# Modal Logic

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# Introduction

- In the previous section uncertainty has been represented in a quantitative way by giving numerical values to beliefs.
- In this section we shall consider a symbolic approach to representing uncertainty in terms of modalities.
- We shall then see how this can be combined with a numerical approach to provide further insight into the relationship between DS-theory and Probability theory.
- The idea is to directly incorporate the notions of *possibly the case* and *necessarily the case* directly into a formal logic.
- This is done by means of modal operators $\Box$ for necessity and $\Diamond$ for possibly.

# Propositional Modal Logic

- Let $W$ be a set of possible worlds and let $E$ be a binary relation on $W$.
- The relation $E$ can have a variety of interpretations including $E(w_1, w_2)$ meaning; $w_1$ and $w_2$ are indistinguishable, $w_2$ is accessible from $w_1$, $w_2$ is imaginable from $w_1$ etc.
- The pair $\langle W, E \rangle$ is refered to as a Kripke interpretation.
- We define a set of basic propositions $A, B, C, A_1, A_2$ etc.
- For each proposition $A$ is either true or false in each of the possible worlds.
- Let $w \models A$ denote that proposition $A$ is true in world $w$.
- For each proposition $A$ we can identify the set of worlds in which $A$ is true:
- $W(A) = \{w \in W : w \models A\}$

# Well-Formed Formula of Modal Logic

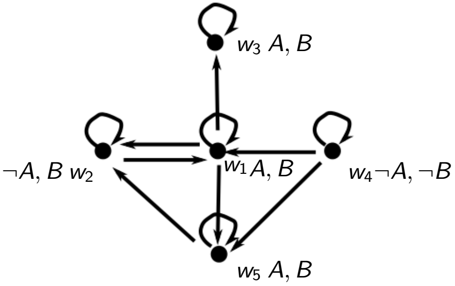
- We can combine propositions to make formula using the standard connectives $\neg$ (not), $\wedge$ (and), $\vee$ (or).
- As well as these connectives we also introduce two modal operators $\Box$ (necessarily) and $\Diamond$ (possibly).
- So for a formula $\theta$, $\Box\theta$ means that '$\theta$ is *necessarily* the case' and $\Diamond\theta$ means that '$\theta$ is *possibly* the case'.
- The well-formed formula (wff) of modal logic are defined recursively as follows:
  - All propositions are wff.
  - If $\theta, \varphi$ are wff so are $\neg\theta$, $\theta \wedge \varphi$, and $\theta \vee \varphi$.
  - If $\theta$ is a wff so are $\Box\theta$ and $\Diamond\theta$.

# Semantics of Modal Logic

- For a Kripke Interpretation $(W, E)$;
- $w \models \neg\theta$ if and only if $w \not\models \theta$ i.e. $\theta$ is *not true* at $w$.
- $w \models \theta \wedge \varphi$ if and only if $w \models \theta$ and $w \models \varphi$.
- $w \models \theta \vee \varphi$ if and only if $w \models \theta$ or $w \models \varphi$.
- $w \models \Box\theta$ if $\theta$ is true in *every world accessible* from $w$.
- i.e. for all $w'$ such that $E(w, w')$, $w' \models \theta$.
- $w \models \Diamond\theta$ if $\theta$ is true in *at least one world accessible* from $w$.
- i.e. there exists $w'$ such that $E(w, w')$ and $w' \models \theta$.

# Example

- Let $W = \{w_1, w_2, w_3, w_4, w_5\}$ be a set of webpages.
- Let $E$ be such that $E(w_i, w_i)$ and $E(w_i, w_j)$ for $i \neq j$ if there is a link from webpage $w_i$ to $w_j$.
- Further suppose we have two propositional variables $A =$'this page is an Eng. Maths page' and $B =$'this page is a UoB page'.



# Example

- For this Kripke interpretation we have:
- $w_1 \models \Box B$, $w_1 \models \Diamond B$, $w_1 \models \Diamond A$, $w_1 \models \Diamond\neg A$, $w_1 \not\models \Box A$.
- $w_2 \models \Box B$, $w_2 \models \Diamond B$, $w_2 \models \Diamond A$, $w_2 \models \Diamond\neg A$, $w_2 \not\models \Box A$.
- $w_3 \models \Box B$, $w_2 \models \Diamond B$, $w_2 \models \Diamond A$, $w_2 \models \Box A$.
- $w_4 \models \Diamond B$, $w_4 \models \Diamond\neg B$, $w_4 \not\models \Box B$, $w_4 \models \Diamond A$, $w_4 \models \Diamond\neg A$, $w_4 \not\models \Box A$.
- $w_5 \models \Box B$, $w_5 \models \Diamond B$, $w_5 \models \Diamond A$, $w_5 \models \Diamond\neg A$, $w_5 \not\models \Box A$.

