Solutions to Additional Probability Worksheet

Q1 $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) \Rightarrow$ by P2 that $P(A \cup B) = P(A \cap B) + P(A \cap B^c) + P(A^c \cap B)$. Also, $A = (A \cap B) \cup (A \cap B^c) \Rightarrow$ by P2 that $P(A) = P(A \cap B) + P(A \cap B^c)$ and $B = (A \cap B) \cup (A^c \cap B) \Rightarrow P(B) = P(A \cap B) + P(A^c \cap B)$. Hence, $P(A) + P(B) = 2P(A \cap B) + P(A \cap B^c) + P(A^c \cap B) = P(A \cup B) + P(A \cap B) \Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Q2 Since $p < \mu(\{r\})$, $p < \mu(\{g\})$ and $p < \mu(\{b\})$ then you would pick Bet1 in each case i.e. you would accept the following three bets:

- Gain $0.6S$ if the wheel lands red and lose $0.4S$ otherwise.
- Gain $0.6S$ if the wheel lands green and lose $0.4S$ otherwise.
- Gain $0.6S$ if the wheel lands blue and lose $0.4S$ otherwise.

In this case your total again would be:

$$\begin{aligned} &S(\chi_{\{r\}}(w^*) - 0.4) + S(\chi_{\{g\}}(w^*) - 0.4) + S(\chi_{\{b\}}(w^*) - 0.4) \\ &= S(\chi_{\{r\}}(w^*) + \chi_{\{g\}}(w^*) + \chi_{\{b\}}(w^*) - 1.2) = S(1 - 1.2) = -0.2S < 1 \end{aligned}$$

Hence, you will lose $-0.2S$ no matter on what colour the wheel lands.

Q3 Let $P(w_i) = p_i$ then from K we have that $p_1 + p_2 = \alpha$ and $p_1 + p_3 = \beta$. Hence, $p_2 = \alpha - p_1$ and $p_3 = \beta - p_1$. Also, $p_4 = 1 - p_1 - p_2 - p_3 = 1 - p_1 - (\alpha - p_1) - (\beta - p_1) = 1 - \alpha + \beta + p_1$. Therefore, entropy is given by;

$$H = -p_1 \log_2(p_1) - (\alpha - p_1)\log_2(\alpha - p_1) - (\beta - p_1)\log_2(\beta - p_1) - (1 - \alpha - \beta + p_1)\log_2(1 - \alpha - \beta + p_1)$$

Hence,

$$\frac{dH}{dp_1} = -log_2(p_1) + \log_2(\alpha - p_1) + \log_2(\beta - p_1) - \log_2(1 - \alpha - \beta + p_1)$$

Therefore,

$$\begin{aligned} \frac{dH}{dp_1} = 0 &\Rightarrow log_2(p_1) + \log_2(1 - \alpha - \beta + p_1) = \log_2(\alpha - p_1) + \log_2(\beta - p_1) \\ &\Rightarrow \log_2(p_1(1 - \alpha - \beta + p_1)) = \log_2((\alpha - p_1)(\beta - p_1)) \\ &\Rightarrow p_1(1 - \alpha - \beta + p_1) = (\alpha - p_1)(\beta - p_1) \Rightarrow p_1 - \alpha p_1 - \beta p_1 + p_1^2 = \alpha\beta - \alpha p_1 - \beta p_1 + p_1^2 \\ &\Rightarrow p_1 = \alpha\beta \end{aligned}$$

From this we have that $P(A \cap B) = P(\{w_1\}) = p_1 = \alpha\beta$

Q4 Notice that since $p_2 = \alpha - p_1$ and $p_3 = \beta - p_1$ it follows that $p_1 \leqslant \min(\alpha, \beta)$ since $p_2 \geqslant 0$ and $p_3 \geqslant 0$. Also, since $p_4 = 1 - \alpha - \beta + p_1$ then $p_4 \geqslant 0 \Rightarrow p_4 \geqslant \max(0, \alpha + \beta - 1)$. Hence, the centre of mass solution is given by;

$$\hat{p}_1 = \frac{\int_{\max(0,\alpha+\beta-1)}^{\min(\alpha,\beta)} p_1 dp_1}{\int_{\max(0,\alpha+\beta-1)}^{\min(\alpha,\beta)} dp_1} = \frac{\frac{1}{2}(\min(\alpha,\beta)^2 - \max(0, \alpha+\beta-1)^2)}{\min(\alpha,\beta) - \max(0,\alpha+\beta-1)}$$
$$= \frac{\frac{1}{2}(\min(\alpha,\beta) - \max(0,\alpha+\beta-1))(\min(\alpha,\beta) + \max(0,\alpha+\beta-1))}{\min(\alpha,\beta) - \max(0,\alpha+\beta-1)}$$
$$= \frac{1}{2}(\min(\alpha,\beta) + \max(0,\alpha+\beta-1))$$

