An Introduction to Wavelets

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The past ten years have seen an explosion of research in the theory of wavelets and their applications. Theoretical accomplishments include development of new bases for many different function spaces and the characterization of orthonormal wavelets with compact support. Applications span the fields of signal processing, image processing and compression, data compression, and quantum mechanics. At the present time however, much of the literature remains highly mathematical, and consequently, a large investment of time is often necessary to develop a general understanding of wavelets and their potential uses. This paper thus seeks to provide an overview of the wavelet transform from an intuitive standpoint. Throughout the paper a signal processing frame of reference is adopted.

Contents

1 Introduction

Over the past ten years much has been accomplished in the development of the theory of wavelets, and people are continuing to find new application domains. Theoretical accomplishments include specification of new bases for many different function spaces and characterization of orthogonal wavelets with compact support. Application areas so far discovered include signal processing, especially for nonstationary signals, image processing and compression, data compression, and quantum mechanics.

However, at the present time most of the literature remains highly mathematical and requires a large investment of time to develop an understanding of wavelets and their potential uses. The purpose of this paper is to provide an overview of wavelet theory by developing, from an intuitive standpoint, the idea of the wavelet transform. Since a complete study of wavelets would encompass both a lengthy mathematical development and consideration of many application domains, we adopt a particular viewpoint that lends itself readily to signal processing applications. Our discussion starts with a comparison of the wavelet and Fourier transforms of an impulse function. This motivates a discussion of the multiresolution decomposition of a function with finite energy. We then give the definition of a wavelet and the wavelet transform. Following is a comparison of the similarities and differences between the wavelet and Fourier transforms. We conclude with some examples of wavelet transforms of "popular" signals. Other introductions to wavelets and their applications may be found in [1], [2], [5], [8], and [10].

2 A Motivation for Wavelets

The short-time Fourier transform is frequently utilized for nonstationary signal analysis. Although a powerful tool, it has some limitations in analyzing time-localized events. The wavelet transform has similarities with the short-time Fourier transform, but it also possesses a time-localization property that generally renders it superior for analyzing nonstationary phenomena. We now review the Fourier and short-time Fourier transforms, discuss some often desirable properties that the short-time Fourier transform does not possess, and introduce the wavelet transform.

The Fourier and Short-Time Fourier Transforms

For any function f with finite energy, the *Fourier transform* of f is defined to be the integral

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt, \tag{1}$$

ω being the angular rate, equal to 2π times frequency. A Fourier transform is often represented by its power spectrum—the square of the modulus of $\hat{f}(\omega)$ *vs.* ω. For example, the power spectrum of an impulse function has a constant value of unity and is independent of the time at which the impulse occurs. Time of occurrence affects only the phase of each frequency component.

The Fourier transform is best suited to analyze stationary periodic functions—those that exactly repeat themselves once per period, without modification. It provides a single spectrum for the whole signal. For nonstationary signals we are interested in the frequencies that are dominant at any given time. For example, we perceive a musical melody as a succession of notes, each with its own frequency spectrum, rather than as one big signal with an overall spectrum. To analyze such signals, we may turn to the short-time Fourier transform.

The *short-time Fourier transform* (or *STFT*) of a function at some time t is the Fourier transform of that function as examined through some time-limited window centered on t. A different Fourier transform exists for each position t of the window. These transforms, produced by sliding the examination window along in time, constitute the STFT.

If the examination window simply omits the signal outside the window, two problems are encountered. One is the sudden change in the power spectrum as a discontinuity enters or leaves the window, compounded by a lack of sensitivity to the position of the discontinuity within the window. The other problem is spectral leakage: if some component of the signal has a cycle time which is not an integral divisor of the window width, the transform exhibits spurious response at many frequencies. These problems are ameliorated by attenuating samples away from the center of the window, by a "windowing function," g. An example of a windowing function is the Gaussian, $g(t) = e^{-at^2}$, for some constant a.[1] Mathematically, the STFT at time τ is given by

The true frequency component of the original periodic signal will be covered by side lobes of rectangular window if the DFT assumed signal does not match the original signal.

τ: the position of window

$$\text{STFT} \longrightarrow \hat{f}_g(\omega, \tau) = \int_{-\infty}^{\infty} f(t)g(t-\tau)e^{-i\omega t}dt. \tag{2}$$

Window function.

The response of the STFT, centered at time $\tau = \tau_0$, to an impulse function $\delta(t - t_0)$ occurring at time $t = t_0$ is given by

$$\begin{aligned} \hat{f}_g(\omega, \tau_0) &= \int_{-\infty}^{\infty} \delta(t-t_0)g(t-\tau_0)e^{-i\omega t}dt \\ &= g(t_0 - \tau_0)e^{-i\omega t_0}. \end{aligned} \tag{3}$$

The power spectrum of the STFT is $\hat{f}_g(\omega, \tau_0) = g^2(t_0 - \tau_0)$. As shown in Figure 1, the power spectrum is the same for all frequencies. The cross-section of the transform at constant frequency produces a time-reversed copy of the windowing function. Thus, the width (standard deviation) of the windowing function limits the accuracy with which the impulse can be located in time.

Although the STFT windowing function's width is constant, its impact varies with frequency. At high frequencies the number of waves in a window is high, producing good accuracy in frequency measurement; yet the window width prevents good localization of signal discontinuities, which the high frequencies otherwise could provide. Narrowing the window width to accommodate more precise time-localization of discontinuities causes other problems. A narrow window width is inappropriate at low frequencies, because a narrow windowing function spans fewer cycles. It distorts the signal noticeably over

[1] The STFT using a Gaussian window is known as the *Gabor transform*.

$\frac{2\pi}{N}$ (distance between frequency bins)

CW).

Frequency Resolution $\rightleftharpoons$ Length of samples in window.

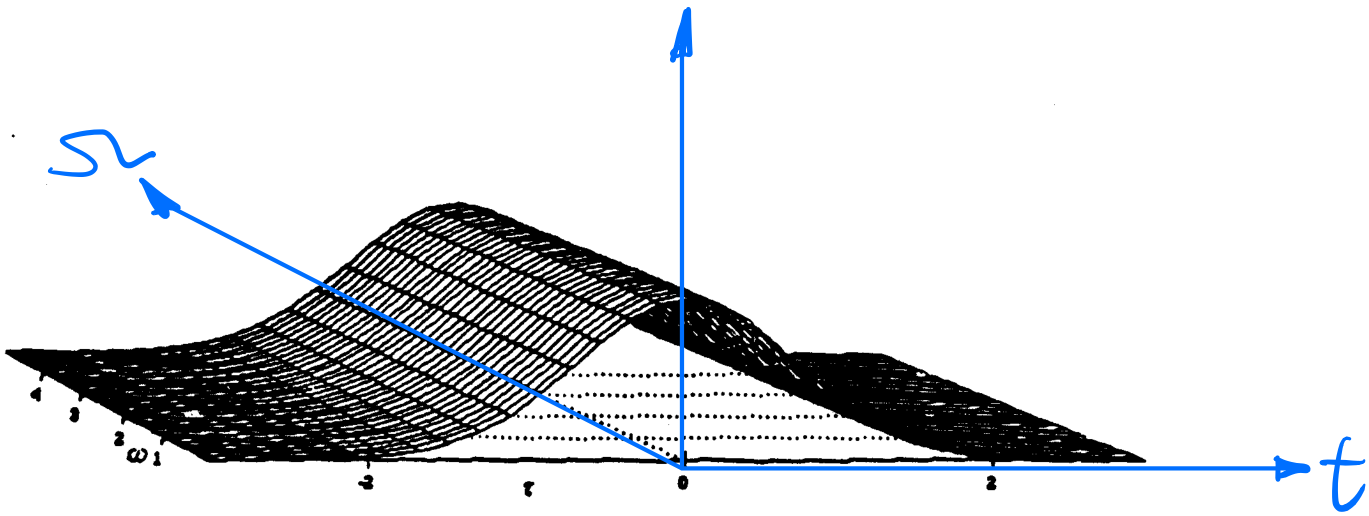


Figure 1: Power spectrum for the short-time Fourier transform of an impulse function using a Gaussian window centered at $\tau_0 = 0$

one wavelength, degrading accuracy of frequency measurement. Indeed, wavelengths longer than the window width cannot be measured. From these considerations it seems advantageous to let the windowing function be broad for analyzing low frequencies and narrow for high frequencies.

Windowing has more impact to low frequency signals.

