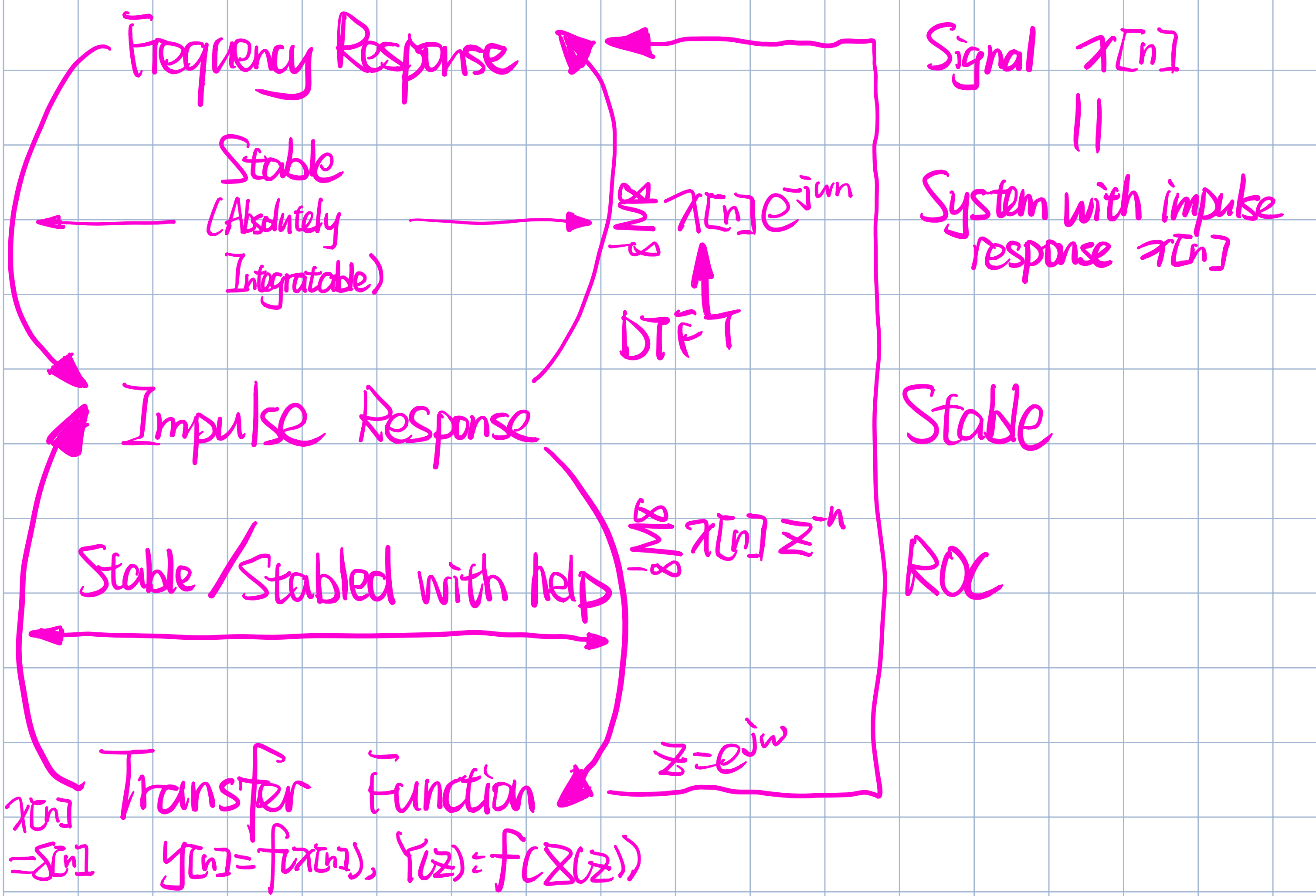


Impulse response, External appearance of system
Stability
(Absolute summable/Integrateable → Signal/System stable)
System's full response to input
(Output = Input * $h(t)$/$h[n]$)

Transfer function, Internal logistic of system
Representation of system with differential equation
Poles, Zeros
Poles = feedback
Zeros = feedforward

$Y(z) = 2X(z) + 3z^{-1}X(z) + 5z^{-2}X(z) + 2z^{-3}X(z)$

FIR filter: $y[n] = 2x[n] + 3x[n-1] + 5x[n-2] + 2x[n-3]$

$\rightarrow h[0] = 2\delta[0] + 3\delta[-1] + 5\delta[-2] + 2\delta[-3]$
$= 2$

$h[1] = 2\delta[1] + 3\delta[0] + 5\delta[-1] + 2\delta[-2]$
$= 3$ ...

Previous output (System's initial condition) Required. (System has feedback).

IIR filter: $y[n] + 2y[n-1] + 3y[n-2] = x[n]$

$\rightarrow h[0] + 2h[-1] + 3h[-2] = \delta(0) = 1$

$\rightarrow h[1] + 2h[0] + 3h[-1] = \delta(1) = 0$

$\rightarrow h[2] + 2h[1] + 3h[0] = \delta(2) = 0$

Assume causal initial condition: $h[n] = 0$ for $n < 0$

$\rightarrow h[0] = 1$

$h[1] = -2$

$h[2] = 1$

$h[3] = \dots$

$\vdots$

A filter is BIBO stable if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

(Which is also the condition for DTFT convergence)

DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Convergence:

$$\underline{\sum_{n=-\infty}^{\infty} |x[n]| < \infty}$$

Not always satisfied for any function

Z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= (e^{j\omega} \cdot a)^{-n} = z^{-n}$$

$$\sum_{n=-\infty}^{\infty} \boxed{x[n] \boxed{a^{-n}} e^{-j\omega n}}$$

Attenuation to be integrated in the system.

Convergence ($\sum_{-\infty}^{\infty} | \quad | < \infty$) Satisfied by most of functions if a is properly selected.

Also:

$$\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \rightarrow \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt$$

$$\int_{-\infty}^{\infty} x(t) e^{-st}$$

$$s = \sigma + j\omega$$

Region of Convergence of Z transform reflect our selection of a.

e.g. $x[n]=(0.5)^n u[n]$ ← Causal

$$X(z)=\sum_{n=0}^{\infty}(0.5z^{-1})^n=(0.5z^{-1})^0\frac{1-(0.5z^{-1})^\infty}{1-(0.5z^{-1})}$$

$$=\frac{1}{1-0.5z^{-1}},\ \text{if } |0.5z^{-1}|<1$$

→ $|z|>\frac{1}{2}$ → $a>\frac{1}{2}$

e.g.2 $x[n]=-(0.5)^n u[-n-1]$ ← Anti-Causal

$$X(z)=\sum_{n=-\infty}^{-1}-(0.5)^n z^{-n}=-\sum_{n=1}^{\infty}(2z)^n$$

$$=-(2z)^1\times\frac{1-(2z)^n}{1-2z}$$

$$=-\frac{2z}{1-2z}\ \text{if } |2z|<1$$

$$=\frac{1}{1-0.5z^{-1}}\ \text{if } |z|<\frac{1}{2}$$ → $0<a<\frac{1}{2}$

→ If Impulse response is causal (Which is required in real life)

↳ ROC will be outside of maximum pole circle ($a>xx$)

→ If Impulse response is stable

↳ FT of Impulse response exist.

↳ ROC should cover unit circle (proper selection of a should include 1)

$$X(z) = \frac{2 - 2.05z^{-1}}{1 - 2.05z^{-1} + z^{-2}}$$, What do we know about its impulse response?

$$= \frac{1}{1-0.8z^{-1}} + \frac{1}{1-1.25z^{-1}}$$ poles: 0.8, 1.25

- We assume $x[n]$ is causal
- ROC is $a > 1.25$ / $|z| > 1.25$
- ROC do not include $a = 1$
- $x[n]$ is not stable. $x[n]$ has no DTFT.
- $x[n] = (0.8)^n u[n] + (1.25)^n u[n]$

What if we let $z = e^{jw}$ and calculate frequency response?

