第三次作业

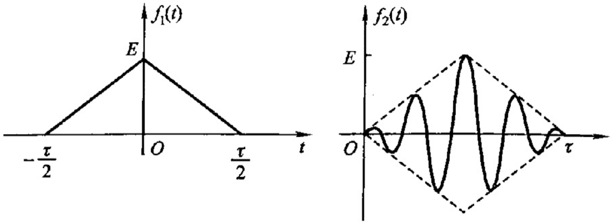
2025 年 4 月 21 日

1. 已知$f_1(t)$，$f_2(t)$的波形如题图 3-1 所示。求解以下问题，写出详细步骤。（20 分）

(1) 求三角脉冲$f_1(t)$的傅里叶变换$F_1(\omega)$；

(2) 对三角脉冲$f_1(t)$以等间隔 $\tau/10$进行冲激采样，求所得采样信号的频谱；

(3) 基于$F_1(\omega)$的结果，求 $f_2(t) = f_1\left(t-\frac{\tau}{2}\right)\cos(\omega_0 t)$ 的傅里叶变换$F_2(\omega)$。



题图 3-1

(1) $\frac{df_1(t)}{dt} = \frac{2E}{\tau}[u(t+\frac{\tau}{2}) - 2u(t) + u(t-\frac{\tau}{2})]$ $\quad \frac{d^2f_1(t)}{dt^2} = \frac{2E}{\tau}[\delta(t+\frac{\tau}{2}) - 2\delta(t) + \delta(u-\frac{\tau}{2})]$

$X_2(w) = \mathcal{F}[\frac{d^2f_1(t)}{dt^2}] = \frac{2E}{\tau}[e^{jw\frac{\tau}{2}} - 2 + e^{-jw\frac{\tau}{2}}] = \frac{2E}{\tau}[2\cos(\frac{w\tau}{2}) - 2] = -\frac{8E}{\tau}\sin^2(\frac{w\tau}{4})$

$X_1(w) = \mathcal{F}[\frac{df_1(t)}{dt}] = \frac{1}{jw}[-\frac{8E}{\tau}\sin^2(\frac{w\tau}{4})] + \pi X_2(0)\delta(w) = \frac{1}{jw}[-\frac{8E}{\tau}\sin^2(\frac{w\tau}{4})]$

$F_1(w) = \mathcal{F}[f_1(t)] = \frac{1}{(jw)^2}[-\frac{8E}{\tau}\sin^2(\frac{w\tau}{4})] + \pi X_1(0)\delta(w) = \frac{E\tau}{2}Sa^2(\frac{w\tau}{4})$

(2) 采样信号 $f_s(t) = f_1(t)\delta_T(t) = \sum_{n=-\infty}^{\infty} f_1(\frac{n\tau}{10})\delta(t-\frac{n\tau}{10})$

$F_s(w) = \mathcal{F}[f_s(t)] = \frac{10}{\tau}\sum_{n=-\infty}^{\infty} F_1(w-\frac{n\pi\tau}{5}) = 5E\sum_{n=-\infty}^{\infty} Sa^2\left(\frac{w-\frac{n\pi\tau}{5}}{4}\cdot\tau\right)$

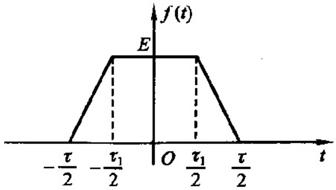
(3) $F_2(w) = \mathcal{F}[f_1(t-\frac{\tau}{2})\cos(w_0 t)] = \frac{1}{2\pi}\mathcal{F}[f_1(t-\frac{\tau}{2})] * \mathcal{F}[\cos(w_0 t)]$

$= \frac{1}{2\pi}[F_1(w)e^{jw\frac{\tau}{2}}] * \pi[\delta(w+w_0) + \delta(w-w_0)]$

$= \frac{1}{2}[F_1(w+w_0)e^{j(w+w_0)\frac{\tau}{2}} + F_1(w-w_0)e^{j(w-w_0)\frac{\tau}{2}}]$