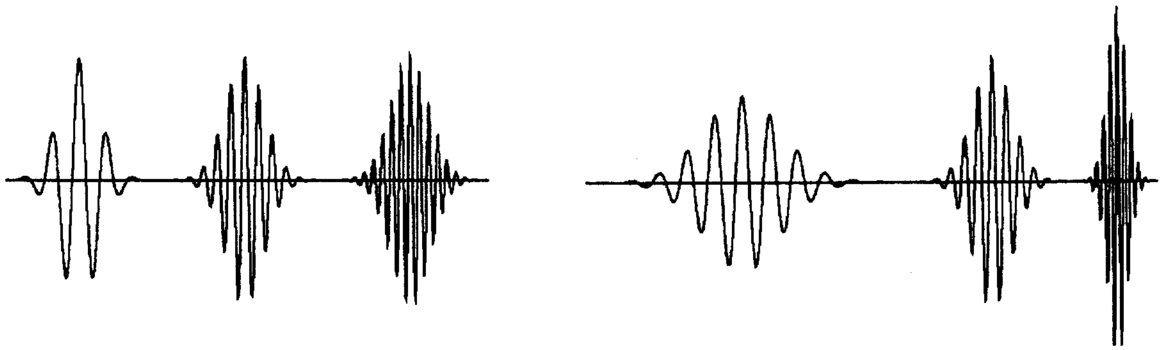
Since human perception of many types of signals has a logarithmic nature, the use of constant-Q filters in signal processing is not uncommon. A number of studies of the human senses find that the perceptual "distance" between two stimuli is dependent upon their ratios (with respect to the appropriate units of measurement for the stimuli). For example, a musical note one octave above another has twice the frequency. Loudness is rated in decibels, a logarithmic measurement. Psychophysical experiments conclude that the contribution of different frequencies to perceived quality of a visual image is logarithmic in frequency. A filter bank used to process these signals naturally consists of constant-Q filters—those whose bandwidths are proportional to their center frequencies. However, interpreting the STFT as a filtering process results in a filter bank whose filters have constant bandwidth.

Wavelet Transforms

Window

Equation 2 shows that the STFT of a signal is the inner product of the signal with an element of the set of basis functions $g(t-\tau)e^{-i\omega t}$, which vary over frequency ω and time τ. As shown by Figure 2a, all basis functions have the same time-amplitude envelope. The STFT decomposes a signal into a set of frequency bands at any given time.

Wavelet transforms also decompose a signal into a set of "frequency bands" (referred to as scales) by projecting the signal onto an element of a set of basis functions. Although the scales do not live in the frequency domain, projection of the signal onto different scales is equivalent to bandpass filtering with a bank of constant-Q filters. The basis functions are called wavelets. Wavelets in a basis are all similar to each other, varying only by dilation and translation, as illustrated in Figure 2b. Wavelet transforms thus accommodate the two shortcomings of the STFT discussed above.



a. STFT **b.** Wavelet

Figure 2: Comparison of basis functions for the STFT and wavelet transforms. The STFT places a varying number of waves under the same modulation envelope. The wavelet basis functions are self-similar: scaled in time to maintain the same number of oscillations and scaled in amplitude to maintain energy.

Figure 3 displays the square of the modulus of the continuous wavelet transform (in analogy with the power spectrum) of an impulse function, using a Gaussian wavelet. The parameter τ represents time, and s represents scale. (The s axis in Figure 3 is actually the base 2 logarithm of the scale.) Location of the impulse is clearly shown at time $\tau = 0$. Narrowness of the fine-scale basis functions trades off frequency resolution in favor of time resolution. And yet, there is no set limit to the broadness of scale at which analysis might be performed, aside from the length of the signal itself. In contrast, the STFT is limited in both time and frequency resolution by the fixed width of its window.

Multiresolution Analysis

In order to analyze a nonstationary signal, we need to determine its behavior at any individual event. Multiresolution analysis provides one means to do this. A *multiresolution analysis* decomposes a signal into a smoothed version of the original signal and a set of detail information at different scales. This type of decomposition is most easily understood by thinking of a picture (which is a two dimensional signal). We remove from the picture information that distinguishes the sharpest edges, leaving a new picture that is slightly blurred. This blurred version of the original picture is a rendering at a slightly coarser scale. We then recursively repeat the procedure. Each time we obtain some detail information and a more and more blurred (or smoothed) version of the original image. Removal of the detail information corresponds to a bandpass filtering, and generation of the smoothed image corresponds to a lowpass filtering. Given the decomposition, we can reconstruct the original image.

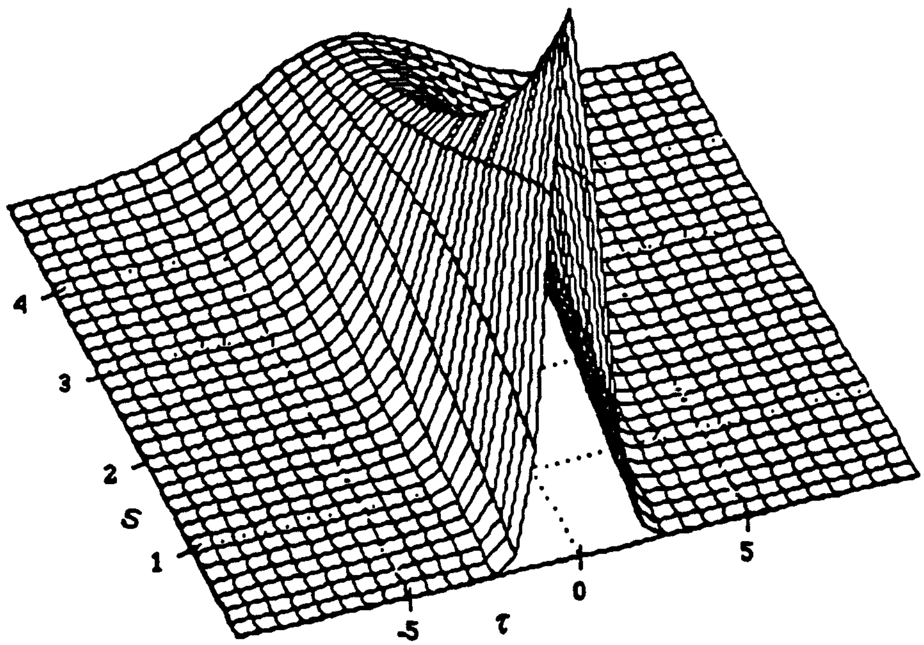


Figure 3: Square of the modulus of the continuous wavelet transform of an impulse function at time $\tau = 0$, using a Gaussian wavelet. (The scale axis s is labeled in exponents of two)

Once we have decomposed a signal this way, we may analyze the behavior of the detail information across the different scales. We can extract the regularity of a singularity, which characterizes the signal's behavior at that point. This provides an effective means of edge detection. Furthermore, noise has a specific behavior across scales, and hence, in many cases we can separate the signal from the noise. Reconstruction then yields a relatively accurate noise free approximation of the original signal. This denoising technique is developed in [7].

The wavelet transform specifies a multiresolution decomposition, with the wavelet defining the bandpass filter that determines the detail information. Associated with the wavelet is a smoothing function, which defines the complementary lowpass filter. Conditions to be described later ensure that the set consisting of the detail information at all scales and the smoothed version of the original signal contains no redundant information.

In lieu of the wavelet transform's ability to localize in time and its ability to specify a multiresolution analysis, many potential application areas have been identified. These include edge characterization, noise reduction, data compression, and sub-band coding. This list is by no means exhaustive—new applications are continually being discovered both in signal processing and in other domains.

3 Wavelets and the Wavelet Transform

Mathematical Definitions

In this section we focus our attention on the mathematical definition of wavelets and the wavelet transform. Table 1 provides a summary of the notation used in the rest of this paper.

