths of sides of rectangles must be positive, so $x > 0$. If $x > 0$ then so is $100/x$, so we need no second condition on x.

We next find $p'(x)$ and set it equal to zero. Write

$$p'(x) = 2 - 200/x^2 = 0.$$

Solving for x gives us $x = \pm 10$. We are interested only in $x > 0$, so only the value $x = 10$ is of interest. Since $p'(x)$ is defined everywhere on the interval $(0, \infty)$, there are no more critical values, and there are no endpoints. Is there a local maximum, minimum, or neither at $x = 10$? The second derivative is $p''(x) = 400/x^3$, and $f''(10) > 0$, so there is a local minimum. Since there is only one critical value, this is also the global minimum, so the rectangle with smallest perimeter is the 10 cm$\times$10 cm square.

■

■ **Example 16.2** You want to sell a certain number n of items in order to maximize your profit. Market research tells you that if you set the price at \$1.50, you will be able to sell 5000 items, and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Suppose that your fixed costs ("start-up costs") total \$2000, and the per item cost of production ("marginal cost") is \$0.50. Find the price to set per item and the number of items sold in order to maximize profit, and also determine the maximum profit you can get. [1]

Solution 16.2 The first step is to convert the problem into a function maximization problem. The revenue for selling n items at x dollars is given by

$$r(x) = nx$$

and the cost of producing n items is given by

$$c(x) = 2000 + 0.5n.$$

However, from the problem we see that the number of items sold is itself a function of x,

$$n(x) = 5000 + 1000(1.5 - x)/0.10$$

So profit is give by:

$$\begin{aligned}
P(x) &= r(x) - c(x) \\
&= nx - (2000 + 0.5n) \\
&= (5000 + 1000(1.5 - x)/0.10)x - 2000 - 0.5(5000 + 1000(1.5 - x)/0.10) \\
&= -10000x^2 + 25000x - 12000.
\end{aligned}$$

We want to know the maximum value of this function when x is between 0 and 1.5. The derivative is $P'(x) = -20000x + 25000$, which is zero when $x = 1.25$. Since $P''(x) = -20000 < 0$, there must be a local maximum at $x = 1.25$, and since this is the only critical value it must be a global maximum as well. Alternately, we could compute $P(0) = -12000$, $P(1.25) = 3625$, and $P(1.5) = 3000$ and note that $P(1.25)$ is the maximum of these. Thus the maximum profit is \$3625, attained when we set the price at \$1.25 and sell 7500 items.

■

■ **Example 16.3** Find the rectangle with largest area that fits inside the graph of the parabola $y = x^2$ below the line $y = a$, where a is an unspecified constant value, with the top side of the rectangle on the horizontal line $y = a$. See Figure 16.2. [1]

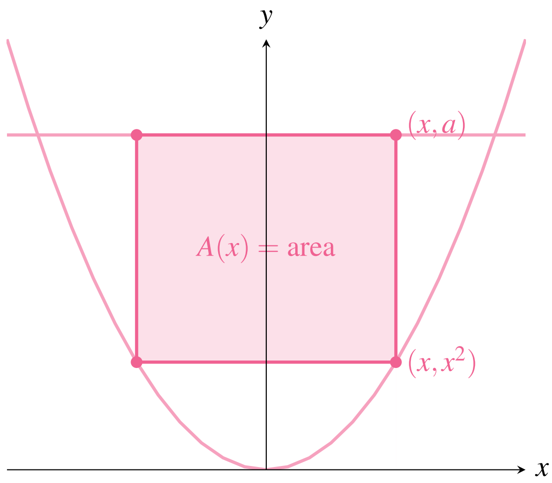


Figure 16.2: A plot of the parabola $y = x^2$ along with the line $y = a$ and the rectangle in question.

Solution 16.3 We want to maximize value of $A(x)$. The lower right corner of the rectangle is at (x, x^2), and once this is chosen the rectangle is completely determined. Then the area is

$$A(x) = (2x)(a - x^2) = -2x^3 + 2ax.$$

We want the maximum value of $A(x)$ when x is in $[0, \sqrt{a}]$. You might object to allowing $x = 0$ or $x = \sqrt{a}$, since then the "rectangle" has either no width or no height, so is not "really" a rectangle. But the problem is somewhat easier if we simply allow such rectangles, which have zero area as we may then apply the Extreme Value Theorem, Theorem 13.1.1.

