第一章 线性规划

第四节 单纯形法

- ➡ 典式
- 迭代原理
- 单纯形法举例
- 两阶段法

$$(LP)\ \min S = CX$$

$$AX = b \qquad R(A) = m$$

$$X \geq 0$$

基变量个数与方程个数一致

$$A = (\underbrace{p_1, p_2, \cdots, p_m}_{B(\text{可行基})}, \underbrace{p_{m+1}, p_{m+2}, \cdots, p_n}_{N}) = (B, N)$$

$$B^{-1}b \geq 0$$

$$X = (\underbrace{x_1, x_2, \cdots, x_m}, \underbrace{x_{m+1}, x_{m+2}, \cdots, x_n})^T = \begin{pmatrix} X_B \\ X_N \end{pmatrix}$$

$$C = (\underbrace{c_1, c_2, \cdots, c_m}_{C_B}, \underbrace{c_{m+1}, c_{m+2}, \cdots, c_n}_{C_N}) = (C_B, C_N)$$

复习

$$(LP)\ \min S = CX \qquad C=(C_B, C_N)$$

$$AX = b \qquad A=(B, N) \qquad X=\begin{pmatrix} X_B \\ X_N \end{pmatrix}$$

$$X \geq 0$$

$$AX = b \rightarrow (B, N)\begin{pmatrix} X_B \\ X_N \end{pmatrix} = b \rightarrow BX_B + NX_N = b$$

$$\rightarrow X_B + B^{-1}NX_N = B^{-1}b \rightarrow X_B = B^{-1}b - B^{-1}NX_N$$

$$S = CX = (C_B, C_N)\begin{pmatrix} X_B \\ X_N \end{pmatrix} = C_B X_B + C_N X_N$$

$$= C_B(B^{-1}b - B^{-1}NX_N) + C_N X_N$$

$$= C_B B^{-1}b + (C_N - C_B B^{-1}N)X_N$$

$$\min S = CX$$
$$AX = b$$
$$X \geq 0$$

复习 $C - C_B B^{-1} A \geq 0 \Leftrightarrow C_N - C_B B^{-1} N \geq 0$

$$\begin{aligned} C - C_B B^{-1} A &= (C_B, C_N) - C_B B^{-1}(B, N) \\ &= (C_B, C_N) - (C_B, C_B B^{-1} N) \\ &= (0, C_N - C_B B^{-1} N) \ \blacksquare \end{aligned}$$

$$X = \begin{pmatrix} X_B \\ X_N \end{pmatrix}$$

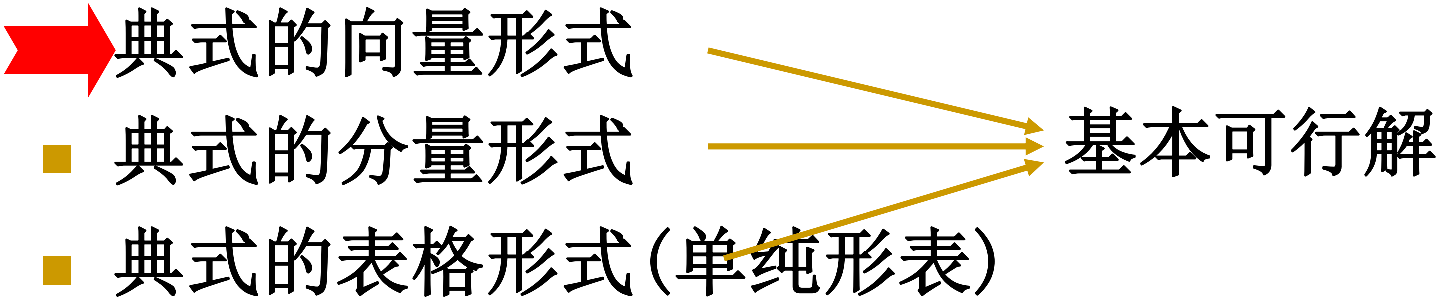
定理1-1（最优性判别定理）

对于 (LP) 的基 B，若有 $X_B^* = \underline{B^{-1} b \geq 0}$ 且 $\underline{C - C_B B^{-1} A \geq 0}$ $\underline{(C_N - C_B B^{-1} N \geq 0)}$ ，则基本可行解 $X^* = \begin{pmatrix} B^{-1} b \\ 0 \end{pmatrix}$ 是 (LP) 的最优解，称为**最优基本可行解**，B 称为**最优基**。

检验数向量　非基变量检验数向量

第一章 线性规划

一. 典式



- 典式的向量形式
- 典式的分量形式
- 典式的表格形式(单纯形表)

基本可行解

1. 典式的向量形式

$S=CX=C_BB^{-1}$

一个典式唯一地对应一个基本可行解

$$(LP)\ \min S = CX$$

$$AX = b$$

$$X \geq 0$$

$$X_B = B^{-1}b - B^{-1}NX_N$$

$$\min S = C_BB^{-1}b + (C_N - C_BB^{-1}N)X_N$$

基本可行解

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = X_B$$

$$X_B + B^{-1}NX_N = B^{-1}b \longrightarrow X = \begin{pmatrix} B^{-1}b \geq 0 \\ 0 \end{pmatrix}$$

$$X_B \geq 0 \quad X_N \geq 0$$

$$S = C_BB^{-1}b$$

称为(*LP*)的以 $x_1, x_2, \cdots, x_m$ 为基变量的典式

典式的向量形式

$$\min S=C_BB^{-1}b+(C_N-C_BB^{-1}N)X_N$$
$$X_B+B^{-1}NX_N=B^{-1}b$$
$$X_B\ge 0,\quad X_N\ge 0$$

例： $$\min S=x_1+x_2+2x_3+2x_4$$
$$\begin{cases}x_1+x_2+0x_3+x_4=1\\ x_1+2x_2+2x_3+x_4=2\end{cases}$$
$$x_j\ge 0,\quad j=1,2,3,4$$

$$A=\begin{array}{c}\begin{pmatrix}1&1&0&1\\ &B&&N\\ 1&2&2&1\end{pmatrix}\\ \begin{array}{cccc}x_1&x_2&x_3&x_4\end{array}\end{array}\quad b=\begin{pmatrix}1\\2\end{pmatrix}\quad R(A)=2$$

$$B^{-1}(A,b)=B^{-1}\begin{pmatrix}1&1&0&1&1\\ &B&&N&b\\ 1&2&2&1&2\end{pmatrix}=\begin{pmatrix}1&0&-2&1&0\\ &E&&B^{-1}N&B^{-1}b\\ 0&1&2&0&1\end{pmatrix}$$

$$\begin{cases}x_1+x_2+0x_3+x_4=1\\ x_1+2x_2+2x_3+x_4=2\end{cases}\Rightarrow\begin{cases}x_1\qquad -2x_3+\ x_4=0\\ \qquad x_2+2x_3+0x_4=1\end{cases}$$

$$C_BB^{-1}b=(1,1)\begin{pmatrix}0\\1\end{pmatrix}=1\quad C_N-C_BB^{-1}N=(2,2)-(1,1)\begin{pmatrix}-2&1\\2&0\end{pmatrix}=(2,1)$$

$$\min S=C_BB^{-1}b+(C_N-C_BB^{-1}N)X_N=1+(2,1)\begin{pmatrix}x_3\\x_4\end{pmatrix}=1+2x_3+x_4$$

例： $\min S = x_1 + x_2 + 2x_3 + 2x_4$

$$\begin{cases} x_1 + x_2 + 0x_3 + x_4 = 1 \\ x_1 + 2x_2 + 2x_3 + x_4 = 2 \end{cases}$$

$$x_j \geq 0, \quad j = 1,2,3,4$$

$$\min S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$
$$X_B + B^{-1} N X_N = B^{-1} b$$
$$X_B \geq 0, \quad X_N \geq 0$$

$$A = \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 & 1 \\ & B & & N \\ 1 & 2 & 2 & 1 \end{pmatrix} \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array} \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad R(A) = 2$$

$$B^{-1}(A,b) = B^{-1} \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ & B & & N & b \\ 1 & 2 & 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ & E & & B^{-1}N & B^{-1}b \\ 0 & 1 & 2 & 0 & 1 \end{pmatrix} \quad C_B B^{-1} b = 1$$

典式(基变量为x_1, x_2)：

$$\min S = 1 + 2x_3 + x_4$$

$$\begin{cases} x_1 \qquad - 2x_3 + \ x_4 = 0 \\ \qquad x_2 + 2x_3 + 0x_4 = 1 \\ x_j \geq 0, j = 1,2,3,4 \end{cases}$$

基本可行解：

$$X = (0,1,0,0)^T$$

目标值：$S = 1$

$$X = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} \geq 0$$

$$S = C_B B^{-1} b$$

基变量：只出现在其中一个方程中，且系数为1。

注： 可行基 ⟺ 典式 ⟺ 基本可行解

2. 典式的分量形式

$$\min S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$
$$X_B + B^{-1} N X_N = B^{-1} b$$
$$X_B \geq 0, \quad X_N \geq 0$$

$$X_B + B^{-1} N X_N = B^{-1} b$$

$$A = (\underbrace{p_1, p_2, \cdots, p_m}_{B\text{(可行基)}}, \underbrace{p_{m+1}, p_{m+2}, \cdots, p_n}_{N}) = (B, N)$$

$$B^{-1} b \geq 0$$

$$B^{-1} N = (B^{-1} p_{m+1}, B^{-1} p_{m+2}, \cdots, B^{-1} p_n)$$

$$B^{-1} b = \begin{pmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{pmatrix} \geq 0 \qquad = \begin{pmatrix} y_{1m+1} & y_{1m+2} & \cdots & y_{1n} \\ y_{2m+1} & y_{2m+2} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{mm+1} & y_{mm+2} & \cdots & y_{mn} \end{pmatrix}$$