→ The corresponding Impulse response of this frequency response is the non-causal solution

$$x[n] = (0.8)^n u[n] - (1.25)^n u[-n-1]$$

Poles and Zeros

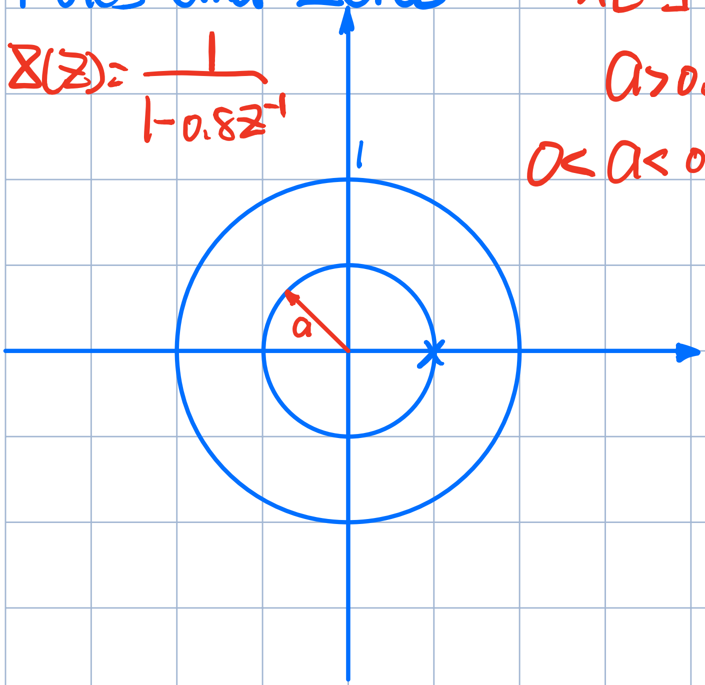
$X(z) = \frac{1}{1-0.8z^{-1}}$

$x[n] = (0.8)^n u[n]$

$a > 0.8 \to$ Convergence

$0 < a < 0.8 \to$ Divergence.

$$\sum_{n=-\infty}^{\infty} \boxed{x[n]\,\boxed{a^{-n}}}\,\boxed{e^{-j\omega n}}$$



It shows that:

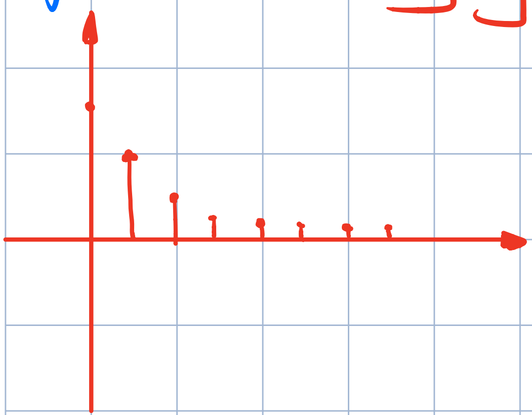
$0.8 < 1$, so $x[n]$ is absolute summable.

↳ Therefore, pole at 0.8 correspond to a exponentially decaying term over time

↳ The closer the pole is to the unit circle, the slower the decay goes.

↳ If the pole is outside of unit circle, then it is a exponentially increasing term

The closer the pole is to the origin point, the faster the decay goes.



Frequency domain:

Low pass, real pole $\to \theta = 0$

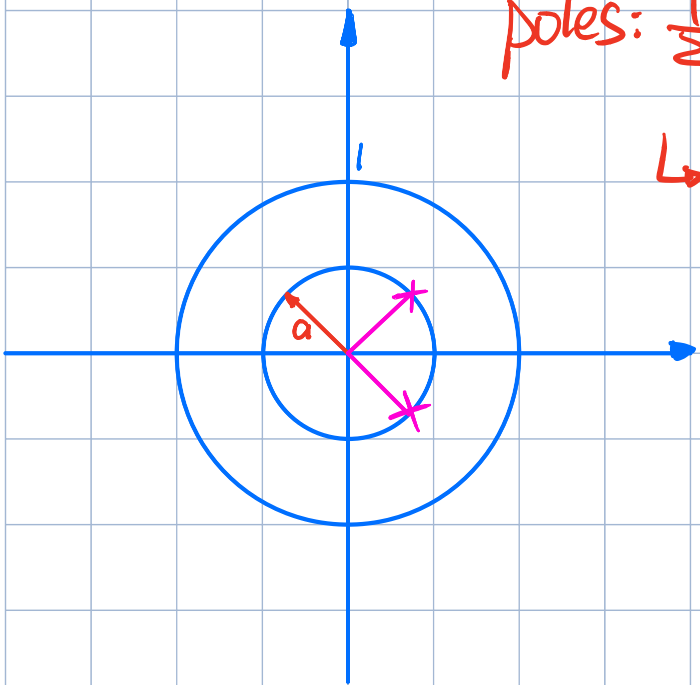
$\to$ Emphasize DC

Slow decay = sharp DC selection

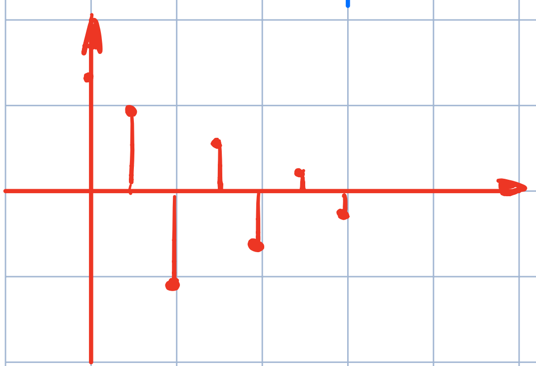
Fast decay $\to x[n] \approx 0$

$$X(z)=\frac{1}{1-z^{-1}+\frac{1}{2}z^{-2}}=\frac{1}{1-(\frac{1}{2}+\frac{1}{2}j)z^{-1}}+\frac{1}{1-(\frac{1}{2}-\frac{1}{2}j)z^{-1}}$$

poles: $\frac{1}{2}\angle\frac{\pi}{4}$, $\frac{1}{2}\angle-\frac{\pi}{4}$ ROC: $a>\frac{\sqrt{2}}{2}$



$$\hookrightarrow x[n]=\left(\frac{\sqrt{2}}{2}e^{j\frac{\pi}{4}}\right)^n+\left(\frac{\sqrt{2}}{2}e^{-j\frac{\pi}{4}}\right)^n u[n]$$
$$=\left(\frac{\sqrt{2}}{2}\right)^n\left[e^{j\frac{\pi}{4}n}+e^{-j\frac{\pi}{4}n}\right]u[n]$$
$$=\left(\frac{\sqrt{2}}{2}\right)^n\times 2\times\cos\left(\frac{\pi}{4}n\right)u[n]$$



It shows that Conjugate poles correspond to a decaying sinusoid.

pole at $A\angle\theta$, $A\angle-\theta$

A: Speed of decay. The closer A is to 1, the slower the decay.

θ: Frequency of sinusoid, $0\sim\pi$

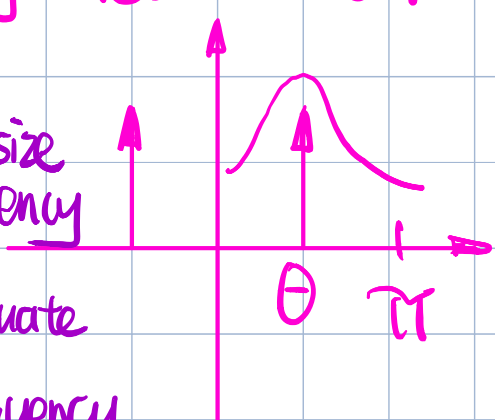
Filtering = Frequency selection + Decay to prevent system divergence.

Slow decay: $x[n]\approx\cos\theta n$

Frequency domain

Pole = Emphasize frequency

Zero = Attenuate frequency



A: Slow decay → sharp frequency selection, closer to divergence

θ: Selected Frequency.

Filter Design = Obtain Transfer function

By Spectrum (Frequency response → Impulse response → Transfer function)

e.g. Truncated ideal filter More order, larger delay, closer to ideal

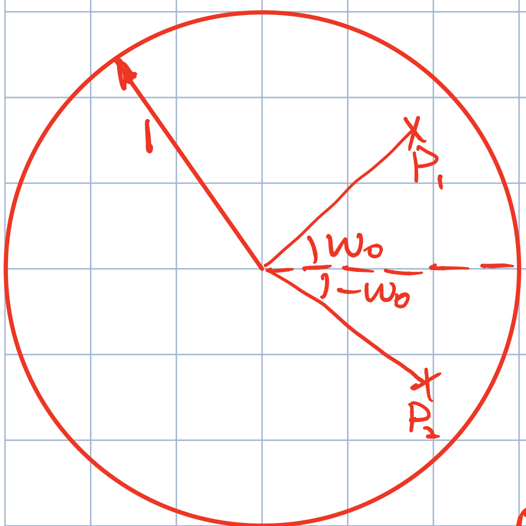
e.g. FIR filter design (Ideal filter truncation + Hamming window)

$H[n] = [h_0, h_1, h_2 \ldots h_n]$

$\rightarrow H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots + h_n z^{-n}$ (No poles)

By Pole Placing (Directly design Transfer function)

e.g. Resonator Filter $P_1 = re^{jw_0}$, $P_2 = re^{-jw_0}$ (Pole Placing)

