1. Hilbert space and function basis

Define distance (Metric Space)
$d(x,y) \in \mathbb{R}$
- if $x=y$, then $d(x,y)=0$; if $x \neq y$, $d(x,y) \geq 0$
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(y,z)$

Define norm (Normed Space)
(Distance to 0) $\|x\| \in \mathbb{R}$
- $\|x\| \geq 0$
- $\|x+y\| \leq \|x\| + \|y\|$
(From definition of distance)
- $\|ax\| = |a| \|x\|$
(Exclusively)

Define linear combination
$+, \times$ (Vector space)
(Elements can be represented with a set of basis)

Vector length can be calculated
(Normed vector space)

Also, a measure of similarity

Define angle between two vectors

(Inner product space) $\langle x, y\rangle \in \mathbb{R}$

- $\langle x, y\rangle = \langle y, x\rangle$ / $\langle x, y\rangle = \overline{\langle y, x\rangle}$
- $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$
- $\langle x, x\rangle \geq 0$, $=0$ only when $x=0$.

(Also known as Euclidean Space $\mathbb{R}^n$)

Add Completeness (result of $\lim_{x \to x'} f(x)$ still in the same space)

Dimension can be infinite

(Hilbert space)

	In Vector Hilbert Space	In Function Hilbert Space
Elements	$[\ldots 0, 1, 2, 3 \ldots]$	functions
+ and x	$a\vec{x} + b\vec{y} = \vec{z}$	$af(x) + bg(x) = h(x)$
Norm	$\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$	$\sqrt{\int_{-\infty}^{\infty} \lvert f(x)\rvert^2 dx}$
Inner product	$\langle \vec{x}, \vec{y}\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$	$\langle f(x), g(x)\rangle = \int_{-\infty}^{\infty} f(x) \cdot \overline{g(x)} dx$.

Orthogonal basis $\{e_k\}$:

$$\langle e_j, e_k\rangle = \begin{cases} q(e_k) & j=k \\ 0 & j \neq k \end{cases}$$

Orthonormal basis $\{e_k\}$:

$$\langle e_j, e_k\rangle = \begin{cases} 1 & j=k \\ 0 & j \neq k \end{cases}$$

2. Orthonormal basis for periodical function

On $L^2[0,2\pi]$, $\langle e_i, e_j\rangle$ is defined as $\int_0^{2\pi} e_i(x)\cdot \overline{e_j(x)}$

Trigonometric basis and exponential basis are orthonormal basis on $L^2([0,2\pi])$

Means: $x\in\mathbb{R}$, $f(x)=f(x+2\pi)$, $\int_0^1 |f(x)|^2 dx<\infty$ — period 2π function with finite energy in period.

↳ $f(x)=\sum_{n\in\mathbb{Z}} C_n e_n$, $e_n=\{\cos nk, \sin nk\}$ (Trigonometric, $f(x)$ real)

$e_n = e^{jnx}$ (Exponential)

i.e. $f(x)$ can be expressed with Fourier series

$L^2[0,2\pi]$ can be transformed into any period function situation by scaling & translation, so it basically represent any periodic function.

① Trigonometric basis for periodic functions (with period T)

$$\{e_n\}=\left\{\frac{\sqrt{1}}{\sqrt{T}}, n=0\right\}\cup\left\{\sqrt{\frac{2}{T}}\cos\frac{2\pi n}{T}x, n=1,2\ldots\right\}\cup\left\{\sqrt{\frac{2}{T}}\sin\frac{2\pi n}{T}x, n=1,2\ldots\right\}$$

Representing functions in the space with basis:

Base: $\frac{2\pi}{T}$

Harmonics: $\frac{2\pi}{T}\times n$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\Big[\underbrace{a_n\cos\frac{2n\pi}{T}x}_{\text{Even function}}+\underbrace{b_n\sin\frac{2n\pi}{T}x}_{\text{Odd function}}\Big]=\sum_{n=0}^{\infty}A_n e_n$$

Prove of orthonormality

$\langle e_m, e_n\rangle$ — As long as a period is covered it is ok.

$$=① \frac{2}{T}\int_0^T \cos\frac{2\pi m}{T}x\,\sin\frac{2\pi n}{T}x\,dx$$

$$=\frac{2}{T}\int_0^T \frac{1}{2}\left[\sin\frac{2\pi(m+n)}{T}x-\sin\frac{2\pi(m-n)}{T}x\right]dx$$

$$=\frac{2}{T}\left[\int_0^{2\pi}\sin(m+n)x-\int_0^{2\pi}\sin(m-n)x\right]=0,\ m,n>0$$

$$=② \frac{2}{T}\int_0^T \cos\frac{2\pi m}{T}x\cdot\cos\frac{2\pi n}{T}x\,dx$$

$$=\frac{2}{T}\int_0^T\frac{1}{2}\left[\cos\frac{2\pi(m+n)}{T}x+\cos\frac{2\pi(m-n)}{T}x\right]dx$$

