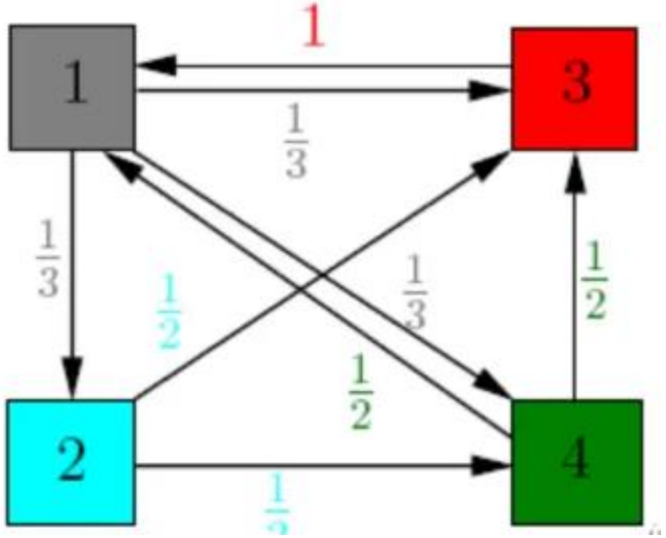
Eigenvalue and Eigenvector

Pagerank



$$A = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

$$\mathrm{v} = \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix}, \mathrm{Av} = \begin{pmatrix} 0.37 \\ 0.08 \\ 0.33 \\ 0.20 \end{pmatrix}, \mathrm{A}^2\mathrm{v} = \mathrm{A}(\mathrm{Av}) = \mathrm{A}\begin{pmatrix} 0.37 \\ 0.08 \\ 0.33 \\ 0.20 \end{pmatrix} = \begin{pmatrix} 0.43 \\ 0.12 \\ 0.27 \\ 0.16 \end{pmatrix}$$

$$\mathrm{A}^3\mathrm{v} = \begin{pmatrix} 0.35 \\ 0.14 \\ 0.29 \\ 0.20 \end{pmatrix}, \mathrm{A}^4\mathrm{v} = \begin{pmatrix} 0.39 \\ 0.11 \\ 0.29 \\ 0.19 \end{pmatrix}, \mathrm{A}^5\mathrm{v} = \begin{pmatrix} 0.39 \\ 0.13 \\ 0.28 \\ 0.19 \end{pmatrix}$$

$$\boxed{A\begin{pmatrix} 0.38 \\ 0.12 \\ 0.29 \\ 0.19 \end{pmatrix} = \begin{pmatrix} 0.38 \\ 0.12 \\ 0.29 \\ 0.19 \end{pmatrix}}$$

$$\mathrm{A}^6\mathrm{v} = \begin{pmatrix} 0.38 \\ 0.13 \\ 0.29 \\ 0.19 \end{pmatrix}, \mathrm{A}^7\mathrm{v} = \begin{pmatrix} 0.38 \\ 0.12 \\ 0.29 \\ 0.19 \end{pmatrix}, \mathrm{A}^8\mathrm{v} = \begin{pmatrix} 0.38 \\ 0.12 \\ 0.29 \\ 0.19 \end{pmatrix}$$

Systems of Linear differential equations

$\dot{Y} = AY$ **Solution???** $\dot{y}(t) = ay(t) \longrightarrow y(t) = ce^{at}$

$$Y = \begin{bmatrix} x_1 e^{\lambda t} \\ x_2 e^{\lambda t} \\ \vdots \\ x_n e^{\lambda t} \end{bmatrix} = e^{\lambda t} x \longrightarrow \dot{Y} = \lambda e^{\lambda t} x = \lambda Y$$

$$Ax = \lambda x$$

$$\longrightarrow AY = e^{\lambda t} Ax = \lambda e^{\lambda t} x = \lambda Y = \dot{Y}$$