As discussed above, a multiresolution analysis of any function f with finite energy decomposes f into a collection of details at different scales and a smoothed version of the original function. If we let $L^2(\mathbf{R})$ denote the set of all functions with finite energy, then the details of f at any scale m is simply a projection of f onto a subspace W_m of $L^2(\mathbf{R})$. This projection may be formally represented by a projection operator $Q_m : L^2(\mathbf{R}) \to W_m$. Furthermore, there exists another projection operator $P_M : L^2(\mathbf{R}) \to V_M$, $V_M \subset L^2(\mathbf{R})$, such that $P_M f$ is the smoothed version of f. Here m takes the values $m = 1, 2, ..., M$, that is, f is decomposed into the smoothed version $P_M f$ and M sets of details at different scales. As M increases the resolution of the smoothed version of f becomes coarser, and consequently, the finer detail information is contained in the scales corresponding to low values of m. For any multiresolution analysis the W_m are orthogonal both to each other and to V_M. In addition, assuming that $V_0 = L^2(\mathbf{R})$, we have $L^2(\mathbf{R}) = \oplus_{m=1}^{M} W_m \oplus V_M$, and hence we may write

$$f = P_M f + \sum_{m=1}^{M} Q_m f. \tag{4}$$

The question now arises as to how to define the Q_m. We desire an orthonormal basis

$L^2(\mathbf{R})$	the space of finite-energy (also known as square-integrable) functions
f	a finite-energy function, that is, $f \in L^2(\mathbf{R})$
m	discrete, integer parameter indicating scale
n	discrete, integer parameter indicating (time) translation along any fixed scale
s	continuous, positive real parameter indicating scale. s is related to m through the discretization $s = s_0^m$
τ	continuous, real parameter indicating (time) translation along any fixed scale. τ is related to n through the discretization $\tau = ns_0^m\tau_0$
s_0	real number greater than 1 used for discretization of the continuous scaling parameter
τ_0	nonzero real number used for discretization of the continuous translation parameter
ψ	an analyzing wavelet
$\psi_{s,\tau}$	wavelets generated by dilation and translation of the analyzing wavelet ψ using the continuous dilation parameter s and the continuous translation parameter τ
ψ_{mn}	wavelets generated by dilation and translation of an analyzing wavelet ψ using the discrete dilation parameter s_0^m, and the discrete translation parameter $\tau = ns_0^m\tau_0$
W_m	subspace of $L^2(\mathbf{R})$ containing the details of f at scale m
V_m	subspace of $L^2(\mathbf{R})$ containing the smoothed version of f at scale m
Q_m	projection operator that projects f onto W_m, that is, Q_m provides the details of f at scale m
P_m	projection operator that projects f onto V_m, that is, P_m provides the smoothed version of f at scale m

Table 1: Summary of notation

of W_m, call it ψ_{mn} (m fixed), so that

$$Q_m f = \sum_{n=-\infty}^{\infty} \langle f, \psi_{mn} \rangle \psi_{mn}, \tag{5}$$

where $\langle \cdot, \cdot \rangle$ denotes inner product. As it turns out, wavelets provide the ψ_{mn}. We therefore turn our attention to the definition of wavelets and the wavelet transform and then show how to define Q_m in terms of them.

Before proceeding however a few remarks may provide some clarification. After removing the first set of detail information from f we are left with a slightly smoothed version of f. Iteratively removing detail information progressively generates more and more smoothed versions of f. We stop the process once we have enough detail information or a smooth enough version to do the analysis we desire. Thus the subspaces W and V are intimately related in the following way: given any m, a function $f \in V_m$ is decomposed into details and a smoothed version at the $m+1$ scale. That is, f is decomposed by projecting it onto orthogonal complements in V_m, one subspace being W_{m+1} and the other V_{m+1}. Formally, we have $V_m = W_{m+1} \oplus V_{m+1}$.

Wavelets consist of the dilations and translations of a single real valued[2] function $\psi \in L^2(\mathbf{R})$, called the *analyzing wavelet* (also known as the *basic wavelet* or *mother wavelet*). By a dilation we mean a scaling of the argument, so that given any function $\psi(t)$ and a parameter $s > 0$, $\frac{1}{\sqrt{s}}\psi(\frac{t}{s})$ is a dilation of ψ. Consequently, a dilation of a function corresponds to a either a spreading out or contraction of the function. We introduce the factor $\frac{1}{\sqrt{s}}$ with the foresight that it yields a normalization necessary to have an orthonormal wavelet basis. Translation simply means a shift of the argument along the real axis, that is, given τ, the translation of $\psi(t)$ by τ is $\psi(t-\tau)$.

For any analyzing wavelet ψ we thus define a family of functions $\psi_{s,\tau}$ by the dilations and translations of ψ,

$$\psi_{s,\tau}(t) = s^{-1/2}\psi(\frac{t-\tau}{s}), \quad s, \tau \in \mathbf{R},\, s > 0. \tag{6}$$

Each $\psi_{s,\tau}$ is called a *wavelet*.[3]

We may represent any function $f \in L^2(\mathbf{R})$ by

$$(Wf)(s,\tau) = \langle f, \psi_{s,\tau} \rangle = \int_{-\infty}^{\infty} f(t)\psi_{s,\tau}(t)\,dt. \tag{7}$$

Wf is called the *continuous wavelet transform* of f.

[2]In general, the analyzing wavelet may be complex valued. For this paper however we assume that it is real valued.

[3]Technically, ψ must also satisfy the *admissibility condition*

$$C_\psi = \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\,d\omega < \infty,$$

where $\hat{\psi}$ denotes the Fourier transform of ψ. See [3, 4, 5].

Given $s_0 > 1$ and $\tau_0 \neq 0$, we may restrict s and τ, respectively, to the discrete lattices $s \in \{s_0^m,\ m \in \mathbf{Z}\}$ and $\tau \in \{ns_0^m\tau_0,\ m,n \in \mathbf{Z}\}$. Then $\psi_{s,\tau}$ becomes

$$\psi_{mn}(t) = s_0^{-m/2}\psi(s_0^{-m}t - n\tau_0), \tag{8}$$

and Wf becomes

$$(Wf)(m,n) = \langle f, \psi_{mn}\rangle = \int_{-\infty}^{\infty} f(t)\psi_{mn}(t)\,dt. \tag{9}$$

Note that the translation parameter τ depends upon the scaling parameter s_0. When m is large and positive, ψ_{mn} is spread out, and the translation steps become large accordingly. When m is large (in absolute value) and negative however the ψ_{mn} are very concentrated, and the translation steps are small.

The choice of τ_0 is arbitrary and by convention is taken to be 1. On the other hand, the choice of s_0 significantly affects the properties of the ψ_{mn}. For doing multiresolution analysis we want the ψ_{mn} to be orthonormal. Taking $s_0 = 2$ allows us to define ψ such that the ψ_{mn} are orthonormal [3, 6]. This is also the conventional choice for s_0. The $\psi_{mn} = 2^{-m/2}\psi(2^{-m}t - n)$ then form an orthonormal basis of $L^2(\mathbf{R})$. Equation 9, with this choice of ψ_{mn}, is called the *dyadic wavelet transform.*

As an example, let the analyzing wavelet be defined by

$$\psi(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}) \\ -1, & t \in [\frac{1}{2}, 1) \\ 0, & otherwise \end{cases} \tag{10}$$

This wavelet, illustrated in Figure 4, is called the *Haar wavelet*, and the corresponding ψ_{mn} form a basis of $L^2(\mathbf{R})$ called the *Haar basis.*

Now we return to Q_m, the operator that projects a function onto its details at scale m. We define Q_m as in Equation 5 by

$$Q_m f = \sum_{n=-\infty}^{\infty} \langle f, \psi_{mn}\rangle\psi_{mn} = \sum_n (Wf)(m,n)\,\psi_{mn}. \tag{11}$$

The details of f at each scale m thus consist of the sum of the projections of f onto the ψ_{mn}. Note that these projections are not onto all the ψ_{mn}, for at each scale m is fixed and only n varies. To summarize, given any function f with finite energy and an analyzing wavelet ψ such that the ψ_{mn} are orthonormal, we may compute, using Equations 4 and 11, the details of f (i.e., $Q_m f$) at the scales $m = 1, 2, ..., M$ and the corresponding smoothed version of f at scale M (i.e., $P_M f = f - \sum_{m=1}^{M} Q_m f$), thus obtaining a multiresolution analysis.

Properties and Examples

Wavelets possess some interesting properties in addition to those already discussed. The admissibility condition (see footnote, p. 8) implies $\hat{\psi}(0) = 0$, and hence

$$\int_{-\infty}^{\infty} \psi(t)\,dt = \hat{\psi}(0) = 0, \tag{12}$$

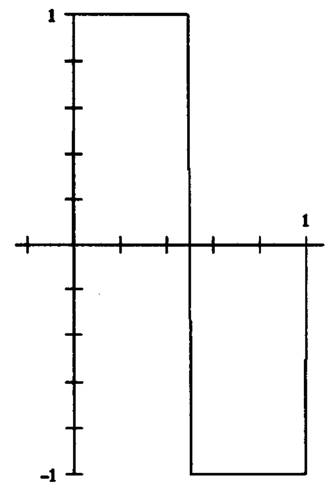


Figure 4: Haar wavelet

that is, wavelets have zero mean. Note that this indicates the equivalence of a wavelet and a bandpass filter.

