Exercise 1: Probability and Random Variables

1. If P is a probability measure use axioms **P1** and **P2** to prove that for any $A, B \subseteq W$

$$P(A \cap B) + P(A \cap B^c) + P(A^c \cap B) + P(A^c \cap B^c) = 1$$

2. Let $W = \{w_1, w_2, w_3, w_4\}$ and $X : W \to \mathbb{R}$ be a random variable defined by $X(w_1) = -2, X(w_2) = -2, X(w_3) = 1, X(4) = 2$. Also, let $P : 2^W \to [0, 1]$ be a probability measure defined by the following probability distribution: $p(w_1) = 0.3, p(w_2) = 0.4, p(w_3) = 0.1, p(w_4) = 0.2$.
(a) Find $P(X = x)$ for $x \in \{-2, 1, 2\}$.
(b) Let $Y : W \to \mathbb{R}$ be a second random variable defined by $Y(w_1) = 10, Y(w_2) = 11, Y(w_3) = 10, Y(w_4) = 12$. Determine the joint distribution $P(X, Y)$. (Give the results in a table)

3. For $A, B \subseteq W$ suppose that $P(A|B) = 0.6$ and $P(A|B^c) = 0.8$. Use the total theorem of probability (i.e $P(A) = P(A|B)P(B) + P(A|B^c)|P(B^c)$) to find upper and lower bounds on $P(A)$

4. For random variables $X : W \to \mathbb{R}$ and $Y : W \to \mathbb{R}$ use axioms **P1** and **P2** to show that $P(X = x) = \sum_y P(X = x, Y = y)$

5. Draw a Bayesian Network to represent the following conditional independence information between random variable $X_1, \ldots, X_5$:

- X_3 is conditionally independent of X_2 given X_1.
- X_4 is conditionally independent of X_1 given X_2 and X_3
- X_5 is conditionally independent of X_4, X_2 and X_1 given X_3

If $X_1, \ldots, X_5$ each take 4 values how many probability values must be specified for this network?

6. Show that for binary variables $X_1, \ldots, X_n$ a Bayesian Network requires between n and $2^n - 1$ probability values to be specified.