$$\boxed{Ax = \lambda x}$$

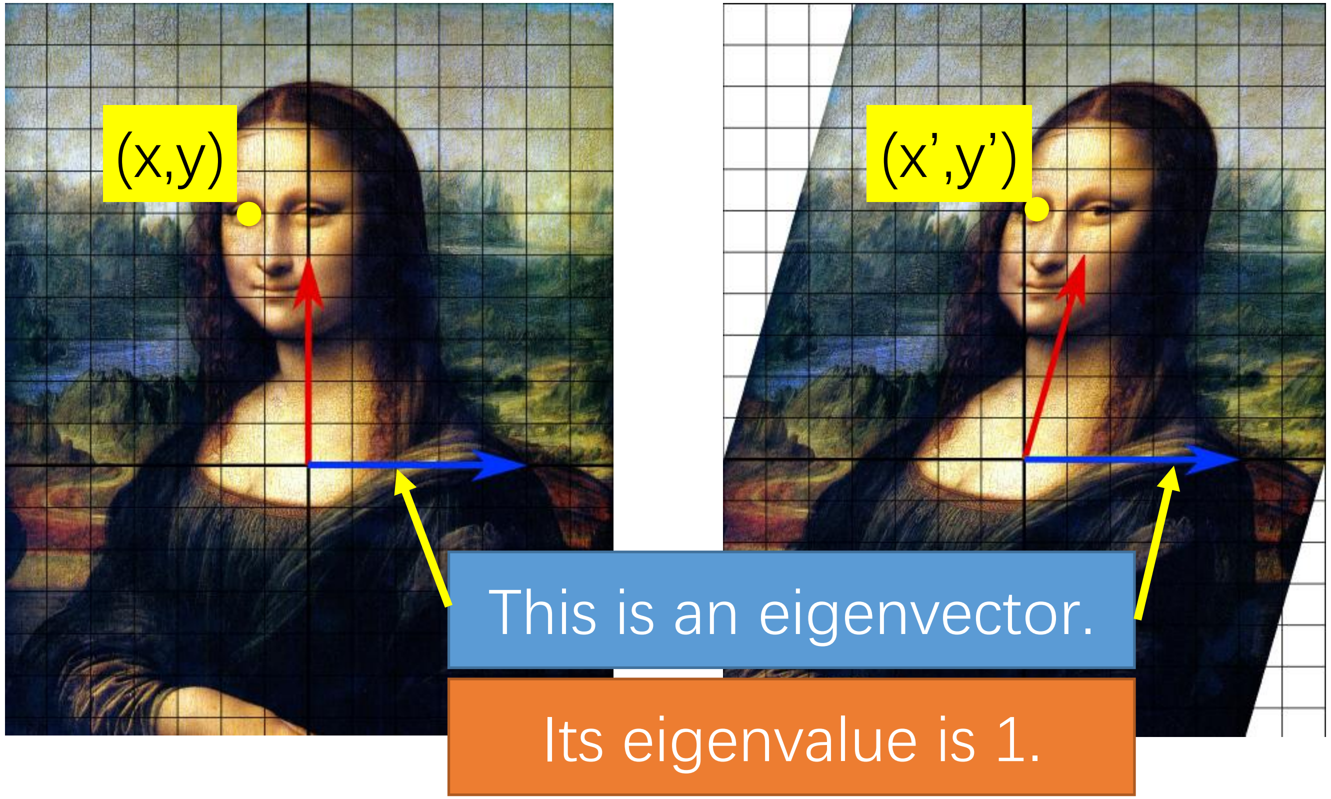
λ eigenvalue

x eigenvector

Example

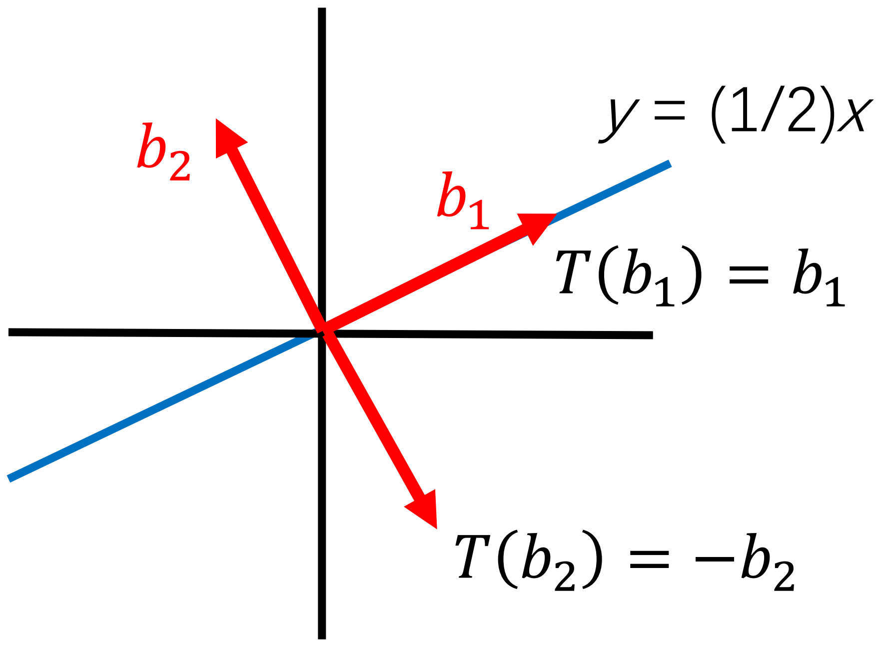
- Shear Transform

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + my \\ y \end{bmatrix}$$



Example

- Reflection operator T about the line $y = (1/2)x$



$\mathbf{b}_1$ is an eigenvector of T

Its eigenvalue is 1.

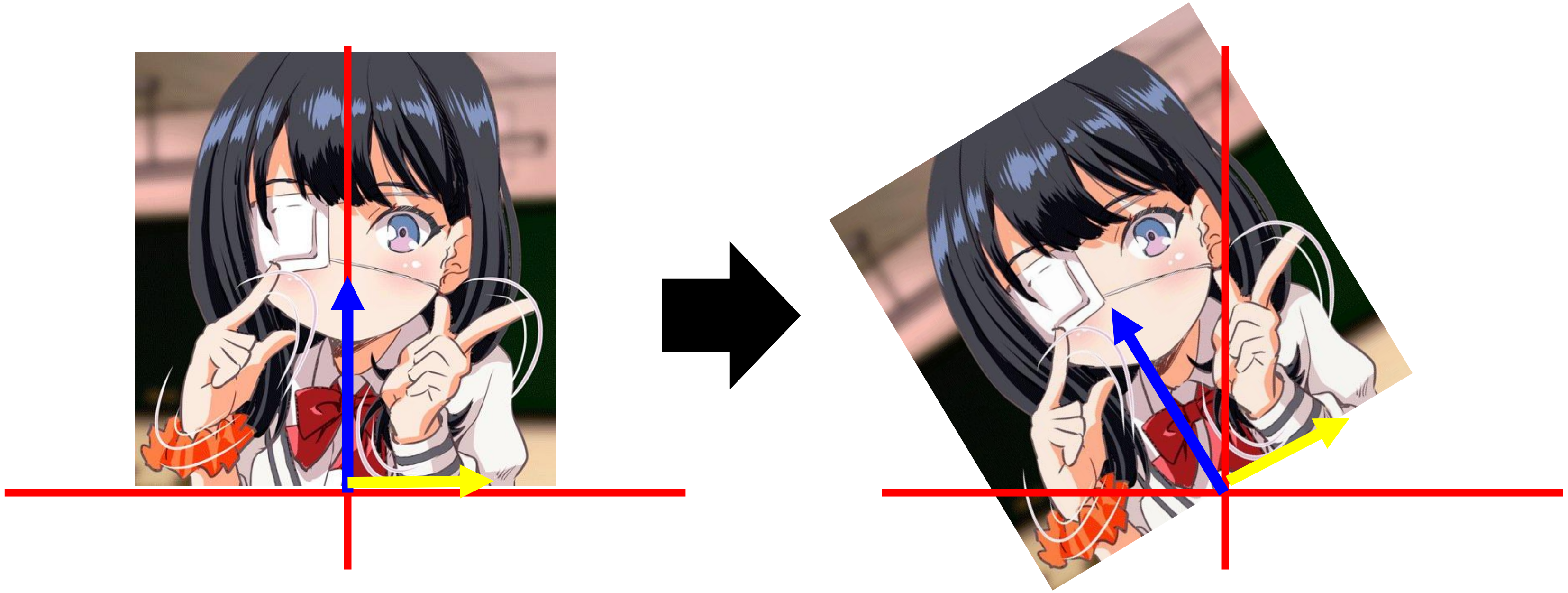
$\mathbf{b}_2$ is an eigenvector of T

Its eigenvalue is -1.

Example

- Rotation

$$M(\theta) = \begin{bmatrix} \cos\theta & -sin\theta \\ sin\theta & \cos\theta \end{bmatrix}$$



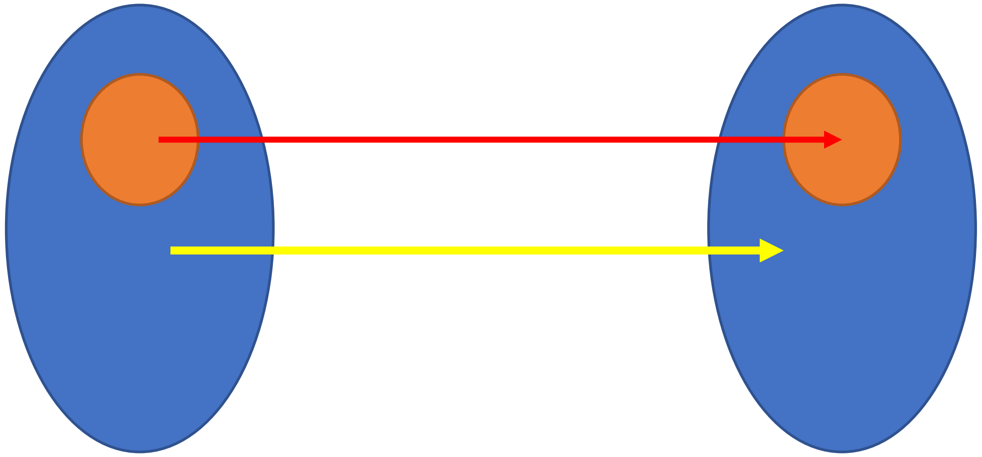
Do any n x n matrix or linear operator have eigenvalues?

Invariant subspace

$$L(V) = L(V,V)$$

direct sum decomposition $\quad V = U_1 \oplus \cdots \oplus U_m$

$T \in L(V)$ $\quad T\big|_{U_j}$ the restriction of T to the smaller domain U_j



Invariant subspace

Definition: Invariant subspace

Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called ***invariant*** under T if $u \in U$ implies $Tu \in U$.

U is invariant under T if $T|_U$ is an operator on U.

Examples

- Invariant subspace of $T \in L(V)$

(a) $\{0\}$;

(b) V;

(c) null T;

(d) range T.

(a) If $u \in \{0\}$, then $u = 0$ and hence $Tu = 0 \in \{0\}$. Thus $\{0\}$ is invariant under T.

(b) If $u \in V$, then $Tu \in V$. Thus V is invariant under T.

(c) If $u \in \operatorname{null} T$, then $Tu = 0$, and hence $Tu \in \operatorname{null} T$. Thus null T is invariant under T.

(d) If $u \in \operatorname{range} T$, then $Tu \in \operatorname{range} T$. Thus range T is invariant under T.