Many wavelets have rapid decay. Y. Meyer [3] constructed a C^∞ wavelet that decays faster than any power. P. G. Lemarié and G. Battle [3] independently constructed a collection of C^k wavelets that decay exponentially.[4]

Orthogonal wavelets may be classified as either having compact support or not having compact support.[5] I. Daubechies [3] characterized all orthogonal wavelets with compact support. She showed that the Haar wavelet is the only such wavelet that is either symmetric or antisymmetric about any point. She also showed that compactly supported wavelets may be chosen with arbitrary regularity; however, the support width varies directly with the regularity. The compact support of Daubechies' wavelets and the rapid decay of the wavelets described by Meyer and Lemarié and Battle help provide for both the time-localization ability and efficient computation of the wavelet transform.

We finally consider some examples of wavelets. The first example is the Haar wavelet discussed above. A second example is a wavelet constructed by Lemarié and Battle [6], illustrated in Figure 5. As an analyzing wavelet it yields an orthonormal basis of $L^2(\mathbf{R})$ but does not have compact support. An example, other than the Haar wavelet, of a compactly supported wavelet that yields an orthornormal basis of $L^2(\mathbf{R})$ is shown in Figure 6. This wavelet was first constructed by Daubechies [3]. Note the lack of symmetry. Although continuous, this wavelet is non-differentiable at an infinite number

[4] C^∞ represents the space of all analytic functions, and C^k represents the space of all k-times continuously differentiable functions.

[5] A function $f:\mathbf{R} \rightarrow \mathbf{R}$ is said to have *compact support* if and only if it is zero everywhere except on a closed, bounded subset of $\mathbf{R}$.

of points. It is supported on the interval $[0,3]$.[6]

4 Comparision of the Fourier and Wavelet Transforms

In this section we discuss some of the similarities and differences between the Fourier and wavelet transforms. Table 2 gives a summary.

Both the Fourier transform and wavelet transform are given by integral equations in the form of a correlation. In the Fourier transform the correlation is with dilations of the function e^{-it}. In the wavelet transform the correlation is with dilations and translations of the analyzing wavelet ψ, which can be any wavelet.

Both the Fourier transform and wavelet transform may take real or complex functions as their input. The output of the Fourier transform is always complex. However, there are both real- and complex-valued wavelets. If a complex-valued wavelet is used as the analyzing wavelet, the wavelet transform is complex-valued. If a real-valued analyzing wavelet is used, the wavelet transform may be real- or complex-valued (real-valued if the input function is real-valued and complex-valued if the input function is complex-valued).

The Fourier transform maps time into frequency and phase; whereas, the wavelet transform maps time into scale and time. For each frequency the Fourier transform

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[6]The Battle-Lemarié wavelet is characterized by its Fourier transform [6]:

$$\hat{\psi}(\omega) = \frac{e^{-i(\omega/2)}}{\omega^4} \frac{\sqrt{\Sigma_8(\frac{\omega}{2}+\pi)}}{\sqrt{\Sigma_8(\omega)\Sigma_8(\frac{\omega}{2})}},$$

where

$$\Sigma_8 = \frac{N_1(\omega)+N_2(\omega)}{105(\sin\frac{\omega}{2})^8},$$

with

$$N_1 = 5 + 30(\cos\frac{\omega}{2})^2 + 30(\sin\frac{\omega}{2})^2(\cos\frac{\omega}{2})^2,$$

and

$$N_2 = 2(\sin\frac{\omega}{2})^4(\cos\frac{\omega}{2})^2 + 70(\cos\frac{\omega}{2})^4 + \frac{2}{3}(\sin\frac{\omega}{2})^6.$$

The Daubechies wavelet is characterized as follows [10]: Given $c_0 = \frac{1}{4}(1+\sqrt{3})$, $c_1 = \frac{1}{4}(3+\sqrt{3})$, $c_2 = \frac{1}{4}(3-\sqrt{3})$, and $c_3 = \frac{1}{4}(1-\sqrt{3})$, we define

$$\psi(t) = \sum_{k=-2}^{1}(-1)^k c_{1-k}\phi(2t-k),$$

where $\phi(t)$ is the limit as $j \to \infty$ of the recursion

$$\phi_j(t) = \sum_{0}^{3} c_k \phi_{j-1}(2t-k),$$

with

$$\phi_0(t) = \begin{cases} 1, & t \in [0,1) \\ 0, & otherwise \end{cases}.$$

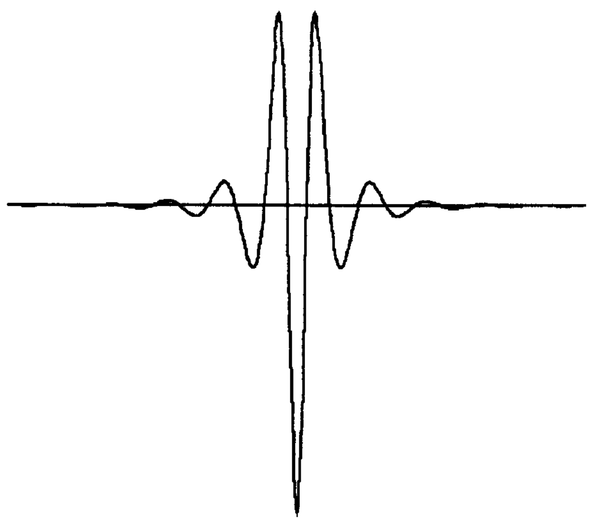

Figure 5: Battle-Lemarié wavelet

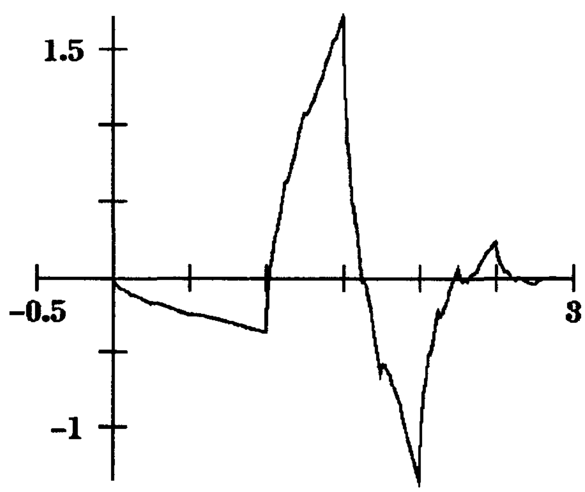

Figure 6: Compactly supported Daubechies wavelet

	Fourier Transform	Wavelet Transform
"root" function	$e^{i\omega t}$	$s^{-1/2}\psi(\frac{t-\tau}{s})$
Continuous transform	$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$	$W_c f(s,\tau) = \int_{-\infty}^{\infty} f(t)s^{-1/2}\psi(\frac{t-\tau}{s})dt$
Inverse transform (up to a proportionality constant)	$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega$	$f(t) = \int_{-\infty}^{\infty}\int_{0}^{\infty} W_c f(s,\tau)s^{-1/2}\psi(\frac{t-\tau}{s})\frac{ds\,d\tau}{s^2}$
Time transformed to	amplitude and phase for each frequency	amplitude for each scale and time
Input domain	**R** or **C**	**R** or **C**
Output range	**C**	**R** or **C**
Localization in frequency	Yes	Yes
Localization in time	No (limited with STFT)	Yes
Time for fast discrete transform	$O(n \log n)$	$O(n)$
Number of nonredundant outputs of discrete transform	n	n

Table 2: Comparison of Fourier and Wavelet Transforms

yields an amplitude and a phase. A signal may then be represented as the sum of sine waves whose phase and amplitude are given by the Fourier transform. Similarly, the wavelet transform yields an amplitude for each scale and time. A signal is represented as the sum across scales of time-centered wavelets whose amplitudes are given by the wavelet transform.

"Scale" is roughly "minus log frequency", in the following limited sense. A single scale s contains information from a band of frequencies. The width and the center frequency of the band are both proportional to $-\log s$. Each scale's band has the same ratio of bandwidth to center frequency, so scales correspond to a set of constant-Q filters. Figure 7 shows the logarithmic progression of band widths for the filters corresponding to different scales.

Both the Fourier transform and wavelet transform are said to "localize in frequency." That is, both produce an output that is nonzero only in a band when given an input that contains only frequencies from that band.

