Fuzzy Set Theory

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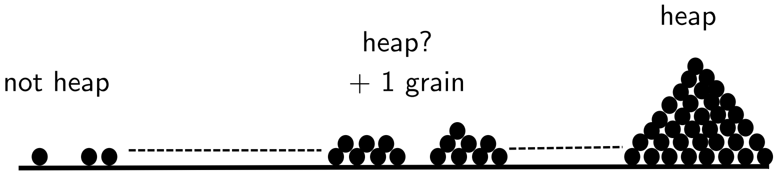
Introduction

- Fuzzy sets were introduced by Lotfi Zadeh in his 1965 seminal paper.
- Although, the fundamental idea dates back to Max Black and Jan Lukasiewicz.
- Fuzzy sets don't quantify beliefs in propositions like probability and Dempster-Shafer theory.
- Instead they allow for the formal representation of vague (or fuzzy) propositions as fuzzy sets.
- Fuzzy sets are sets to which elements can belong to some partial degree.

Sorities Paradoxes

- Many natural language concepts are *vague* in that the dividing line between where the concept does and does not hold is uncertainty.
- Vague concepts are prone to sorities paradoxes.



Membership Functions

- A (crisp) set $A \subseteq W$ can be characterised by a membership function $\chi_A : W \to \{0, 1\}$ so that $\chi_A(w) = \begin{cases} 1 : w \in A \\ 0 : w \notin A \end{cases}$
- A fuzzy set $\tilde{A}$ is characterised by a membership function $\chi_{\tilde{A}} : W \to [0, 1]$ so that $\chi_{\tilde{A}}(w) = \alpha$ means that w is a member of $\tilde{A}$ to degree α.
- Let $W = \{1, \ldots, 6\}$ denote the scores on a dice. Let $\tilde{A}$ be the set of score which can be described as *high*.
- A possible membership function for $\tilde{A}$ could give $\chi_{\tilde{A}}(w) = 0$ for all $w \leq 3$, $\chi_{\tilde{A}}(4) = 0.3$, $\chi_{\tilde{A}}(5) = 0.7$, and $\chi_{\tilde{A}}(6) = 1$

Notation

- Support: The support of a fuzzy set $\tilde{A}$ is given by $Supp(\tilde{A}) = \{w \in W : \chi_{\tilde{A}}(w) > 0\}$
- A simple notation for (finite) fuzzy sets is:

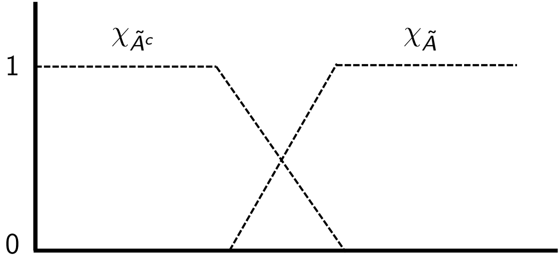
$$\tilde{A} = \sum_{w \in Supp(\tilde{A})} w/\chi_{\tilde{A}}(w)$$

- So for the fuzzy set representing *high* on the previous slide we have:

$$\tilde{A} = 4/0.3 + 5/0.7 + 6/1$$

Avoiding Sorities

- Let W be the set of positive integers, representing grains of sand.
- Let $\tilde{A}$ be the fuzzy set of numbers of grains of sand which can be described as a *heap*.
- Let $\chi_{\tilde{A}}$ be an increasing function of w and $\chi_{\tilde{A}}$ be a decreasing function of w



Fuzzy Set Calculus

- Consider crisp sets $A, B \subseteq W$. $w \in A \cap B$ iff $w \in A$ and $w \in B$.
- For membership functions $\chi_{A \cap B}(w) = 1$ iff $\chi_A(w) = 1$ and $\chi_B(w) = 1$.
- Q: Is there a function $T : \{0,1\}^2 \to \{0,1\}$ such that $\chi_{A \cap B}(w) = T(\chi_A(w), \chi_B(w))$?
- A: Lots of them, but when restricted to $\{0,1\}^2$ they all give the same answer!
- **Examples:** $T(x,y) = \min(x,y)$, $T(x,y) = x \times y$, $T(x,y) = \max(0, x+y-1)$
- We use the same trick for fuzzy sets.

