Exercise Solution 1: Probability and Random Variables

Problem 1:

$$\because \forall A \subseteq W, A \cup A^c = W$$

$$\therefore 1 = P(W) = P(A \cup A^c)$$

$$\because P2 \; and \; A \cap A^c = \emptyset$$

$$\therefore P(A \cup A^c) = P(A) \cup P(A^c)$$

$$\because \forall A, B \subseteq W, A = (A \cap B) \cup (A \cap B^c) \; and \; A \cap B \cap A \cap B^c = \emptyset$$

$$\therefore P(A) = P(A \cap B) \cup P(A \cap B^c) \; and \; P(A^c) = P(A^c \cap B) \cup P(A^c \cap B^c)$$

Also

$$(A \cap B) \cup (A \cap B^c) = \emptyset \; and \; (A \cap B) \cup (A^c \cap B) = \emptyset$$

$$(A \cap B) \cup (A^c \cap B^c) = \emptyset \; and \; (A \cap B^c) \cup (A^c \cap B) = \emptyset$$

$$(A \cap B^c) \cup (A^c \cap B^c) = \emptyset \; and \; (A^c \cap B) \cup (A^c \cap B^c) = \emptyset$$

Therefore $1 = P(A \cap B) + P(A \cap B^c) + P(A^c \cap B) + P(A^c \cap B^c)$

Problem 2:

A)

- $P(X = -2) = P(\{w \,|\, X(w) = -2\} = P(\{w_1, w_2\}) = 0.7$
- $P(X = 1) = P(\{w \,|\, X(w) = 1\} = P(w_3) = 0.1$
- $P(X = 2) = P(\{w \,|\, X(w) = 2\} = P(w_4) = 0.2$

B)

X\Y	10	11	12
-2	0.3	0.4	0
1	0.1	0	0
2	0	0	0.2

Problem 3:

$P(A) = 0.6P(B) + 0.8P(B^c)$

$= 0.6P(B) + 0.8 - 0.8P(B)$

$= 0.8 - 0.2P(B)$

$\because 0 \leq P(B) \leq 1$

$\therefore 0.6 \leq P(A) \leq 0.8$

Problem 4:

$\because P(X = x) = P(\{w \,|\, X(w) = x, w \in W\})$

$\{w \,|\, X(w) = x, w \in W\} = \cup_y \{w \,|\, X(w) = x, Y(w) = y, w \in W\}$

Also in $y \neq y'$

$\{w \,|\, X(w) = x, Y(w) = y, w \in W\} \cap \{w \,|\, X(w) = x, Y(w) = y', w \in W\} = \emptyset$

Therefore $P(X = x) == P(\cup_y \{w \,|\, X(w) = x, Y(w) = y, w \in W\})$

$$= \sum_y P(\{w \,|\, X(w) = x, Y(w) = y, w \in W\}) = \sum_y P(X = x, Y = y)$$

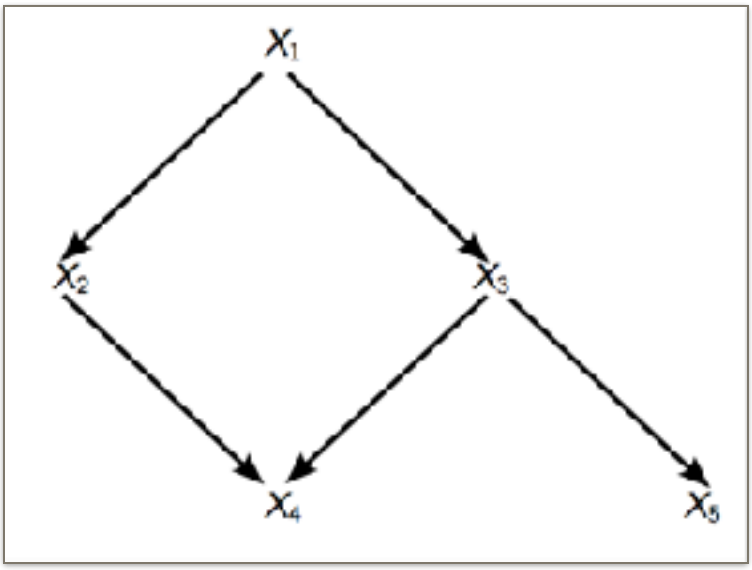
Problem 5:

$P(X_1, X_2, X_3, X_4, X_5) = P(X_1)P(X_2 \,|\, X_1)P(X_3 \,|\, X_1)P(X_4 \,|\, X_2, X_3)P(X_5 \,|\, X_3)$

$P(X_1) \rightarrow 3,\ P(X_2 \,|\, X_1) \rightarrow 3 * 4 = 12,\ P(X_3 \,|\, X_1) \rightarrow 3 * 4 = 12$

$P(X_4 | X_2, X_3) \rightarrow 3*4*4 = 48$, $P(X_5 | X_3) \rightarrow 3*4 = 12$

The number of probability values required is $3 + 12 + 12 + 48 + 12 = 87$.



Problem 6:

$$P(X_1, \ldots, X_n) = = \prod_{n=1}^{n} P(X_i | \sqcap (X_i))$$

$$\sum_{i=1}^{n} 2^{|\sqcap(x_i)|} \geq (\sum_{i=1}^{n} 2^0 = n)$$

$$\sum_{i=1}^{n} 2^{|\sqcap(x_i)|} \leq \sum_{i=1}^{n} 2^{i-1} = 2^n - 1$$