Q1. **(a)** **(i)** State the axioms of a probability measure.

(2 marks)

(ii) Use the axioms in Q1(a)(i) to show that probability measures are monotonic i.e. if $A \subseteq B$ then $P(A) \leq P(B)$.

(3 marks)

(b) Suppose that we consider three hypotheses, H_1, H_2 and H_3, as possible explanations for evidence E. Furthermore, suppose that we have the following likelihoods and prior probabilities:

$$P(E|H_1) = 0.8,\ P(E|H_2) = 0.6,\ P(E|H_3) = 0.5$$

and

$$P(H_1) = 0.2,\ P(H_2) = 0.7,\ P(H_3) = 0.1$$

(i) Use Bayes theorem to identify which hypothesis is the most probable given E.

(4 marks)

(ii) We now learn that $P(E|H_1^c) = 0.3$. Use this new information together with the above likelihood and prior to evaluate $P(H_1|E)$.

(3 marks)

(c) A rational agent is asked to choose between the following two bets on proposition A, with stake $S > 0$ and odds $p \in [0, 1]$:

Bet 1: Gain $S(1 - p)$ if A is true and lose Sp if A is false.

Bet 2: Lose $S(1 - p)$ if A is true and gain Sp if A is false.

(i) Show that the gain from Bet 1 and Bet 2 can be written as $S(\chi_A(w^*) - p)$ and $-S(\chi_A(w^*) - p)$ respectively, where χ_A is the membership(characteristic) function for A and w^* is the true state of the world.

(4 marks)

(ii) If when offered odds p an agent chooses Bet 1, use the formulae for gain given in Q1(c)(i) to show that the same agent should also choose Bet 1 for odds p' where $p' \leq p$.

(4 marks)

Q2. (a) The government is considering implementing one of the following three economic policies: $A =$'more austerity', $S =$'more spending' and $C =$'no change to current austerity or spending'. As part of their decision making process the government asks a group of economic experts for their opinion on which is the best policy. The responses were as follows: 20% said that they did not know, 40% said that there should not be more austerity, 30% said that there should be more spending and 10% said that there should not be more spending.

(i) Write down the mass assignment representing these responses.

(4 marks)

(ii) Determine the belief and plausibility that there should not be more austerity.

(2 marks)

(b) (i) Give the definition of the posterior probability $P(A|m)$ for $A \subseteq W$ and m a mass function on 2^W.

(2 marks)

(ii) Prove that for all $A \subseteq W$, $Bel(A) \leq P(A|m) \leq Pl(A)$.

(5 marks)

(c) (i) Consider the following two mass assignments on 2^W where $W = \{w_1, w_2, w_3, w_4\}$:

$$m_1 := \{w_1, w_2\} : 0.4,\ \{w_1, w_3\} : 0.4,\ \{w_3, w_4\} : 0.2$$
$$m_2 := \{w_1, w_2, w_4\} : 0.3,\ \{w_4\} : 0.5,\ \{w_2, w_3\} : 0.2$$

Use Dempster's rule of combination to determine the combined mass assignment $m_1 \oplus m_2$.

(3 marks)

(ii) Let Pl be a plausibility measure defined by a mass assignment m. We then define a conditional plausibility measure such that for $A, B \subseteq W$, $Pl(A|B) = \sum_{C:A\cap C\neq\emptyset} m \oplus m_B(C)$ where $m_B(B) = 1$. Prove that

$$Pl(A|B) = \frac{Pl(A \cap B)}{Pl(B)}$$

(4 marks)

Q3. (a) Consider a wheel of fortune with three outcomes red, green and blue. Suppose we know that $P(red|not\ blue) = 0.6$.

(i) Draw a geometric representation of the set of probability distributions on $W = \{red, blue, green\}$ which are consistent with this knowledge.

(4 marks)

(ii) Use the maximum entropy inference process to obtain a precise value for $P(green)$.

(6 marks)

(b) (i) Prove that for a Bayesian network with variables $X_1, \ldots, X_n$

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i|\Pi(X_i))$$

where $\Pi(X_i)$ are the parents of X_i.

(3 marks)

(ii) Suppose that $X_i : i = 1, \ldots, n$ are binary variables and that for a certain Bayesian network $|\Pi(X_1)| = 0$ and $|\Pi(X_i)| = i - 2$ for $i = 2, \ldots n$. Determine the number of probability values which must be specified for this Bayesian network.

(3 marks)

(iii) Consider the random variables $X_1, \ldots, X_5$ with the following conditional independence relations:

$$P(X_2|X_1) = P(X_2)$$
$$P(X_3|X_1, X_2) = P(X_3|X_1)$$
$$P(X_4|X_1, X_2, X_3) = P(X_4|X_1, X_2)$$
$$P(X_5|X_1, X_2, X_3, X_4) = P(X_5|X_3, X_4)$$

Draw the corresponding Bayesian network.

(4 marks)

Q4. (a) For the following fuzzy sets $\tilde{A}$ and $\tilde{B}$ use the α-cut method to determine $\frac{\tilde{A}}{\tilde{B}}$:

$$\tilde{A} = 5/1 + 6/0.8 + 7/0.5$$
$$\tilde{B} = 1/1 + 2/0.5 + 3/0.2$$

(5 marks)

(b) Suppose we define $\chi_{\tilde{A}\cap\tilde{B}}(w) = \sup\{\alpha : w \in \tilde{A}_\alpha \cap \tilde{B}_\alpha\}$ for fuzzy sets $\tilde{A}$ and $\tilde{B}$. In this case prove that $\chi_{\tilde{A}\cap\tilde{B}}(w) = \min(\chi_{\tilde{A}}(w), \chi_{\tilde{B}}(w))$.

(4 marks)

(c) State the axioms of a t-norm.

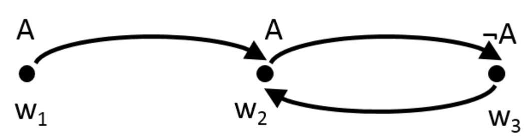
(4 marks)

(d) Prove that for any t-norm $T : [0,1]^2 \to [0,1]$ it holds that $\forall x, y \in [0,1]$, $drastic(x,y) \leq T(x,y) \leq \min(x,y)$ where

$$drastic(x,y) = \begin{cases} x : y = 1 \\ y : x = 1 \\ 0 : \text{otherwise} \end{cases}$$

(4 marks)

(e) Consider the following Kripke interpretation:



Identify in which worlds $\Box A$ is true.

(3 marks)