Q5 Let $P(w_i) = p_i$ then from K we have that $p_1 + p_2 = \alpha$ and $p_3 + p_4 = \beta$. Also, $p_5 + p_6 = 1 - p_1 - p_2 - p_3 - p_4 = 1 - \alpha - \beta$. From this we have that $p_2 = \alpha - p_1$, $p_4 = \beta - p_3$ and $p_6 = 1 - \alpha - \beta - p_5$. Therefore, entropy is given by;

$$H = -p_1 \log_2(p_1) - (\alpha - p_1)\log_2(\alpha - p_1) - p_3 \log_2(p_3)$$
$$-(\beta - p_3)\log_2(\beta - p_3) - p_5 \log_2 p_5 - (1 - \alpha - \beta - p_5)\log_2(1 - \alpha - \beta - p_5)$$

The partial derivatives of H are then;

$$\frac{\partial H}{\partial p_1} = -\log_2(p_1) + \log_2(\alpha - p_1)$$
$$\frac{\partial H}{\partial p_3} = -\log_2(p_3) + \log_2(\beta - p_3)$$
$$\frac{\partial H}{\partial p_5} = -\log_2(p_5) + \log_2(1 - \alpha - \beta - p_5)$$

Hence,

$$\frac{\partial H}{\partial p_1} = \frac{\partial H}{\partial p_3} = \frac{\partial H}{\partial p_5} = 0 \Rightarrow p_1 = \alpha - p_1, p_3 = \beta - p_3, p_5 = 1 - \alpha - \beta - p_5$$
$$\Rightarrow p_1 = \frac{\alpha}{2}, p_3 = \frac{\beta}{2}, p_5 = \frac{1-\alpha-\beta}{2}$$

Then substituting we have that $p_2 = \frac{\alpha}{2}, p_4 = \frac{\beta}{2}, p_6 = \frac{1-\alpha-\beta}{2}$ as required.

Q6 Let $P(w_i) = p_i$ then from K we have that $p_1 + p_2 = 5(p_1 + p_3) \Rightarrow p_2 = 4p_1 + 5p_3$. Also, $p_4 = 1 - p_1 - p_2 - p_3 = 1 - p_1 - 4p_1 - 5p_3 - p_3 = 1 - 5p_1 - 6p_3$. Now since $p_2 \leqslant 1 \Rightarrow 5p_3 + 4p_1 \leqslant 1 \Rightarrow p_3 \leqslant \frac{1}{5}(1 - 4p_1)$. Also, $p_4 \geqslant 0 \Rightarrow 1 - 5p_1 - 6p_3 \geqslant 0 \Rightarrow p_3 \leqslant \frac{1}{6}(1 - 5p_1)$. Therefore, $p_3 \leqslant \min(\frac{1}{5}(1 - 4p_1), \frac{1}{6}(1 - 5p_1))$. But $\frac{1}{6}(1 - 5p_1) \leqslant \frac{1}{5}(1 - 4p_1)$ for $p_1 \in [0, 1]$ and hence $p_3 \leqslant \frac{1}{6}(1 - 5p_1)$. From this we also require that $\frac{1}{6}(1 - 5p_1) \geqslant 0 \Rightarrow p_1 \leqslant \frac{1}{5}$.

Hence, the centre of mass is given by;

$$\hat{p}_1 = \frac{\int_0^{\frac{1}{5}} \int_0^{\frac{1}{6}(1-5p_1)} p_1 dp_3 dp_1}{\int_0^{\frac{1}{5}} \int_0^{\frac{1}{6}(1-5p_1)} dp_3 dp_1} = \frac{1}{15}$$
$$\hat{p}_3 = \frac{\int_0^{\frac{1}{5}} \int_0^{\frac{1}{6}(1-5p_1)} p_3 dp_3 dp_1}{\int_0^{\frac{1}{5}} \int_0^{\frac{1}{6}(1-5p_1)} dp_3 dp_1} = \frac{1}{18}$$

Q7

$$\underline{P}(\{w_1, w_2\}) = \min\{P_1(\{w_1, w_2\}), P_2(\{w_1, w_2\}), P_3(\{w_1, w_2\}), P_4(\{w_1, w_2\})\}$$
$$= \min\{0.5, 0.4, 0.6, 0.3\} = 0.3$$

This bound is realised by P_4.

$$\overline{P}(\{w_3, w_4\}) = \max\{P_1(\{w_3, w_4\}), P_2(\{w_3, w_4\}), P_3(\{w_3, w_4\}), P_4(\{w_3, w_4\})\}$$
$$= \max\{0.5, 0.6, 0.4, 0.7\} = 0.7$$

This bound is realised by P_4.

$$\underline{P}(\{w_2, w_3\}) = \min\{P_1(\{w_2, w_3\}), P_2(\{w_2, w_3\}), P_3(\{w_2, w_3\}), P_4(\{w_2, w_3\})\}$$
$$= \min\{0.6, 0.8, 0.2, 0.4\} = 0.2$$

This bound is realised by P_3.

$$\overline{P}(\{w_1, w_4\}) = \max\{P_1(\{w_1, w_4\}), P_2(\{w_1, w_4\}), P_3(\{w_1, w_4\}), P_4(\{w_1, w_4\})\}$$
$$= \min\{0.4, 0.2, 0.8, 0.6\} = 0.8$$

This bound is realised by P_3.

Q8

$$\overline{P}(A) = \max\{P(A) : P \in \mathcal{P}\} = \max\{1 - P(A^c) : P \in \mathcal{P}\}$$
$$= 1 - \min\{P(A^c) : P \in \mathcal{P}\} = 1 - \underline{P}(A^c)$$