$$=\begin{cases}0 & m\neq n\\ 1 & m=n>0\end{cases}$$

$$④=\frac{1}{T}\int_0^T dx$$

$$=1\quad m=n=0$$

$$=③ \frac{2}{T}\int_0^T \sin\frac{2\pi m}{T}x\cdot\sin\frac{2\pi n}{T}x\,dx$$

$$=\frac{2}{T}\int_0^T\frac{1}{2}\left[\cos\frac{2\pi(m-n)}{T}x-\cos\frac{2\pi(m+n)}{T}x\right]dx$$

$$=\begin{cases}0 & m\neq n\\ 1 & m=n>0\end{cases}$$

↳ $\langle e_m, e_n\rangle = \begin{cases} 1 & e_m=e_n \end{cases}$

$\llcorner$ 0 otherwise

$\hookrightarrow A_n = \int_0^T f(x) e_n$, $a_n = \frac{2}{T}\int_0^T f(x)\cos\frac{2\pi k}{T}x\,dx$

$b_n = \frac{2}{T}\int_0^T f(x)\sin\frac{2\pi k}{T}x\,dx$

② Exponential basis for periodic functions (with period T)

$\{e_n\} = \{\frac{1}{\sqrt{T}} e^{j\frac{2\pi n}{T}x}, n = \cdots -2, -1, 0, 1, 2 \cdots\}$

Representing functions in the space for thesis

$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{j\frac{2n\pi}{T}x} = \sum_{n=-\infty}^{\infty} A_n e_n$ $\quad f(x)$ real $\rightarrow C_{-n} = \bar{C_n}$ $\quad \begin{cases} |C_{-n}| = |C_n| \\ \angle C_{-n} = -\angle C_n \end{cases}$

Prove of orthonormality $\quad f(x)$ real and even $\rightarrow C_{-n} = C_n$

$\langle e_m, e_n \rangle$

$= \int_0^T e_m \cdot \bar{e}_n dx$

$= \frac{1}{T}\int_0^T e^{j\frac{2\pi(m-n)}{T}x} dx$

$= \frac{1}{T}\int_0^T [\cos\frac{2\pi}{T}(m-n)x + j\sin\frac{2\pi}{T}(m-n)x] dx$

$= \begin{cases} 1 & m=n \\ 0 & \text{otherwise} \end{cases}$

$\hookrightarrow C_n = \frac{1}{T}\int_0^T f(x) e^{-j\frac{2\pi n}{T}x} dx$

Conditions of convergence for ① and ② $S_N(x)=\sum_{n=-N}^{N} C_n e^{j\frac{2n\pi}{T}x}$

Functions having A_n exist / L^2 convergence / Energy equality / Equal "length"

↳ $\lim_{N\to\infty} \int_0^T |f(x)-S_N(x)|^2 dx = 0$ i.e. two functions cannot be distinguished with L^2 norm

↳ functions whose definition in one period T satisfies:

$\int_0^T |f(x)|^2 dx < \infty$ $f(x)=\sum_{n=-\infty}^{\infty} C_n e^{j\frac{2n\pi}{T}x} = \sum_{n=-\infty}^{\infty} A_n e_n$ i.e. the "vector" in function space must have finite length

i.e. $\in L^2[0,T]$ ⇒ ↳ Parseval's theorem: Same as vector length under orthonormal basis

i.e. finite energy $\int_0^T |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} A_n^2 = \sum_{n=-\infty}^{\infty} \frac{1}{T} C_n^2$

Point-wise equality

↳ $f(x)=S_N(x)|_{N\to\infty}$ at continuous points.

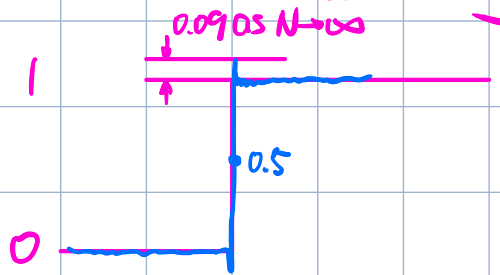
↳ At jump discontinuity, $S_N(x)$ converges to average of upper and lower values at discontinuity as $N\to\infty$

↳ Dirichlet conditions

- $\int_0^T |f(x)| dx < \infty$ Absolute integrable
- $f(x)$ has bounded variation in any bounded interval
- $f(x)$ has finite number of discontinuities in any bounded interval. The discontinuities cannot be infinite.

