First edition, March 2019

1.1 Introduction to Functions

Many everyday phenomena involve two quantities that are related to each other by some rule of correspondence. The mathematical term for such a rule of correspondence is a **relation**. In mathematics, relations are often represented by mathematical equations and formulas. For instance, the simple interest I earned on \$1000 for 1 year is related to the annual interest rate r by the formula $I = 1000r$.

The formula $I = 1000r$ represents a special kind of relation that matches each item from one set with *exactly one* item from a different set. Such a relation is called a **function**.

Definition 1.1.1 — Function. A **function** f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the **domain** (or set of inputs) of the function f, and the set B contains the **range** (or set of outputs).

Functions are commonlt represented in four ways.

Proposition 1.1.1 — Four ways to represent a function.

1. *Verbally* by a sentence that describes how the input variable is related to the output variable
2. *Numerically* by a table or a list of ordered pairs that matches input values with output values
3. *Graphically* by points on a graph in a coordinate plane in which the input values are represented by the horizontal axis and the output values are represented by the vertical axis
4. *Analytically* by an equation in two variables

To determine whether or not a relation is a function, you must decide whether each input value is matched with exactly one output value. When any input value is matched with two or more output values, the relation is not a function.

■ **Example 1.1 — Testing for Functions.**
Determine whether the relation represents y as a function of x.

1. The input value x is the number of representatives from a state, and the output value y is the number of senators.
2. Function f is defined as the following table

Input, x	**Output,** y
2	11
2	10
3	8
4	5
5	1

Solution 1.1

1. This verbal description *does* describe y as a function of x. Regardless of the value of x, the value of y is always 2. Such functions are called *constant functions*.

2. This table *does not* describe y as a function of x. The input value 2 is matched with two different y-values.

■

1.2 Function Notation

When an equation is used to represent a function, it is convenient to name the function so that it can be referenced easily. For example, you know that the equation $y = 3x+4$ describes y as a function of x. Suppose you give this function the name "f." Then you can use the following **function notation**.

Input	**Output**	**Equation**
x	$f(x)$	$f(x) = 3x+4$

The symbol $f(x)$ is read as the value of f at x or simply f of x. The symbol $f(x)$ corresponds to the y-value for a given x. So, you can write $y = f(x)$. Keep in mind that f is the name of the function, whereas $f(x)$ is the value of the function at x. For instance, the function given by

$$f(x) = 3x+4 \tag{1.1}$$

has function values denoted by $f(0)$, $f(1)$, $f(2)$, and so on. To find these values, substitute the specified input values into the given equation.

$$\begin{aligned} f(-1) &= 3(-1)+4 = -3+4 = 1 && x = -1 \\ f(0) &= 3(0)+4 = 0+4 = 4 && x = 0 \\ f(2) &= 3(2)+4 = 6+4 = 10 && x = 2 \end{aligned}$$

Although f is often used as a convenient function name and x is often used as the independent variable, you can use other letters. For instance,

$$f(x) = 3x+4, \quad f(t) = 3t+4, \quad g(s) = 3s+4 \tag{1.2}$$

all define the same function. In fact, the role of the independent variable is that of a "placeholder" that can be replaced by *any real number or algebraic expression*.

■ **Example 1.2 — Evaluating a Function.**
Let $g(x) = -7x + 20$. Find each function value

1. $g(2)$
2. $g(t)$
3. $g(x+2)$

Solution 1.2

1. Replacing x with 2 in $g(x) = -7x + 20$ yields the following.

$$\begin{aligned} g(2) &= -7(2) + 20 && (1.3)\\ &= -14 + 20 && (1.4)\\ &= 6 && (1.5) \end{aligned}$$

2. Replacing x with t yields the following.

$$\begin{aligned} g(t) &= -7(t) + 20 && (1.6)\\ &= -7t + 20 && (1.7) \end{aligned}$$

3. Replacing x with $x+2$ yields the following.

$$\begin{aligned} g(x+2) &= -7(x+2) + 20 && (1.8)\\ &= -7x - 14 + 20 && (1.9)\\ &= 7x + 20 && (1.10) \end{aligned}$$

■

1.3 Piecewise Function

A function defined by two or more equations over a specified domain is called a **piecewise-defined function**.

■ **Example 1.3 — A Piecewise-Defined Function.**
Evaluate the function when $x = -1$, 0, and 1.

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases} \qquad (1.11)$$

Solution 1.3 Because $x = -1$ is less than 0, use $f(x) = x^2 + 1$ to obtain

$$f(-1) = (-1)^2 + 1 = 2 \qquad (1.12)$$

For $x = 0$, use $f(x) = x - 1$ to obtain

$$f(0) = (0) - 1 = -1 \qquad (1.13)$$

For $x = 1$, use $f(x) = x - 1$ to obtain

$$f(1) = (1) - 1 = 0 \qquad (1.14)$$

■

2.1 Introduction

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5 Linear and Piecewise Function

5.1 Introduction

5.1 Introduction

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6 Linear and Piecewise Functior

6.1 Introduction

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7 Linear and Piecewise Function

7.1 Introduction

7.1 Introduction

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8.1 good

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8.2 Citation

This statement requires citation [**book_key**]; this one is more specific [**article_key**].