例： $\min S = x_1 + x_2 + 2x_3 + 2x_4$

$$\begin{cases} x_1 + x_2 + 0x_3 + x_4 = 1 \\ x_1 + 2x_2 + 2x_3 + x_4 = 2 \end{cases}$$

$$x_j \geq 0, \quad j = 1,2,3,4$$

$$\boxed{\begin{gathered} \min S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N \\ X_B + B^{-1} N X_N = B^{-1} b \\ X_B \geq 0, \quad X_N \geq 0 \end{gathered}}$$

$$A = \left(\begin{array}{cccc} 1 & 1 & 0 & 1 \\ B & & N & \\ 1 & 2 & 2 & 1 \end{array}\right) \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad R(A) = 2$$

$$\begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \end{array}$$

$$B^{-1}(A,b) = B^{-1} \left(\begin{array}{ccccc} 1 & 1 & 0 & 1 & 1 \\ B & & N & & b \\ 1 & 2 & 2 & 1 & 2 \end{array}\right) = \left(\begin{array}{ccccc} 1 & 0 & -2 & 1 & 0 \\ E & & B^{-1}N & & B^{-1}b \\ 0 & 1 & 2 & 0 & 1 \end{array}\right) \quad C_B B^{-1} b = 1$$

典式(基变量为x_1, x_2)：

$$\min S = 1 + 2x_3 + x_4$$

$$\begin{cases} x_1 \quad - 2x_3 + x_4 = 0 \\ \quad x_2 + 2x_3 + 0x_4 = 1 \\ x_j \geq 0, j = 1, 2, 3, 4 \end{cases}$$

$$B^{-1} b = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} B^{-1} N &= B^{-1}(p_3, p_4) \\ &= (B^{-1} p_3, B^{-1} p_4) \\ &= \begin{pmatrix} y_{13} & y_{14} \\ y_{23} & y_{24} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2 & 0 \end{pmatrix} \end{aligned}$$

$$X_B + B^{-1}NX_N = B^{-1}b$$

$$B^{-1}b = \begin{pmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{pmatrix} \geq 0 \qquad B^{-1}N = \begin{pmatrix} y_{1m+1} & y_{1m+2} & \cdots & y_{1n} \\ y_{2m+1} & y_{2m+2} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{mm+1} & y_{mm+2} & \cdots & y_{mn} \end{pmatrix}$$

$$X = (\underbrace{x_1, x_2, \cdots, x_m}_{}, \underbrace{x_{m+1}, x_{m+2}, \cdots, x_n}_{})^T = \begin{pmatrix} X_B \\ X_N \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} + \begin{pmatrix} y_{1m+1} & y_{1m+2} & \cdots & y_{1n} \\ y_{2m+1} & y_{2m+2} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{mm+1} & y_{mm+2} & \cdots & y_{mn} \end{pmatrix} \begin{pmatrix} x_{m+1} \\ x_{m+2} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{pmatrix}$$

$X_B + B^{-1}NX_N = B^{-1}b$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} + \begin{pmatrix} y_{1m+1} & y_{1m+2} & \cdots & y_{1n} \\ y_{2m+1} & y_{2m+2} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{mm+1} & y_{mm+2} & \cdots & y_{mn} \end{pmatrix} \begin{pmatrix} x_{m+1} \\ x_{m+2} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{pmatrix}$$

$$\begin{cases} x_1 + y_{1m+1}x_{m+1} + y_{1m+2}x_{m+2} + \cdots + y_{1n}x_n = y_{10} \\ x_2 + y_{2m+1}x_{m+1} + y_{2m+2}x_{m+2} + \cdots + y_{2n}x_n = y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m + y_{mm+1}x_{m+1} + y_{mm+2}x_{m+2} + \cdots + y_{mn}x_n = y_{m0} \end{cases}$$

$$\min S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$
$$X_B + B^{-1} N X_N = B^{-1} b$$
$$X_B \ge 0,\quad X_N \ge 0$$

例：
$$\min S = x_1 + x_2 + 2x_3 + 2x_4$$
$$\begin{cases} x_1 + x_2 + 0x_3 + x_4 = 1 \\ x_1 + 2x_2 + 2x_3 + x_4 = 2 \end{cases}$$
$$x_j \ge 0,\quad j = 1,2,3,4$$

$$A = \left(\underbrace{\begin{matrix} 1 & 1 \\ 1 & 2 \end{matrix}}_{B}\ \ \underbrace{\begin{matrix} 0 & 1 \\ 2 & 1 \end{matrix}}_{N}\right) \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad R(A) = 2$$
$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

$$B^{-1}(A,b) = B^{-1}\left(\underbrace{\begin{matrix} 1 & 1 \\ 1 & 2 \end{matrix}}_{B}\ \ \underbrace{\begin{matrix} 0 & 1 \\ 2 & 1 \end{matrix}}_{N}\ \ \underbrace{\begin{matrix} 1 \\ 2 \end{matrix}}_{b}\right) = \left(\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{E}\ \ \underbrace{\begin{matrix} -2 & 1 \\ 2 & 0 \end{matrix}}_{B^{-1}N}\ \ \underbrace{\begin{matrix} 0 \\ 1 \end{matrix}}_{B^{-1}b}\right) \quad C_B B^{-1} b = 1$$

典式(基变量为x_1, x_2)：

$$\min S = 1 + 2x_3 + x_4$$
$$\begin{cases} x_1 \qquad\ - 2x_3 + \ x_4 = 0 \\ \qquad x_2 + 2x_3 + 0x_4 = 1 \\ x_j \ge 0, j = 1, 2, 3, 4 \end{cases} \Longleftarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -2 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\min S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$

$$\min S = CX$$
$$AX = b$$
$$X \geq 0$$

$$X_B + B^{-1} N X_N = B^{-1} b$$

$$X_B \geq 0 \quad X_N \geq 0$$

$$C = (\underbrace{c_1, c_2, \cdots, c_m}_{C_B}, \underbrace{c_{m+1}, c_{m+2}, \cdots, c_n}_{C_N}) = (C_B, C_N)$$

$$B^{-1} b = \begin{pmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{pmatrix} \qquad B^{-1} N = \begin{pmatrix} y_{1m+1} & y_{1m+2} & \cdots & y_{1n} \\ y_{2m+1} & y_{2m+2} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{mm+1} & y_{mm+2} & \cdots & y_{mn} \end{pmatrix}$$

$$C_B B^{-1} b = y_{00} \qquad (C_N - C_B B^{-1} N) = (y_{0m+1}, y_{0m+2}, \cdots, y_{0n})$$

例：

$$\min S = x_1 + x_2 + 2x_3 + 2x_4$$

$$\begin{cases} x_1 + x_2 + 0x_3 + x_4 = 1 \\ x_1 + 2x_2 + 2x_3 + x_4 = 2 \end{cases}$$

$$x_j \geq 0, \quad j = 1,2,3,4$$

$$\min S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$
$$X_B + B^{-1} N X_N = B^{-1} b$$
$$X_B \geq 0, \quad X_N \geq 0$$

$$A = \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 & 1 \\ & {\color{red}B} & & {\color{red}N} \\ 1 & 2 & 2 & 1 \end{pmatrix} \\ \begin{matrix} {\color{red}x_1} & {\color{red}x_2} & x_3 & x_4 \end{matrix} \end{array} \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad R(A) = 2$$

$$B^{-1}(A,b) = B^{-1} \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ & {\color{red}B} & & {\color{red}N} & {\color{red}b} \\ 1 & 2 & 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ & {\color{red}E} & & {\color{red}B^{-1}N} & {\color{red}B^{-1}b} \\ 0 & 1 & 2 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 + 0x_3 + x_4 = 1 \\ x_1 + 2x_2 + 2x_3 + x_4 = 2 \end{cases} \Longrightarrow \begin{cases} x_1 \qquad\quad - 2x_3 + \ x_4 = 0 \\ \qquad x_2 + 2x_3 + 0x_4 = 1 \end{cases}$$

$${\color{red}C_B B^{-1} b =} (1,1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \quad {\color{red}C_N - C_B B^{-1} N =} (2,2) - (1,1)\begin{pmatrix} -2 & 1 \\ 2 & 0 \end{pmatrix} = (2,1)$$

$$\min S = C_B B^{-1} b + (C_N -$$

$$X_B + B^{-1} N X_N = B$$

$$X_B \geq 0 \quad X_N \geq 0$$

1. 每个方程都有且仅有一个基变量，基变量仅出现在一个方程中且系数为1。

2. 非基变量的系数是其检验数。

$$\min S = y_{00} + y_{0m+1} x_{m+1} + y_{0m+2} x_{m+2} + \cdots + y_{0n} x_n$$

典式的分量形式

$$\begin{cases} x_1 + y_{1m+1} x_{m+1} + y_{1m+2} x_{m+2} + \cdots + y_{1n} x_n = y_{1} \\ x_2 + y_{2m+1} x_{m+1} + y_{2m+2} x_{m+2} + \cdots + y_{2n} x_n = y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m + y_{mm+1} x_{m+1} + y_{mm+2} x_{m+2} + \cdots + y_{mn} x_n = y_{m0} \end{cases}$$

$$x_j \geq 0, j = 1,2,\cdots n$$

$C_B B^{-1} b = y_{00}$ $\qquad$ $C_N - C_B B^{-1} N = (y_{0m+1}, y_{0m+2}, \cdots, y_{0n})$

$$\min S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$
$$X_B + B^{-1} N X_N = B^{-1} b$$
$$X_B \geq 0, \quad X_N \geq 0$$