Examples

Example Suppose that $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is defined by $Tp = p'$.

$$\mathcal{P}_4(\mathbf{R}) = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \mid a_i \in R, i = 0,1,2,3,4\}$$

Eigenvalues and Eigenvectors

Invariant subspaces with dimension 1

$U = \{\lambda v : \lambda \in \mathbf{F}\} = \operatorname{span}(v)$ $\qquad$ $v \in V$ with $v \neq 0$

If U is invariant under an operator $T \in \mathcal{L}(V)$, $\qquad$ then $Tv \in U$

$$\boxed{Tv = \lambda v.}$$

Conversely, if $Tv = \lambda v$ for some $\lambda \in \mathbf{F}$, then $\operatorname{span}(v)$ is a 1-dimensional subspace of V invariant under T.

Eigenvalue

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbf{F}$ is called an ***eigenvalue*** of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

Comment: **T** has a 1-dimensional invariant subspace if and only if **T** has an eigenvalue

$$v \neq 0$$

Equivalent conditions to be an eigenvalue

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in F$. Then the following are equivalent:

(a) λ is an eigenvalue of T;

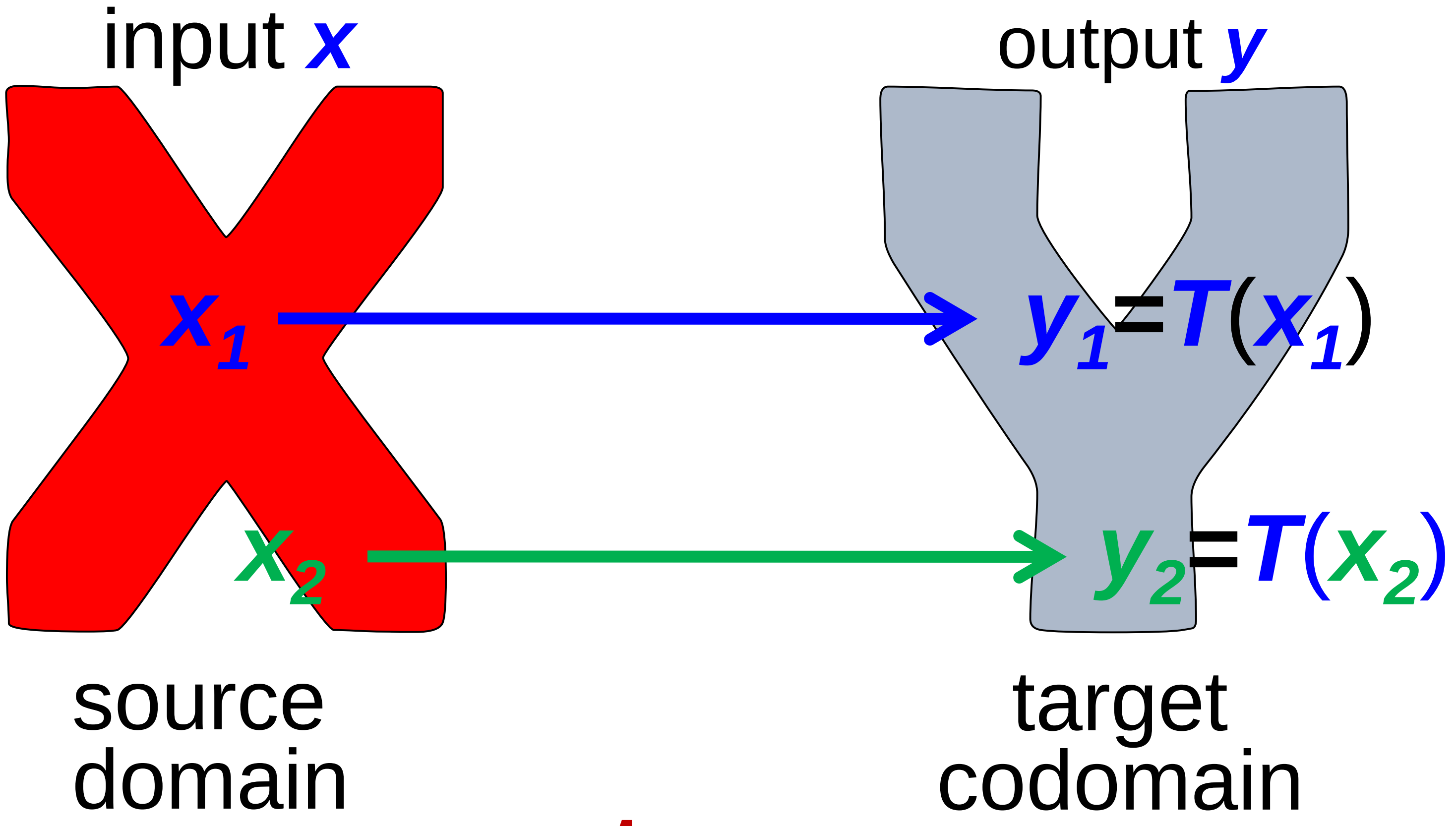
(b) $T - \lambda I$ is not injective;

(c) $T - \lambda I$ is not surjective;

(d) $T - \lambda I$ is not invertible. $\Longleftrightarrow$ $det(T - \lambda I) = 0$

Recall that $I \in \mathcal{L}(V)$ is the identity operator defined by $Iv = v$ for all $v \in V$.

What is one-to-one (injective):



one to one

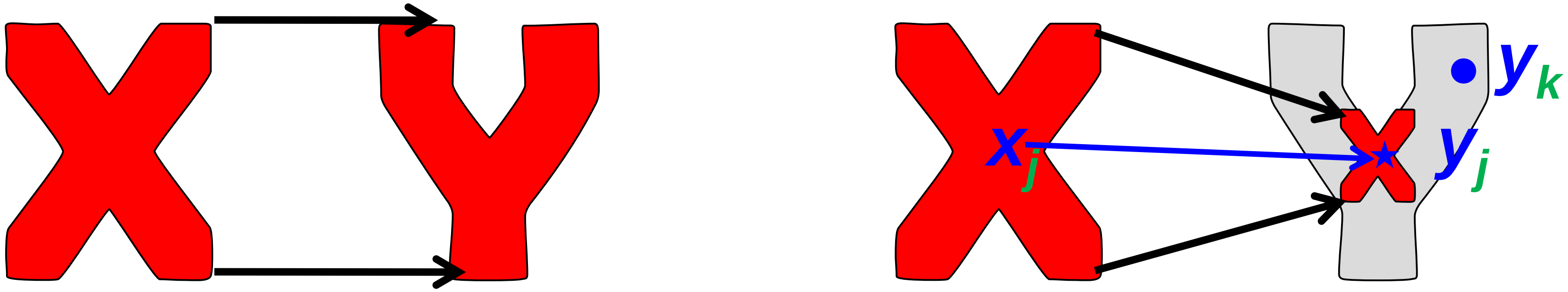
T is one-to-one iff for all x_1 and x_2 in X,
$T(x_1) = T(x_2)$ implies that $x_1 = x_2$.

What is onto (surjective):

Let $T: X \to Y$ be a map. T is onto if its **range** is the whole **target set**. **More** specifically, this means

$$\forall\, y \in Y,\quad \exists\, x \in X,\quad \text{such that}\quad T(x) = y.$$

Intuitively, we may think of a **map** as a way of "**shooting**" from **source** to **target**. The map is onto if any element of the **target set** is "**hit**" by some element of the **source**.



Onto **Not onto**

If T is both **injective** and **surjective**, we call it **bijective**.