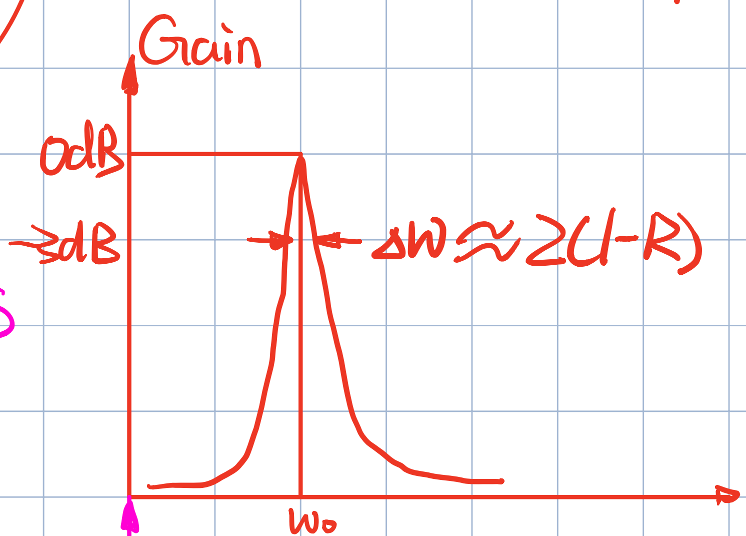


$$H(z) = \frac{Gain}{(1-re^{jw_0}z^{-1})(1-re^{-jw_0}z^{-1})}$$

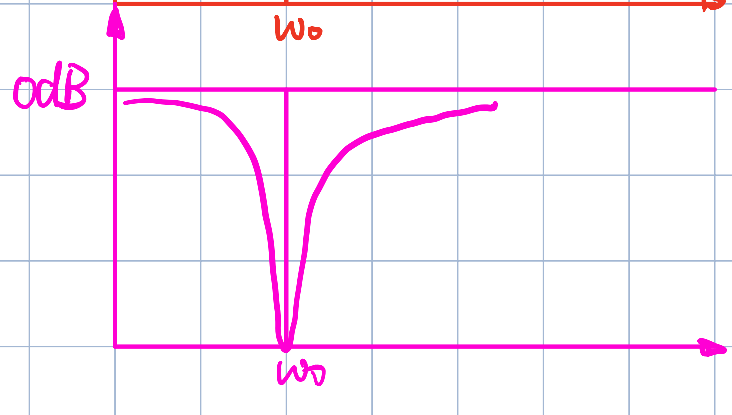
$$= \frac{Gain}{1-2r\cos w_0 z^{-1} + r^2 z^{-2}}$$

$H(w_0) = 1 \rightarrow Gain = (1-r)\sqrt{1-2r\cos(2w_0)+R^2}$

Effect of poles



Effect of zeros

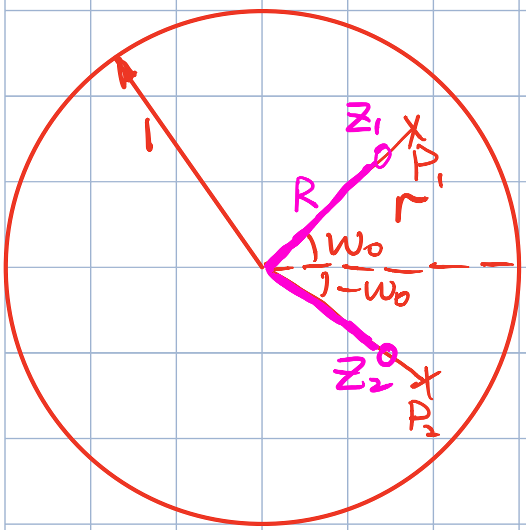


Adding zeros : $z_1 = Re^{j\omega_0}$, $z_2 = Re^{-j\omega_0}$ (Zeros Placing)

$P_1 = re^{j\omega_0}$, $P_2 = re^{-j\omega_0}$ (Pole Placing)

$$H(z) = \frac{(1-Re^{j\omega_0}z^{-1})(1-Re^{-j\omega_0}z^{-1})}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}$$

$$= \frac{1-2R\cos\omega_0 z^{-1}+R^2z^{-2}}{1-2r\cos\omega_0 z^{-1}+r^2z^{-2}}$$

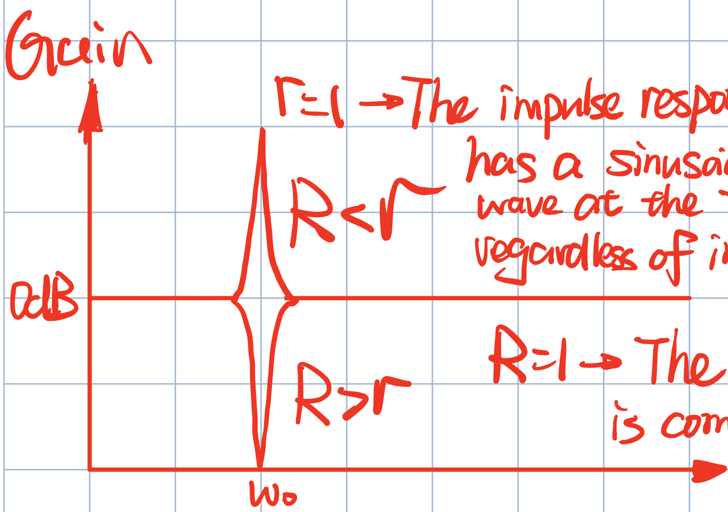


$R \simeq r \rightarrow H(z)$ close to 1

$1 > R > r$ zero take advantage

$1 > r > R$ Pole take advantage.

Useful to flat-Out frequency response at frequencies further from ω_0. (Set to 0dB)

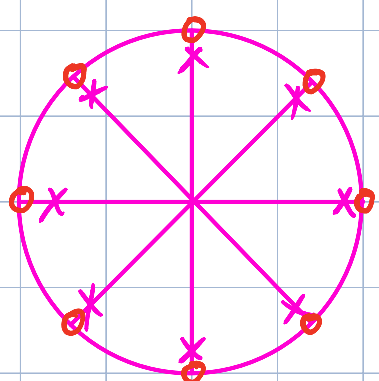


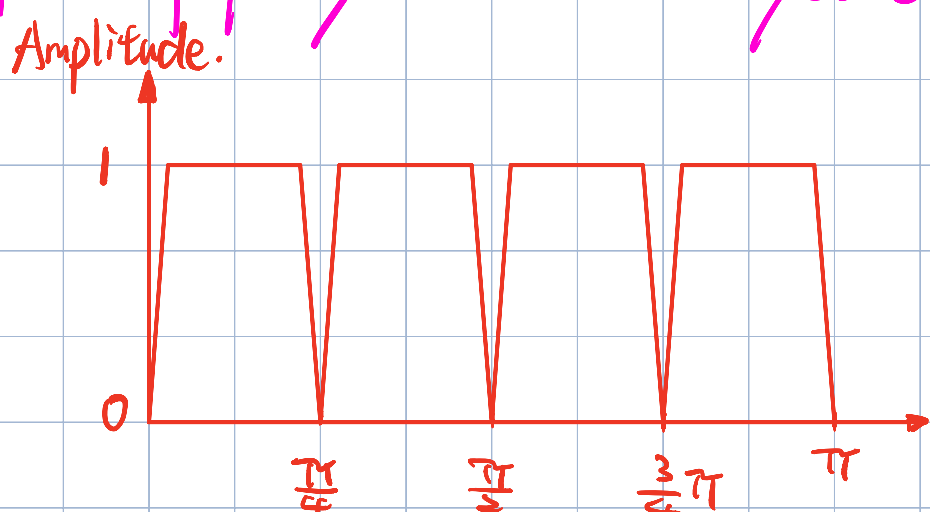
$r=1 \rightarrow$ The impulse response has a sinusoid wave at the frequency, regardless of input.

So other $\rightarrow$ frequencies are lessly affected.

$R=1 \rightarrow$ The impulse response is completly free of the frequency component, regardless of input.

Adding More pairs of poles/zeros $\rightarrow$ Notch/Comb Filter

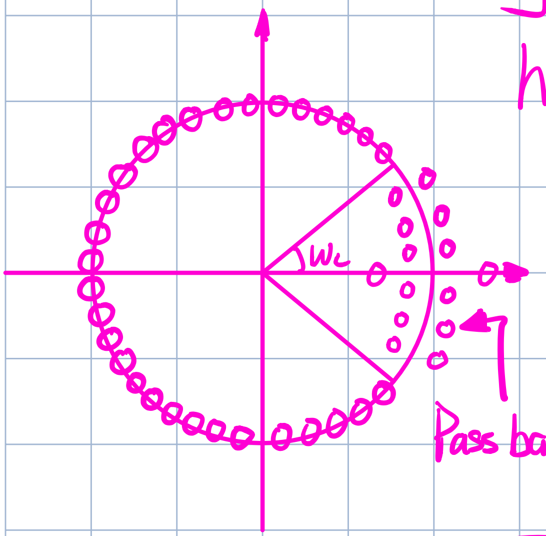




FIR: $b(n) = w(n)\,h(n)$

$h(n)$: Ideal filter's impulse response

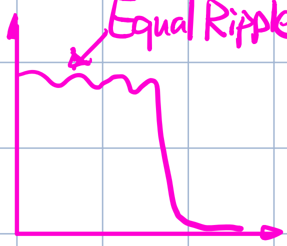
$w(n)$: Hamming window.