↳ Gibbs' phenomenon

↳ The overshoot at discontinuity does not vanish even for $N\to\infty$



Going from $L^2([0,2\pi])$ to $L^2(R)$, periodic to non-periodic

↳ Fourier Series change to Fourier transform, $C_n \to C(\Omega)$. We can no more use sum of Fourier series to represent functions, so the Fourier basis is not the orthonormal basis on $L^2(R)$

↳ But Wavelet basis ψ are the orthonormal basis on $L^2(R)$

3. Fourier transform for non-periodic function

$\mathcal{F}[\mathcal{F}f(-x)] = f(x)$

Duality: $\mathcal{F}f(x) = \mathcal{F}^{-1}f(-x)$
$= F(f)$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{j\frac{2\pi n}{T}x}$$

$$C_n = \frac{1}{T}\int_0^T f(x) e^{-j\frac{2\pi n}{T}x} dx$$

$$\xrightarrow[\text{Continuous}]{\text{Making } f=\frac{n}{T}}$$

$$\left.\begin{aligned} f(x) &= \int_{-\infty}^{\infty} F(f) e^{j2\pi f x} df \\ F(f) &= \int_{-\infty}^{\infty} f(x) e^{-j2\pi f x} dx \end{aligned}\right\}$$

Let $2\pi f = w$
$dw = 2\pi df$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w) e^{jwx} dw$$

$$F(w) = \int_{-\infty}^{\infty} f(x) e^{-jwx} dx$$

Similarly:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(f)|^2 df = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(w)|^2 dw$$

f is real and even $\to$ $F(w)$ is real

f is real and odd $\to$ $F(w)$ is imaginary

Convergence condition:

Same as Fourier series, but changes $\int_0^T \square\, dx$ to $\int_{-\infty}^{\infty} \square\, dx$

4. Discrete Time Fourier transform

$f(x) \to f(nT) \to f[n]$ (non-periodic)

$\hookrightarrow F(f) = \sum_{n=-\infty}^{\infty} f(nT) e^{-j2\pi f nT}$ $\quad T = \frac{1}{f_s}$ ← Sampling frequency

$= \sum_{n=-\infty}^{\infty} f[n] e^{-j2\pi f nT}$ $\hookrightarrow$ $F(f)$ becomes periodic: $F(f+f_s) = F(f_s)$

$$f[n] = \frac{1}{f_s} \int_0^{f_s} F(f) e^{j2\pi f nT} df$$

Taking the other form:

$\sum_{n=-\infty}^{\infty} f[n] e^{-j2\pi f nT}$ $\quad w = 2\pi f,\ T = \frac{1}{f_s}$

$= \sum_{n=-\infty}^{\infty} f[n] e^{-j\frac{2\pi f}{f_s} n}$, Let $\Omega = \frac{2\pi f}{f_s}$

$\hookrightarrow F(\Omega) = \sum_{n=-\infty}^{\infty} f[n] e^{-j\Omega n}$ $\hookrightarrow$ $F(\Omega)$ is periodic: $F(\Omega + 2\pi) = F(\Omega)$

$$f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\Omega) e^{j\Omega n} d\Omega$$

Sampling theorem:

to prevent aliasing and makes $f(x)$ reconstructable from $F(\Omega)$ by

$\underset{\text{periodic}}{F(\Omega)\,(F(f))} \to \underset{\text{one period}}{F(f)} \to f(x)$:

① $f(x) \xrightarrow{F} F(f)$, $F(f)$ need to be band-Limited:

$F(f) = 0$ for $|f| > f_{max}$

② $f_{max} < \frac{1}{2} f_s$

5. Discrete Fourier transform

$f[n]$ periodical / Extend $f[n]$ with finite length to make it periodical

$f[n+N] = f[n]$

↳ $F(\Omega)$ becomes discrete (and periodical)

↳ $F(\Omega)$ has finite non-zero values in a period.

↳ $F(\Omega)$ can be represented with F_k, a finite series i.e. transform => series

i.e. $F(\Omega) = 0$, for $\Omega \neq \Omega_k$

$$f[n] = \frac{1}{N}\sum_{k=0}^{N-1} F_k e^{j\Omega_k n}$$

$$F_k = \sum_{n=0}^{N-1} x[n] e^{-j\Omega_k n}, \quad \Omega_k = \frac{2\pi k}{N}$$

$F_k = F(\Omega_k)$

$f(x)$ is		$F(\Omega)$ is
	periodic	discrete
	non-periodic	continuous
	discrete	periodic
	continuous	non-periodic

zeroing fft coefficient vs zeroing wavelet coefficient

- FFT lacks temporal information: zeroing any fft coefficient will affect the entire signal

  pesudo-periodic signal, proved by PCA

  Why HF better performs than LF?

- FFT is only an approximation of DTFT

  After modifying any coefficient.

Filtering → Designed at DTFT level

Continuous Wavelet → Discrete. Analysis / Synthesis filters.

Peak extraction

= locate time domain location of frequency component

pick desired peak from undesired ones.

↳ inspecting 1st derivative

= highpass filtering