The wavelet transform also "localizes in time." A signal that is nonzero only during a finite span of time has a wavelet transform whose nonzero elements are concentrated around that time. On the other hand, the Fourier transform does not localize in time. For example, the Fourier transform of an impulse function contains high amplitudes at all frequencies. A wavelet transform of an impulse is either contained in a finite band of time or decreases expontentially with distance from the time of the impulse. STFTs, as discussed above, can be used to provide some localization in time, although the STFT

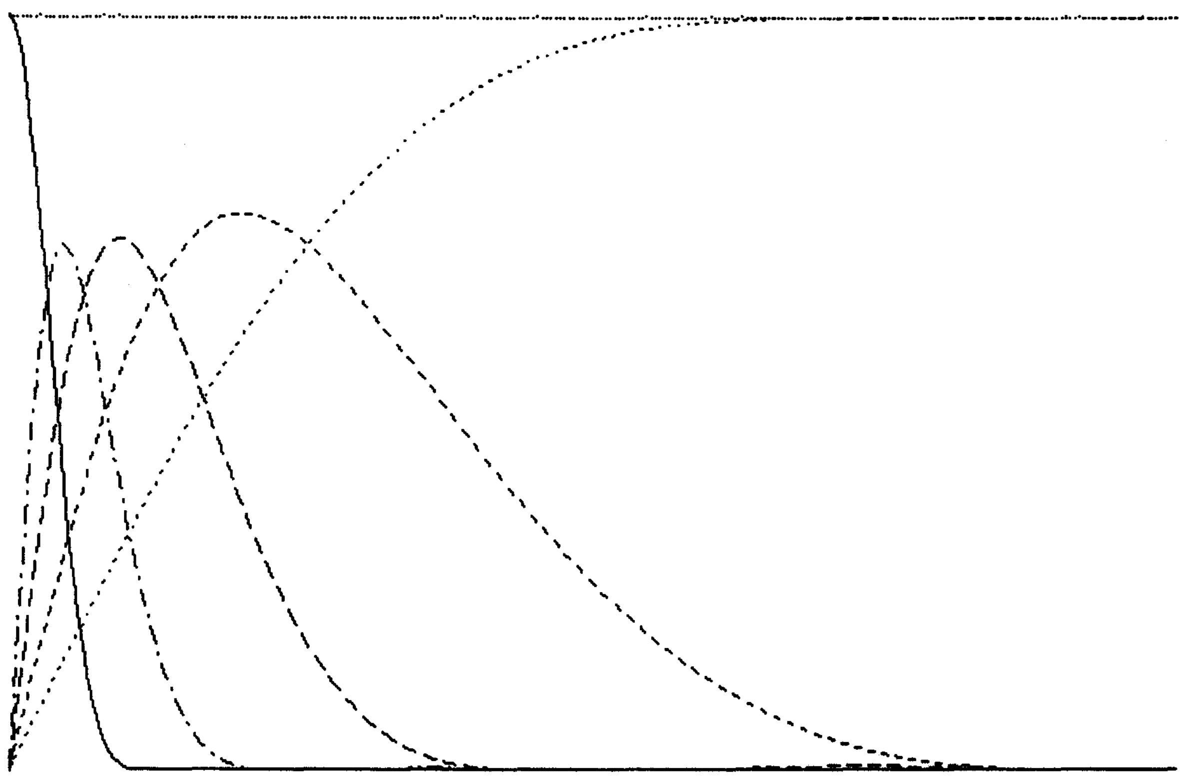

Figure 7: Logarithmic progression of frequency band width for filters corresponding to different scales, using the cubic spline analyzing wavelet. (frequency vs. logarithm of amplitude)

gains its localization in time by trading off both frequency resolution and bandwidth. No such tradeoff is required when using the wavelet transform.

There are discrete versions of both the Fourier transform and the wavelet transform. The discrete Fourier transform (DFT) is obtained from the Fourier transform by replacing the integral with a finite sum. The discrete wavelet transform (DWT) is obtained from the wavelet transform in the same way. Both the DFT and DWT take $O(n^2)$ time, where n is the number of input values. A DFT is usually computed using the Fast Fourier Transform (FFT) algorithm, which takes advantage of certain symmetries inherent in convolving with e^{iwt} when n is a power of two to reduce the time to $O(n \log n)$. Similarly, there is a Fast Wavelet Transform (FWT) algorithm that takes advantage of symmetries that are inherent in convolving with wavelets when n is a power of two. A FWT requires $O(n)$ time. The FFT takes n inputs and produces n outputs ($\frac{n}{2}$ complex numbers, or $\frac{n}{2}$ frequencies and $\frac{n}{2}$ phases). The FWT also takes n inputs and produces n outputs: 1 for $s = \log n$, 2 for $s = \log n - 1, \ldots, \frac{n}{2}$ for $s = 1$.

5 Examples

We now show wavelet transforms of some simple signals. All of the examples were computed with the FWT using $s_0 = 2$ (cf. Section 3).

Figure 8 shows the wavelet transform of an impulse, using the four coefficient Daubechies wavelet W_4 as the analyzing wavelet. Note that the maximum amplitude of the transform decreases at each scale after Scale 2. At first glance each scale of the transform resembles a dilated W_4. However, each scale has more oscillations than the analyzing wavelet. Just like W_4, each scale is finitely-supported and has an infinite number of non-differentiable points. Since W_4 is neither symmetric or antisymmetric, neither is the transform.

Figure 9 shows the wavelet transform of an impulse, using the eight coefficient Daubechies wavelet W_8 as the analyzing wavelet. With this analyzing wavelet the maximum amplitude of the transform decreases at every scale. Each scale of the transform has more oscillations than the corresponding scale in Figure 8, because W_8 has twice as many oscillations as W_4. Just as W_8 is smoother than W_4 (for example, W_8 is differentiable), this transform is smoother than the one in Figure 8. Since W_8 is neither symmetric or antisymmetric, neither is the transform.

Figures 10 and 11 show the wavelet transforms of a step function using W_4 and W_8. Many of the properties seen in the previous two figures still hold. For example, the transform under W_4 is non-differentiable, but the transform under W_8 is. Note, however, that the maximum amplitudes of the transform at different scales are all roughly equal.

Figure 12 shows the wavelet transform of a triangle wave, using the cubic spline wavelet. Note that the amplitude of the transform increases from scale to scale. Also, observe that where the signal is a straight line, the wavelet transform is constant in every scale.

Figure 13 shows the wavelet transform of two steps and an impulse. The analyzing wavelet is the so-called "cublic spline wavelet". This wavelet is a twice continuously

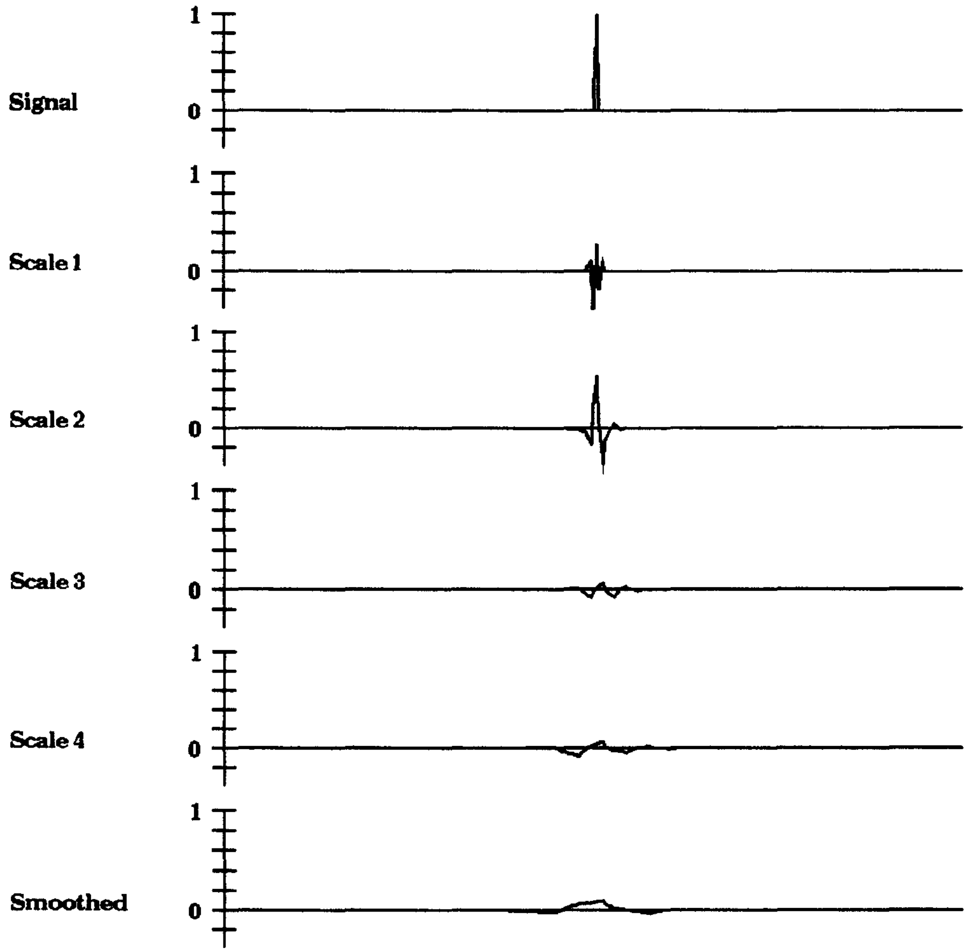

Figure 8: Wavelet transform of an impulse using the four coefficient Daubechies wavelet, W_4

