MILES:
MATLAB package for solving Mixed Integer LEast Squares problems, Version 2.0

Users' Guide

Xiao-Wen Chang, Xiaohu Xie and Tianyang Zhou

Scientific Computing Laboratory

School of Computer Science
McGill University

October 2011

1 Introduction

Let the sets of all real and integer $m \times n$ matrices be denoted by $\mathbb{R}^{m\times n}$ and $\mathbb{Z}^{m\times n}$, respectively, and the sets of real and integer n-vectors by $\mathbb{R}^n$ and $\mathbb{Z}^n$, respectively.

Given $\boldsymbol{A} \in \mathbb{R}^{m\times k}$, $\boldsymbol{B} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{y} \in \mathbb{R}^m$. Suppose that $[\boldsymbol{A}, \boldsymbol{B}]$ has full column rank. This MATLAB package provides a function to produce p optimal solutions to the mixed integer least squares (MILS) problem

$$\min_{\boldsymbol{x}\in\mathbb{R}^k, \boldsymbol{z}\in\mathbb{Z}^n} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x} - \boldsymbol{B}\boldsymbol{z}\|_2^2, \tag{1}$$

in the sense that a pair $\{\boldsymbol{x}^{(j)}, \boldsymbol{z}^{(j)}\} \in \mathbb{R}^k \times \mathbb{Z}^n$ is the j-th optimal solution if its corresponding residual norm $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^{(j)} - \boldsymbol{B}\boldsymbol{z}^{(j)}\|_2$ is the j-th smallest (some of these p residual norms can be equal), thus

$$\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^{(1)} - \boldsymbol{B}\boldsymbol{z}^{(1)}\|_2 \leq \cdots \leq \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^{(j)} - \boldsymbol{B}\boldsymbol{z}^{(j)}\|_2 \leq \cdots \leq \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^{(p)} - \boldsymbol{B}\boldsymbol{z}^{(p)}\|_2.$$

Here p is a parameter to be provided by a user and its default value is 1.

If the matrix $\boldsymbol{A}$ is nonexistent, (1) becomes an ordinary integer least squares (ILS) problem:

$$\min_{\boldsymbol{z}\in\mathbb{Z}^n} \|\boldsymbol{y} - \boldsymbol{B}\boldsymbol{z}\|_2^2. \tag{2}$$

This package also provides a function to produce p optimal solutions to (2).

Some (mixed) integer least squares problems in other forms can be transformed to (1) or (2), and then can be solved by this package. For example, suppose one wants to solve

$$\min_{\boldsymbol{z}\in\mathbb{Z}^n} (\boldsymbol{y} - \boldsymbol{B}\boldsymbol{z})^T \boldsymbol{V}^{-1} (\boldsymbol{y} - \boldsymbol{B}\boldsymbol{z}), \tag{3}$$

where $\boldsymbol{V}$ is symmetric positive definite. One can first compute the Cholesky factorization $\boldsymbol{V} = \boldsymbol{R}^T\boldsymbol{R}$ by MATLAB's built-in function `chol`, then solve $\boldsymbol{R}^T\bar{\boldsymbol{B}} = \boldsymbol{B}$ for $\bar{\boldsymbol{B}}$ and solve $\boldsymbol{R}^T\bar{\boldsymbol{y}} = \boldsymbol{y}$ for $\bar{\boldsymbol{y}}$ by using MATLAB's backslash command. Then (3) becomes $\min_{\boldsymbol{z}\in\mathbb{Z}^n} \|\bar{\boldsymbol{y}} - \bar{\boldsymbol{B}}\boldsymbol{z}\|_2^2$, the same form as (2).

The main purpose of this document is to show how to use this package.

2 System Requirements

The package has been fully tested on Windows XP, Linux and Macintosh with MATLAB 7.x, and should work on any platform supporting MATLAB. For the system requirements of running MATLAB, please refer to:

`http://www.mathworks.com`

3 Obtaining and Installing Package

The package can be downloaded from the web page

`http://www.cs.mcgill.ca/~chang/software.php`

The package is provided as a compressed file with extension "zip". To extract the package, an uncompress tool should be used. Suppose that the package is extracted to "C:/MILES" on Windows, then you can enter the directory and use the provided functions or examples. If you want to use the package in a directory rather than "C:/MILES", you have to add "C:/MILES" to the MATLAB path.