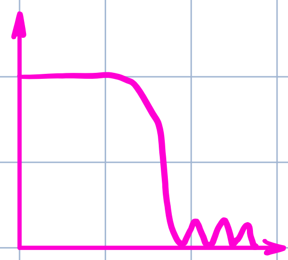


$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}$$
$$= (1-q_1 z^{-1})(1-q_2 z^{-1})\cdots(1-q_n z^{-1})$$

Filter order $= n$

Total # of zeros $= n$





IIR

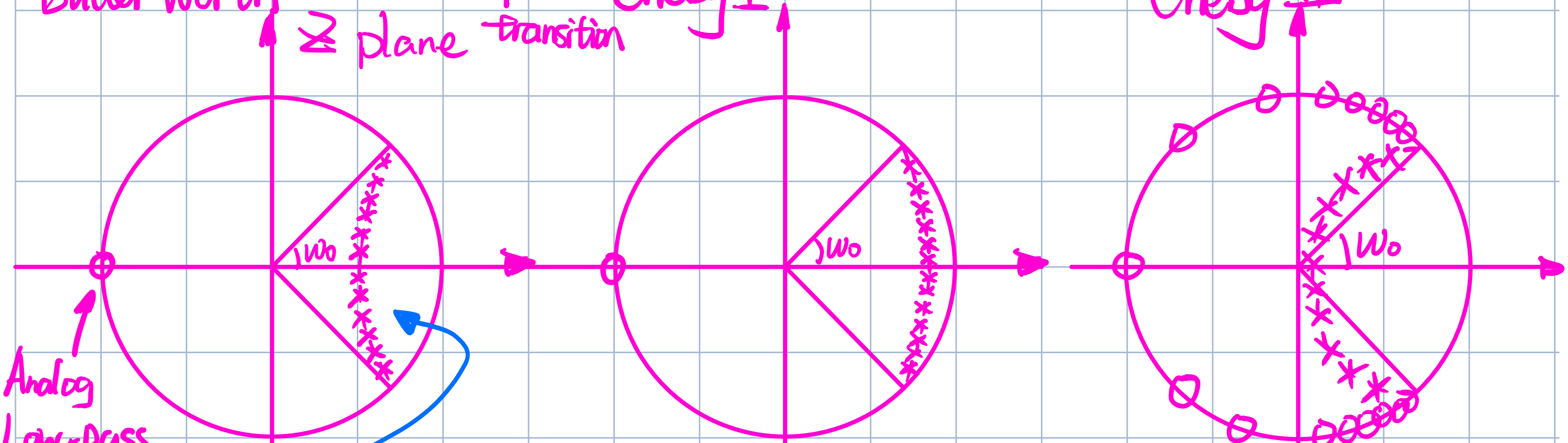
Butterworth

Smaller order, steeper transition

Cheby I

Cheby II

Z plane

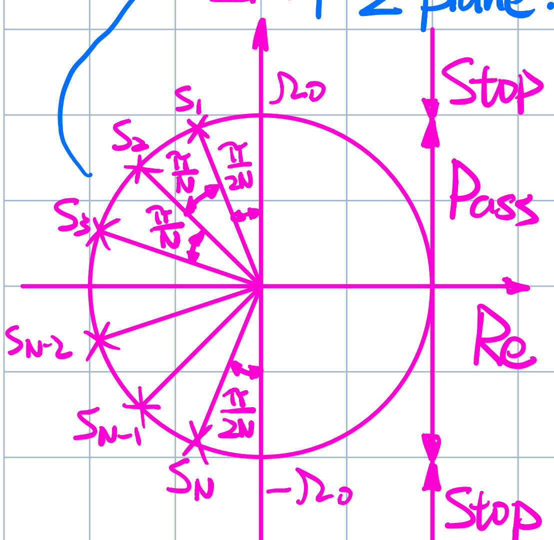


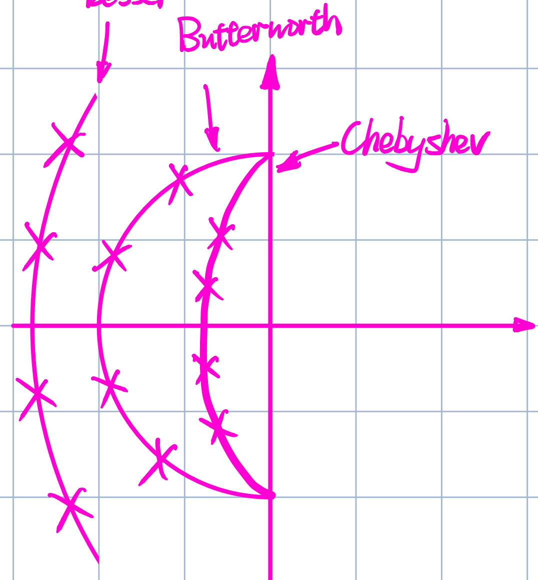
Analog Low-pass filter has a hidden zero at $\Omega = \infty$

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon_{pass}^2 C_N^2\left(\frac{\Omega}{\Omega_0}\right)}$$

$$|H(\Omega)|^2 = \frac{1}{1+\frac{1}{\varepsilon_{pass}^2 C_N^2}\left(\frac{\Omega}{\Omega_0}\right)^2}$$

Bilateral transformation from S plane to Z plane.





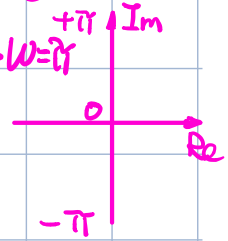
Butterworth

N: Filter order

$\Omega = \tan\frac{w}{2}$

$S = j\Omega$, $z = e^{jw}$ → Bilateral Transformation $S = \frac{1-z^{-1}}{1+z^{-1}} \to \Omega = \tan\frac{w}{2}$

$\Omega = \infty \to w = \pi$



$$|H(s)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_0}\right)^{2N}}$$

More order, Better filter, always -3dB at Ω_0.

$\Omega_0 = \tan\frac{w_0}{2}$ (Cut-off frequency).

$$H(s)H^*(-s) = \frac{1}{1+\left(\frac{s}{j\Omega_0}\right)^{2N}}$$

$\left(|H(s)|^2 = H(s)\,H(s)^*\right.$
$\left.H(s)^* = H^*(-s) \quad \text{E.g. } H(s) = \sum a_n s^n \to H(s)^* = \sum a_n^*(-s)^n\right)$

$$D(s)D^*(-s) = 1 + (-1)^N\left(\frac{s}{\Omega_0}\right)^{2N}$$

→ poles: $S_i = \Omega_0 e^{j\theta_i}$, $\theta_i = \frac{\pi}{2N}(N-1+2i)$

$$\hookrightarrow H(s) = H_0(s) \times \prod_{i=1}^{K} H_i(s),$$

$$H_0(s) = \begin{cases} 1 & \text{if } N = 2K \\ \frac{1}{1+\frac{s}{\Omega_0}} & \text{if } N = 2K+1 \end{cases}$$

$$H_i(s) = \frac{1}{1 - 2\frac{\cos\theta_i}{\Omega_0}s + \frac{s^2}{\Omega_0^2}} \Rightarrow S = \frac{1-z^{-1}}{1+z^{-1}} \to H(z).$$

Ideal Differentiator

order = 31

$x'[n] = \frac{x(n) - x(n-\alpha)}{\alpha}$ $\quad\alpha$: The step size

→ $H(z) = \frac{1-z^{-\alpha}}{\alpha}$

$H(w) = \frac{1-e^{-j\alpha w}}{\alpha}$ $\quad e^{\theta} = 1 + \sum_{k=1}^{\infty}\frac{\theta^k}{k!}$

→ $H(w) = jw + \sum_{k=1}^{\infty} a_k \alpha^k$

$\alpha \to 0 \to H(w) = jw = A(w)e^{j\theta(w)}$

$A(w) = w$, $\theta(w) = \frac{\pi}{2}$

diff: $[1\ -1] \to H(z) = 1 - z^{-1} \to H(w) = 1 - e^{-jw} = A(w)e^{j\theta(w)}$

$A(w) = 2 - 2\cos w \quad \theta(w) = \frac{\pi}{2} - \frac{1}{2}w$

All-pass filter (Phase manipulation)

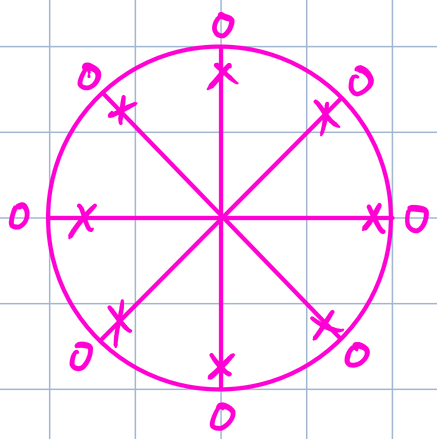
→ $|H(\omega)|=1$

→ $H(z)= A\prod \frac{z^{-1}-p_k}{1-p_k z^{-1}} \prod \frac{(z^{-1}-p_m)(z^{-1}-p_m^*)}{(1-p_m z^{-1})(1-p_m^* z^{-1})}$

↳ Real valued poles.