- Proof:

(a) $\longleftrightarrow$ (b) $\quad Tv = \lambda v \iff (T - \lambda I)v = 0.$

(b) $\longleftrightarrow$ (c) $\longleftrightarrow$ (d) $\quad T - \lambda I$ is not injective. Thus $null(T - \lambda I) \neq \{0\}$

$\dim \ \mathrm{range}(T - \lambda I) = \dim V - \dim \mathrm{null}(T - \lambda I) < \dim V$ $\quad T - \lambda I$ is not surjective

Eigenvector

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$ is an eigenvalue of T. A vector $v \in V$ is called an ***eigenvector*** of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

$Tv = \lambda v$ if and only if $(T - \lambda I)v = 0$, $\Longrightarrow$ $v \in \operatorname{null}(T - \lambda I)$

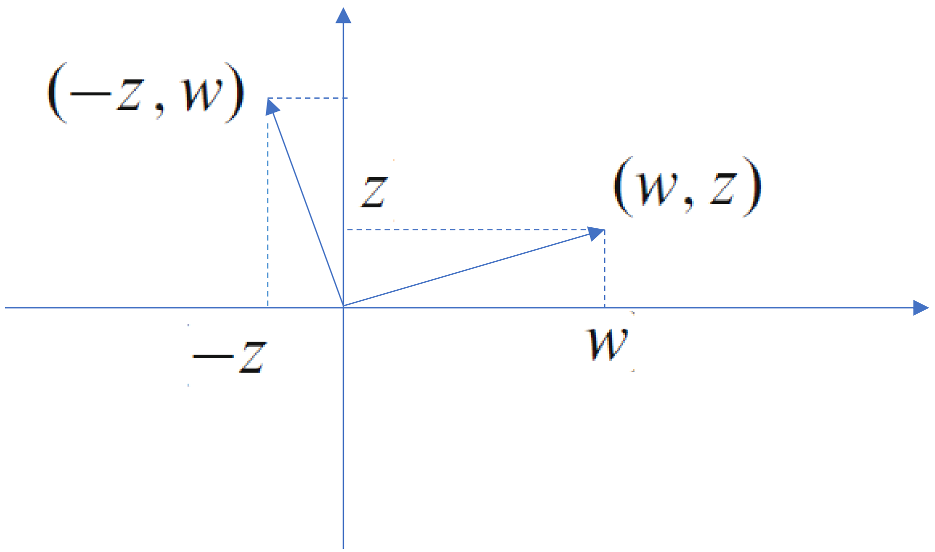
Example

Suppose $T \in \mathcal{L}(\mathbf{F}^2)$ is defined by

$$T(w, z) = (-z, w).$$

(a) Find the eigenvalues and eigenvectors of T if $\mathbf{F} = \mathbf{R}$.

(b) Find the eigenvalues and eigenvectors of T if $\mathbf{F} = \mathbf{C}$.



Solution

(a) T has no eigenvalues

(b) $-z = \lambda w, \quad w = \lambda z \Longrightarrow -z = \lambda^2 z \Longrightarrow -1 = \lambda^2$

$$\lambda = i \text{ and } \lambda = -i$$

Eigenvectors corresponding to $\lambda = i$ is $(w, -wi)$

Eigenvectors corresponding to $\lambda = -i$ is (w, wi)

example

- Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

Suppose λ is an eigenvalue of T with an eigenvector q, then

$$q' = Tq = \lambda q.$$

Note that in general $\deg p' < \deg p$(because we consider $\deg 0 = -\infty$). If $\lambda \neq 0$, then $\deg \lambda q > \deg q'$. We get a contradiction. If $\lambda = 0$, then $q = c$ for nonzero $c \in \mathbb{R}$. Hence the only eigenvalue of T is zero with nonzero constant polynomials as eigenvectors.

Eigenvectors and linearly independent

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are corresponding eigenvectors. Then $v_1, \ldots, v_m$ is linearly independent.

Proof Suppose $v_1, \ldots, v_m$ is linearly dependent. Let k be the smallest positive integer such that

5.11
$$v_k \in \operatorname{span}(v_1, \ldots, v_{k-1});$$

the existence of k with this property follows from the Linear Dependence Lemma (2.21). Thus there exist $a_1, \ldots, a_{k-1} \in \mathbf{F}$ such that

5.12
$$v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}.$$

Apply T to both sides of this equation, getting

$$\lambda_k v_k = a_1 \lambda_1 v_1 + \cdots + a_{k-1} \lambda_{k-1} v_{k-1}.$$

Multiply both sides of 5.12 by λ_k and then subtract the equation above, getting

$$0 = a_1(\lambda_k - \lambda_1) v_1 + \cdots + a_{k-1}(\lambda_k - \lambda_{k-1}) v_{k-1}.$$

Corollary

Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Existence of Eigenvalues

5.21 Operators on complex vector spaces have an eigenvalue

Every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue.

Proof Suppose V is a complex vector space with dimension $n > 0$ and $T \in \mathcal{L}(V)$. Choose $v \in V$ with $v \neq 0$. Then

$$v, Tv, T^2 v, \ldots, T^n v$$

is not linearly independent, because V has dimension n and we have $n + 1$ vectors. Thus there exist complex numbers $a_0, \ldots, a_n$, not all 0, such that

$$0 = a_0 v + a_1 T v + \cdots + a_n T^n v.$$

Note that $a_1, \ldots, a_n$ cannot all be 0, because otherwise the equation above would become $0 = a_0 v$, which would force a_0 also to be 0.

Make the a's the coefficients of a polynomial, which by the Fundamental Theorem of Algebra (4.14) has a factorization

$$a_0 + a_1 z + \cdots + a_n z^n = c(z - \lambda_1) \cdots (z - \lambda_m),$$

where c is a nonzero complex number, each λ_j is in $\mathbf{C}$, and the equation holds for all $z \in \mathbf{C}$ (here m is not necessarily equal to n, because a_n may equal 0). We then have

$$\begin{aligned} 0 &= a_0 v + a_1 T v + \cdots + a_n T^n v \\ &= (a_0 I + a_1 T + \cdots + a_n T^n) v \\ &= c(T - \lambda_1 I) \cdots (T - \lambda_m I) v. \end{aligned}$$

Thus $T - \lambda_j I$ is not injective for at least one j. In other words, T has an eigenvalue. ∎

Matrix of an operator

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. The ***matrix of*** T with respect to this basis is the n-by-n matrix

$$\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \ldots & A_{1,n} \\ \vdots & & \vdots \\ A_{n,1} & \ldots & A_{n,n} \end{pmatrix}$$

whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}v_1 + \cdots + A_{n,k}v_n.$$

If the basis is not clear from the context, then the notation $\mathcal{M}\big(T, (v_1, \ldots, v_n)\big)$ is used.

Example

5.23 **Example** Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(x, y, z) = (2x + y, 5y + 3z, 8z)$. Then

$$\mathcal{M}(T) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}.$$

upper-triangular matrix

A matrix is called ***upper triangular*** if all the entries below the diagonal equal 0.

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix} \qquad \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Conditions for upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then the following are equivalent:

(a) the matrix of T with respect to $v_1, \ldots, v_n$ is upper triangular;

(b) $Tv_j \in \operatorname{span}(v_1, \ldots, v_j)$ for each $j = 1, \ldots, n$;

(c) $\operatorname{span}(v_1, \ldots, v_j)$ is invariant under T for each $j = 1, \ldots, n$.

operator and upper-triangular matrix

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.

Proof 1 We will use induction on the dimension of V. Clearly the desired result holds if $\dim V = 1$.

Suppose now that $\dim V > 1$ and the desired result holds for all complex vector spaces whose dimension is less than the dimension of V. Let λ be any eigenvalue of T (5.21 guarantees that T has an eigenvalue). Let

$$U = \operatorname{range}(T - \lambda I).$$

Because $T - \lambda I$ is not surjective (see 3.69), $\dim U < \dim V$. Furthermore, U is invariant under T. To prove this, suppose $u \in U$. Then

$$Tu = (T - \lambda I)u + \lambda u.$$

Proof

Obviously $(T - \lambda I)u \in U$ (because U equals the range of $T - \lambda I$) and $\lambda u \in U$. Thus the equation above shows that $Tu \in U$. Hence U is invariant under T, as claimed.