Setting $0 = A'(x) = -6x^2 + 2a$ we find $x = \sqrt{a/3}$ as the only critical point. Testing this and the two endpoints, we have $A(0) = A(\sqrt{a}) = 0$ and $A(\sqrt{a/3}) = (4/9)\sqrt{3}a^{3/2}$. Hence, the maximum area thus occurs when the rectangle has dimensions $2\sqrt{a/3} \times (2/3)a$.

■

■ **Example 16.4** If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by "cone" we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.) [1]

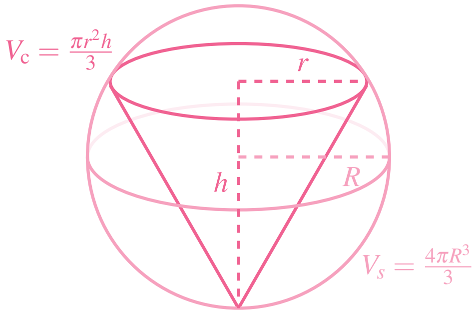


Figure 16.3: A cone inside a sphere.

Solution 16.4 Let R be the radius of the sphere, and let r and h be the base radius and height of the cone inside the sphere. Our goal is to maximize the volume of the cone: $V_c = \pi r^2 h/3$. The largest r could be is R and the largest h could be is $2R$.

Notice that the function we want to maximize, $\pi r^2 h/3$, depends on *two* variables. Our next step is to find the relationship and use it to solve for one of the variables in terms of the other, so as to have a function of only one variable to maximize. In this problem, the condition is apparent in the figure, as the upper corner of the triangle, whose coordinates are $(r, h-R)$, must be on the circle of radius R. Write

$$r^2 + (h-R)^2 = R^2.$$

Solving for r^2, since r^2 is found in the formula for the volume of the cone, we find

$$r^2 = R^2 - (h-R)^2.$$

Substitute this into the formula for the volume of the cone to find

$$\begin{aligned} V_c(h) &= \pi(R^2 - (h-R)^2)h/3 \\ &= -\frac{\pi}{3}h^3 + \frac{2}{3}\pi h^2 R \end{aligned}$$

We want to maximize $V_c(h)$ when h is between 0 and $2R$. We solve

$$V_c'(h) = -\pi h^2 + (4/3)\pi h R = 0,$$

finding $h = 0$ or $h = 4R/3$. We compute

$$V_c(0) = V_c(2R) = 0 \qquad \text{and} \qquad V_c(4R/3) = (32/81)\pi R^3.$$

The maximum is the latter. Since the volume of the sphere is $(4/3)\pi R^3$, the fraction of the sphere occupied by the cone is

$$\frac{(32/81)\pi R^3}{(4/3)\pi R^3} = \frac{8}{27} \approx 30\%.$$

■

■ **Example 16.5** You are making cylindrical containers to contain a given volume. Suppose that the top and bottom are made of a material that is N times as expensive (cost per unit area) as the material used for the lateral side of the cylinder. Find (in terms of N) the ratio of height to base radius of the cylinder that minimizes the cost of making the containers. [1]

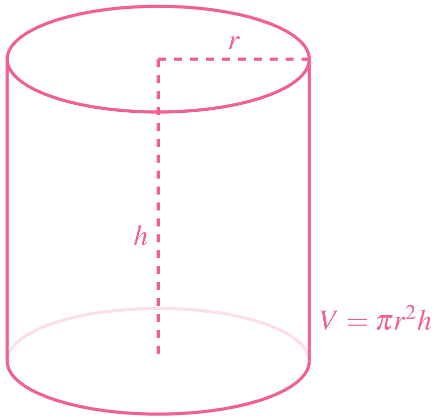


Figure 16.4: A cylinder with radius r, height h, volume V, c for the cost per unit area of the lateral side of the cylinder.