↳ Conjugate pole pairs

→ If P_0 is a pole → P_0^* is a pole, $\frac{1}{P_0}$, $\frac{1}{P_0^*}$ are zeros.



Inverse System.

→ $H(z)\times H'(z)=1$

→ $H'(z)=\frac{1}{H(z)}$

If $H(z)=\frac{b_0+b_1 z^{-1}+b_2 z^{-2}+\cdots}{1+a_1 z^{-1}+a_2 z^{-2}+\cdots}$ → zeros, poles position interchange.

Linear Phase Response for FIR filter = Constant Phase Delay

$H(\omega) = |H(\omega)| e^{j\beta(\omega)}$, $|H(\omega)| > 0$, $\beta(\omega)$ discontinue

$= A(\omega) e^{j\theta(\omega)}$ — Magnitude response

Amplitude response , $\theta(\omega)$ continuous

Linear phase $\Rightarrow \theta(\omega) = M \cdot \omega + B$, $M < 0$

$x[n] = \cos(\omega_1 n + \phi_1) \rightarrow \boxed{H(\omega) = A(\omega) e^{j\theta(\omega)}}$

$y[n] = A(\omega_1) \cos[\omega_1 n + \theta(\omega_1) + \phi_1]$

Constant Term in $\theta(\omega)$ goes here.

$= A(\omega_1) \cos[\omega_1 (n + \frac{\theta(\omega_1)}{\omega_1}) + \phi_1]$

If $\frac{\theta(\omega_1)}{\omega_1} = M \rightarrow y[n] = A(\omega_1) \cos[\omega_1 (n + M) + \phi_1]$

$y[n]$ is just $x[n]$ delayed by $-M$ with a new amplitude

If M is not constant $\rightarrow$ The delay of frequency component ω_1 is $M(\omega_1)$.

$M(\omega) = \frac{\theta(\omega)}{\omega}$ is the Phase Delay of Filter.

4 Types of FIR filter (real impulse response) has linear phase response:

1. Symmetric, Odd length
2. Symmetric, Even length

$\Big\}\ h[n] = h[N-1-n]$

$0 \le n \le N-1$

$+\frac{\pi}{2}$ phase shift $\Big\{$

3. Anti-Symmetric, Odd length
4. Anti-Symmetric, Even length.

$\Big\}\ h[n] = -h[N-1-n]$

Phase Delay $= \frac{N-1}{2}$

Order of FIR filter $= N-1$ $=$ # of zeros

Type I e.g. $h[n] = [h_0, h_1, h_2, h_1, h_0]$, $N=5$

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + h_1 z^{-3} + h_0 z^{-4}$$

$$\hookrightarrow H(\omega) = h_0 + h_1 e^{-j\omega} + h_2 e^{-2j\omega} + h_1 e^{-3j\omega} + h_0 e^{-4j\omega}$$

$$= e^{-2j\omega}[h_0 e^{2j\omega} + h_1 e^{j\omega} + h_2 + h_1 e^{-j\omega} + h_0 e^{-2j\omega}]$$

$$= e^{-2j\omega}[\ h_2 + 2h_1 \cos\omega + 2h_0 \cos 2\omega]$$

$$= [2h_0 \cos 2\omega + 2h_1 \cos\omega + h_2]\, e^{-2j\omega}$$

$$= A(\omega) e^{j\theta(\omega)},$$

$\theta(\omega) = -2\omega$, Phase Delay $= 2 = \frac{N-1}{2}$

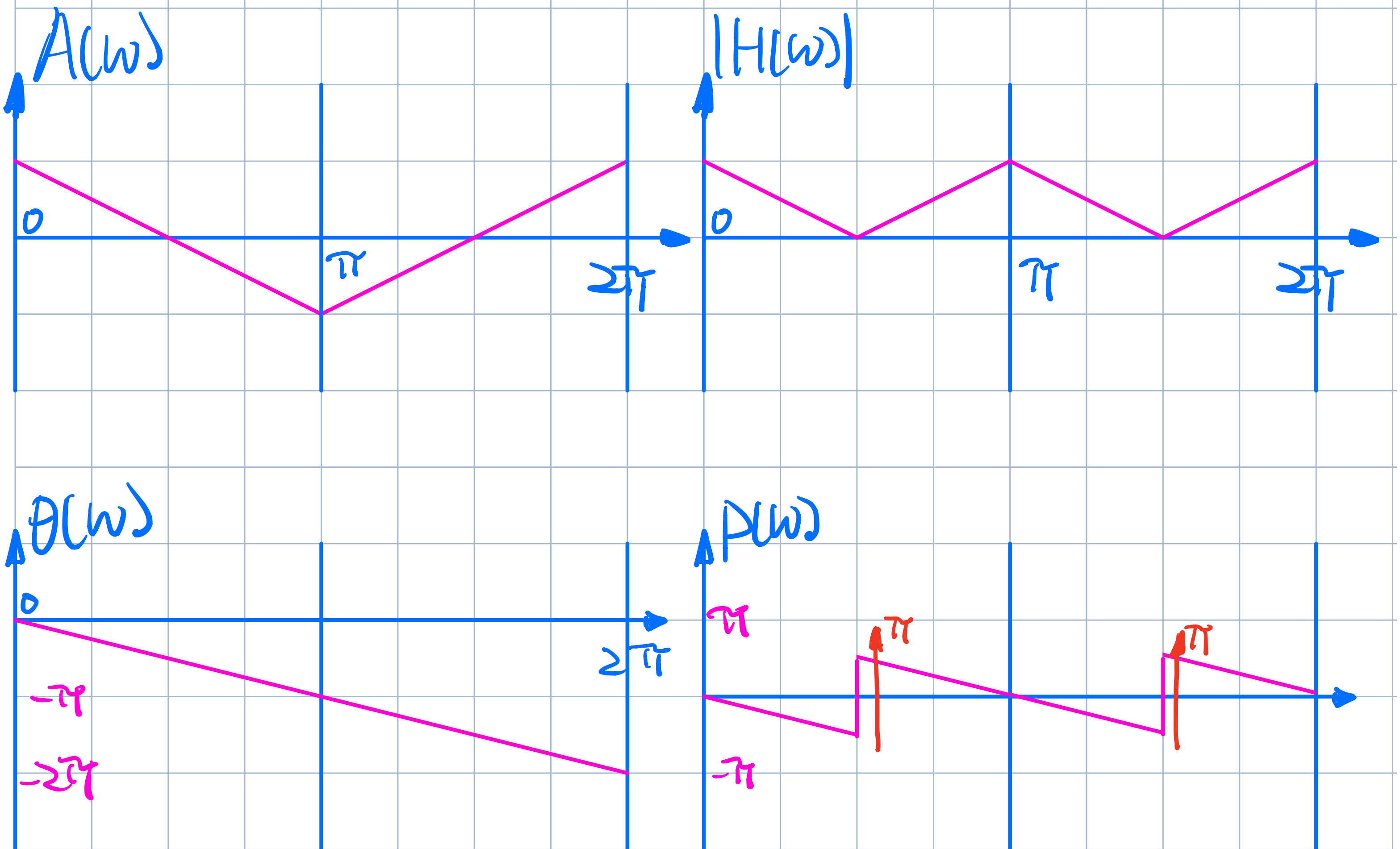
$$A(\omega) = 2h_0 \cos 2\omega + 2h_1 \cos\omega + h_2$$

$$= 2h_0 \cos\left(\tfrac{N-1}{2}\right)\omega + 2h_1 \cos\left(\tfrac{N-1}{2}-1\right)\omega + \cdots + 2h_{m-1}\cos\omega + h_m, \quad m = \frac{N-1}{2}$$

$$\hookrightarrow A(\omega) = A(-\omega)$$

$$A(\pi+\omega) = A(\pi-\omega)$$

$$A(\omega+2\pi) = A(\omega)$$



Type 2 e.g. $h[n] = [h_0, h_1, h_1, h_0]$ $N=4$

$\theta(\omega) = -\frac{3}{2}\omega$, Phase Delay $= \frac{3}{2} = \frac{N-1}{2}$