Thus $T|_U$ is an operator on U. By our induction hypothesis, there is a basis $u_1, \dots, u_m$ of U with respect to which $T|_U$ has an upper-triangular matrix. Thus for each j we have (using 5.26)

5.28 $$Tu_j = (T|_U)(u_j) \in \operatorname{span}(u_1, \dots, u_j).$$

Extend $u_1, \dots, u_m$ to a basis $u_1, \dots, u_m, v_1, \dots, v_n$ of V. For each k, we have

$$Tv_k = (T - \lambda I)v_k + \lambda v_k.$$

The definition of U shows that $(T - \lambda I)v_k \in U = \operatorname{span}(u_1, \dots, u_m)$. Thus the equation above shows that

5.29 $$Tv_k \in \operatorname{span}(u_1, \dots, u_m, v_1, \dots, v_k).$$

From 5.28 and 5.29, we conclude (using 5.26) that T has an upper-triangular matrix with respect to the basis $u_1, \dots, u_m, v_1, \dots, v_n$ of V, as desired. ■

Determination of invertibility from upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then T is invertible if and only if all the entries on the diagonal of that upper-triangular matrix are nonzero.

Determination of eigenvalues from upper-triangular matrix

5.32 Determination of eigenvalues from upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.

Eigenspaces and Diagonal Matrices

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. The ***eigenspace*** of T corresponding to λ, denoted $E(\lambda, T)$, is defined by

$$E(\lambda, T) = \operatorname{null}(T - \lambda I).$$

In other words, $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector.

5.35 **Example**

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$E(8, T) = \operatorname{span}(v_1), \quad E(5, T) = \operatorname{span}(v_2, v_3)$$

Sum of eigenspaces is a direct sum

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Suppose also that $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum. Furthermore,

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V.$$

Proof To show that $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ is a direct sum, suppose

$$u_1 + \cdots + u_m = 0,$$

where each u_j is in $E(\lambda, T)$. Because eigenvectors corresponding to distinct eigenvalues are linearly independent (see 5.10), this implies that each u_j equals 0. This implies (using 1.44) that $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ is a direct sum, as desired.

Now

$$\begin{aligned} \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) &= \dim\bigl(E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)\bigr) \\ &\leq \dim V, \end{aligned}$$

where the equality above follows from Exercise 16 in Section 2.C. ■

diagonalizable

5.39 Definition ***diagonalizable***

An operator $T \in \mathcal{L}(V)$ is called ***diagonalizable*** if the operator has a diagonal matrix with respect to some basis of V.

5.40 Example Define $T \in \mathcal{L}(\mathbf{R}^2)$ by

$$T(x, y) = (41x + 7y, -20x + 74y).$$

The matrix of T with respect to the standard basis of $\mathbf{R}^2$ is

$$\begin{pmatrix} 41 & 7 \\ -20 & 74 \end{pmatrix},$$

which is not a diagonal matrix. However, T is diagonalizable, because the matrix of T with respect to the basis $(1, 4), (7, 5)$ is

$$\begin{pmatrix} 69 & 0 \\ 0 & 46 \end{pmatrix},$$

as you should verify.

Conditions equivalent to diagonalizability

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent:

(a) T is diagonalizable;

(b) V has a basis consisting of eigenvectors of T;

(c) there exist 1-dimensional subspaces $U_1, \ldots, U_n$ of V, each invariant under T, such that

$$V = U_1 \oplus \cdots \oplus U_n;$$

(d) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$;

(e) $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

Proof An operator $T \in \mathcal{L}(V)$ has a diagonal matrix

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

with respect to a basis $v_1, \dots, v_n$ of V if and only if $Tv_j = \lambda_j v_j$ for each j. Thus (a) and (b) are equivalent.

Suppose (b) holds; thus V has a basis $v_1, \dots, v_n$ consisting of eigenvectors of T. For each j, let $U_j = \operatorname{span}(v_j)$. Obviously each U_j is a 1-dimensional subspace of V that is invariant under T. Because $v_1, \dots, v_n$ is a basis of V, each vector in V can be written uniquely as a linear combination of $v_1, \dots, v_n$. In other words, each vector in V can be written uniquely as a sum $u_1 + \cdots + u_n$, where each u_j is in U_j. Thus $V = U_1 \oplus \cdots \oplus U_n$. Hence (b) implies (c).

Suppose now that (c) holds; thus there are 1-dimensional subspaces $U_1, \dots, U_n$ of V, each invariant under T, such that $V = U_1 \oplus \cdots \oplus U_n$. For each j, let v_j be a nonzero vector in U_j. Then each v_j is an eigenvector of T. Because each vector in V can be written uniquely as a sum $u_1 + \cdots + u_n$, where each u_j is in U_j (so each u_j is a scalar multiple of v_j), we see that $v_1, \dots, v_n$ is a basis of V. Thus (c) implies (b).

Suppose (b) holds; thus V has a basis consisting of eigenvectors of T. Hence every vector in V is a linear combination of eigenvectors of T, which implies that

$$V = E(\lambda_1, T) + \cdots + E(\lambda_m, T).$$

Now 5.38 shows that (d) holds.

That (d) implies (e) follows immediately from Exercise 16 in Section 2.C.

Finally, suppose (e) holds; thus

5.42 $$\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T).$$

Choose a basis of each $E(\lambda_j, T)$; put all these bases together to form a list $v_1, \ldots, v_n$ of eigenvectors of T, where $n = \dim V$ (by 5.42). To show that this list is linearly independent, suppose

$$a_1 v_1 + \cdots + a_n v_n = 0,$$

where $a_1, \ldots, a_n \in \mathbf{F}$. For each $j = 1, \ldots, m$, let u_j denote the sum of all the terms $a_k v_k$ such that $v_k \in E(\lambda_j, T)$. Thus each u_j is in $E(\lambda_j, T)$, and

$$u_1 + \cdots + u_m = 0.$$

Because eigenvectors corresponding to distinct eigenvalues are linearly independent (see 5.10), this implies that each u_j equals 0. Because each u_j is a sum of terms $a_k v_k$, where the v_k's were chosen to be a basis of $E(\lambda_j, T)$, this implies that all the a_k's equal 0. Thus $v_1, \ldots, v_n$ is linearly independent and hence is a basis of V (by 2.39). Thus (e) implies (b), completing the proof. ■

5.43 Example Show that the operator $T \in \mathcal{L}(\mathbf{C}^2)$ defined by

$$T(w, z) = (z, 0)$$

is not diagonalizable.

Solution As you should verify, 0 is the only eigenvalue of T and furthermore $E(0, T) = \{(w, 0) \in \mathbf{C}^2 : w \in \mathbf{C}\}$.

Thus conditions (b), (c), (d), and (e) of 5.41 are easily seen to fail (of course, because these conditions are equivalent, it is only necessary to check that one of them fails). Thus condition (a) of 5.41 also fails, and hence T is not diagonalizable.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix}$$

Enough eigenvalues implies diagonalizability

If $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, then T is diagonalizable.

Proof Suppose $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues $\lambda_1, \ldots, \lambda_{\dim V}$. For each j, let $v_j \in V$ be an eigenvector corresponding to the eigenvalue λ_j. Because eigenvectors corresponding to distinct eigenvalues are linearly independent (see 5.10), $v_1, \ldots, v_{\dim V}$ is linearly independent. A linearly independent list of $\dim V$ vectors in V is a basis of V (see 2.39); thus $v_1, \ldots, v_{\dim V}$ is a basis of V. With respect to this basis consisting of eigenvectors, T has a diagonal matrix. ■

5.45 Example Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(x, y, z) = (2x + y, 5y + 3z, 8z)$. Find a basis of $\mathbf{F}^3$ with respect to which T has a diagonal matrix.

Solution With respect to the standard basis, the matrix of T is

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}.$$

$$T(x, y, z) = \lambda(x, y, z)$$

for $\lambda = 2$, then for $\lambda = 5$, and then for $\lambda = 8$. These simple equations are easy to solve: for $\lambda = 2$ we have the eigenvector $(1, 0, 0)$; for $\lambda = 5$ we have the eigenvector $(1, 3, 0)$; for $\lambda = 8$ we have the eigenvector $(1, 6, 6)$.