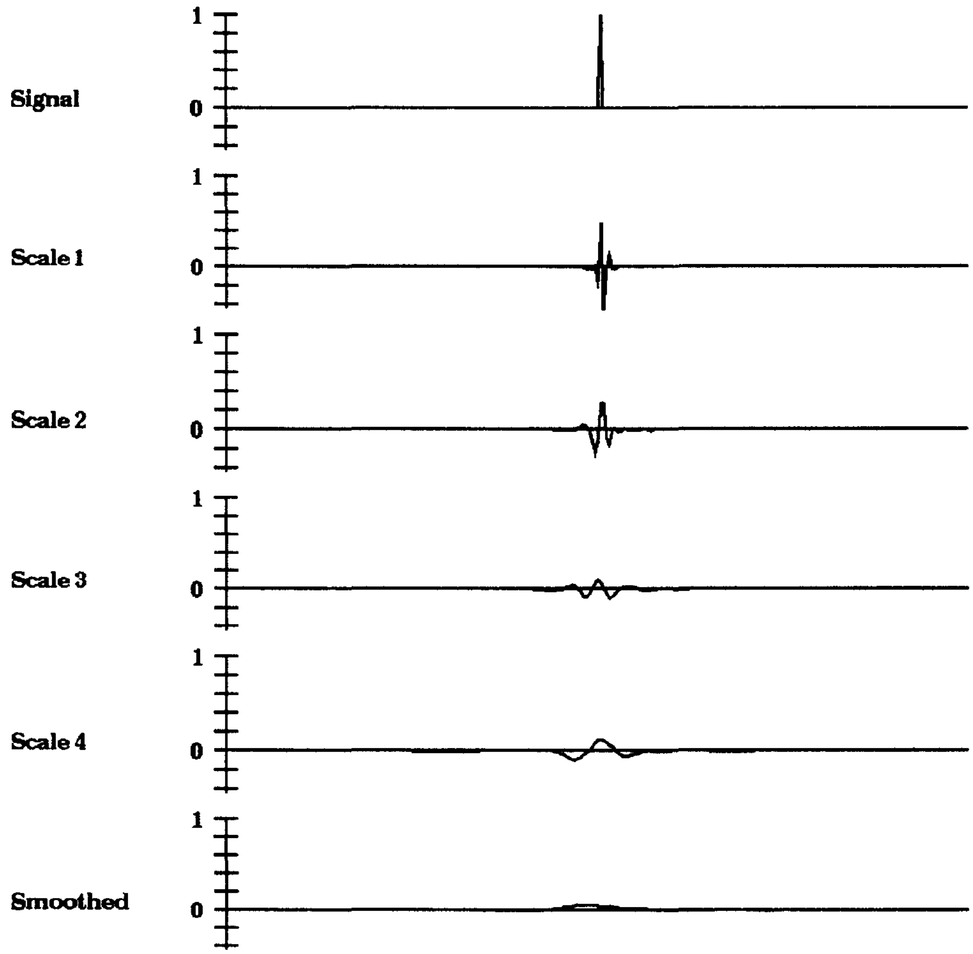

Figure 9: Wavelet transform of an impulse using the eight coefficient Daubechies wavelet, W_8