$= \frac{E\tau}{4}\{Sa^2[\frac{(w+w_0)\tau}{4}]e^{j(w+w_0)\frac{\tau}{2}} + Sa^2[\frac{(w-w_0)\tau}{4}]e^{j(w-w_0)\frac{\tau}{2}}\}$

2. 求如题图 3-2 所示的梯形脉冲的傅里叶变换，并大致画出 $\tau = 2\tau_1$ 情况下该脉冲的频谱图，并在图中标注频谱密度函数的最大幅值和第一次过零点的坐标。（20 分）



题图 3-2

$$\frac{df(t)}{dt} = \frac{2E}{\tau-\tau_1}\left[u\left(t+\frac{\tau}{2}\right) - u\left(t+\frac{\tau_1}{2}\right) - u\left(t-\frac{\tau_1}{2}\right) + u\left(t-\frac{\tau}{2}\right)\right]$$

$$F(\omega) = \mathcal{F}[f(t)] \xlongequal{\text{微分性质}} \frac{1}{j\omega}\mathcal{F}\left[\frac{df(t)}{dt}\right]$$

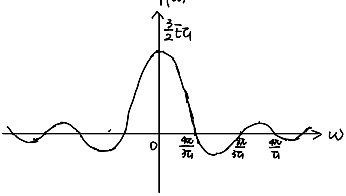
$$\xlongequal{\text{线性性质}} \frac{1}{j\omega}\frac{2E}{\tau-\tau_1}\cdot\mathcal{F}\left[u\left(t+\frac{\tau}{2}\right) - u\left(t+\frac{\tau_1}{2}\right) - u\left(t-\frac{\tau_1}{2}\right) + u\left(t-\frac{\tau}{2}\right)\right]$$

$$\xlongequal{\text{时移性质}} \frac{2E}{j\omega(\tau-\tau_1)}\cdot\frac{1}{j\omega}\left(e^{j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau_1}{2}} - e^{-j\omega\frac{\tau_1}{2}} + e^{-j\omega\frac{\tau}{2}}\right)$$

$$\xlongequal{\text{欧拉公式}} \frac{8E}{\omega^2(\tau-\tau_1)}\sin\frac{\omega(\tau+\tau_1)}{4}\sin\frac{\omega(\tau-\tau_1)}{4}$$

$\tau = 2\tau_1$ 时，$F(\omega) = \frac{8E}{\omega^2\tau_1}\sin\left(\frac{3}{4}\omega\tau_1\right)\sin\left(\frac{1}{4}\omega\tau_1\right) = \frac{3}{2}E\tau_1\, Sa\left(\frac{3\omega\tau_1}{4}\right) Sa\left(\frac{\omega\tau_1}{4}\right)$

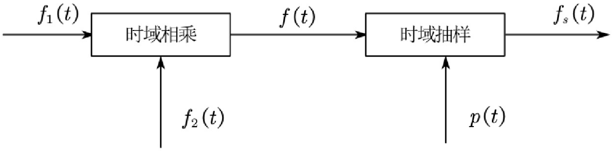
在 $\omega=0$ 取峰值 $F(0) = \frac{3}{2}E\tau_1$ 过零点为 $\omega = \frac{4k\pi}{3\tau_1}$ 或 $\frac{4k\pi}{\tau_1}$，第一次过零点 $\omega_0 = \frac{4\pi}{3\tau_1}$



3. 已知系统如题图 3-3 所示，已知信号$f_1(t)=Sa(1000\pi t)$，$f_2(t)=Sa(2000\pi t)$，$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$，$f(t)=f_1(t)f_2(t)$，$f_s(t)=f(t)p(t)$。（20 分）

(1) 为从$f_s(t)$无失真恢复$f(t)$，求最大采样间隔$T_{\max}$；

(2) 当$T=T_{\max}$时，画出$f_s(t)$的幅度谱$|F_s(\omega)|$。



题图 3-3

1) 设单矩形脉冲为$g(t)$，则 $G(\omega)=E\tau\, Sa(\frac{\omega\tau}{2})$. 则 $\mathcal{F}[Sa(t)]=\begin{cases}\pi, & |\omega|<1\\ 0, & |\omega|>1\end{cases}$