Thus $(1, 0, 0), (1, 3, 0), (1, 6, 6)$ is a basis of $\mathbf{F}^3$, and with respect to this basis the matrix of T is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Systems of Linear differential equations

$\dot{Y} = AY$ **Solution???** $\dot{y}(t) = ay(t) \Longrightarrow y(t) = ce^{at}$

$$Ax = \lambda x$$

$$Y = \begin{bmatrix} x_1 e^{\lambda t} \\ x_2 e^{\lambda t} \\ \vdots \\ x_n e^{\lambda t} \end{bmatrix} = e^{\lambda t} x \Longrightarrow \dot{Y} = \lambda e^{\lambda t} x = \lambda Y \Longrightarrow AY = e^{\lambda t} Ax = \lambda e^{\lambda t} x = \lambda Y = \dot{Y}$$

Y_1 Y_2 are solutions, then $\alpha Y_1 + \beta Y_2$ is also a solution

Example

$$\dot{y}_1 = 3y_1 + 4y_2$$
$$\dot{y}_2 = 3y_1 + 2y_2$$

$$A = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} \xrightarrow{\det(A-\lambda I)=0} \lambda_1 = 6,\quad \lambda_2 = -1 \xrightarrow{(A-6I)x=0,\,(A+I)x=0} x_1 = (4,3)^T,\, x_2 = (1,-1)^T$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = \begin{pmatrix} 4c_1 e^{6t} + c_2 e^{-t} \\ 3c_1 e^{6t} - c_2 e^{-t} \end{pmatrix}$$

$$\mathrm{Y}(0) = \begin{pmatrix} 4c_1 + c_2 \\ 3c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} \longrightarrow c_1 = 1, c_2 = 2$$

Complex eigenvalues

Let A be a real $n \times n$ matrix with a complex eigenvalue $\lambda = a + bi$, and let x be an eigenvector belonging to λ. The vector $\mathbf{x}$ can be split up into its real and imaginary parts.

$$\mathbf{x} = \begin{bmatrix} \text{Re } x_1 + i \text{ Im } x_1 \\ \text{Re } x_2 + i \text{ Im } x_2 \\ \vdots \\ \text{Re } x_n + i \text{ Im } x_n \end{bmatrix} = \begin{bmatrix} \text{Re } x_1 \\ \text{Re } x_2 \\ \vdots \\ \text{Re } x_n \end{bmatrix} + i \begin{bmatrix} \text{Im } x_1 \\ \text{Im } x_2 \\ \vdots \\ \text{Im } x_n \end{bmatrix} = \text{Re } \mathbf{x} + i \text{Im } \mathbf{x}$$

Since the entries of A are all real, it follows that $\bar{\lambda} = a - bi$ is also an eigenvalue of A with eigenvector

$$\bar{\mathbf{x}} = \begin{bmatrix} \text{Re } x_1 - i \text{ Im } x_1 \\ \text{Re } x_2 - i \text{ Im } x_2 \\ \vdots \\ \text{Re } x_n - i \text{ Im } x_n \end{bmatrix} = \text{Re } \mathbf{x} - i \text{ Im } \mathbf{x}$$

and hence $e^{\lambda t}\mathbf{x}$ and $e^{\bar{\lambda}t}\bar{\mathbf{x}}$ are both solutions of the first-order system $\mathbf{Y}' = A\mathbf{Y}$. Any linear combination of these two solutions will also be a solution. Thus, if we set

$$\mathbf{Y}_1 = \frac{1}{2}\left(e^{\lambda t}\mathbf{x} + e^{\bar{\lambda}t}\bar{\mathbf{x}}\right) = \mathrm{Re}\left(e^{\lambda t}\mathbf{x}\right)$$

and

$$\mathbf{Y}_2 = \frac{1}{2i}\left(e^{\lambda t}\mathbf{x} - e^{\bar{\lambda}t}\bar{\mathbf{x}}\right) = \mathrm{Im}\left(e^{\lambda t}\mathbf{x}\right)$$

then the vector functions $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are real-valued solutions of $\mathbf{Y}' = A\mathbf{Y}$. Taking the real and imaginary parts of

$$\begin{aligned} \mathbf{e}^{\lambda \mathbf{t}}\mathbf{x} &= e^{(a+ib)t}\mathbf{x} \\ &= e^{at}(\cos bt + i \sin bt)(\mathrm{Re}\,\mathbf{x} + i\,\mathrm{Im}\,\mathbf{x}) \end{aligned}$$

we see that

$$\begin{aligned} \mathbf{Y}_1 &= e^{at}[(\cos bt)\ \mathrm{Re}\,\mathbf{x} - (\sin bt)\ \mathrm{Im}\,\mathbf{x}] \\ \mathbf{Y}_2 &= e^{at}[(\cos bt)\ \mathrm{Im}\,\mathbf{x} + (\sin bt)\ \mathrm{Re}\,\mathbf{x}] \end{aligned}$$

Example

$$\begin{aligned} y_1' &= \quad y_1 + \ y_2 \\ y_2' &= -2y_1 + 3y_2 \end{aligned}$$

Let

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$$

The eigenvalues of A are $\lambda = 2 + i$ and $\bar{\lambda} = 2 - i$, with eigenvectors $\mathbf{x} = (1, 1 + i)^T$ and $\bar{\mathbf{x}} = (1, 1 - i)^T$, respectively.

$$\begin{aligned} e^{\lambda t}\mathbf{x} &= \begin{bmatrix} e^{2t}(\cos t + i \sin t) \\ e^{2t}(\cos t + i \sin t)(1+i) \end{bmatrix} \\ &= \begin{bmatrix} e^{2t}\cos t + ie^{2t}\sin t \\ e^{2t}(\cos t - \sin t) + ie^{2t}(\cos t + \sin t) \end{bmatrix} \end{aligned}$$

Let

$$\mathbf{Y}_1 = \mathrm{Re}\left(e^{\lambda t}\mathbf{x}\right) = \begin{bmatrix} e^{2t}\cos t \\ e^{2t}(\cos t - \sin t) \end{bmatrix}$$

and

$$\mathbf{Y}_2 = \mathrm{Im}\left(e^{\lambda t}\mathbf{x}\right) = \begin{bmatrix} e^{2t}\sin t \\ e^{2t}(\cos t + \sin t) \end{bmatrix}$$

Any linear combination

$$\mathbf{Y} = c_1\mathbf{Y}_1 + c_2\mathbf{Y}_2$$

will be a solution of the system.

Higher-Order Systems

Given a second-order system of the form

$$\mathbf{Y}'' = A_1\,\mathbf{Y} + A_2\,\mathbf{Y}'$$

we may translate it into a first-order system by setting

$$\begin{aligned} y_{n+1}(t) &= y_1'(t) \\ y_{n+2}(t) &= y_2'(t) \\ &\vdots \\ y_{2n}(t) &= y_n'(t) \end{aligned}$$

If we let

$$\mathbf{Y}_1 = \mathbf{Y} = (y_1, y_2, \ldots, y_n)^T$$

and

$$\mathbf{Y}_2 = \mathbf{Y}' = (y_{n+1}, \ldots, y_{2n})^T$$

then

$$\mathbf{Y}_1' = O\mathbf{Y}_1 + I\mathbf{Y}_2$$

and

$$\mathbf{Y}_2' = A_1\,\mathbf{Y}_1 + A_2\,\mathbf{Y}_2$$

The equations can be combined to give the $2n \times 2n$ first-order system

$$\begin{bmatrix} \mathbf{Y}_1' \\ \mathbf{Y}_2' \end{bmatrix} = \begin{bmatrix} O & I \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix}$$