Solution 16.5 First we draw a picture, see Figure 16.4. Now we can write an expression for the cost of materials:

$$C = 2\pi crh + 2\pi r^2 Nc.$$

Since we know that $V = \pi r^2 h$, we can use this relationship to eliminate h (we could eliminate r, but it's a little easier if we eliminate h, which appears in only one place in the above formula for cost). We find

$$\begin{aligned} C(r) &= 2c\pi r \frac{V}{\pi r^2} + 2Nc\pi r^2 \\ &= \frac{2cV}{r} + 2Nc\pi r^2. \end{aligned}$$

We want to know the minimum value of this function when r is in $(0,\infty)$. Setting

$$C'(r) = -2cV/r^2 + 4Nc\pi r = 0$$

we find $r = \sqrt[3]{V/(2N\pi)}$. Since $C''(r) = 4cV/r^3 + 4Nc\pi$ is positive when r is positive, there is a local minimum at the critical value, and hence a global minimum since there is only one critical value.

Finally, since $h = V/(\pi r^2)$,

$$\begin{aligned} \frac{h}{r} &= \frac{V}{\pi r^3} \\ &= \frac{V}{\pi(V/(2N\pi))} \\ &= 2N, \end{aligned}$$

so the minimum cost occurs when the height h is $2N$ times the radius. If, for example, there is no difference in the cost of materials, the height is twice the radius.

■

■ **Example 16.6** Suppose you want to reach a point A that is located across the sand from a nearby road, see Figure 16.5. Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A? [1]

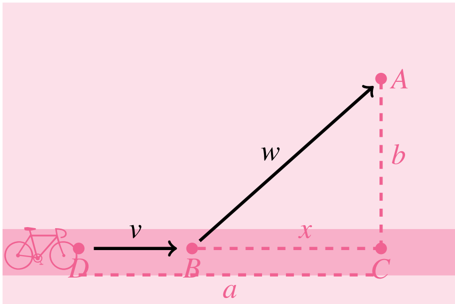


Figure 16.5: A road where one travels at rate v, with sand where one travels at rate w. Where should one turn off of the road to minimize total travel time from D to A?

Solution 16.6 Let x be the distance short of C where you turn off, the distance from B to C. We want to minimize the total travel time. Recall that when traveling at constant velocity, time is distance divided by velocity.

You travel the distance from D to B at speed v, and then the distance from B to A at speed w. The distance from D to B is $a-x$. By the Pythagorean theorem, the distance from B to A is

$$\sqrt{x^2+b^2}.$$

Hence the total time for the trip is

$$T(x) = \frac{a-x}{v} + \frac{\sqrt{x^2+b^2}}{w}.$$

We want to find the minimum value of T when x is between 0 and a. As usual we set $T'(x) = 0$ and solve for x. Write

$$T'(x) = -\frac{1}{v} + \frac{x}{w\sqrt{x^2+b^2}} = 0.$$

We find that

$$x = \frac{wb}{\sqrt{v^2-w^2}}$$

Notice that a does not appear in the last expression, but a is not irrelevant, since we are interested only in critical values that are in $[0,a]$, and $wb/\sqrt{v^2-w^2}$ is either in this interval or not. If it is, we can use the second derivative to test it:

$$T''(x) = \frac{b^2}{(x^2+b^2)^{3/2}w}.$$

Since this is always positive there is a local minimum at the critical point, and so it is a global minimum as well.

If the critical value is not in $[0,a]$ it is larger than a. In this case the minimum must occur at one of the endpoints. We can compute

$$T(0) = \frac{a}{v} + \frac{b}{w}$$
$$T(a) = \frac{\sqrt{a^2+b^2}}{w}$$

but it is difficult to determine which of these is smaller by direct comparison. If, as is likely in practice, we know the values of v, w, a, and b, then it is easy to determine this. With a little cleverness, however, we can determine the minimum in general. We have seen that $T''(x)$ is always positive, so the derivative $T'(x)$ is always increasing. We know that at $wb/\sqrt{v^2-w^2}$ the derivative is zero, so for values of x less than that critical value, the derivative is negative. This means that $T(0) > T(a)$, so the minimum occurs when $x = a$.

So the upshot is this: If you start farther away from C than $wb/\sqrt{v^2-w^2}$ then you always want to cut across the sand when you are a distance $wb/\sqrt{v^2-w^2}$ from point C. If you start closer than this to C, you should cut directly across the sand.