$A(\omega) = 2h_0 \cos\frac{3}{2}\omega + 2h_1 \cos\frac{1}{2}\omega$

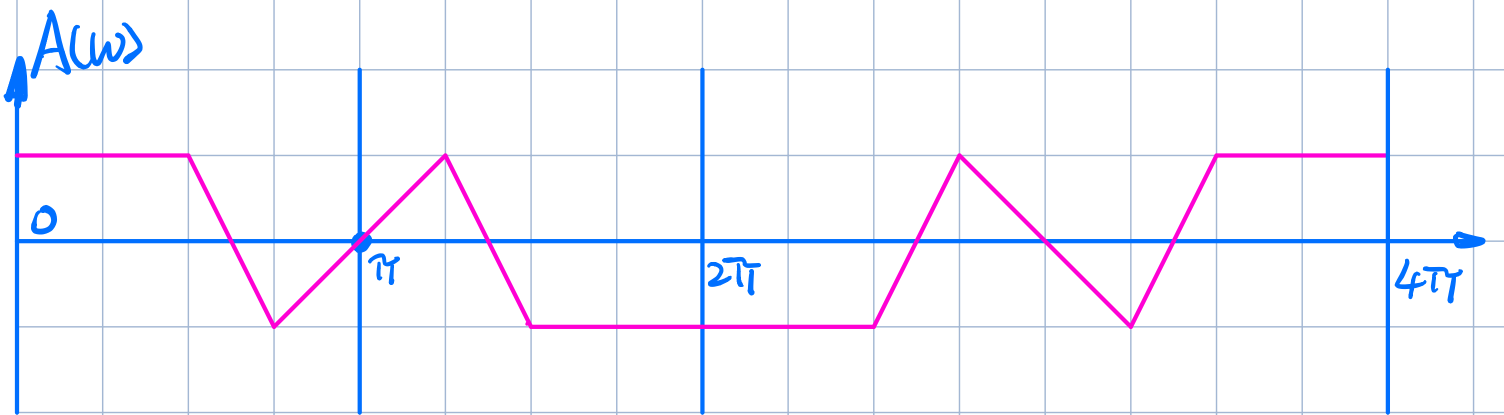
$= 2h_0 \cos(\frac{N-1}{2})\omega + 2h_1 \cos(\frac{N-1}{2}-1)\omega$

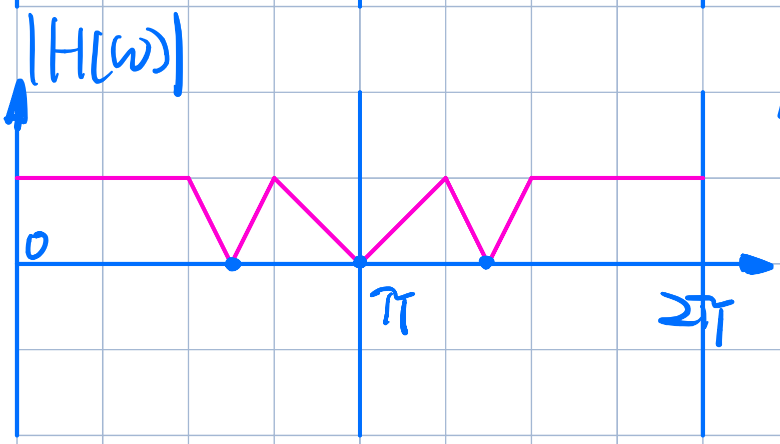
$+ \cdots + 2h_m \cos\frac{1}{2}\omega$, $m = \frac{N}{2} - 1$

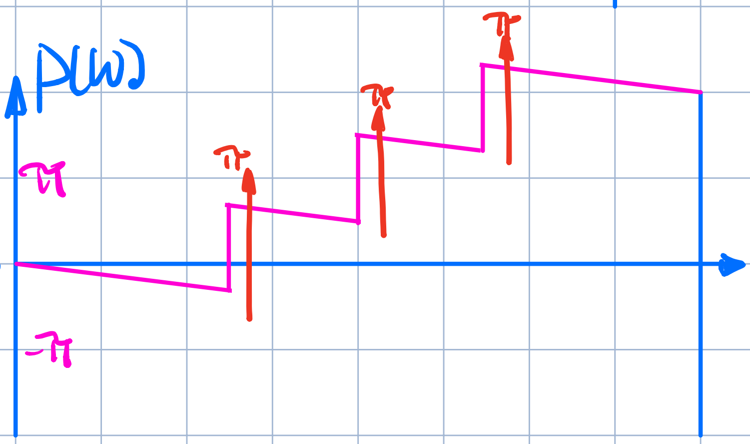
$\hookrightarrow A(\omega) = A(-\omega)$ $A(\pi) = 0$

$A(\pi+\omega) = -A(\pi-\omega)$ $A(\omega+2\pi) = -A(\omega)$

$A(\omega+4\pi) = A(\omega)$







Type 3 e.g. $h[n]=[h_0, h_1, 0, -h_1, -h_0]$ $N=5$

$\hookrightarrow H(\omega)=h_0+h_1e^{-j\omega}-h_1e^{-3j\omega}-h_0e^{-4j\omega}$

$=e^{-2j\omega}[h_0e^{2j\omega}+h_1e^{j\omega}-h_1e^{-j\omega}-h_0e^{-2j\omega}]$

$=e^{-2j\omega}[2jh_1\sin\omega+2jh_0\sin 2\omega]$

$=[2h_0\sin 2\omega+2h_1\sin\omega]\, j\, e^{-2j\omega}$

$=A(\omega)e^{j\theta(\omega)}$,

$\theta(\omega)=-2\omega+\frac{\pi}{2}$, Phase Delay $=2=\frac{N-1}{2}$ and $+\frac{\pi}{2}$ phase shift.

$A(\omega)=2h_0\sin 2\omega+2h_1\sin\omega$

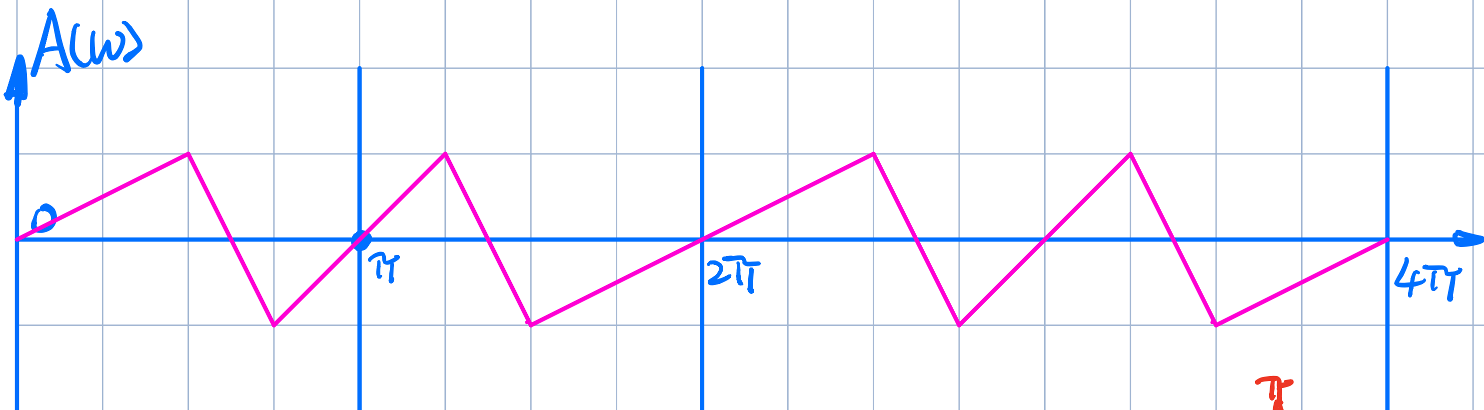
$=2h_0\sin(\frac{N-1}{2})\omega+2h_1\sin(\frac{N-1}{2}-1)\omega$