例： $\min S = x_1 + x_2 + 2x_3 + 2x_4$

$$\begin{cases} x_1 + x_2 + 0x_3 + x_4 = 1 \\ x_1 + 2x_2 + 2x_3 + x_4 = 2 \end{cases}$$

$$x_j \geq 0, \quad j = 1,2,3,4$$

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ & B & & N \\ 1 & 2 & 2 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad R(A) = 2$$

$$B^{-1}(A,b) = B^{-1}\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ & B & & N & b \\ 1 & 2 & 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ & E & & B^{-1}N & B^{-1}b \\ 0 & 1 & 2 & 0 & 1 \end{pmatrix} \quad C_B B^{-1} b = 1$$

典式(基变量为x_1, x_2)：

$$\min S = 1 + 2x_3 + x_4$$

$$\begin{cases} x_1 \quad - 2x_3 + x_4 = 0 \\ x_2 + 2x_3 + 0x_4 = 1 \\ x_j \geq 0, j = 1,2,3,4 \end{cases}$$

基本可行解：

$$X = (0, 1, 0, 0)^T$$

目标值： $S = 1$

$$\min S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$

$$X_B + B^{-1} N X_N = B^{-1} b$$

$$X_B \geq 0 \quad X_N \geq 0$$

$$X_N = \begin{pmatrix} x_{m+1} \\ x_{m+2} \\ \vdots \\ x_n \end{pmatrix}$$

⇓

$$\min S = y_{00} + y_{0m+1} x_{m+1} + y_{0m+2} x_{m+2} + \cdots + y_{0n} x_n$$

典式的分量形式

$$\begin{cases} x_1 + y_{1m+1} x_{m+1} + y_{1m+2} x_{m+2} + \cdots + y_{1n} x_n = y_{10} \\ x_2 + y_{2m+1} x_{m+1} + y_{2m+2} x_{m+2} + \cdots + y_{2n} x_n = y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m + y_{mm+1} x_{m+1} + y_{mm+2} x_{m+2} + \cdots + y_{mn} x_n = y_{m0} \end{cases}$$

$$x_j \geq 0, j = 1,2,\cdots n$$

$$C_B B^{-1} b = y_{00} \qquad C_N - C_B B^{-1} N = (y_{0m+1}, y_{0m+2}, \cdots, y_{0n})$$

$$\min S = CX$$
$$AX = b$$
$$X \geq 0$$

$$C_N - C_B B^{-1} N = (y_{0m+1}, y_{0m+2}, \cdots, y_{0n})$$

$$\begin{aligned} & C_N - C_B B^{-1} N \\ &= (c_{m+1}, c_{m+2}, \cdots c_n) - C_B B^{-1}(p_{m+1}, p_{m+2}, \cdots, p_n) \\ &= (c_{m+1}, c_{m+2}, \cdots c_n) - (C_B B^{-1} p_{m+1}, \quad C_B B^{-1} p_{m+2}, \cdots, \quad C_B B^{-1} p_n) \\ &= (\underline{c_{m+1} - C_B B^{-1} p_{m+1}}, \quad \underline{c_{m+2} - C_B B^{-1} p_{m+2}}, \cdots, \quad \underline{c_n - C_B B^{-1} p_n}) \end{aligned}$$

非基变量 x_j 的检验数

$$\left.\begin{aligned} y_{0m+1} &= c_{m+1} - C_B B^{-1} p_{m+1} \\ y_{0m+2} &= c_{m+2} - C_B B^{-1} p_{m+2} \\ &\vdots \\ y_{0n} &= c_n - C_B B^{-1} p_n \end{aligned}\right\} \quad \Downarrow$$

$$y_{0j} = c_j - C_B B^{-1} p_j$$

$$j = m+1, m+2, \cdots, n$$

典式的分量形式:

$$y_{00}=C_BB^{-1}b \qquad y_{0j}=c_j-C_BB^{-1}p_j$$

$$\min S=y_{00}+y_{0m+1}x_{m+1}+y_{0m+2}x_{m+2}+\cdots+y_{0n}x_n$$

$$\begin{cases} x_1+y_{1m+1}x_{m+1}+y_{1m+2}x_{m+2}+\cdots+y_{1n}x_n=y_{10} \\ x_2+y_{2m+1}x_{m+1}+y_{2m+2}x_{m+2}+\cdots+y_{2n}x_n=y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m+y_{mm+1}x_{m+1}+y_{mm+2}x_{m+2}+\cdots+y_{mn}x_n=y_{m0} \end{cases}$$

$$x_j\geq 0,\ j=1,2,\cdots n$$

基本可行解:

$$\begin{array}{lccccccccl} & x_1 & x_2 & \cdots & x_m & x_{m+1} & x_{m+2} & \cdots & x_n & \\ & \Downarrow & \Downarrow & & \Downarrow & \Downarrow & \Downarrow & & \Downarrow & \\ X^0=(& y_{10}, & y_{20}, & \cdots, & y_{m0}, & 0, & 0, & \cdots & ,0 &)^T=\begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}\geq 0 \end{array}$$

目标值:

$$S^0=y_{00}=C_BB^{-1}b$$

3. $-y_{00}=-S+y_{0m+1}x_{m+1}+y_{0m+2}x_{m+2}+\cdots+y_{0n}x_n$

$$\min S=y_{00}+y_{0m+1}x_{m+1}+y_{0m+2}x_{m+2}+\cdots+y_{0n}x_n$$

$$\begin{cases} x_1+y_{1m+1}x_{m+1}+y_{1m+2}x_{m+2}+\cdots+y_{1n}x_n=y_{10} \\ x_2+y_{2m+1}x_{m+1}+y_{2m+2}x_{m+2}+\cdots+y_{2n}x_n=y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m+y_{mm+1}x_{m+1}+y_{mm+2}x_{m+2}+\cdots+y_{mn}x_n=y_{m0} \end{cases}$$

单纯形表

		x_1	x_2	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_n
y_{0j}	$-y_{00}$	0	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	y_{0n}
x_1	y_{10}	1	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1n}
x_2	y_{20}	0	1	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$	y_{2n}
$\vdots$	$\vdots$								
x_m	y_{m0}	0	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mn}

单纯形表：

$y_{0j}=c_j-C_BB^{-1}p_j$ $\quad$ $C-C_BB^{-1}A$

$-C_BB^{-1}b$		x_1	x_2	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_n
y_{0j}	$-y_{00}$	0	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	y_{0n}
x_1	y_{10}	1	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1n}
x_2	y_{20}	0	1	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$ $B^{-1}N$	y_{2n}
$\vdots$	$\vdots$								
x_m	y_{m0}	0	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mn}

$B^{-1}b$

$$C-C_BB^{-1}A=(C_B,C_N)-C_BB^{-1}(B,N)$$

$$=(C_B,C_N)-(C_B,C_BB^{-1}N)$$

$$=(0,C_N-C_BB^{-1}N)$$

$$=(0,0,\cdots,0,y_{0m+1},y_{0m+2},\cdots,y_{0n})$$

单纯形表：$S = y_{00} + y_{0m+1}x_{m+1} + y_{0m+2}x_{m+2} + \cdots + y_{0n}x_n$

$-C_B B^{-1}b$		x_1	x_2	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_n
y_{0j}	$-y_{00}$	0	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1n}
x_1	y_{10}	1	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	y_{0n}
x_2	y_{20}	0	1	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$	y_{2n}
$\vdots$	$\vdots$								
x_m	y_{m0}	0	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mn}

$B^{-1}b \ge 0$

$$\begin{array}{cccccccc} x_1 & x_2 & \cdots & x_m & x_{m+1} & x_{m+2} & \cdots & x_n \\ \Downarrow & \Downarrow & & \Downarrow & \Downarrow & \Downarrow & & \Downarrow \end{array}$$

基本可行解: $X^0 = (y_{10}, y_{20}, \cdots, y_{m0}, 0, \quad 0, \quad \cdots \quad ,0)^T = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} \ge 0$

目标值: $S^0 = y_{00} = C_B B^{-1}b$

例： $\min S = x_1 + x_2 + 2x_3 + 2x_4$

$$\begin{cases} x_1 + x_2 + 0x_3 + x_4 = 1 \\ x_1 + 2x_2 + 2x_3 + x_4 = 2 \end{cases}$$

$$x_j \geq 0, \quad j = 1,2,3,4$$

$$\xrightarrow{\text{典式}}$$

$\min S = 1 + 2x_3 + x_4$

$$\begin{cases} x_1 \qquad - 2x_3 + \ x_4 = 0 \\ \qquad x_2 + 2x_3 + 0x_4 = 1 \\ x_j \geq 0, j = 1,2,3,4 \end{cases}$$