Higher-Order Systems

In general, if we have an mth-order system of the form

$$\mathbf{Y}^{(m)} = A_1\mathbf{Y} + A_2\mathbf{Y}' + \ldots + A_m\mathbf{Y}^{(m-1)}$$

where each A_i is an $n \times n$ matrix, we can transform it into a first-order system by setting

$$\mathbf{Y}_1 = \mathbf{Y},\ \mathbf{Y}_2 = \mathbf{Y}_1', \ldots, \mathbf{Y}_m = \mathbf{Y}_{m-1}'$$

We will end up with a system of the form

$$\begin{bmatrix} \mathbf{Y}_1' \\ \mathbf{Y}_2' \\ \vdots \\ \mathbf{Y}_{m-1}' \\ \mathbf{Y}_m' \end{bmatrix} = \begin{bmatrix} O & I & O & \ldots & O \\ O & O & I & \ldots & O \\ \vdots & & & & \\ O & O & O & \ldots & I \\ A_1 & A_2 & A_3 & \ldots & A_m \end{bmatrix} \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \vdots \\ \mathbf{Y}_{m-1} \\ \mathbf{Y}_m \end{bmatrix}$$

If the system is simply of the form $\mathbf{Y}^{(m)} = A\mathbf{Y}$, it is usually not necessary to introduce new variables. In this case, we need only calculate the mth roots of the eigenvalues of A. If λ is an eigenvalue of A, $\mathbf{x}$ is an eigenvector belonging to λ, σ is an mth root of λ, and $\mathbf{Y} = e^{\sigma t}\mathbf{x}$, then

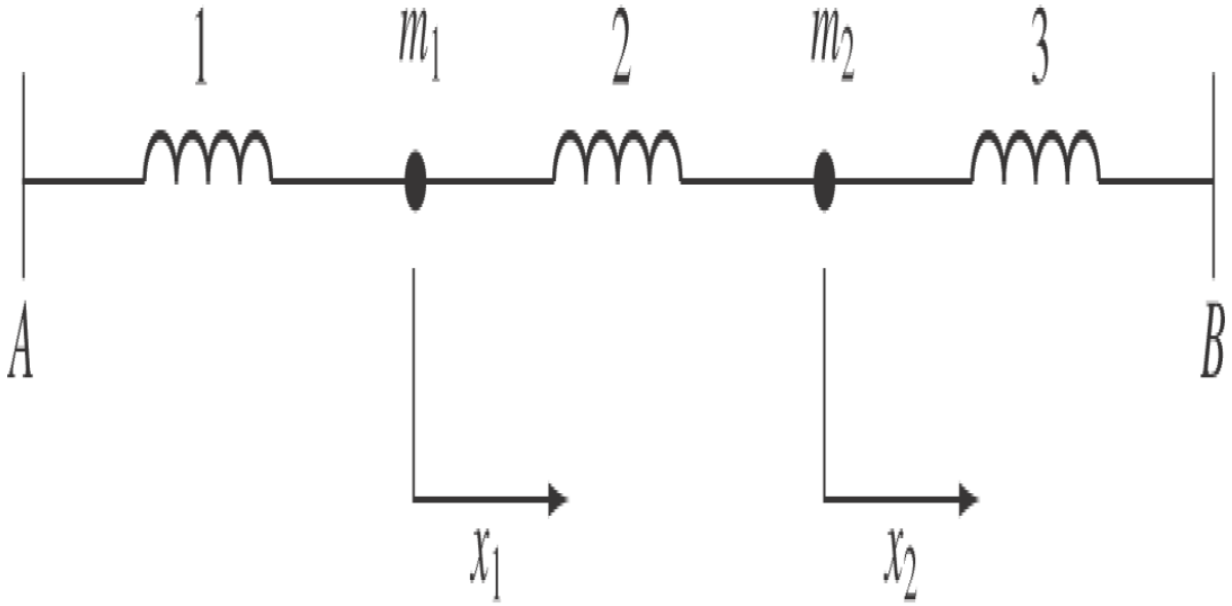
$$\mathbf{Y}^{(m)} = \sigma^m e^{\sigma t}\mathbf{x} = \lambda\mathbf{Y}$$

and

$$A\mathbf{Y} = e^{\sigma t}A\mathbf{x} = \lambda e^{\sigma t}\mathbf{x} = \lambda\mathbf{Y}$$

Therefore, $\mathbf{Y} = e^{\sigma t}\mathbf{x}$ is a solution to the system.

Applications



$$m_1 x_1''(t) = -kx_1 + k(x_2 - x_1)$$

$$m_2 x_2''(t) = -k(x_2 - x_1) - kx_2$$

$$x_1'' = -\frac{k}{m_1}(2x_1 - x_2)$$

$$x_2'' = -\frac{k}{m_2}(-x_1 + 2x_2)$$

Suppose now that $m_1 = m_2 = 1, k = 1$

$$\mathbf{X}'' = A\mathbf{X} \qquad\qquad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3$. Corresponding to λ_1, we have the eigenvector $\mathbf{v}_1 = (1, 1)^T$ and $\sigma_1 = \pm i$. Thus, $e^{it}\mathbf{v}_1$ and $e^{-it}\mathbf{v}_1$ are both solutions

$$\frac{1}{2}\left(e^{it} + e^{-it}\right)\mathbf{v}_1 = \left(\text{Re } e^{it}\right)\mathbf{v}_1 = (\cos t)\mathbf{v}_1$$

and

$$\frac{1}{2i}\left(e^{it} - e^{-it}\right)\mathbf{v}_1 = \left(\text{Im } e^{it}\right)\mathbf{v}_1 = (\sin t)\mathbf{v}_1$$

are also both solutions of (2). Similarly, for $\lambda_2 = -3$, we have the eigenvector $\mathbf{v}_2 = (1, -1)^T$ and $\sigma_2 = \pm\sqrt{3}i$. It follows that

$$\left(\operatorname{Re} e^{\sqrt{3}it}\right)\mathbf{v}_2 = \left(\cos\sqrt{3}t\right)\mathbf{v}_2$$

and

$$\left(\operatorname{Im} e^{\sqrt{3}it}\right)\mathbf{v}_2 = \left(\sin\sqrt{3}t\right)\mathbf{v}_2$$

are also solutions of (2). Thus, the general solution will be of the form

$$\begin{aligned}\mathbf{X}(t) &= c_1(\cos t)\mathbf{v}_1 + c_2(\sin t)\mathbf{v}_1 + c_3\left(\cos\sqrt{3}t\right)\mathbf{v}_2 + c_4\left(\sin\sqrt{3}t\right)\mathbf{v}_2 \\ &= \begin{bmatrix} c_1 \cos t + c_2 \sin t + c_3 \cos\sqrt{3}t + c_4 \sin\sqrt{3}t \\ c_1 \cos t + c_2 \sin t - c_3 \cos\sqrt{3}t - c_4 \sin\sqrt{3}t \end{bmatrix}\end{aligned}$$

At time $t = 0$, we have

$$\mathbf{X}(t) = \begin{bmatrix} 2\ \mathrm{sin}t \\ 2\ \mathrm{sin}t \end{bmatrix}$$

$$x_1(0) = x_2(0) = 0 \quad \text{and} \quad x_1'(0) = x_2'(0) = 2$$

The Exponential of a Matrix

Given a scalar a, the exponential e^a can be expressed in terms of a power series

$$e^a = 1 + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \dots$$

Similarly, for any $n \times n$ matrix A, we can define the *matrix exponential* e^A in terms of the convergent power series

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

the matrix exponential is easy to compute:

$$\begin{aligned} e^D &= \lim_{m\to\infty}\left(I + D + \frac{1}{2!}D^2 + \ \ldots + \frac{1}{m!}D^m\right) \\ &= \lim_{m\to\infty}\begin{bmatrix} \sum_{k=0}^{m}\frac{1}{k!}\lambda_1^k & & \\ & \ddots & \\ & & \sum_{k=0}^{m}\frac{1}{k!}\lambda_n^k \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix} \end{aligned}$$