由尺度变换性质得 $F_1(\omega)=\begin{cases}\frac{1}{1000}, & |\omega|<1000\pi\\ 0, & 其他\end{cases}$ $F_2(\omega)=\begin{cases}\frac{1}{2000}, & |\omega|<2000\pi\\ 0, & 其他\end{cases}$

$f(t)=f_1(t)f_2(t)$. 则

$F(\omega)=\frac{1}{2\pi}F_1(\omega)*F_2(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\omega}F_1(\tau)d\tau*\frac{d}{d\omega}F_2(\omega)$

$=\frac{1}{2\pi}\cdot\frac{1}{1000}\cdot\frac{1}{2000}\cdot[r(t+1000\pi)-r(t-1000\pi)]*[\delta(t+2000\pi)-\delta(t-2000\pi)]$

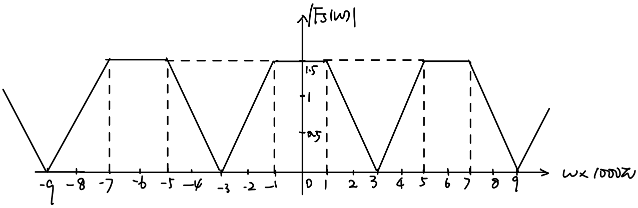
$=\frac{1}{4\pi\times10^6}[r(t+3000\pi)-r(t+1500\pi)-r(t-1000\pi)+r(t-3000\pi)]$

因此 $F(\omega)$ 只占据 $-3000\pi\sim3000\pi$ 范围，即 $\omega_m=3000\pi$ rad/s. $f_m=\frac{\omega_m}{2\pi}=1500$ Hz

由 Shannon 定理，最大采样间隔 $T_{max}=\frac{1}{2f_m}=\frac{1}{3000}$ s

(2) $\delta_T(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)$ 的 Fourier 系数 $P_n=\frac{1}{T_s}$

则 $F_s(\omega)=\sum_{n=-\infty}^{\infty}P_nF(\omega-n\omega_s)=\frac{1}{T_s}\sum_{n=-\infty}^{\infty}F(\omega-n\frac{2\pi}{T_{max}})=3000\sum_{n=-\infty}^{\infty}F(\omega-6000\pi n)$



4. 根据以下给出的序列，判断：序列是否为周期性的？给出原因。如是周期序列，确定其周期。（20 分）

(1) $x(n) = 5\cos\left(\frac{3\pi}{7}n - \frac{\pi}{8}\right)$

(2) $x(n) = 2\cos\left(\frac{\pi}{4}n\right) + \sin\left(\frac{\pi}{8}n\right) - 2\cos\left(\frac{\pi}{2}n + \frac{\pi}{6}\right)$

(3) $x(n) = \sin\left(\frac{1}{2}n - \pi\right)$

(4) $x(n) = e^{j\left(\frac{n}{8} - \pi\right)}$

(5) $x(n) = \cos\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right)$

(1) $\Omega_0 = \frac{3\pi}{7}$ $N = \frac{2k\pi}{\Omega_0} = \frac{14k}{3}$ 当 $k=3$ 时 $N=14$. 故序列 $x(n)$ 是周期的且周期 $N=14$

(2) 对 $\cos(\frac{\pi}{4}n)$, $\Omega_1 = \frac{\pi}{4}$. $N_1 = \frac{2k\pi}{\Omega_1} = 8k$. 当 $k=1$ 时 $N_1 = 8$

对 $\sin\frac{\pi}{8}n$. $\Omega_2 = \frac{\pi}{8}$, $N_2 = \frac{2k\pi}{\Omega_2} = 16k$. 当 $k=1$ 时 $N_2 = 16$

对 $\cos(\frac{\pi}{2}n + \frac{\pi}{6})$, $\Omega_3 = \frac{\pi}{2}$, $N_3 = \frac{2k\pi}{\Omega_3} = 4k\pi$, 当 $k=1$ 时 $N_3 = 4$.