基本可行解：

$X = (0, 1, 0, 0)^T$

单纯形表：

		x_1	x_2	x_3	x_4
y_{0j}	-1	0	0	2	1
x_1	0	1	0	-2	1
x_2	1	0	1	2	0

目标值： $S = 1$

第一章 线性规划

第四节 单纯形法

- ✓ 典式
- ➡ 迭代原理
- 单纯形法举例
- 两阶段法

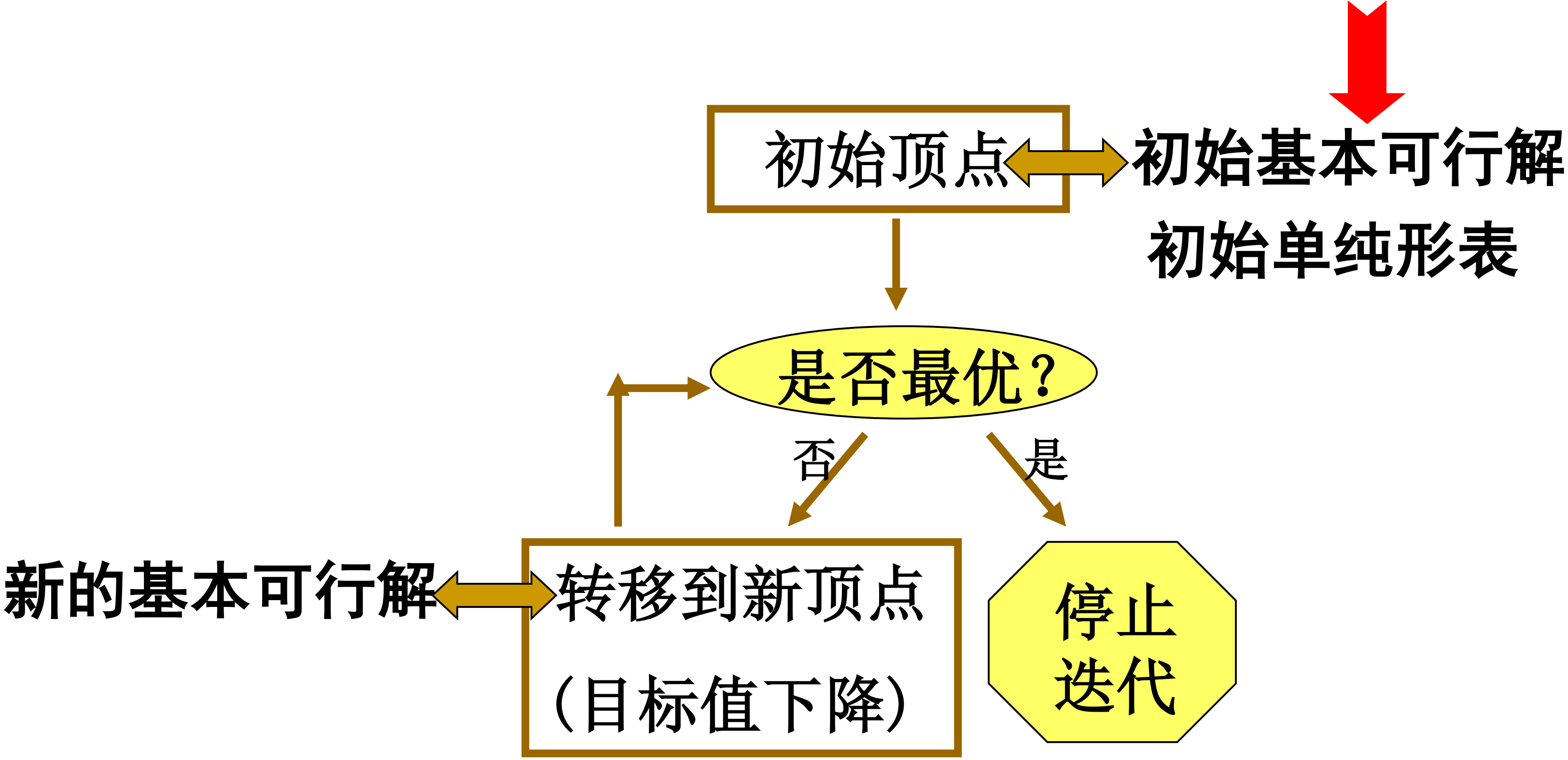
单纯形法的迭代思想：

顶点1 → 顶点2 → 顶点3 →…… 目标值↓

基本可行解1 → 基本可行解2 → 基本可行解3 →…… 目标值↓

单纯形表1 → 单纯形表2 → 单纯形表3 →…… 目标值↓

二. 迭代原理：



初始顶点 ⟷ **初始基本可行解**

初始单纯形表

是否最优？

否

是

新的基本可行解 ⟷ 转移到新顶点

（目标值下降）

停止迭代

1. 初始单纯形表

		x_1	x_2	$\cdots$	x_p	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_q	$\cdots$	x_n
y_{0j}	$-y_{00}$	0	0	$\cdots$	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	y_{0q}	$\cdots$	y_{0n}
x_1	y_{10}	1	0	$\cdots$	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1q}	$\cdots$	y_{1n}
x_2	y_{20}	0	1	$\cdots$	0	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$	y_{2q}	$\cdots$	y_{2n}
x_p	y_{p0}	0	0	$\cdots$	1	$\cdots$	0	y_{pm+1}	y_{pm+2}	$\cdots$	y_{pq}	$\cdots$	y_{pn}
x_m	y_{m0}	0	0	$\cdots$	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mq}	$\cdots$	y_{mn}

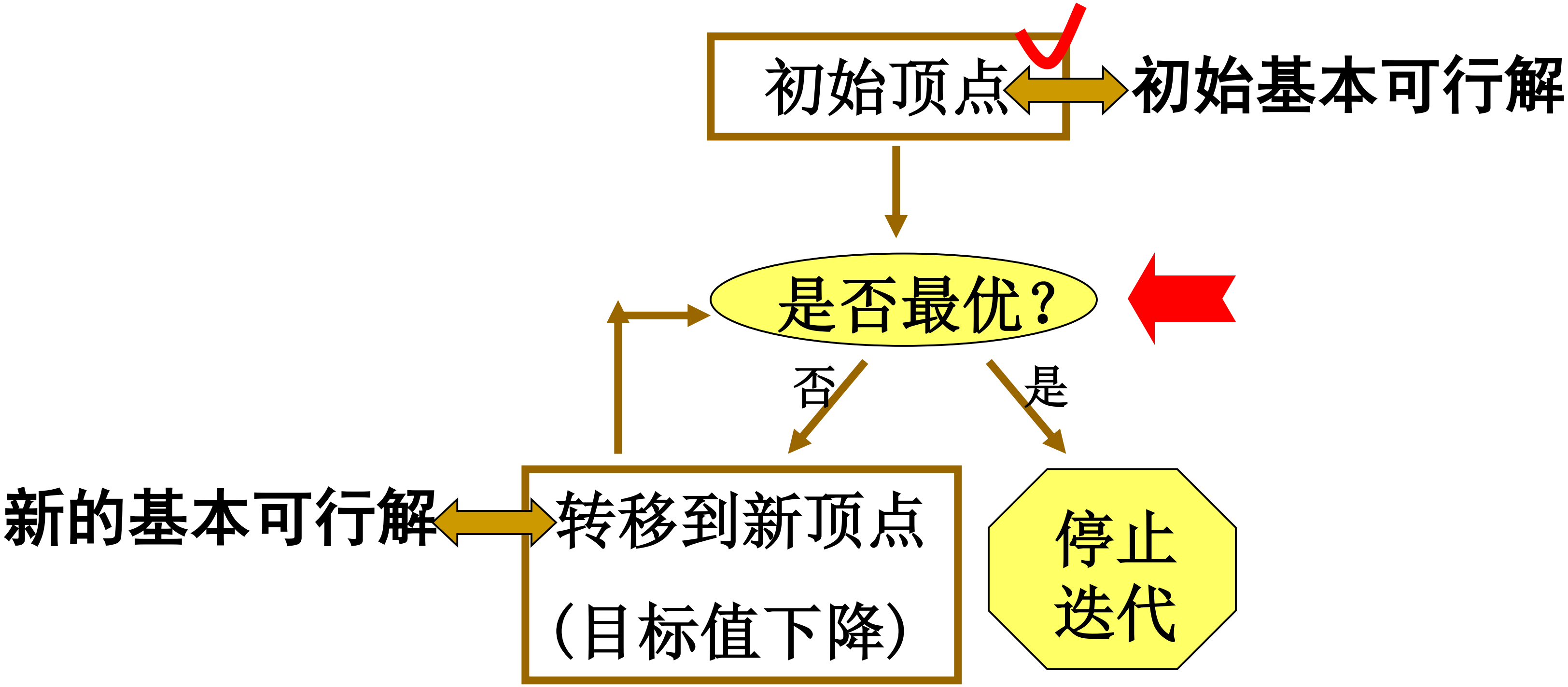
$B^{-1}b \geq 0$ $\quad x_1 \ x_2 \cdots x_p \cdots x_m \ x_{m+1} \ x_{m+2} \cdots x_q \cdots x_n$

$$X^0 = (y_{10}, y_{20}, \cdots, y_{p0}, \cdots y_{m0}, 0, \quad 0, \ \cdots \ 0, \cdots 0)^T$$

⇓

初始基本可行解

二. 迭代原理：



初始顶点 ⟺ 初始基本可行解

是否最优？

否

是

新的基本可行解 ⟺ 转移到新顶点

（目标值下降）

停止迭代

2. 判断当前基本可行解是否是最优解：

1）若 $\underline{C - C_B B^{-1} A \geq 0}$ 或 $\underline{C_N - C_B B^{-1} N \geq 0}$,

即非基变量 x_j 的检验数 $y_{0j} = c_j - C_B B^{-1} p_j$ 都 ≥ 0,

则 X^0 是最优解。

2）若有某些检验数 $\underline{y_{0j} < 0}$ ，例如：$y_{0q} < 0\ (m+1 \leq q \leq n)$,

则 X^0 不是最优解（在非退化情况下）。

1）单纯形表

$y_{0j}=c_j-C_BB^{-1}p_j\ge 0$　　$C-C_BB^{-1}A\ge 0$

		x_1	x_2	$\cdots$	x_p	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_q	$\cdots$	x_n
y_{0j}	$-y_{00}$	0	0	$\cdots$	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	y_{0q}	$\cdots$	y_{0n}
x_1	y_{10}	1	0	$\cdots$	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1q}	$\cdots$	y_{1n}
x_2	y_{20}	0	1	$\cdots$	0	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$	y_{2q}	$\cdots$	y_{2n}
x_p	y_{p0}	0	0	$\cdots$	1	$\cdots$	0	y_{pm+1}	y_{pm+2}	$\cdots$	y_{pq}	$\cdots$	y_{pn}
x_m	y_{m0}	0	0	$\cdots$	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mq}	$\cdots$	y_{mn}