4 Citation

MILES is a freely available software package provided on the Internet and is copyrighted. If you are using it in research work to be published, please include explicit mention of out work in your publication. You may state something like:

"To solve the (mixed) integer least squares problem, we use MILES [1]"

with the following corresponding entry in your bibiography:

[1] Xiao-Wen Chang, Xiaohu Xie and Tianyang Zhou, MILES: MATLAB package for solving Mixed Integer LEast Squares problems, Version 2.0, http://www.cs.mcgill.ca/∼chang/software.php, October 2011.

5 Support

The MILES project supports the package in the sense that reports of errors or poor performance will gain immediate attention from the developers. Any comment and suggestion for improvement of the code or the document is also welcome. It may still be possible to improve the efficiency by using some programming tricks or MATLAB build-in functions, but for research purpose we try to keep the code simple and clear. Error reports and also descriptions of interesting applications and other comments should be sent to:


```
Prof. Xiao-Wen Chang
Email: chang@cs.mcgill.ca
```


6 Routines

In this section, we will introduce the input and output of some routines in this package. Each routine can be used separately without invoking the main function `mils.m`. Those who are not patient enough to read the following part can start directly from reading the example files `mils_example.m` and `ils_example.m`.

6.1 qrmcp.m

Given $\boldsymbol{B} \in \mathbb{R}^{m\times n}$ with $m \geq n$ and $\boldsymbol{y} \in \mathbb{R}^m$. This routine computes the QR factorization of $\boldsymbol{B}$ with minimum-column pivoting: $\boldsymbol{Q}^T\boldsymbol{B}\boldsymbol{P} = \left[\begin{smallmatrix}\boldsymbol{R}\\ \boldsymbol{0}\end{smallmatrix}\right]$ and computes $\boldsymbol{y} := \boldsymbol{Q}^T\boldsymbol{y}$. The Q-factor is not produced. The permutation matrix $\boldsymbol{P}$ is encoded in an integer vector $\boldsymbol{piv}$. The MATLAB function is:

```
function [R,piv,y] = qrmcp(B,y)
```

Input arguments
`B` — m by n real matrix to be QR factorized
`y` — m by p real matrix to be transformed to `Q'y`
Output arguments
`R` — n by n real upper triangular matrix
`piv` – n-vector storing the information of the permutation matrix `P`
`y` — m by p matrix transormed from the input `y` by `Q'`, i.e., `y:= Q'*y`

6.2 reduction.m

Given $\boldsymbol{B} \in \mathbb{R}^{m\times n}$ with full column rank and $\boldsymbol{y} \in \mathbb{R}^{m}$. This routine computes the LLL-QRZ factorization of $\boldsymbol{B}$: $\boldsymbol{Q}^T\boldsymbol{B}\boldsymbol{Z} = \left[\begin{smallmatrix}\boldsymbol{R}\\ \boldsymbol{0}\end{smallmatrix}\right]$ and computes $\boldsymbol{y} := \boldsymbol{Q}^T\boldsymbol{y}$. The Q-factor is not produced. Its goal is to reduce a general integer least squares problem to an upper triangular one. The MATLAB function is:

```
function [R,Z,y] = reduction(B,y)
```

Input arguments
`B` — m by n real matrix with full column rank
`y` — m-dimensional real vector to be transformed to `Q'y`
Output arguments
`R` — n by n LLL-reduced upper triangular matrix
`Z` — n by n unimodular matrix, i.e., an integer matrix with $|\det(\texttt{Z})| = 1$
`y` — m-vector transformed from the input `y` by `Q'`, i.e., `y:= Q'*y`

6.3 search.m

Given a nonsingular upper triangular $\boldsymbol{R} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{y} \in \mathbb{R}^{n}$. This routine produces p optimal solutions to the ILS problem $\min_{\boldsymbol{z}\in\mathbb{Z}^n} \|\boldsymbol{y} - \boldsymbol{R}\boldsymbol{z}\|_2^2$ by a search algorithm. The MATLAB function is:

```
function Zhat = search(R,y,p)
```