Combination Operators

- We define function $T : [0,1]^2 \to [0,1]$, $S : [0,1]^2 \to [0,1]$ and $C : [0,1] \to [0,1]$ such that for any fuzzy sets $\tilde{A}$ and $\tilde{B}$ we define: $\forall w \in W$
- $\chi_{\tilde{A} \cap \tilde{B}}(w) = T(\chi_{\tilde{A}}(w), \chi_{\tilde{B}}(w))$, $\chi_{\tilde{A} \cup \tilde{B}}(w) = S(\chi_{\tilde{A}}(w), \chi_{\tilde{B}}(w))$ and $\chi_{\tilde{A}^c}(w) = C(\chi_{\tilde{A}}(w))$
- But in this case the crisp intersection functions do not all give the same value:
- **Example:** Suppose $\chi_{\tilde{A}}(w) = 0.3$ and $\chi_{\tilde{B}}(w) = 0.5$.
- If $T(x,y) = \min(x,y)$ then $\chi_{\tilde{A} \cap \tilde{B}}(w) = 0.3$
- If $T(x,y) = x \times y$ then $\chi_{\tilde{A} \cap \tilde{B}}(w) = 0.15$
- What T (and S and C) function should be choose?

T-norms

- T-norms (Triangular norms) are functions $T : [0,1]^2 \to [0,1]$ which satisfy the following properties:
- **T1:** $\forall x \in [0,1] \;\; T(x,1) = x$.
- **T2:** $\forall x,y,z \in [0,1]$, if $y \leq z$ then $T(x,y) \leq T(x,z)$.
- **T3:** $\forall x,y \in [0,1], \;\; T(x,y) = T(y,x)$.
- **T4:** $\forall x,y,z \in [0,1], \;\; T(x,T(y,z)) = T(T(x,y),z)$.
- **Examples:** $T(x,y) = \min(x,y)$ (minimum), $T(x,y) = x \times y$ (product), $T(x,y) = \max(0, x+y-1)$ (bounded difference),

$$T(x,y) = \begin{cases} x : y = 1 \\ y : x = 1 \\ 0 : \text{otherwise} \end{cases}$$

Ordering of T-norms

- We can order t-norms in terms of the degree to which they preserve membership. i.e. some t-norms are more conservative than others.
- $T_1 \leq T_2$ if $\forall x, y \in [0,1] \;\; T_1(x,y) \leq T_2(x,y)$
- **Theorem:** For any t-norm T $\textit{drastic} \leq T \leq \min$
- **Examples:**
  $\textit{drastic}(x,y) \leq \max(0, x+y-1) \leq x \times y \leq \min(x,y)$
- But are there other critierion for choosing a t-norm?

Idempotence

- **T5:** $\forall x \in [0,1]\ T(x,x) = x$
- **Theorem:** min is the only idempotent t-norm.
- **Proof:** Clearly min is idempotent since $\min(x,x) = x$.
- Now assume that there exists an idempotent t-norm T such that $\forall x \in [0,1]\ T(x,x) = x$.
- If $x \leq y$ then by **T2** and **T1** it holds that:

$$\min(x,y) = x = T(x,x) \leq T(x,y) \leq T(x,1) = x = \min(x,y)$$

- If $y \leq x$ then by **T2** and **T1** it holds that:

$$\min(x,y) = y = T(y,y) \leq T(y,x) \leq T(y,1) = y = \min(x,y)$$

Families of T-norms

- **Frank's t-norms:** $T_s(x,y) = \log_s \left[1 + \frac{(s^x-1)(s^y-1)}{s-1}\right]$ where $s > 0$ and $s \neq 1$.
- **Special cases:** $\lim_{s\to 1} T_s(x,y) = x \times y$,
  $\lim_{s\to 0} T_s(x,y) = \min(x,y)$,
  $\lim_{S\to\infty} T_s(x,y) = \max(0, x+y-1)$.
- **Dombi Family:**
  $T_\lambda(x,y) = \left\{1 + \left[(\frac{1}{x} - 1)^\lambda + (\frac{1}{y} - 1)^\lambda\right]^{\frac{1}{\lambda}}\right\}^{-1}$ where $\lambda > 0$.
- **Special cases:** $\lim_{\lambda\to 0} T_\lambda(x,y) = \mathit{drastic}(x,y)$,
  $T_1(x,y) = \frac{xy}{x+y-xy}$, $\lim_{\lambda\to\infty} T_\lambda(x,y) = \min(x,y)$

T-conorms

- **S1:** $\forall x \in [0,1]\ S(x,0) = x$.
- **S2:** $\forall x, y, z \in [0,1]$ if $y \leq z$ then $S(x,y) \leq S(x,z)$.
- **S3:** $\forall x, y \in [0,1]\ S(x,y) = S(y,x)$.
- **S4:** $\forall x, y, z \in [0,1]\ S(x, S(y,z)) = S(S(x,y), z)$.
- **Examples:** $S(x,y) = \max(x,y)$ (maximum), $S(x,y) = x + y - xy$ (algebraic sum), $S(x,y) = \min(1, x+y)$ (bounded sum), $S(x,y) = \begin{cases} x : y = 0 \\ y : x = 0 \\ 1 : \text{otherwise} \end{cases}$
- **S5:** $\forall x \in [0,1]\ S(x,x) = x$ (idempotence)
- **Theorem:** max is the only idempotent T-conorm.