$B^{-1}b\ge 0$　$x_1\ \ x_2 \cdots x_p \cdots x_m\ x_{m+1}\ \ x_{m+2} \cdots x_q \cdots x_n$

$$X^0=(y_{10},y_{20},\cdots,y_{p0},\cdots y_{m0},0,\ 0,\ \ldots\ 0,\ldots 0)^T$$

⇩

最优解

2）单纯形表

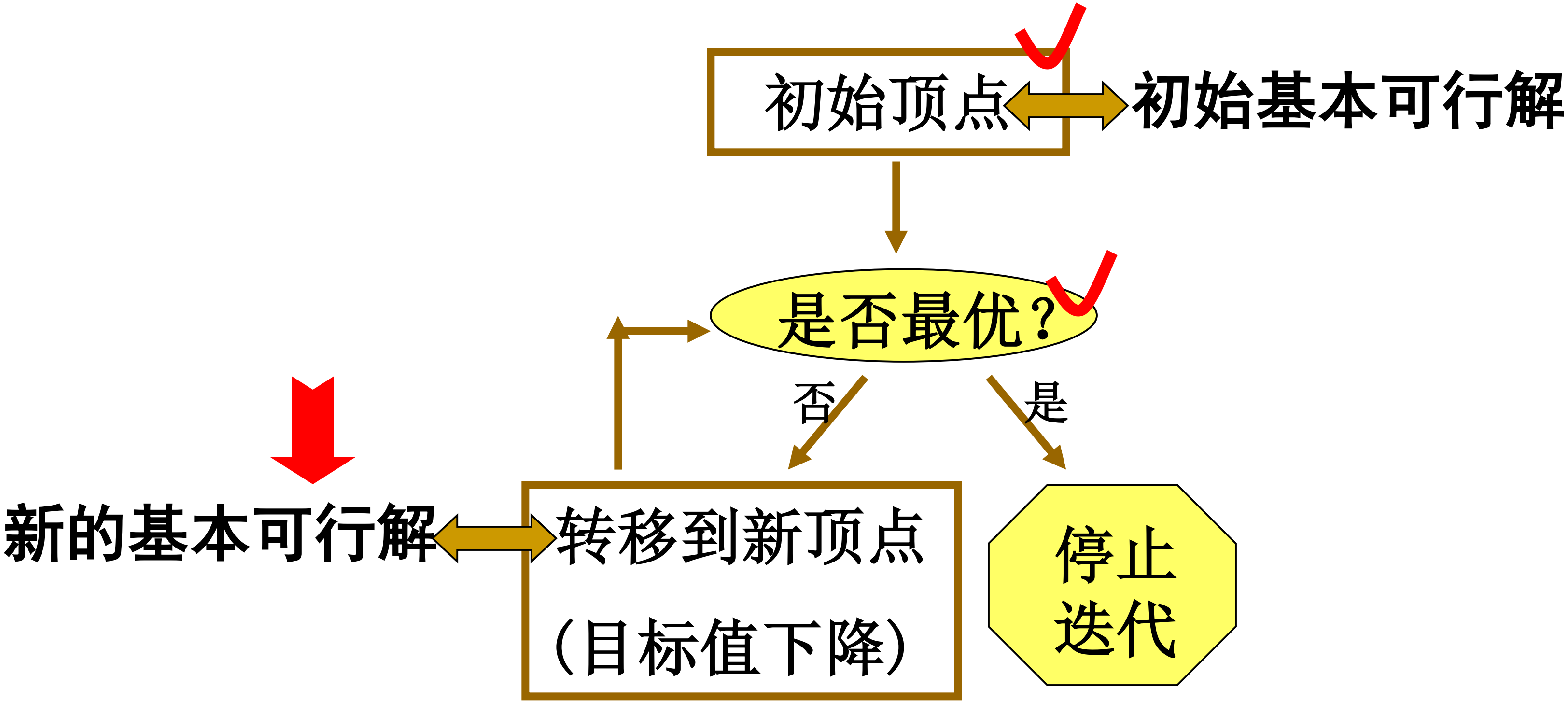
		x_1	x_2	$\cdots$	x_p	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_q	$\cdots$	x_n
y_{0j}	$-y_{00}$	0	0	$\cdots$	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	$y_{0q} < 0$		y_{0n}
x_1	y_{10}	1	0	$\cdots$	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1q}	$\cdots$	y_{1n}
x_2	y_{20}	0	1	$\cdots$	0	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$	y_{2q}	$\cdots$	y_{2n}
x_p	y_{p0}	0	0	$\cdots$	1	$\cdots$	0	y_{pm+1}	y_{pm+2}	$\cdots$	y_{pq}	$\cdots$	y_{pn}
x_m	y_{m0}	0	0	$\cdots$	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mq}	$\cdots$	y_{mn}

$B^{-1}b \geq 0$ $\quad x_1 \quad x_2 \cdots x_p \cdots x_m \; x_{m+1} \quad x_{m+2} \cdots \; x_q \cdots \; x_n$

$$X^0 = (y_{10}, y_{20}, \cdots, y_{p0}, \cdots y_{m0}, 0, \quad 0, \quad \cdots \quad 0, \cdots 0)^T$$

不是最优解

二. 迭代原理:



初始顶点 ⟺ 初始基本可行解

是否最优？

否

是

新的基本可行解 ⟺ 转移到新顶点（目标值下降）

停止迭代

3. 转移到新的基本可行解：

- ➡ 确定进基变量
- 确定离基变量
- 进行换基运算

1.典式的向量形式：

复习

$$\min S = CX$$
$$AX = b$$
$$X \geq 0$$

$$\min S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$
$$X_B + B^{-1} N X_N = B^{-1} b$$
$$X_B \geq 0 \quad X_N \geq 0$$

2.典式的分量形式：

$$\min S = y_{00} + y_{0m+1} x_{m+1} + y_{0m+2} x_{m+2} + \cdots + y_{0n} x_n$$

$$\begin{cases} x_1 + y_{1m+1} x_{m+1} + y_{1m+2} x_{m+2} + \cdots + y_{1n} x_n = y_{10} \\ x_2 + y_{2m+1} x_{m+1} + y_{2m+2} x_{m+2} + \cdots + y_{2n} x_n = y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m + y_{mm+1} x_{m+1} + y_{mm+2} x_{m+2} + \cdots + y_{mn} x_n = y_{m0} \end{cases}$$

$$x_j \geq 0, j = 1,2,\cdots n$$

1）确定进基变量：

$$\min S = y_{00} + y_{0m+1}x_{m+1} + y_{0m+2}x_{m+2} + \cdots + \overset{<0}{y_{0q}}x_q + \cdots + y_{0n}x_n$$

$$x_1 \quad x_2 \;\ldots\; x_p \;\ldots\; x_m \quad x_{m+1}\, x_{m+2} \quad x_q \ldots x_n$$

$$X^0 = (y_{10}, y_{20}, \cdots, y_{p0}, \cdots y_{m0},\; 0,\; 0\cdots, 0, \cdots 0)^T \qquad S^0 = y_{00}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$X^1 = (y'_{10}, y'_{20}, \cdots, y'_{p0}, \cdots y'_{m0},\; 0,\; 0\cdots, \theta > 0\; 0)^T$$

$S^1 = y_{00} + y_{0q}\theta \;< S^0 = y_{00} \downarrow$ x_q称为**进基变量**

确定进基变量的准则：

将检验数 <0 的非基变量进基做基变量，可使新的基本可行解目标函数值下降。

确定进基变量的准则：

如果有不止一个负检验数，则有两种方法确定进基变量 x_q：

1\. *Bland* 规则： $q = \min\{ j \mid y_{0j} < 0, j = 1,2,\cdots,n\}$

2\. 最负检验数法： $y_{0q} = \min\{ y_{0j} \mid y_{0j} < 0, j = 1,2,\cdots,n\}$

单纯形表

		x_1	$x_2\cdots$	$x_p\cdots$	x_m	x_{m+1}	$x_{m+2}\cdots$	x_q	$\cdots$	x_n
y_{0j}	$-y_{00}$	0	$0\cdots$	$0\cdots$	0	y_{0m+1}	$y_{0m+2}\cdots$	$y_{0q}<0$		y_{0n}
x_1	y_{10}	1	$0\cdots$	$0\cdots$	0	y_{1m+1}	$y_{1m+2}\cdots$	y_{1q}	$\cdots$	y_{1n}
x_2	y_{20}	0	$1\cdots$	$0\cdots$	0	y_{2m+1}	$y_{2m+2}\cdots$	y_{2q}	$\cdots$	y_{2n}
x_p	y_{p0}	0	$0\cdots$	$1\cdots$	0	y_{pm+1}	$y_{pm+2}\cdots$	y_{pq}	$\cdots$	y_{pn}
x_m	y_{m0}	0	$0\cdots$	$0\cdots$	1	y_{mm+1}	$y_{mm+2}\cdots$	y_{mq}	$\cdots$	y_{mn}

$$\begin{array}{lcccccccccccc} & x_1 & x_2 & \cdots & x_p & \cdots & x_m & x_{m+1} & x_{m+2} & \cdots & x_q & \cdots & x_n \\ X^0=(& y_{10}, & y_{20}, & \cdots, & y_{p0}, & \cdots & y_{m0}, & 0, & 0, & \cdots & 0, & \cdots & 0)^T \end{array}$$

采用**Bland规则**方法确定进基变量 x_q.