■

Exercises For Chapter 16

1. Find the dimensions of the rectangle of largest area having fixed perimeter 100. [1]
2. A box with square base and no top is to hold a volume 100. Find the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base. [1]
3. A box with square base and no top is to hold a volume V. Find (in terms of V) the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base. (This ratio will not involve V.) [1]
4. Marketing tells you that if you set the price of an item at \$10 then you will be unable to sell it, but that you can sell 500 items for each dollar below \$10 that you set the price. Suppose your fixed costs total \$3000, and your marginal cost is \$2 per item. What is the most profit you can make? [1]
5. In Example 16.6, what happens if $w \geq v$? [1]
6. For a cylinder with surface area 50, including the top and the bottom, find the ratio of height to base radius that maximizes the volume. [1]
7. Given a right circular cone, you put an upside-down cone inside it so that its vertex is at the center of the base of the larger cone and its base is parallel to the base of the larger cone. If you choose the upside-down cone to have the largest possible volume, what fraction of the volume of the larger cone does it occupy? (Let H and R be the height and base radius of the larger cone, and let h and r be the height and base radius of the smaller cone. Hint: Use similar triangles to get an equation relating h and r.) [1]
8. Two electrical charges, one a positive charge A of magnitude a and the other a negative charge B of magnitude b, are located a distance c apart. A positively charged particle P is situated on the line between A and B. Find where P should be put so that the pull away from A towards B is minimal. Here assume that the force from each charge is proportional to the strength of the source and inversely proportional to the square of the distance from the source. [1]
9. If you fit the cone with the largest possible surface area (lateral area plus area of base) into a sphere, what percent of the volume of the sphere is occupied by the cone? [1]

■

Answer Key

Chapter 1

1. domain; range; function
2. Verbally; Numerically; Graphically; Analytically
3. independent; dependent
4. piecewise
5. (a) -1 (b) -9 (c) $2x-5$
6. (a) -7 (b) 4 (c) 9
7. (a) 19 (b) 17 (c) 0

Chapter 2

1. (a) ± 6 (b) Odd multiplicity; number of turning points: 1
2. (a) $0, 2\pm\sqrt{3}$ (b) Odd multiplicity; number of turning points: 2
3. (a) $\pm 2, -3$ (b) Odd multiplicity; number of turning points: 2
4. (a) $0, \pm\sqrt{3}$ (b) 0, odd multiplicity; $\pm\sqrt{3}$, even multiplicity; number of turning points: 4
5. x^2-8x
6. $x^4-4x^3-9x^2+36x$
7. x^2-2x-2
8. x^3+9x^2+20x
9. $x=-2$; $y=0$
10. $x=2, x=\pm 1$; $y=0$
11. $x=2$; $y=1$
12. None; None

Chapter 3

1. (a) x^2+4x-5 (b) x^2-4x+5 (c) $4x^3-5x^2$ (d) $\dfrac{x^2}{4x-5}$
2. (a) $\dfrac{x+1}{x^2}$ (b) $\dfrac{x-1}{x^2}$ (c) $\dfrac{1}{x^3}$
3. 3
4. 74
5. $\dfrac{3}{5}$

Chapter 4

1. $f(g(x)) = f(x/2) = 2(x/2) = x$; $g(f(x)) = g(2x) = (2x)/2 = x$
2. $f(g(x)) = f(1/x) = 1/(1/x) = x$; $g(f(x)) = g(1/x) = 1/(1/x) = x$
3. $g^{-1}(x) = 8x$
4. No inverse
5. $f^{-1}(x) = \dfrac{x^2-3}{2}$
6. No inverse

Chapter 5

1. Let $\varepsilon > 0$. Set $\delta = \varepsilon$. If $0 < |x-0| < \delta$, then $|x \cdot 1| < \varepsilon$, since $\sin\left(\frac{1}{x}\right) \leq 1$, $\left|x\sin\left(\frac{1}{x}\right) - 0\right| < \varepsilon$.
2. Let $\varepsilon > 0$. No matter what I choose for δ, if x is within δ of -2, then π is within ε of π.
3. Let $\varepsilon > 0$. Set $\delta = 3\varepsilon$. Assume $0 < |x-9| < \delta$. Divide both sides by 3 to get $\frac{|x-9|}{3} < \varepsilon$. Note that $\sqrt{x}+3 > 3$, so $\frac{|x-9|}{\sqrt{x}+3} < \varepsilon$. This can be rearranged to conclude $\left|\frac{x-9}{\sqrt{x}-3} - 6\right| < \varepsilon$.
4. Consider what happens when x is near zero and positive, as compared to when x is near zero and negative.