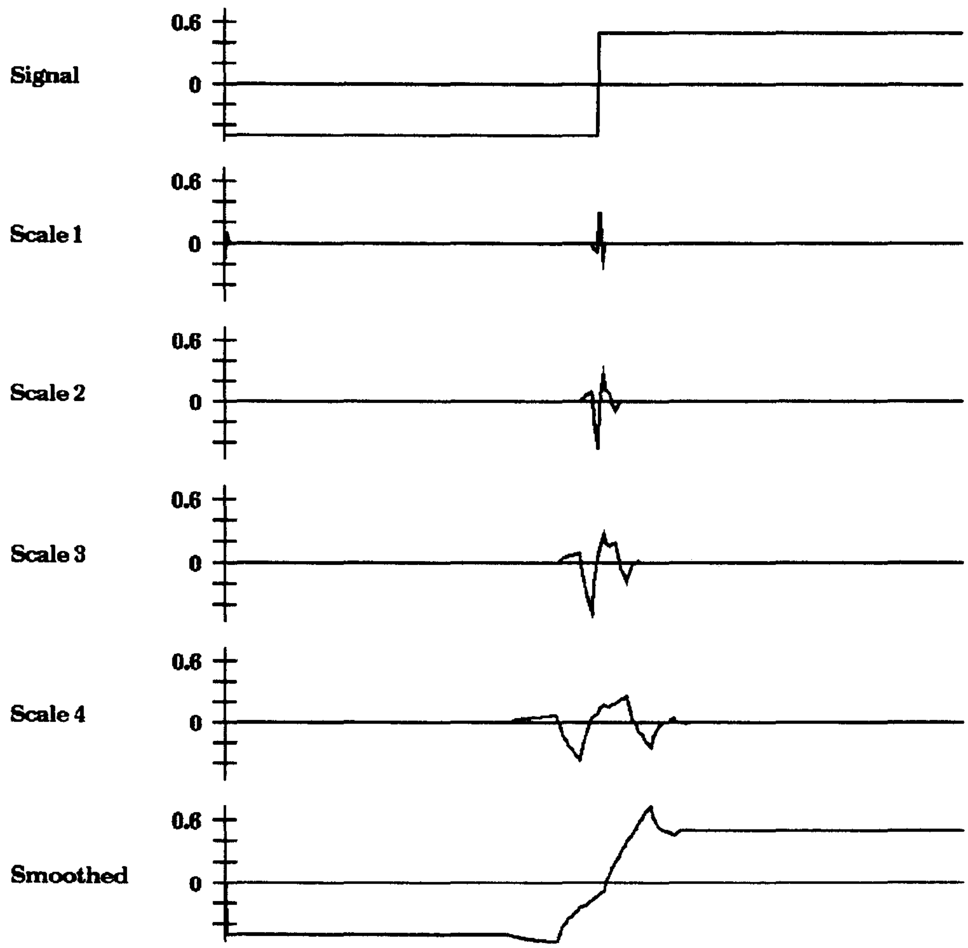

Figure 10: Wavelet transform of a step using W_4

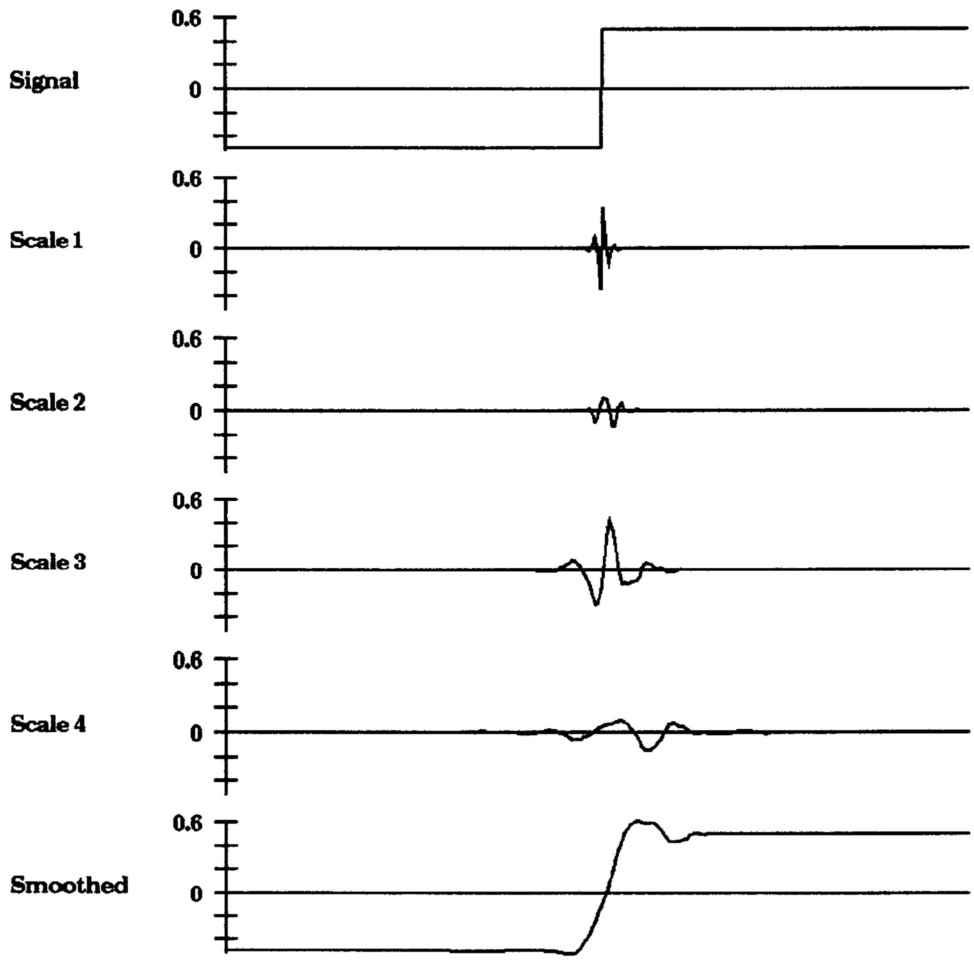

Figure 11: Wavelet transform of a step using W_8

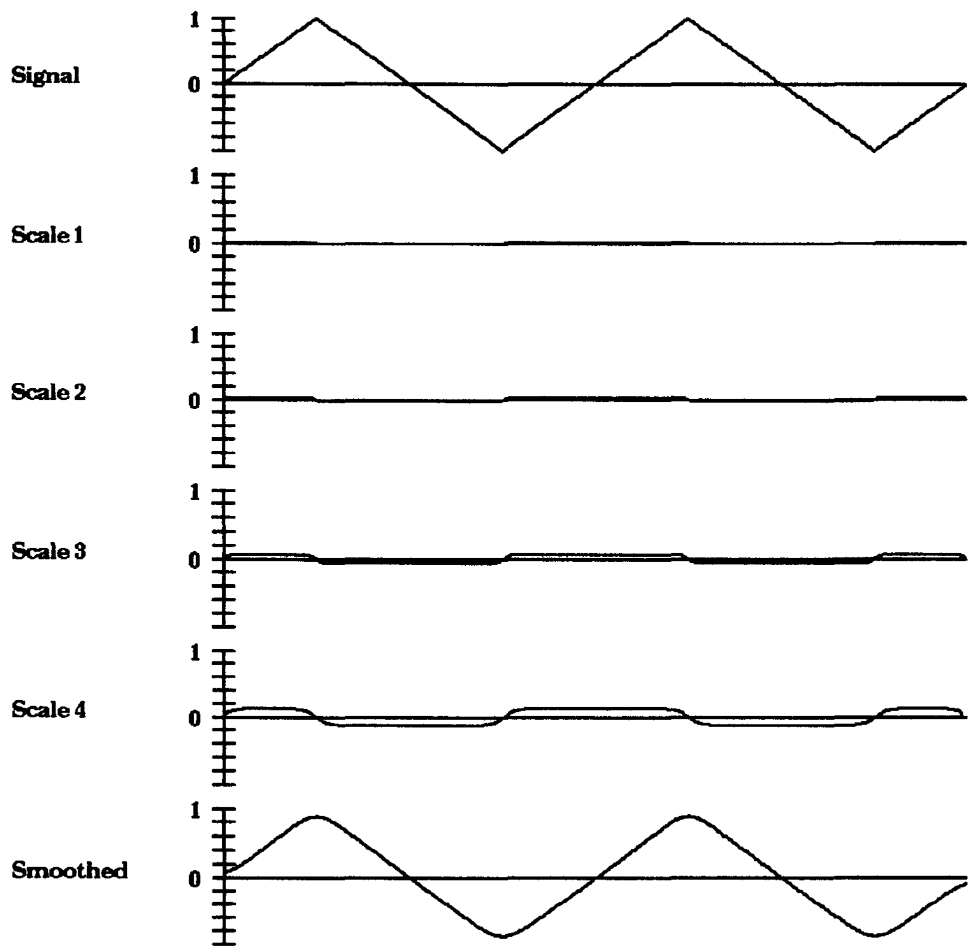

Figure 12: Wavelet transform of a triangle wave using a cubic spline wavelet

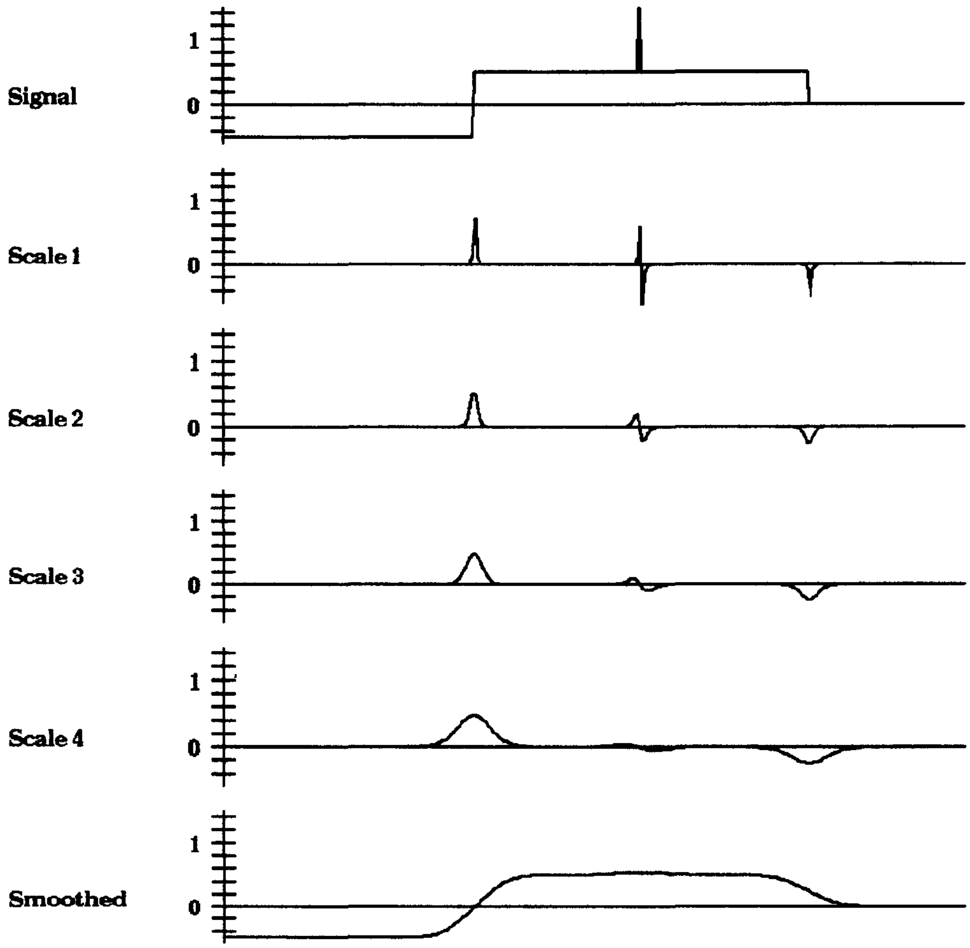

Figure 13: Wavelet transform of the sum of an impulse and two steps using a cubic spline wavelet

differentiable, piecewise cubic function. Unlike W_4 and W_8, it is antisymmetric. This antisymmetry causes the wavelet transform of symmetric signal features to be symmetric. This is visible in Figure 13, where the transform is symmetric about the steps and impulse.

The cubic spline wavelet does *not* have finite support. However, it does decay exponentially. In Figure 13, each scale of the transform contains small amplitude oscillations that extend infintely in both directions. These oscilations are invisible in the figure due to the rapid decay of the wavelet

Figure 13 also demonstrates the linearity of wavelet transforms. The wavelet transform of the two steps and impulse is the sum of the wavelet transforms of each step and of the impulse. In the signal the upwards step goes from -0.5 to 0.5, and the downwards step goes from 0.5 to 0. In the wavelet transform, at each scale, the peak resulting from the downwards step is opposite in sign and half the magnitude of the peak resulting from the upwards step.

Again, note that the wavelet transform of the impulse decays from scale to scale, but the wavelet transform of the steps stay roughly constant.

The signal in Figure 14 is the signal of Figure 13 plus noise uniformly distributed on $[-0.3, 0.3]$. Note that the transform peaks generated by the noise decay like the wavelet transform of an impulse. By Scale 4 the peaks generated by the step edges are pronounced, but the noise peaks are quite small. This observation is the basis of several signal denoising algorithms in the literature. (See, for example, [7]).

6 Conclusion

Due to their capability to localize in time, wavelet transforms readily lend themselves to nonstationary signal analysis. Detection of short duration events, on the other hand, are limited in Fourier analysis by the width of the windowing function used in the short-time Fourier transform. Wavelet transforms exist that project a finite energy function onto to an orthonormal basis of $L^2(\mathbf{R})$. The corresponding multiresolution analysis decomposes the function into a set of details at different resolutions and a smoothed version of the original function. As with the Fourier transform, a "fast wavelet transform" exists. However, the fast wavelet transform generates a multiresolution analysis in $O(n)$ time; whereas, a fast Fourier transform takes $O(n \log n)$ time.