3. 转移到新的基本可行解：

- ✓ 确定进基变量
- ➡ 确定离基变量
- 进行换基运算

1.典式的向量形式：

$$\min S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$

$$X_B + B^{-1} N X_N = B^{-1} b$$

$$X_B \geq 0 \quad X_N \geq 0$$

复习

$$\min S = CX$$

$$AX = b$$

$$X \geq 0$$

⇩

2.典式的分量形式：

$$\min S = y_{00} + y_{0m+1} x_{m+1} + y_{0m+2} x_{m+2} + \cdots + y_{0n} x_n$$

$$\begin{cases} x_1 + y_{1m+1} x_{m+1} + y_{1m+2} x_{m+2} + \cdots + y_{1n} x_n = y_{10} \\ x_2 + y_{2m+1} x_{m+1} + y_{2m+2} x_{m+2} + \cdots + y_{2n} x_n = y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m + y_{mm+1} x_{m+1} + y_{mm+2} x_{m+2} + \cdots + y_{mn} x_n = y_{m0} \end{cases}$$

$$x_j \geq 0, j = 1,2,\cdots n$$

2) 确定离基变量：

$$\begin{array}{cccccccccccc}
 & x_1 & x_2 & \cdots & x_p & \cdots & x_m & x_{m+1} & x_{m+2} & x_q & \cdots & x_n \\
X^0 = (& y_{10}, & y_{20}, & \cdots, & y_{p0}, & \cdots & y_{m0}, & 0, & 0\cdots, & 0, & \cdots & 0)^T \\
\Downarrow & & & & \Downarrow & & & & & \Downarrow & & \\
X^1 = (& y'_{10}, & y'_{20}, & \cdots, & y'_{p0}, & \cdots & y'_{m0}, & 0, & 0\cdots, & \theta > 0 & & 0)^T
\end{array}
\qquad S^0 = y_{00}$$

$$S^1 = y_{00} + y_{0q}\theta < S^0 = y_{00} \downarrow \qquad y_{0q} < 0$$

$$\left\{\begin{array}{ll}
x_1 + y_{1m+1}x_{m+1} + \cdots + y_{1q}x_q + \cdots + y_{1n}x_n = y_{10} & x_1 = y_{10} - y_{1q}\theta \geq 0 \\
x_2 + y_{2m+1}x_{m+1} + \cdots + y_{2q}x_q + \cdots + y_{2n}x_n = y_{20} & x_2 = y_{20} - y_{2q}\theta \geq 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
x_m + y_{mm+1}x_{m+1} + \cdots + y_{mq}x_q + \cdots + y_{mn}x_n = y_{m0} & y_m = y_{m0} - y_{mq}\theta \geq 0
\end{array}\right.$$

$$x_j \geq 0, j = 1,2,\cdots n$$

2) 确定离基变量：

$$x_1 \quad x_2 \quad \ldots \quad x_p \quad \ldots x_m \quad x_{m+1}\, x_{m+2} \quad x_q \ldots x_n$$

$$X^0 = (y_{10}, y_{20}, \cdots, y_{p0}, \cdots y_{m0}, \; 0, \; 0\cdots, 0, \cdots 0)^T \qquad S^0 = y_{00}$$

$$X^1 = (\underset{>0}{y'_{10}}, \underset{>0}{y'_{20}}, \cdots, \underset{=0}{y'_{p0}}, \cdots \underset{>0}{y'_{m0}}, \; 0, \; 0\cdots, \theta > 0 \; 0)^T \qquad S^1 = y_{00} + y_{0q}\theta \downarrow$$

$$y_{0q} < 0$$

$$X^1 = (y'_{10}, y'_{20}, \cdots, 0, \cdots \; y'_{m0}, 0, \; 0\cdots, \frac{y_{p0}}{y_{pq}}, 0)^T$$

$${}'_{2q} > 0)$$

$$y_{2q}$$

$$\begin{cases} x_p = y_{p0} - y_{pq}\theta \geq 0 \longrightarrow \theta \leq \dfrac{y_{p0}}{y_{pq}} (y_{pq} > 0) \\ x_m = y_{m0} - y_{mq}\theta \geq 0 \longrightarrow \theta \leq \dfrac{y_{m0}}{y_{mq}} (y_{mq} > 0) \end{cases}$$

$$\boxed{\theta = \min\{\frac{y_{i0}}{y_{iq}} \Big| y_{iq} > 0, 1 \leq i \leq m\} = \frac{y_{p0}}{y_{pq}}} \Rightarrow$$

x_p称为**离基变量**

最小非负比值准则

单纯形表

		x_1	x_2	$\cdots$	x_p	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_q	$\cdots$	x_n
y_{0j}	$-y_{00}$	0	0	$\cdots$	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	$y_{0q} < 0$		y_{0n}
x_1	y_{10}	1	0	$\cdots$	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1q}	$\cdots$	y_{1n}
x_2	y_{20}	0	1	$\cdots$	0	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$	y_{2q}	$\cdots$	y_{2n}
x_p	y_{p0}	0	0	$\cdots$	1	$\cdots$	0	y_{pm+1}	y_{pm+2}	$\cdots$	y_{pq}	$\cdots$	y_{pn}
x_m	y_{m0}	0	0	$\cdots$	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mq}	$\cdots$	y_{mn}

第一种情况： 假设 $y_{1q}, y_{2q}, y_{pq}, y_{mq} > 0$

$$\theta = \min\{\frac{y_{10}}{y_{1q}}, \frac{y_{20}}{y_{2q}}, \frac{y_{p0}}{y_{pq}}, \frac{y_{m0}}{y_{mq}}\} = \frac{y_{p0}}{y_{pq}} \Rightarrow x_p$$

x_p称为**离基变量**

单纯形表

		x_1	x_2	$\cdots$	x_p	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_q	$\cdots$	x_n
y_{0j}	$-y_{00}$	0	0	$\cdots$	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	$y_{0q}<0$		y_{0n}
x_1	y_{10}	1	0	$\cdots$	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1q}	$\cdots$	y_{1n}
x_2	y_{20}	0	1	$\cdots$	0	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$	y_{2q}	$\cdots$	y_{2n}
x_p	y_{p0}	0	0	$\cdots$	1	$\cdots$	0	y_{pm+1}	y_{pm+2}	$\cdots$	y_{pq}	$\cdots$	y_{pn}
x_m	y_{m0}	0	0	$\cdots$	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mq}	$\cdots$	y_{mn}

第二种情况： 若 $y_{iq} \le 0, i = 1, 2, \cdots, m$, 则对 θ 没限制，

而 $S^1 = y_{00} + y_{0q}\theta \xrightarrow[\theta\to+\infty]{} -\infty$，

即(LP)没有有限的最优解。

$$y'_{10} = y_{10} - y_{1q}\theta \ge 0$$

$$y'_{20} = y_{20} - y_{2q}\theta \ge 0$$

$$y'_{m0} = y_{m0} - y_{mq}\theta \ge 0$$

第三种情况：无穷多最优解判别条件：

$$\begin{array}{l} \qquad x_1 \quad x_2 \ \cdots \ x_p \ \cdots x_m \quad x_{m+1} x_{m+2} \quad x_q \cdots x_n \\ X^0 = (y_{10}, y_{20}, \cdots, y_{p0}, \cdots y_{m0}, \ 0, \ 0 \cdots, 0, \cdots 0)^T \quad S^0 = y_{00} \\ \Downarrow \\ X^1 = (y'_{10}, y'_{20}, \cdots, y'_{p0}, \cdots y'_{m0}, \ 0, \ 0 \cdots, \theta > 0 \ 0)^T \ S^1 = y_{00} + y_{0q}\theta \end{array}$$

$$\begin{cases} y'_{10} = y_{10} - y_{1q}\theta \geq 0 \\ y'_{20} = y_{20} - y_{2q}\theta \geq 0 \\ y'_{p0} = y_{p0} - y_{pq}\theta \geq 0 \\ y'_{m0} = y_{m0} - y_{mq}\theta \geq 0 \end{cases}$$

非基变量 x_q 的检验数 $y_{0q} = 0$,

则 $S^1 = y_{00} + y_{0q}\theta = y_{00} \Rightarrow$ 最优值

例：

若对于某个基本可行解，所有检验数都非负，且存在一个非基变量的检验数=0，则(*LP*) 有无穷多个最优解。

3. 转移到新的基本可行解：

- ✓ 确定进基变量
- ✓ 确定离基变量
- ➡ 进行换基运算

3) 换基运算：

$$y'_{10} = y_{10} - y_{1q}\frac{y_{p0}}{y_{pq}}$$

		x_1	x_2	···	x_p	···	x_m	x_{m+1}	x					
y_{0j}	$-y_{00}$	0	0	···	0	···	0	y_{0m+1}	y_{0m+2}	···	y_{0q}	0		y_{0n}
x_1	y_{10}	1	0	···	0	···	0	y_{1m+1}	y_{1m+2}	···	y_{1q}	0	··	y_{1n}
x_2	y_{20}	0	1	···	0	···	0	y_{2m+1}	y_{2m+2}	···	y_{2q}	0	··	y_{2n}
← x_q	y_{p0}	0	0	···	1	···	0	y_{pm+1}	y_{pm+2}	···	y_{pq}	1	··	y_{pn}
x_m	y_{m0}	0	0	···	0	···	1	y_{mm+1}	y_{mm+2}	··	y_{mq}	0	··	y_{mn}