$x(n)$ 是周期序列

$N = lcm(N_1, N_2, N_3) = 16$

(3) $\Omega_0 = \frac{1}{2}$. $N = \frac{2k\pi}{\Omega_0} = 4k\pi$ 不存在整数 k 使得 N 为正整数, 故 $x(n)$ 是非周期的

(4) 设 $x(n+N) = x(n)$. 即 $e^{j(\frac{n+N}{8} - \pi)} = e^{j(\frac{n}{8} - \pi)}$

则 $e^{j\frac{N}{8}} = 1$. $N = 16k\pi$. 不存在整数 k 使得 N 为正整数, 故 $x(n)$ 是非周期的

(5) $x(n) = \cos(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n) = \frac{1}{2}\cos(\frac{\pi}{4}n) + \frac{1}{2}\cos(\frac{3\pi}{4}n)$

对 $\cos(\frac{\pi}{4}n)$. $\Omega_1 = \frac{\pi}{4}$, $N_1 = \frac{2k\pi}{\Omega_1} = 8k$. 当 $k=1$ 时 $N_1 = 8$

对 $\cos(\frac{3\pi}{4}n)$. $\Omega_2 = \frac{3\pi}{4}$, $N_2 = \frac{2k\pi}{\Omega_2} = \frac{8k}{3}$. 当 $k=3$ 时 $N_2 = 8$

$x(n)$ 是周期序列

$N = lcm(N_1, N_2) = 8$

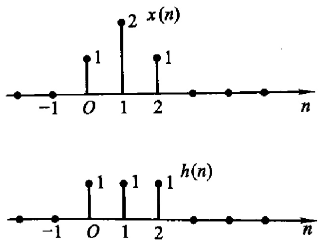
5. 以下各序列中，$x(n)$是系统的输入（或称激励信号、激励函数），$h(n)$是线性时不变（LTI）系统的单位脉冲响应（或称单位样值响应），要求基于卷积和 $y(n) = x(n) * h(n)$，分别求出各$y(n)$，画出$y(n)$的图形。（20 分）

(1) $x(n)$、$h(n)$见题图 3-5(a)；

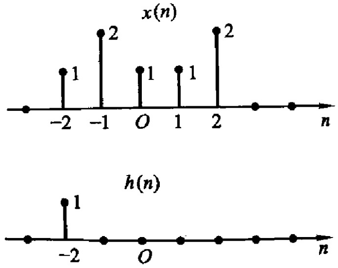
(2) $x(n)$、$h(n)$见题图 3-5(b)；

(3) $x(n) = \alpha^n u(n)$，$0 < \alpha < 1$，$h(n) = \beta^n u(n)$，$0 < \beta < 1$，$\beta \neq \alpha$；

(4) $x(n) = u(n)$，$h(n) = \delta(n-2) - \delta(n-3)$。



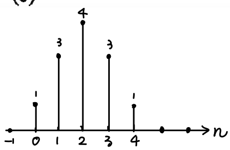
(a)



(b)

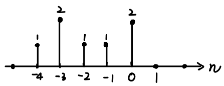
(1) $y(n) = [\delta(n) + 2\delta(n-1) + \delta(n-2)] * [\delta(n) + \delta(n-1) + \delta(n-2)]$

$= \delta(n) + 3\delta(n-1) + 4\delta(n-2) + 3\delta(n-3) + \delta(n-4)$



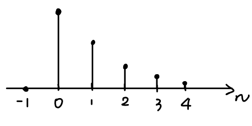
(2) $y(n) = [\delta(n+2) + 2\delta(n+1) + \delta(n) + \delta(n-1) + 2\delta(n-2)] * \delta(n+2)$

$= \delta(n+4) + 2\delta(n+3) + \delta(n+2) + \delta(n+1) + 2\delta(n)$



(3) $y(n) = \alpha^n u(n) * \beta^n u(n) = \sum_{m=0}^{n} \alpha^m \beta^{n-m}$

$= \beta^n \sum_{m=0}^{n} (\frac{\alpha}{\beta})^m = \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} u(n)$



(4) $y(n) = u(n) * [\delta(n-2) - \delta(n-3)]$

$= u(n-2) - u(n-3) = \delta(n-2)$