Input arguments
`R` — n by n real nonsingular upper triangular matrix
`y` — n-dimensional real vector
`p` — the number of optimal solutions to be found with a default value of 1
Output argument
`Zhat` — n by p integer matrix (in double precision). Its j-th column is the j-th optimal solution, i.e., its corresponding residual norm is the j-th smallest, so
$\|\texttt{y-R*Zhat(:,1)}\|_2 \leq \cdots \leq \|\texttt{y-R*Zhat(:,p)}\|_2$

6.4 ils.m

Given $\boldsymbol{B} \in \mathbb{R}^{m\times n}$ with full column rank and $\boldsymbol{y} \in \mathbb{R}^{m\times n}$. This routine produces p optimal solutions to the ILS problem $\min_{\boldsymbol{z}\in\mathbb{Z}^n} \|\boldsymbol{y} - \boldsymbol{B}\boldsymbol{z}\|_2^2$. The MATLAB function is:

```
function Zhat = ils(B,y,p)
```

Input arguments

`B` — m by n real matrix with full column rank
`y` — m-dimensional real vector
`p` — the number of optimal solutions to be found and its default value is 1

Output arguments

`Zhat` — n by p integer matrix (in double precision). Its j-th column is the j-th optimal solution, i.e., its corresponding residual norm is the j-th smallest, so $\|$`y-B*Zhat(:,1)`$\|_2 \leq \cdots \leq \|$`y-B*Zhat(:,p)`$\|_2$

6.5 mils.m

Given $\boldsymbol{A} \in \mathbb{R}^{m\times k}$, $\boldsymbol{B} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{y} \in \mathbb{R}^m$. Suppose that $[\boldsymbol{A}, \boldsymbol{B}]$ is of full column rank. This routine produces p optimal solution to $\min_{\boldsymbol{x}\in\mathbb{R}^k, \boldsymbol{z}\in\mathbb{Z}^n} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x} - \boldsymbol{B}\boldsymbol{z}\|_2^2$. The MATLAB function is:

```
function [xhat,zhat] = mils(A,B,y,p)
```

Input arguments

`A` — m by k real matrix
`B` — m by n real matrix, `[A,B]` has full column rank
`y` — m-dimensional real vector
`p` — the number of optimal solutions to be found and its default value is 1

Output arguments

`Xhat` — k by p real matrix
`Zhat` — n by p integer matrix (in double precision). The pair {`Xhat(:,j)`,`Zhat(:,j)`} is the j-th optimal solution, i.e., its corresponding residual norm is the j-th smallest, so $\|$`y-A*Xhat(:,1)-B*Zhat(:,1)`$\|_2 \leq \cdots \leq \|$`y-A*Xhat(:,p)-B*Zhat(:,p)`$\|_2$

6.6 Examples

The package provides two examples of using the routines.

- `ils_example.m` is a script M-file which gives a simple example of using this package to solve an ILS problem.
- `mils_example.m` is a script M-file which gives a simple example of using this package to solve an MILS problem.

7 Troubleshooting

Here we list some problems a user may encounter.

7.1 Common errors in calling MILES routines

A user should always carefully read the leading comments of a routine before using it. The leading comments describe what the routine can do and give a detailed description of all input/output arguments. For the benefit of users, here we list the most common programming errors in calling a routine. These errors may cause the MILES routines or MATLAB to report a failure, or may lead to wrong results without a warning message.

- Wrong number of arguments
- Arguments in wrong order
- Wrong dimensions for an array argument
- The input matrix is rank deficient
- MATLAB path is not set up appropriately.

7.2 Poor performance in efficiency

One should note that the integer least squares problem is NP-hard. If the dimension of the integer least squares problem is big, then the computation can be very time-consuming. Another thing we should mention is, MATLAB is slower than some high level programming languages, such as C/C++.

7.3 Integer overflow

If an integer number produced in the computation is outside of the interval $[-2^{53}+1, 2^{53}-1]$, then its floating point representation in double precision may not be accurate. This may lead wrong integer solutions. For many practical applications, however, this integer overflow phenomena may not be a concern. This package does not check integer overflow and so does not give an warning message if this occurs.