Complementation Functions

- **C1:** $C(0) = 1$ and $C(1) = 0$.
- **C2:** C is a decreasing function.
- **C3:** C is a continuous function on $[0, 1]$.
- **C4:** $\forall x \in [0, 1]\ C(C(x)) = x$.
- **Examples:** $\forall x \in [0, 1]\ C(x) = 1 - x$
- **Sugeno Class:** $\forall x \in [0, 1]\ C_\lambda(x) = \frac{1-x}{1+\lambda x}$ where $\lambda > -1$.
- These axioms restrict C to a rescaling of $C(x) = 1 - x$.

The Nature of Complement

- **Theorem:** If C satisfies **C1** to **C4** then $([0,1], C, <)$ is isomorphic to $([0,1], 1-x, <)$
- **Proof:** Since $C(1) = 0$ and $C(0) = 1$ then by continuity (**C4**) it follows that there is some value k such that $C(k) = k$.
- Define $f(x) = \begin{cases} \frac{x}{2k} & : x \leq k \\ 1 - \frac{C(x)}{2k} & : x > k \end{cases}$.
- Then $f(0) = 0$, $f(1) = 1$, $f(k) = 0.5$ and it is easy to check that f is strictly increasing and continuous, hence a bijection.
- For $x \leq k$, $C(x) \leq C(k) = k$ so $f(C(x)) = 1 - \frac{C(C(x))}{2k} = 1 - \frac{x}{2k} - 1 - f(x)$.
- Similarly for $x > k$.

Duality

- We might want fuzzy sets to satisfy de Morgan's Laws, $(\tilde{A} \cup \tilde{B}) = (\tilde{A}^c \cap \tilde{B}^c)^c$.
- If so then we need: $S(\chi_{\tilde{A}}(w), \chi_{\tilde{B}}(w)) = 1 - \chi_{\tilde{A}^c \cap \tilde{B}^c}(w) = 1 - T(\chi_{\tilde{A}^c}(w), \chi_{\tilde{B}^c}(w)) = 1 - T(1 - \chi_{\tilde{A}}(w), 1 - \chi_{\tilde{B}}(w))$.
- So t-conorm $S(x, y) = 1 - T(1 - x, 1 - y)$ is the *dual* of t-norm T.
- **Examples:** $\max(x, y)$ is the dual of $\min(x, y)$, $x + y - xy$ is the dual of $x \times y$, $\min(1, x + y)$ is the dual of $\max(0, x + y - 1)$ and the drastic t-conorm is the dual of the drastic t-norm.

Frank's Equation

- We might want membership functions to be additive so that:

$$\chi_{\tilde{A}\cup\tilde{B}}(w) = \chi_{\tilde{A}}(w) + \chi_{\tilde{B}}(w) - \chi_{\tilde{A}\cap\tilde{B}}(w)$$

- This gives:

$$\forall x, y \in [0,1]\ S(x,y) = x + y - T(x,y)$$

- Combining this with duality gives:

$$\begin{gathered}\forall x, y \in [0,1]\ 1 - T(1-x, 1-y) = x + y - T(x,y) \\ \Rightarrow T(x,y) - T(1-x,1-y) = x + y - 1\end{gathered}$$

- Frank's equation is only (well almost) satisfied by Frank's t-norms.

Excluded Middle and Non-Contradiction

- **Law of Excluded Middle:** $\forall w \in W \ \chi_{\tilde{A} \cup \tilde{A}^c}(w) = 1$.
- **Law of Non-Contradiction:** $\forall w \in W \ \chi_{\tilde{A} \cap \tilde{A}^c}(w) = 0$.
- Fuzzy logic is not guaranteed to satisfy these laws.
- Take $T = \min$, $S = \max$ and $\chi_{\tilde{A}}(x) = 0.5$.
- Then $\chi_{\tilde{A} \cup \tilde{A}^c}(w) = \max(0.5, 0.5) = 0.5$ and $\chi_{\tilde{A} \cap \tilde{A}^c}(w) = \min(0.5, 0.5) = 0.5$
- Taking $T(x, y) = \max(0, x + y - 1)$ and $S = \min(1, x + y)$ then both laws are satisfied.
- There is no pair S, T which satisfy both idempotence and the laws of excluded middle and non-contradiction.