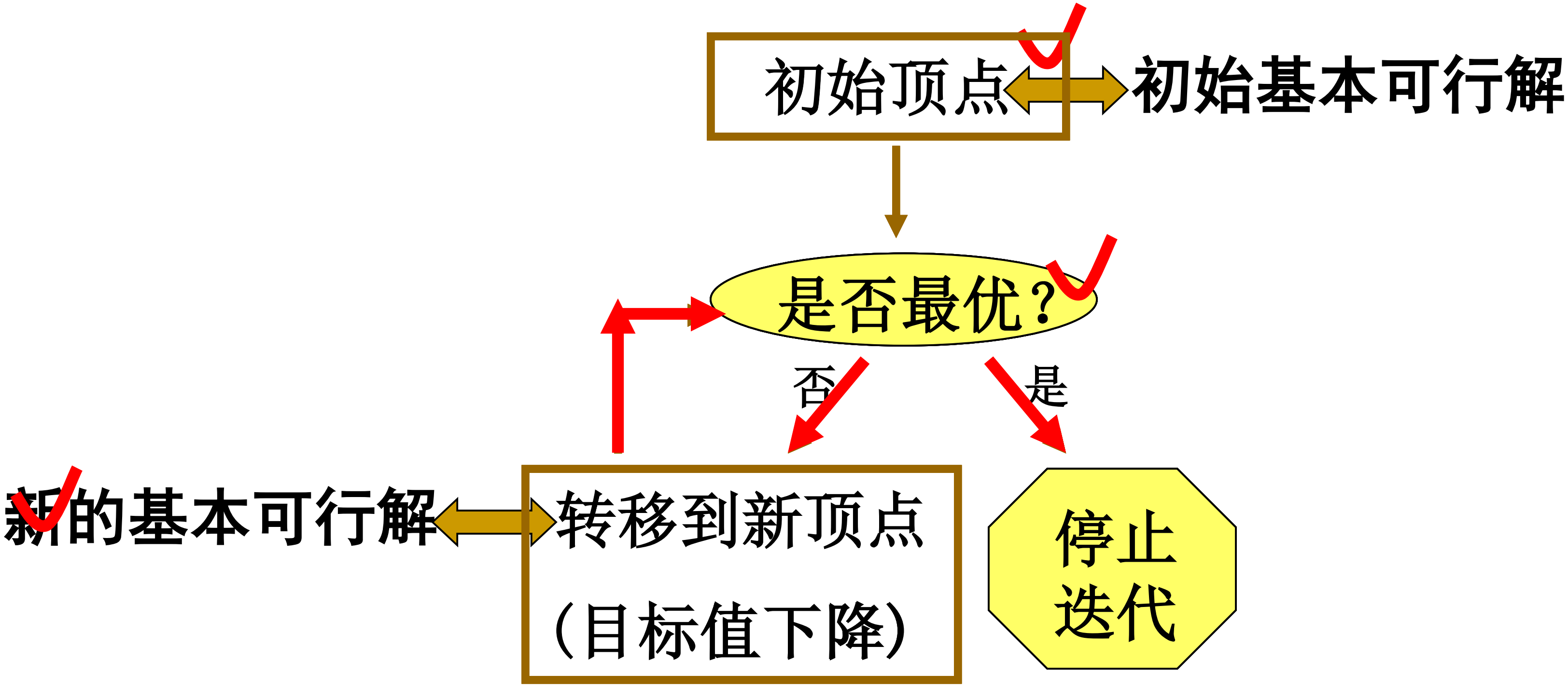
因为 x_q 代替 x_p 成为第p 个方程的基变量，所以它只能出现在第p个方程中，且系数为1，不能出现在其他方程中，检验数也为0。

$$\begin{array}{ccccccccccccccc} & x_1 & & x_2 & \cdots & x_p & \cdots & x_m & & x_{m+1} & x_{m+2} & \cdots & x_q & \cdots & x_n \\ X^1 = (& y'_{10}, & & y'_{20}, & \cdots, & 0, & \cdots & y'_{m0}, & & 0, & 0 & \cdots, & \frac{y_{p0}}{y_{pq}}, & \cdots & 0)^T \end{array}$$

3. 转移到新的基本可行解：

- ✓确定进基变量
- ✓确定离基变量
- ✓进行换基运算

二. 迭代原理：



初始顶点 ⟷ 初始基本可行解

是否最优？

否

是

新的基本可行解 ⟷ 转移到新顶点

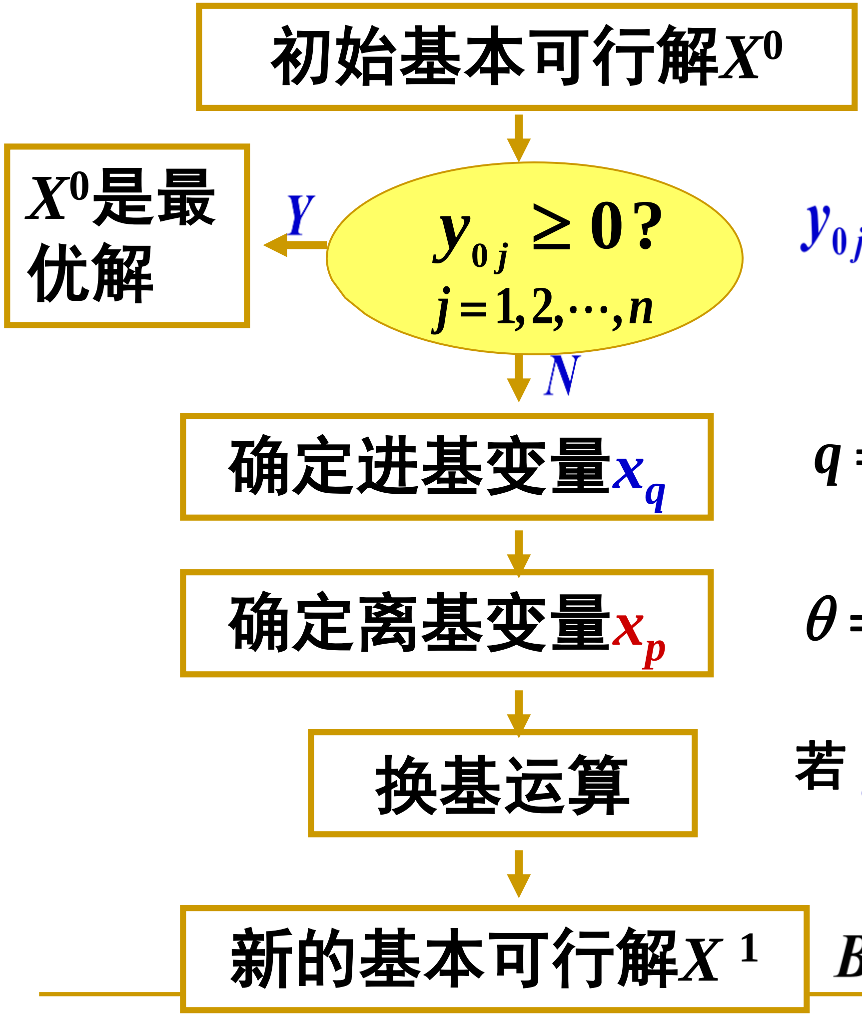
（目标值下降）

停止
迭代

迭代原理：

$$S^1 = y_{00} + y_{0q}\theta < S^0 = y_{00} \downarrow$$

$$\begin{array}{lllllllllll} & x_1 & x_2 & & x_p & & x_m & x_{m+1} & x_{m+2} & x_q & x_n \\ X^0 = (& y_{10}, & y_{20}, & \cdots, & y_{p0}, & \cdots & y_{m0}, & 0, & 0 \cdots, & 0, & \cdots 0)^T \\ X^1 = (& y'_{10}, & y'_{20}, & \cdots, & 0, & \cdots & y'_{m0}, & 0, & 0 \cdots, & \frac{y_{p0}}{y_{pq}}, & 0)^T \end{array}$$



初始基本可行解X^0

$$B_0 = (p_1, p_2, \cdots p_p \cdots p_m)$$

$y_{0j} \geq 0$?
$j = 1, 2, \cdots, n$

Y

X^0是最优解

N

$$y_{0j} = c_j - C_B B^{-1} p_j \geq 0$$

确定进基变量x_q

$$q = \min\{ j \mid y_{0j} < 0, j = 1, 2, \cdots, n \}$$

确定离基变量x_p

$$\theta = \min\{ \frac{y_{i0}}{y_{iq}} \mid y_{iq} > 0, 1 \leq i \leq m \} = \frac{y_{p0}}{y_{pq}}$$

若 $y_{iq} \leq 0$, (LP)没有有限的最优解。

换基运算

新的基本可行解X^1

$$B_1 = (p_1, p_2, \cdots p_q \cdots p_m)$$

第一章 线性规划

第四节 单纯形法

- ✓ 典式
- ✓ 迭代原理
- ➡ 单纯形法举例
- 两阶段法

例1-10 求解线性规划问题：

典式 $X_0 = (6,0,0,0,3,4)^T$ $S_0 = C_B B^{-1} b$

$\sigma_j = c_j - C_B B^{-1} p_j$

$$\min S = x_1 - 2x_2 + x_3 - 3x_4$$

$$B_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$+ x_4 = 6$

$+ x_4 \le 3$

$- x_4 \le 4$

$= 1,2,3,4$

标准形 $\longrightarrow$

$$\min S = x_1 - 2x_2 + x_3 - 3x_4$$

$$\begin{cases} x_1 + x_2 + 3x_3 + x_4 = 6 \\ -2x_2 + x_3 + x_4 + x_5 = 3 \\ -x_2 + 6x_3 - x_4 + x_6 = 4 \\ x_j \ge 0, j = 1,2,3,4,5,6 \end{cases}$$