Our intent in this paper was to present the basic concept of the wavelet transform from a viewpoint that targets signal analysis applications. Much of the current literature utilizes a high level of mathematical terminology. Our hope was to provide an brief introduction to the primary underlying ideas in a relatively intuitive manner.

For those who are interested, we provide an annotated bibliography that includes some of the key papers in the field. With each listing is a short description of the contents. Most of the papers require an understanding of Fourier analysis and sometimes an understanding of more general functional analysis principles. To help specify the mathematical sophistication of a paper, we adopt a relative rating scale, based upon our experience of reading the papers, 1 meaning little or no mathematical sophistication

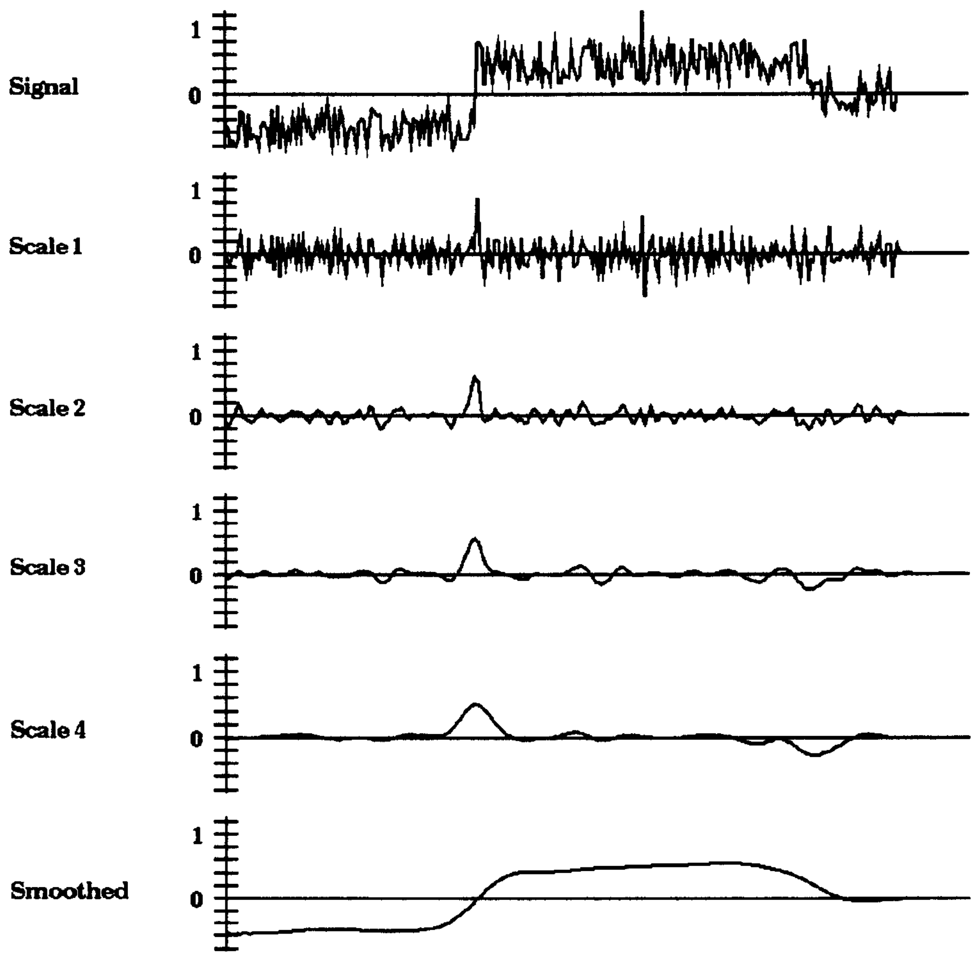

Figure 14: Wavelet transform of the sum of an impulse, a step and uniform noise using a cubic spline wavelet

required and 5 meaning a high level of mathematical sophistication required, relative to the other papers.

References

[1] Cody, Mac A. "The Fast Wavelet Transform," *Dr. Dobb's Journal*, April 1992.

This paper provides a quick overview of wavelets and the fast wavelet transform. It also contains and discusses some C code that implements a wavelet transform. Mathematical level–1.

[2] *Wavelets: Time-Frequency Methods and Phase Space.* Proceedings of the International Conference, Marseille, France, December 14-18, 1987. Combes, J. M., et al. eds. Springer-Verlag.

This book presents some of the earlier papers in the development of wavelets. The papers are organized into the five categories:

Part I: Introduction to Wavelet Transforms
Part II: Some Topics in Signal Analysis
Part III: Wavelets and Signal Processing
Part IV: Mathematics and Mathematical Physics
Part V: Implementations

Mathematical level–1 through 5

[3] Daubechies, Ingrid. "Orthonormal Bases of Compactly Supported Wavelets," *Communications on Pure and Applied Mathematics*, Vol. 41, No. 7, 1988, pp. 909-996.

In this paper Daubechies reviews multiresolution analysis, the Laplacian pyramid decomposition scheme, and the fast wavelet transform algorithm for doing a multiresolution analysis using a wavelet basis. She then demonstrates the classification of all orthonormal bases of wavelets with compact support. Mathematical level–4

[4] Grossman, A. and Morlet, J. "Decomposition of Hardy Functions into Square Integrable Wavelets of Constant Shape," *SIAM Journal of Mathematical Analysis*, Vol. 15, No. 4, July 1984, pp. 723-736.

This is the paper primarily responsible for the current interest in wavelet transforms. They discuss the definition of the transform, its inverse, the admissibility condition, and they prove that the wavelet transform is an isometry. Mathematical level–5

[5] Mallat, S. "Multifrequency Channel Decompositions of Images and Wavelet Models," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, Vol. 37, No. 12, December 1989, pp. 2091-2110.

We found this paper to be a good introductory paper to the wavelet transform. It starts by discussing multifrequency channel decompositions in psychophysics, reviews the short-time Fourier transform, and introduces the wavelet transform. It then proceeds to discuss pyramidal multiresolution decompositions, two dimensional wavelet transforms, and zero-crossings of multifrequency channels. Mathematical level–3

[6] Mallat, S. "A Theory for Multiresolution Signal Decomposition: the Wavelet Representation," *IEEE Transactions on Pattern Analysis and Machine Intellegence*, Vol. 11, No. 7, July 1989, pp. 674-693.

Mallat here presents his $O(n)$ algorithm for computing the fast wavelet transform. Furthermore, he discusses some of the technical conditions imposed upon the lowpass and bandpass filters necessary for them to correspond to the projection operators that generate the multiresolution analysis. (Daubechies further discusses these conditions in her paper cited above). He concludes by discussing image processing applications of the wavelet transform and by giving an example of a multiresolution approximation. Mathematical level–3

[7] Mallat, S. and Hwang, W. L. "Singularity Detection and Processing with Wavelets," *IEEE Transactions on Information Theory*, Vol. 38, No. 2, March 1992, pp. 617-643.

In this paper Mallat and Hwang discuss a means of using the wavelet transform to characterize the Lipschitz regularity of a function at a point. They discuss the detection and measurement of oscillating and nonoscillating singularities and illustrate a technique for denoising a signal based upon its wavelet transform modulus local maxima. Mathematical level–4

[8] Rioul, O. and Vetterli, M. "Wavelets and Signal Processing," *IEEE Signal Processing Magazine*, October 1991, pp. 14-38.

This is an introductory paper to wavelet transforms. It well illustrates the difference between the short-time Fourier transform and the wavelet transform. The paper contains a relatively complete set of references that we found very useful. Mathematical level–2

[9] *Wavelet and Their Applications.* Ruskai, M. B. et al. eds. Jones and Bartlett, 1992.

This book consists of a collection of papers organized into the categories

Signal Analysis
Numerical Analysis
Other Applications
Theoretical Developments

and contains many of the more recent papers. Mathematical level–3 through 5

[10] Strang, G. "Wavelets and Dilation Equations: A Brief Introduction," *SIAM Review*, Vol. 31, No. 4, December 1989, pp. 614-627.

Strang presents a brief mathematical introduction to multiresolution analysis and the fast wavelet tranform algorithm. However, the paper is rather terse in its explanations. Mathematical level–3

[11] *IEEE Transactions on Information Theory*, Vol. 38, No. 2, March 1992, Part II.

Part II of this issue is dedicated to wavelets and their applications. It contains many recent papers on the applications of wavelets in information theory related areas. Mathematical level–3 through 4