From Fuzzy to Crisp Sets

- α-cuts provide a way of representing fuzzy sets as a nested sequence of crisp sets.
- For fuzzy set $\tilde{A}$:

$$\tilde{A}_\alpha = \{w \in W : \chi_{\tilde{A}}(w) \geq \alpha\}$$

- If $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$ then $\tilde{A}_{\alpha_1} \supseteq \tilde{A}_{\alpha_2} \supseteq \ldots \supseteq \tilde{A}_{\alpha_n}$
- **Example:** Let $W = \{w_1, w_2, w_3\}$ and $\tilde{A} = w_1/1 + w_2/0.5 + w_3/0.2$ then:

$$\tilde{A}_\alpha = \begin{cases} \{w_1, w_2, w_3\} : \alpha \in [0, 0.2] \\ \{w_1, w_2\} : \alpha \in (0.2, 0.5] \\ \{w_1\} : \alpha \in (0.5, 1] \end{cases}$$

α-cuts to Fuzzy Sets

- The alpha cuts of a fuzzy sets can be used to determine the membership function as follows:
- $\chi_{\tilde{A}}(w) = \int_{\alpha : w \in \tilde{A}_\alpha} d\alpha$
- Or alternatively $\chi_{\tilde{A}}(w) = \sup\{\alpha : w \in \tilde{A}_\alpha\}$
- Dubois and Prade, have proposed using the alpha cut approach to extend functions from crisp to fuzzy sets.
- Let $f : 2^W \to \mathbb{R}$ then f can be extended to fuzzy sets according to:
- $f(\tilde{A}) = \int_0^1 f(\tilde{A}_\alpha) d\alpha$ or
- $f(\tilde{A}) = \sup\{f(\tilde{A}_\alpha) : \alpha \in (0, 1]\}$

Fuzzy Set Valued Functions

- If $f : 2^W \times 2^W \to 2^W$ then we can extend f to fuzzy sets as follows:
- Let $f(\tilde{A}, \tilde{B})_\alpha = f(\tilde{A}_\alpha, \tilde{B}_\alpha)$ and then take $\chi_{f(\tilde{A},\tilde{B})}(w) = \sup\{\alpha : w \in f(\tilde{A}, \tilde{B})_\alpha\}$
- Let $W = \mathbb{N}$ and let f be defined by $f(A, B) = \{w_1 + w_2 : w_1 \in A, w_2 \in B\}$ e.g. $f(\{1, 2\}, \{4, 5\}) = \{5, 6, 7\}$
- Let $\tilde{A} = 2/0.3 + 3/1 + 4/0.5$ and $\tilde{B} = 5/0.6 + 6/0.8 + 7/1$

$$\tilde{A}_\alpha = \begin{cases} \{2, 3, 4\} : \alpha \in [0, 0.3] \\ \{3, 4\} : \alpha \in (0.3, 0.5] \\ \{3\} : \alpha \in (0.5, 1] \end{cases} \qquad \tilde{B}_\alpha = \begin{cases} \{5, 6, 7\} : \alpha \in [0, 0.6] \\ \{6, 7\} : \alpha \in (0.6, 0.8] \\ \{7\} : \alpha \in (0.8, 1] \end{cases}$$

Fuzzy Set Valued Functions:2

$\tilde{A}+\tilde{B}$	$\{2,3,4\}$ $[0,0.3]$	$\{3,4\}$ $(0.3,0.5]$	$\{3\}$ $(0.5,1]$
$\{5,6,7\}$ $[0,0.6]$	$\{7,8,9,10,11\}$ $[0,0.3]$	$\{8,9,10,11\}$ $(0.3,0.5]$	$\{8,9,10\}$ $(0.5,0.6]$
$\{6,7\}$ $(0.6,0.8]$	$\emptyset$	$\emptyset$	$\{9,10\}$ $(0.6,0.8]$
$\{7\}$ $(0.8,1]$	$\emptyset$	$\emptyset$	$\{10\}$ $(0.8,1]$

$$(\tilde{A}+\tilde{B})_\alpha = \begin{cases} \{7,8,9,10,11\} : \alpha \in [0,0.3] \\ \{8,9,10,11\} : \alpha \in (0.3,0.5] \\ \{8,9,10\} : \alpha \in (0.5,0.6] \\ \{9,10\} : \alpha \in (0.6,0.8] \\ \{10\} : \alpha \in (0.8,1] \end{cases}$$

$$\tilde{A}+\tilde{B} = 10/1 + 9/0.8 + 8/0.6 + 11/0.5 + 7/0.3$$