Chapter 6

1. 8
2. 1/5
3. -1/9
4. 1/6
5. $3x^2$
6. 4
7. b

Chapter 7

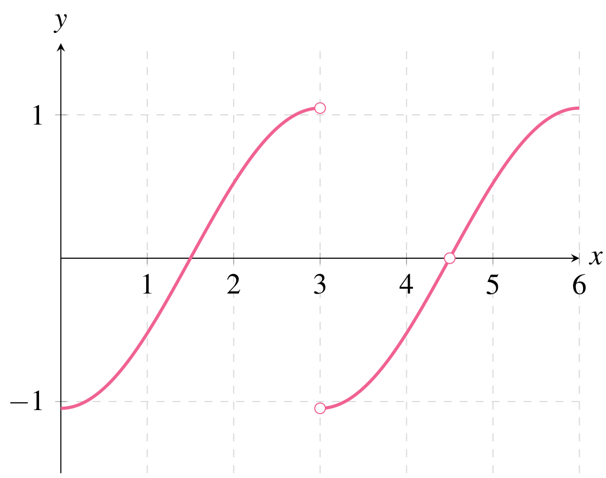
1. Continuous for all real x
2. Nonremovable discontinuity at $x = 1$
   Removable discontinuity at $x = 0$
3. Removable discontinuity at $x = -2$
   Nonremovable discontinuity at $x = 5$
4. Nonremovable discontinuity at $x = -7$
5. $a = 7$
6. $a = -1, b = 1$

Chapter 8

1. $x = 1$ and $x = -3$
2. $x = -4$
3. $y = 0$
4. $y = 17$
5. After 10 years, $\approx$ 174 cats; after 50 years, $\approx$ 199 cats; after 100 years, $\approx$ 200 cats; after 1000 years, $\approx$ 200 cats; in the sense that the population of cats cannot grow indefinitely this is somewhat realistic.
6. The amplitude goes to zero.

Chapter 9

1. $f(2) = 10$ and $f'(2) = 7$
2. $f'(-2) = 4$
3. $f(1.2) \approx 2.2$
4. $(0, 4.5) \cup (4.5, 6)$
5. $(0, 3) \cup (3, 4.5) \cup (4.5, 6)$
6. See figure on the right.



Answer to Question 6. A sketch of $f'(x)$. [1]

Chapter 10

1. 0
2. 0
3. $\pi x^{\pi - 1}$
4. $-(9/7)x^{-16/7}$
5. $15x^2 + 24x$
6. $-5x^{-6} - x^{-3/2}/2$
7. $3x^2(x^3 - 5x + 10) + x^3(3x^2 - 5)$
8. $(x^2 + 5x - 3)(5x^4 - 18x^2 + 6x - 7) + (2x + 5)(x^5 - 6x^3 + 3x^2 - 7x + 1)$
9. $\frac{3x^2}{x^3-5x+10} - \frac{x^3(3x^2-5)}{(x^3-5x+10)^2}$
10. $\frac{2x+5}{x^5-6x^3+3x^2-7x+1} - \frac{(x^2+5x-3)(5x^4-18x^2+6x-7)}{(x^5-6x^3+3x^2-7x+1)^2}$
11. $6 + 18x$

12. $\frac{1}{2}\left(\frac{-169}{x^2}-1\right)\Big/\sqrt{\frac{169}{x}-x}$
13. $\frac{300x}{(100-x^2)^{5/2}}$
14. $\left(4x(x^2+1)+\frac{4x^3+4x}{2\sqrt{1+(x^2+1)^2}}\right)\Big/ 2\sqrt{(x^2+1)^2+\sqrt{1+(x^2+1)^2}}$
15. $6x(2x-4)^3+6(3x^2+1)(2x-4)^2$
16. $-5/(3x-4)^2$
17. $56x^6+72x^5+110x^4+100x^3+60x^2+28x+6$
18. $4x^3-9x^2+x+7$

Chapter 11

1. $-x/y$
2. $-(2x+y)/(x+2y)$
3. $(2xy-3x^2-y^2)/(2xy-3y^2-x^2)$
4. $-\sqrt{y}/\sqrt{x}$
5. $\frac{y^{3/2}-2}{1-y^{1/2}3x/2}$
6. $-y^2/x^2$
7. $-4/y^3$
8. $(3x)/(4y)$