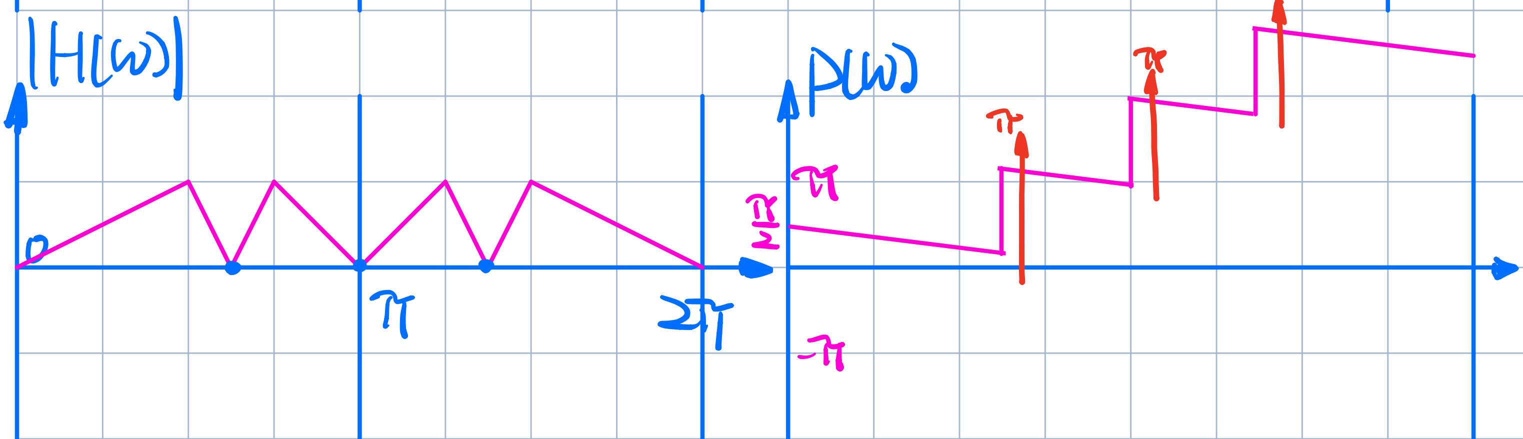
$+\cdots+2h_{m-1}\sin\omega$ $\quad m=\frac{N-1}{2}$

$\hookrightarrow A(\omega)=-A(-\omega)$ $\quad A(\pi)=0, A(0)=0$

$A(\pi+\omega)=-A(\pi-\omega)$.

$A(\omega+2\pi)=A(\omega)$





Type 4 e.g. $h[n]=[h_0, h_1, -h_1, -h_0]$ $N=4$

$\theta(\omega)=-\frac{3}{2}\omega+\frac{\pi}{2}$, Phase Delay $=\frac{3}{2}=\frac{N-1}{2}$ and $+\frac{\pi}{2}$ phase shift.

$A(\omega)=2h_0\sin\frac{3}{2}\omega+2h_1\sin\frac{1}{2}\omega$

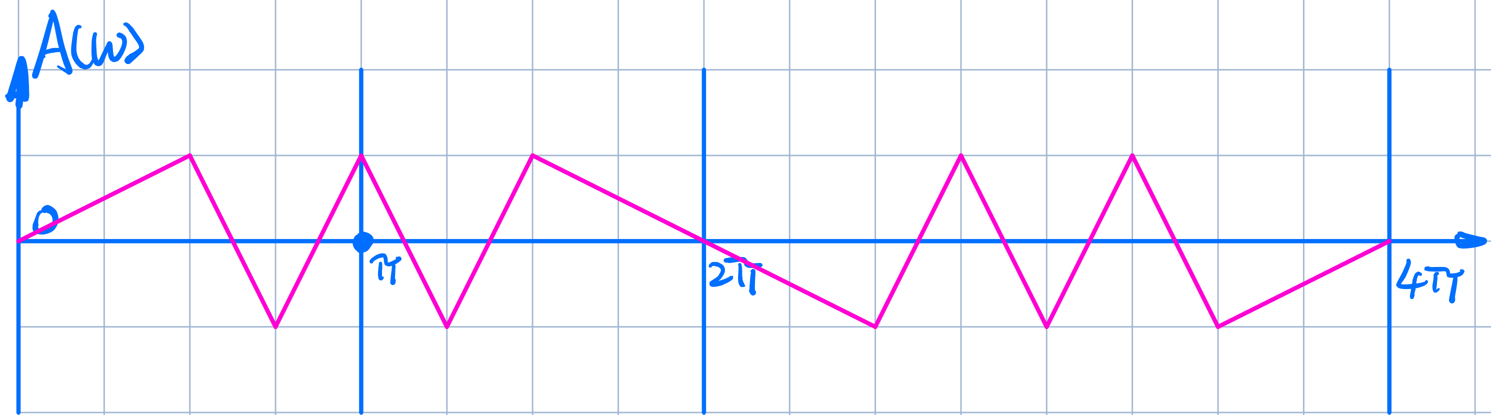
$=2h_0\sin(\frac{N-1}{2})\omega+2h_1\sin(\frac{N-1}{2}-1)\omega$

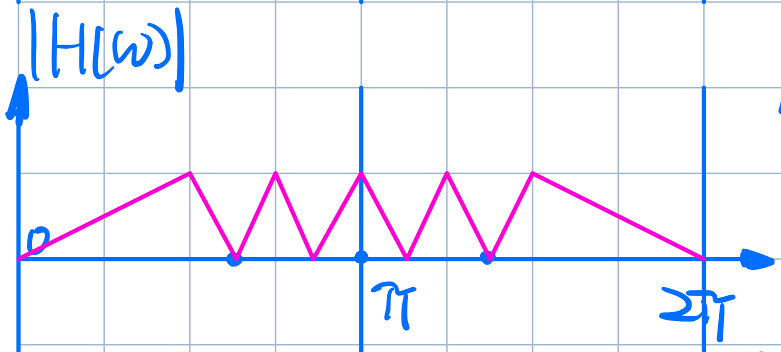
$+\cdots+2h_m\sin\frac{1}{2}\omega$, $m=\frac{N}{2}-1$

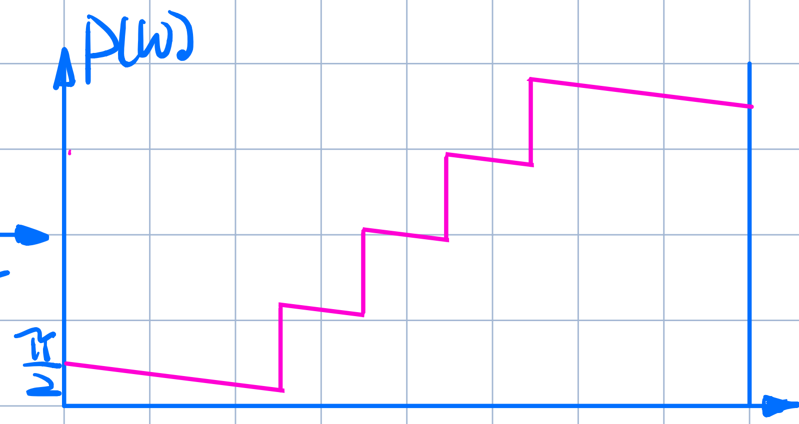
↳ $A(\omega)=-A(-\omega)$ $A(0)=0$

$A(\pi+\omega)=A(\pi-\omega)$. $A(\omega+2\pi)=-A(\omega)$

$A(\omega+4\pi)=A(\omega)$







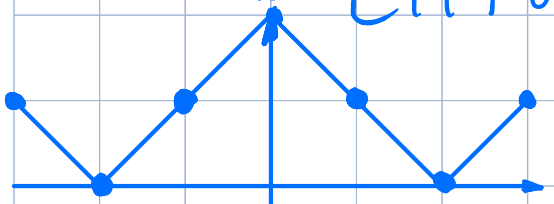
I Symmetric, Odd length, Even order, LP, HP, BP, BS

II Symmetric, Even length, Odd order $A(\pi)=0$ LP, BP

III Anti-Symmetric, Odd length, Even order $A(\pi)=0$ } $\frac{\pi}{2}$ phase shift

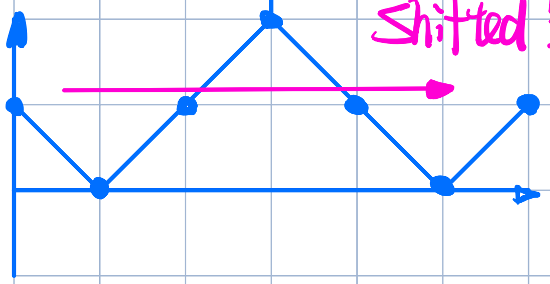
IV Anti-Symmetric, Even length, Odd order

$H(n)$ (FFT of $H(n)$ is real)



$\rightarrow \theta(\omega) = 0$

shifted $\frac{N-1}{2}$ for causality



$\rightarrow \theta(\omega) = -3\omega$

Zeros location of FIR filter

Type I, II:

$h[n] = h[N-1-n]$

$\rightarrow H(z) = z^{-(N-1)} H(z^{-1})$

$h[n] \rightarrow H(z)$

$h[-n] \rightarrow H(z^{-1})$

$h[-n+N-1] \rightarrow z^{-(N-1)} H(z^{-1})$

z_0 is a zero $\rightarrow$ z_0^* are also zeros

$1/z_0$

$1/z_0^*$

$\rightarrow$ Zeros on real axis appear in pairs $z_0, \frac{1}{z_0}$

Zeros on unit circle appear in pairs z_0, z_0^*

General zeros appear in set of 4

IIR filter can only have almost-linear phase response in pass band.