8.3 Lists

Lists are useful to present information in a concise and/or ordered way[1].

8.3.1 Numbered List

1. The first item
2. The second item
3. The third item

8.3.2 Bullet Points

- The first item
- The second item

[1] Footnote example...

- The third item

8.3.3 Descriptions and Definitions

Name Description
Word Definition
Comment Elaboration

9.1 Theorems

This is an example of theorems.

9.1.1 Several equations

This is a theorem consisting of several equations.

Theorem 9.1.1 — Name of the theorem. In $E = \mathbb{R}^n$ all norms are equivalent. It has the properties:

$$\big| ||\mathbf{x}|| - ||\mathbf{y}|| \big| \leq ||\mathbf{x} - \mathbf{y}|| \tag{9.1}$$

$$|| \sum_{i=1}^{n} \mathbf{x}_i || \leq \sum_{i=1}^{n} ||\mathbf{x}_i|| \quad \text{where } n \text{ is a finite integer} \tag{9.2}$$

9.1.2 Single Line

This is a theorem consisting of just one line.

Theorem 9.1.2 A set $\mathscr{D}(G)$ in dense in $L^2(G)$, $|\cdot|_0$.

9.2 Definitions

This is an example of a definition. A definition could be mathematical or it could define a concept.

Definition 9.2.1 — Definition name. Given a vector space E, a norm on E is an application, denoted $||\cdot||$, E in $\mathbb{R}^+ = [0,+\infty[$ such that:

$$||\mathbf{x}|| = 0 \Rightarrow \mathbf{x} = \mathbf{0} \tag{9.3}$$
$$||\lambda \mathbf{x}|| = |\lambda| \cdot ||\mathbf{x}|| \tag{9.4}$$
$$||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}|| \tag{9.5}$$

9.3 Notations

Notation 9.1. *Given an open subset G of $\mathbb{R}^n$, the set of functions φ are:*

1. *Bounded support G;*
2. *Infinitely differentiable;*

a vector space is denoted by $\mathscr{D}(G)$.

9.4 Remarks

This is an example of a remark.

R The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field $\mathbb{K} = \mathbb{R}$, however, established properties are easily extended to $\mathbb{K} = \mathbb{C}$.

9.5 Corollaries

This is an example of a corollary.

Corollary 9.5.1 — Corollary name. The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field $\mathbb{K} = \mathbb{R}$, however, established properties are easily extended to $\mathbb{K} = \mathbb{C}$.

9.6 Propositions

This is an example of propositions.

9.6.1 Several equations

Proposition 9.6.1 — Proposition name. It has the properties:

$$\big| ||\mathbf{x}|| - ||\mathbf{y}|| \big| \leq ||\mathbf{x}-\mathbf{y}|| \tag{9.6}$$
$$||\sum_{i=1}^{n} \mathbf{x}_i|| \leq \sum_{i=1}^{n} ||\mathbf{x}_i|| \quad \text{where } n \text{ is a finite integer} \tag{9.7}$$

9.6.2 Single Line

Proposition 9.6.2 Let $f,g \in L^2(G)$; if $\forall \varphi \in \mathscr{D}(G)$, $(f,\varphi)_0 = (g,\varphi)_0$ then $f = g$.

9.7 Examples

This is an example of examples.

9.7.1 Equation and Text

■ **Example 9.1** Let $G = \{x \in \mathbb{R}^2 : |x| < 3\}$ and denoted by: $x^0 = (1,1)$; consider the function:

$$f(x) = \begin{cases} e^{|x|} & \text{si } |x - x^0| \leq 1/2 \\ 0 & \text{si } |x - x^0| > 1/2 \end{cases} \tag{9.8}$$

The function f has bounded support, we can take $A = \{x \in \mathbb{R}^2 : |x - x^0| \leq 1/2 + \varepsilon\}$ for all $\varepsilon \in]0; 5/2 - \sqrt{2}[$. ■

9.7.2 Paragraph of Text

■ **Example 9.2 — Example name.** Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris. ■

9.8 Exercises

This is an example of an exercise.

Exercises For Chapter 1 This is a good place to ask a question to test learning progress or further cement ideas into students' minds. ■

9.9 Problems

Problem 9.1 What is the average airspeed velocity of an unladen swallow?

9.10 Vocabulary

Define a word to improve a students' vocabulary.
Vocabulary 9.1 — Word. Definition of word.

10.1 Table

Treatments	**Response 1**	**Response 2**
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table 10.1: Table caption

10.2 Figure

Figure 10.1: Figure caption

Books

Articles