单纯形表

		C_j	1	-2	1	-3	0	0
			x_1	x_2	x_3	x_4	x_5	x_6
C_B	X_B	-6	0	-3	-2	-4	0	0
1	x_1	6	1	1	3	1	0	0
0	x_5	3	0	-2	1	1	1	0
0	x_6	4	0	-1	6	-1	0	1

例1–10

$y_{0j} = c_j - C_B B^{-1} p_j \ge 0$ $C - C_B B^{-1} A \ge 0$

表一		x_1	x_2	x_3	x_4	x_5	x_6
X_B	-6	0	-3	-2	-4	0	0
x_2	6	1	1	3	1	0	0
x_5	15	2	0	7	3	1	0
x_6	4	0	-1	6	-1	0	1

$\times 2$

$$X^0 = \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ (6, & 0, & 0, & 0, & 3, & 4)^T \end{array}$$

$$B_0 = (p_1, p_5, p_6) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

例1-10

表一		x_1	x_2	x_3	x_4	x_5	x_6
X_B	-6	0	-3	-2	-4	0	0
x_2	6	1	1	3	1	0	0
x_5	15	2	0	7	3	1	0
x_6	10	1	0	9	0	0	1

$$X^0 = \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ (6, & 0, & 0, & 0, & 3, & 4)^T \end{array} \qquad S^0 = 6$$

例1-10

表一		x_1	x_2	x_3	x_4	x_5	x_6
X_B	-6	0	-3	-2	-4	0	0
x_2	6	1	1	3	1	0	0
x_5	15	2	0	7	3	1	0
x_6	10	1	0	9	0	0	1

$$X^0 = \left(\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 6, & 0, & 0, & 0, & 3, & 4 \end{array}\right)^T \qquad S^0 = 6$$

例1-10

表一		x_1	x_2	x_3	x_4	x_5	x_6
X_B	12	3	0	7	-1	0	0
x_2	6	1	1	3	1	0	0
x_5	15	2	0	7	3	1	0
x_6	10	1	0	9	0	0	1

×3

$$X^0=(\overset{x_1}{6},\ \overset{x_2}{0},\ \overset{x_3}{0},\ \overset{x_4}{0},\ \overset{x_5}{3},\ \overset{x_6}{4})^T \qquad S^0=6$$

例1-10

$y_{0j} = c_j - C_B B^{-1} p_j \ge 0$ $C - C_B B^{-1} A \ge 0$

表二		x_1	x_2	x_3	x_4	x_5	x_6
X_B	**12**	**3**	**0**	**7**	**-1**	**0**	**0**
x_2	**6**	**1**	**1**	**3**	**1**	**0**	**0**
x_4	**5**	$2/3$	**0**	$7/3$	**1**	$1/3$	**0**
x_6	**10**	**1**	**0**	**9**	**0**	**0**	**1**

$\times \frac{1}{3}$

$$A = \begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 1 & 0 \\ 0 & -1 & 6 & -1 & 0 & 1 \end{pmatrix}, B_1 = (p_2, p_5, p_6) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

例1-10

表二		x_1	x_2	x_3	$\downarrow$ x_4	x_5	x_6
X_B	12	3	0	7	-1	0	0
x_2	1	$1/3$	1	$2/3$	0	$-1/3$	0
x_4	5	$2/3$	0	$7/3$	$\boxed{1}$	$1/3$	0
x_6	10	1	0	9	0	0	1

$\times(-1)$

$$\begin{array}{rcl} & & \quad\; x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \\ X^0 & = & (6,\ 0,\ 0,\ 0,\ 3,\ 4\ )^T \qquad S^0 = 6 \\ X^1 & = & (0,\ 6,\ 0,\ 0,\ 15,10\ )^T \qquad S^1 = -12 \downarrow \end{array}$$

例1-10

表二		x_1	x_2	x_3	x_4	x_5	x_6
X_B	**17**	$11/3$	**0**	$28/3$	**0**	$1/3$	**0**
x_2	**1**	$1/3$	**1**	$2/3$	**0**	$-1/3$	**0**
x_4	**5**	$2/3$	**0**	$7/3$	**1**	$1/3$	**0**
x_6	**10**	**1**	**0**	**9**	**0**	**0**	**1**

$$\begin{array}{lcccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ X^0 = & (& 6, & 0, & 0, & 0, & 3, & 4 &)^T \quad S^0 = 6 \\ X^1 = & (& 0, & 6, & 0, & 0, & 15, & 10 &)^T \quad S^1 = -12 \downarrow \end{array}$$

$$\min S = x_1 - 2x_2 + x_3 - 3x_4 \quad\quad \min S = x_1 - 2x_2 + x_3 - 3x_4$$

$$\begin{cases} x_1 + x_2 + 3x_3 + x_4 = 6 \\ -2x_2 + x_3 + x_4 \le 3 \\ -x_2 + 6x_3 - x_4 \le 4 \\ x_j \ge 0, j = 1,2,3,4 \end{cases} \xrightarrow{\text{标准形}} \begin{cases} x_1 + x_2 + 3x_3 + x_4 = 6 \\ -2x_2 + x_3 + x_4 + x_5 = 3 \\ -x_2 + 6x_3 - x_4 + x_6 = 4 \\ x_j \ge 0, j = 1,2,3,4,5,6 \end{cases}$$

x_2	1	$1/3$	1	$2/3$	0	$-1/3$	0
x_4	5	$2/3$	0	$7/3$	1	$1/3$	0
x_6	10	1	0	9	0	0	1

$$A = \begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 1 & 0 \\ 0 & -1 & 6 & -1 & 0 & 1 \end{pmatrix}, B^* = (p_2, p_4, p_6) = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

例1-11 求解线性规划问题：

$y_{0j}=c_j-C_BB^{-1}p_j$ $y_{00}=C_BB^{-1}b$

$$\max S=x_1+3x_2+4x_3 \qquad \min(-S)=-x_1-3x_2-4x_3$$

$$\begin{cases}3x_1+5x_2-4x_3\le 10\\ -2x_1+3x_2+x_3\le 5\\ x_j\ge 0,\ j=1,2,3\end{cases} \xrightarrow{\text{标准形}} \begin{cases}3x_1+5x_2-4x_3+x_4=10\\ -2x_1+3x_2+x_3+x_5=5\\ x_j\ge 0,\ j=1,2,3,4,5\end{cases}$$

单纯形表

		C_j	-1	-3	-4	0	0
			x_1	x_2	x_3	x_4	x_5
C_B	X_B	0	-1	-3	-4	0	0
0	x_4	10	3	5	-4	1	0
0	x_5	5	-2	3	1	0	1

例1–11

$y_{0j}=c_j-C_BB^{-1}p_j\geq 0$ $C-C_BB^{-1}A\geq 0$

		x_1	x_2	x_3	x_4	x_5
X_B	0	-1	-3	-4	0	0
x_1	$10/3$	1	$5/3$	$-4/3$	$1/3$	0
x_5	5	-2	3	1	0	1

$\times\frac{1}{3}$

$$\begin{array}{ccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & \\ X^0=(& 0, & 0, & 0, & 10, & 5 &)^T \end{array} \quad S^0=0$$

例1−11

		x_1	x_2	x_3	x_4	x_5
X_B	$10/3$	0	$-4/3$	$-16/3$	$1/3$	0
x_1	$10/3$	1	$5/3$	$-4/3$	$1/3$	0
x_5	$35/3$	0	$19/3$	$-5/3$	$2/3$	1

$$\begin{array}{cccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & \\ X^0 & =(& 0, & 0, & 0, & 10, & 5 &)^T \end{array} \quad S^0 = 0$$

例1−11

$y_{0j}=c_j-C_BB^{-1}p_j\geq 0 \quad C-C_BB^{-1}A\geq 0$

		x_1	x_2	x_3	x_4	x_5
X_B	$10/3$	0	$-4/3$	$-16/3$	$1/3$	0
x_1	$10/3$	1	$5/3$	$-4/3$	$1/3$	0
x_5	$35/3$	0	$19/3$	$-5/3$	$2/3$	1

$$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$$

$$X^0=(0,\ 0,\ 0,\ 10,\ 5)^T \quad S^0=0$$

$$X^1=(10/3,\ 0,\ 0,\ 0,\ 35/3)^T \quad S^1=-10/3$$ ↓

例1−11

$y_{0j}=c_j-C_BB^{-1}p_j\geq 0 \quad C-C_BB^{-1}A\geq 0$

		x_1	x_2	x_3	x_4	x_5
X_B	$10/3$	0	$-4/3$	$-16/3$	$1/3$	0
x_1	$10/3$	1	$5/3$	$-4/3$	$1/3$	0
x_5	$35/3$	0	$19/3$	$-5/3$	$2/3$	1

$$X^2 \quad S^2=y_{00}+y_{0q}\theta=-\frac{10}{3}-\frac{16}{3}\theta \xrightarrow[\theta\to+\infty]{} -\infty$$

该问题没有有限的最优解

$$X^0=(0,\ 0,\ 0,\ 10,\ 5)^T \quad S^0=0$$

$$X^1=(10/3,\ 0,\ 0,\ 0,\ 35/3)^T \quad S^1=-10/3$$ ↓ ■

第一章 线性规划

第四节 单纯形法

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- ✓ 迭代原理
- ✓ 单纯形法举例
- 两阶段法

作业：第1章 6 (1) (2) (3)