# Entailment and Equivalence

- $\theta \models \varphi$ ($\theta$ entails $\varphi$) if and only if, $w \models \theta$ implies that $w \models \varphi$; alternatively $W(\theta) \subseteq W(\varphi)$.
- $\theta \equiv \varphi$ if and only if $\theta \models \varphi$ and $\varphi \models \theta$; alternatively $W(\theta) = W(\varphi)$
- In our example $A \models A \wedge B$, $B \equiv \Box B$, $B \equiv \Box\Box B$.
- In general, $\neg\Box\neg\theta \equiv \Diamond\theta$ (similarly, $\neg\Diamond\neg\theta \equiv \Box\theta$)

$$
\begin{aligned}
W(\neg\Box\neg\theta) = W(\Box\neg\theta)^c &= \{w : \text{ for all } w', E(w,w') \Rightarrow w' \models \neg\theta\}^c \\
&= \{w : \text{ for all } w', E(w,w') \Rightarrow w' \not\models \theta\}^c \\
&= \{w : \text{ there exists } w', E(w,w') \text{ and } w' \models \theta\} = W(\Diamond\theta)
\end{aligned}
$$

# Properties of Relations

- Recall the following properties of relations:
- **Reflexive**: $E(w, w)$ for all $w \in W$.
- **Symmetric**: If $E(w, w')$ then $E(w', w)$.
- **Transitive:** If $E(w_1, w_2)$ and $E(w_2, w_3)$ then $E(w_1, w_3)$.
- **Dense:** If $E(w_1, w_3)$ then there exists $w_2$ such that $E(w_1, w_2)$ and $E(w_2, w_3)$.
- **Serial:** For all $w$, there exists $w'$ such that $E(w, w')$.
- **Identity Relation:** $E(w, w')$ if and only if $w = w'$.

# Properties of $E$

- The properties of $E$ determine the properties of the entailment relation $\models$.
- If $E$ is *reflexive* then $\Box\theta \models \theta$.
- If $E$ is *transitive* then $\Box\theta \models \Box\Box\theta$.
- If $E$ is *dense* then $\Box\Box\theta \models \Box\theta$.
- If $E$ is *serial* then $\Box\theta \models \Diamond\theta$.
- If $E$ is *symmetric* then $\theta \models \Box\Diamond\theta$.
- If $E$ is a subset of the *identity relation* then $\theta \models \Box\theta$

# Properties of $E$

- Show that: If $E$ is *reflexive* then $\Box\theta \models \theta$.
- Suppose $w \models \Box\theta \Rightarrow$ for all $w'$ such that $E(w, w')$, $w' \models \theta$. Since $E$ is reflexive $E(w, w) \Rightarrow w \models \theta$.
- Show that: If $E$ is *serial* then $\Box\theta \models \Diamond\theta$.
- Suppose $w \models \Box\theta \Rightarrow$ for all $w'$ such that $E(w, w')$, $w' \models \theta$. Since $E$ is serial there exists $w'$ such that $E(w, w')$. Hence, $w' \models \theta \Rightarrow w \models \Diamond\theta$.

# Probability, Modal Logic and DS-Theory

- Let $P$ be a probability measure on $2^W$.
- Let $R(w) = \{w' : E(w, w')\}$ i.e. $R(w)$ is the set of worlds which are reachable from $w$.
- $R$ is a *random set* from $W$ into $2^W$.
- Notice that $w \models \Box\theta$ if and only if $R(w) \subseteq W(\theta)$.
- Also, $w \models \Diamond\theta$ if and only if $R(w) \cap W(\theta) \neq \emptyset$.
- We can define a *mass function* such that $m(F) = P(\{w : R(w) = F\}) = \sum_{w:R(w)=F} P(w)$.
- If $E$ is serial then for all $w$, $R(w) \neq \emptyset$ and hence $m(\emptyset) = 0$.
- In this case $P(\Box\theta) = Bel(\theta)$ and $P(\Diamond\theta) = Pl(\theta)$

# Probability of Necessary and Possible

$$P(\Box\theta) = P(W(\Box\theta)) = \sum_{w:R(w)\subseteq W(\theta)} P(w) = \sum_{F\subseteq W(\theta)} \sum_{w:R(w)=F} P(w)$$

$$= \sum_{F\subseteq W(\theta)} m(F) = Bel(W(\theta)) = Bel(\theta)$$

Also,

$$P(\Diamond\theta) = P(\neg\Box\neg\theta) = 1 - P(\Box\neg\theta) = 1 - Bel(\neg\theta) = Pl(\theta)$$