Chapter 12

1. $c=\dfrac{6\pm\sqrt{3}}{3}$
2. Not differentiable at $x=0$
3. $c=-2+\sqrt{5}$
4. $c1/2$
5. $c=\pm 1/\sqrt{3}$
6. f is not differentiable at $x=-\dfrac{1}{2}$

Chapter 13

1. min at $x=1/2$
2. min at $x=-1$, max at $x=1$
3. max at $x=2$, min at $x=4$
4. min at $x=\pm 1$, max at $x=0$.
5. min at $x=0$, max at $x=\frac{3\pm\sqrt{17}}{2}$
6. none
7. relative min at $x=0$

Chapter 14

1. min at $x=1/2$
2. min at $x=-1$, max at $x=1$
3. max at $x=2$, min at $x=4$
4. max at $x=0$, min at $x=\pm 11$
5. $f'(x)=2ax+b$, this has only one root and hence one critical point; $a<0$ to guarantee a maximum.
6. y-intercept at $(0,0)$; no vertical asymptotes; critical points: $x=\pm 1/\sqrt[4]{5}$; local max at $x=-1/\sqrt[4]{5}$, local min at $x=-1/\sqrt[4]{5}$; increasing on $(-\infty,-1/\sqrt[4]{5})$, decreasing on $(-1/\sqrt[4]{5},1/\sqrt[4]{5})$, increasing on $(1/\sqrt[4]{5},\infty)$; concave down on $(-\infty,0)$, concave up on $(0,\infty)$; root at $x=0$; no horizontal asymptotes; interval for sketch: $[-1.2,1.2]$ (answers may vary)
7. y-intercept at $(0,0)$; no vertical asymptotes; critical points: $x=1$; local max at $x=1$; increasing on $[0,1)$, decreasing on $(1,\infty)$; concave down on $[0,\infty)$; roots at $x=0$, $x=4$; no horizontal asymptotes; interval for sketch: $[0,6]$ (answers may vary)
8. y-intercept at $(0,0)$; no vertical asymptotes; critical points: $x=0$, $x=1$, $x=3$; local max at $x=1$, local min at $x=3$; increasing on $(-\infty,0)$ and $(0,1)$, decreasing on $(1,3)$, increasing on $(3,\infty)$; concave down on $(-\infty,0)$, concave up on $(0,(3-\sqrt{3})/2)$, concave down on $((3-\sqrt{3})/2,(3+\sqrt{3})/2)$, concave up on $((3+\sqrt{3})/2,\infty)$; roots at $x=0$, $x=\frac{5\pm\sqrt{5}}{2}$; no horizontal asymptotes; interval for sketch: $[-1,4]$ (answers may vary)
9. no y-intercept; vertical asymptote at $x=0$;

critical points: $x=0, x=\frac{1}{\sqrt[3]{2}}$; local min at $x=\frac{1}{\sqrt[3]{2}}$; decreasing on $(-\infty,0)$, decreasing on $(0,\frac{1}{\sqrt[3]{2}})$, increasing on $(\frac{1}{\sqrt[3]{2}},\infty)$; concave up on $(-\infty,-1)$, concave down on $(-1,0)$, concave up on $(0,\infty)$; root at $x=-1$; no horizontal asymptotes; interval for sketch: $[-3,2]$ (answers may vary)

Chapter 15

1. $20/(3\pi)$ cm/s
2. $5\sqrt{10}/2$ m/s
3. $1/4$ m/s
4. tip: 6 ft/s, length: $5/2$ ft/s
5. $3/256$ m/s^2
6. $-4.9t^2+100$, (ignoring air resistance)

Chapter 16

1. 25×25
2. $w=l=2\cdot 5^{2/3}$, $h=5^{2/3}$, $h/w=1/2$
3. $w=l=2^{1/3}V^{1/3}$, $h=V^{1/3}/2^{2/3}$, $h/w=1/2$
4. \$5000
5. Go direct from A to D.
6. $h/r=2$
7. $4/27$
8. P should be at distance $c\sqrt[3]{a}/(\sqrt[3]{a}+\sqrt[3]{b})$ from charge A.
9. The ratio of the volume of the sphere to the volume of the cone is $1033/4096+33/4096\sqrt{17}\approx 0.2854$, so the cone occupies approximately 28.54% of the sphere.

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