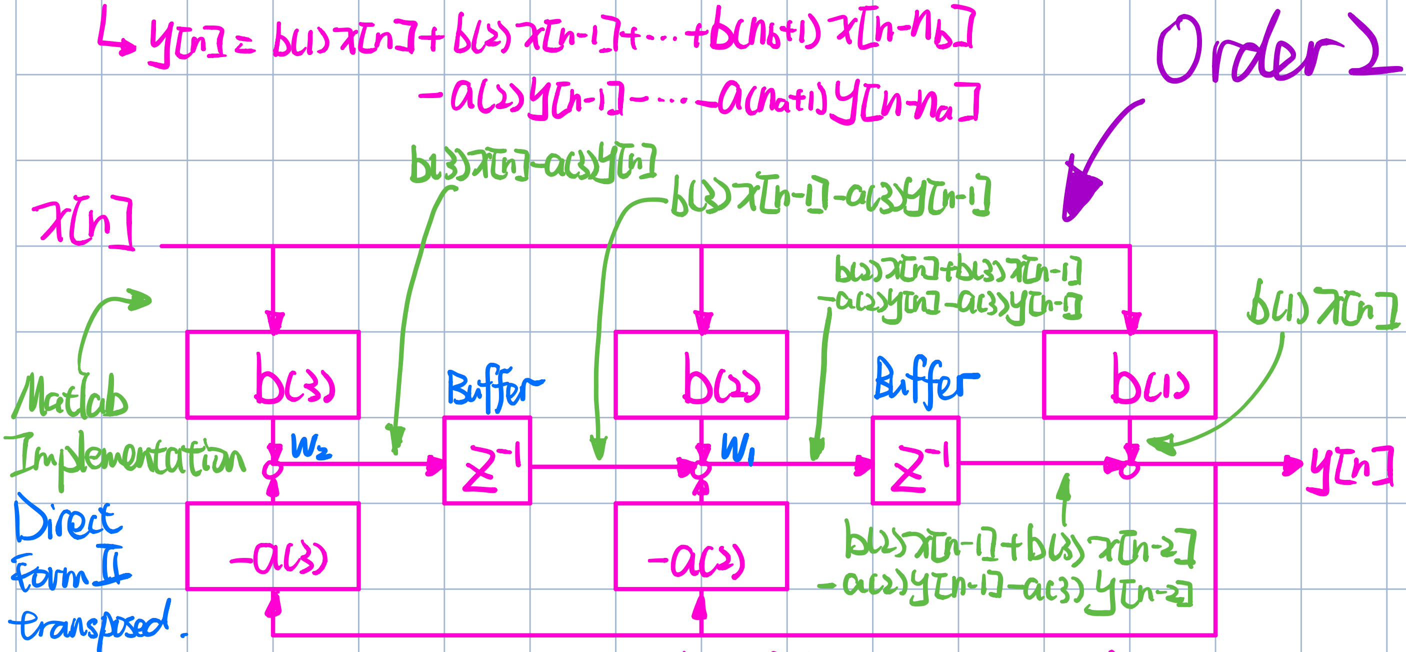
Matlab algorithms

1. filter

$b=[b(1), b(2), b(3)]\ldots \quad a=[1,a(2),a(3)\ldots]$

$$Y(z)=\frac{b(1)+b(2)z^{-1}+\cdots+b(n_b+1)\,z^{-n_b}}{1+a(2)z^{-1}+\cdots+a(n_a+1)\,z^{-n_a}}$$

$$\hookrightarrow y[n]=b(1)x[n]+b(2)x[n-1]+\cdots+b(n_b+1)x[n-n_b]-a(2)y[n-1]-\cdots-a(n_a+1)y[n-n_a]$$



$$y(z)=\frac{b_1+b_2z^{-1}+b_3z^{-2}+b_4z^{-3}}{1+a_2z^{-1}+a_3z^{-2}+a_4z^{-3}} \quad (\text{order } 3)$$

$\rightarrow y(i)=b_1\times x(i)+w(1)$

$w=[w(2:\text{end});0]$

$w=w+b(2:\text{end})\times x(i)-a(2:\text{end})\times y(i)$

W_1	W_2	W_3
m	n	q
n	q	0
$n+b_2x$ $-a_2y$	$q+b_3x$ $-a_3y$	b_4x $-a_4y$

Order 3

Signal | b_1 | Initial State

x_5	x_4	x_3	x_2	x_1	0	0	0
x_6	x_5	x_4	x_3	x_2	b_2x_1	b_3x_1	b_4x_1
	x_6	x_5	x_4	x_3	$b_2x_2+b_3x_1$	$b_3x_2+b_4x_1$	b_4x_2
		x_6	x_5	x_4	$b_2x_3+b_3x_2+b_4x_1$	$b_3x_3+b_4x_2$	b_4x_3
			x_6	x_5	$b_2x_4+b_3x_3+b_4x_2$	$b_3x_4+b_4x_3$	b_4x_4
				x_6	$b_2x_5+b_3x_4+b_4x_3$	$b_3x_5+b_4x_4$	b_4x_5

Final state $\rightarrow$ | $b_2x_6+b_3x_5+b_4x_4$ | $b_3x_6+b_4x_5$ | b_4x_6

y

x_1b_1
$x_2b_1+x_1b_2$
$x_3b_1+x_2b_2+x_1b_3$
$x_4b_1+x_3b_2+x_2b_3+x_1b_4$
$x_5b_1+x_4b_2+x_3b_3+x_2b_4$
$x_6b_1+x_5b_2+x_4b_3+x_3b_4$

Transient (Will change If initial condition is different)

Not dependent of initial condition

← Order 2.

Filter: $\boxed{h_0}\boxed{h_1}\boxed{h_2}$

$x_4 \quad x_3 \quad x_2 \quad x_1 \quad x_0 \quad 0 \quad 0$

$y_0 = h_0 x_0$

$y_1 = h_0 x_1 + h_1 x_0$

$y_2 = h_0 x_2 + h_1 x_1 + h_2 x_0$

$y_3 = h_0 x_3 + h_1 x_2 + h_2 x_1$

$y_4 = h_0 x_4 + h_1 x_3 + h_2 x_2$

Non-Causal Filtering $\boxed{h_{-1}}\boxed{h_0}\boxed{h_1}$

$0 \quad x_4 \quad x_3 \quad x_2 \quad x_1 \quad x_0 \quad 0$

Stable IIR filter designed to be causal, so is is impossible for IIR

$y_0 = h_{-1} x_1 + h_0 x_0$

$y_1 = h_{-1} x_2 + h_0 x_1 + h_1 x_0$

$y_2 = h_{-1} x_3 + h_0 x_2 + h_1 x_1$

$y_3 = h_{-1} x_4 + h_0 x_3 + h_1 x_2$

$y_4 = h_0 x_4 + h_1 x_3$

Butterworth filter order selection

2. filtfilt - zero phase response, non-causal

→ $H(z) = |H_0(z)|^2$

→ $H(z) = A(\omega)$, $\theta(\omega) = 0$, zero phase response, zero phase delay

$x[n] \to$ filter $\to y[n] \to y[N-n] \to$ filter $\to z[N-n]$

↳ $y[-n]$

$z[-n]$

$z[n]$

↳ $z[n]$

$x(z) \to \boxed{H(z)} \to H(z)x(z) \to z^{-n}H(\frac{1}{z})x(\frac{1}{z}) \to \boxed{H(z)}$

$\to z^{-n}H(z)H(\frac{1}{z})x(\frac{1}{z}) \to z^{n}z^{-n}x(z)H(z)H(z^{-1})$

$\Rightarrow x[n] \to \boxed{H(z)H(z^{-1})} \to z[n]$

$H'(\omega) = H(e^{j\omega})H(e^{j\omega})$ ← Coefficients are real

$= |H(e^{j\omega})|^2$ ← Real frequency response

Determine Initial Condition.

↳ If general initial condition:

F-B $x[n] \to y[n] \to y[N-n] \to z[N-n] \to z[n]$

B-F $x[n] \to x[N-n] \to y[N-n] \to y[n] \to z[n]$

$\neq$

F-B has transient at the end

B-F has transient at the beginning

- Transient can be optimized by setting the initial condition
  - Goal: if the input is [1, 1, 1 . . . . 1]
    Then the transient output should directly be:

    $$\underbrace{1, 1, 1 \ldots 1}_{N = \text{Filter order}}$$

  - Solve for initial conditions.