It is more difficult to compute the matrix exponential for a general $n \times n$ matrix A. If, however, A is diagonalizable, then

$$\begin{aligned} A^k &= XD^kX^{-1} \quad \text{for} \quad k = 1, 2, \ldots \\ e^A &= X\left(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \ldots\right)X^{-1} \\ &= Xe^DX^{-1} \end{aligned}$$

$$y' = ay, \quad y(0) = y_0$$

the solution is

$$y = e^{at} y_0$$

The matrix exponential can be applied to the initial value problem

$$\mathbf{Y}' = A\mathbf{Y}, \quad \mathbf{Y}(0) = \mathbf{Y}_0$$

$$\boxed{\mathbf{Y}(t) = e^{tA}\mathbf{Y}_0}$$

then

$$\mathbf{Y}' = Ae^{tA}\mathbf{Y}_0 = A\mathbf{Y}$$

and

$$\mathbf{Y}(0) = \mathbf{Y}_0$$

Thus, the solution of

$$\mathbf{Y}' = A\mathbf{Y}, \quad \mathbf{Y}(0) = \mathbf{Y}_0$$

is simply

$$\mathbf{Y} = e^{tA}\mathbf{Y}_0$$

$$c_1e^{\lambda_1 t}\mathbf{x}_1 + c_2e^{\lambda_2 t}\mathbf{x}_2 + \ldots + c_ne^{\lambda_n t}\mathbf{x}_n$$

If A is diagonalizable,

$$\mathbf{Y} = Xe^{tD}X^{-1}\mathbf{Y}_0$$

Thus,

$$\begin{aligned}
\mathbf{Y} &= Xe^{tD}\mathbf{c} \\
&= (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)\begin{bmatrix} c_1e^{\lambda_1 t} \\ c_2e^{\lambda_2 t} \\ \vdots \\ c_ne^{\lambda_n t} \end{bmatrix} \\
&= c_1e^{\lambda_1 t}\mathbf{x}_1 + \ldots + c_ne^{\lambda_n t}\mathbf{x}_n
\end{aligned}$$

Singular Value Decomposition

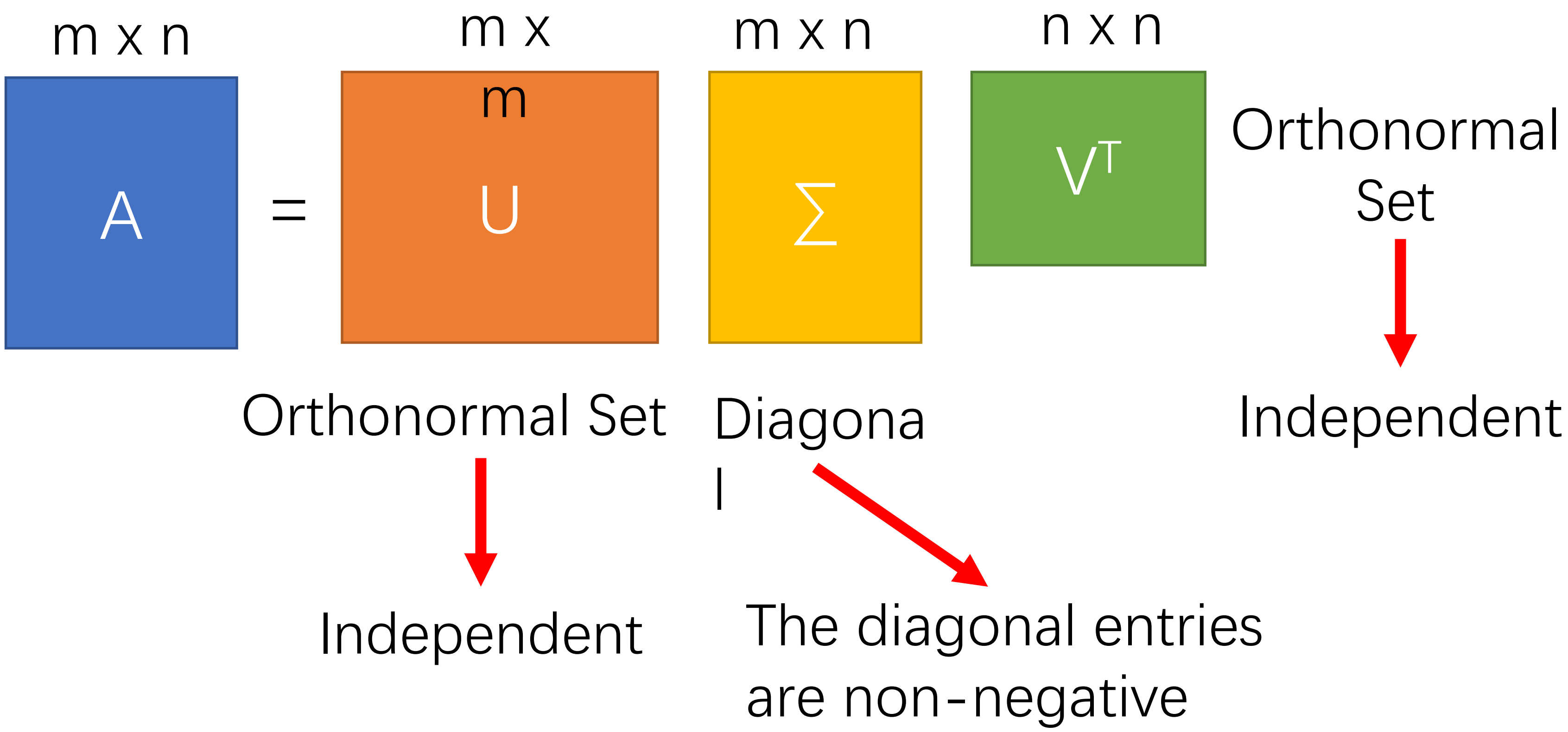
- Diagonalization can only apply on some square matrices.

$$A = W\Sigma W^{-1}$$

- Singular value decomposition (SVD) can apply on any matrix.

SVD

- Any m x n matrix A



SVD

(We can exchange some rows and columns to achieve that)

$$\left[\begin{array}{cccc|cccc}
\sigma_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
0 & \sigma_2 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & & \sigma_k & 0 & 0 & \dots & 0 \\
\hline
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0
\end{array}\right]$$

- Any m x n matrix A

m x n		m x m	m x n	n x n	
A	=	U	Σ	V^T	Independent
		Independent	Diagonal		

What is the rank of A?

If A is a m x n matrix, and B is a n x k matrix.

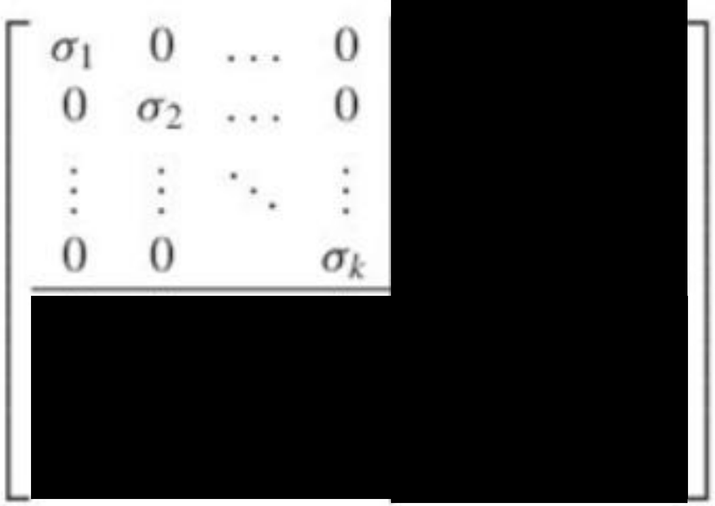
$$Rank(AB) \le min\big(Rank(A), Rank(B)\big)$$

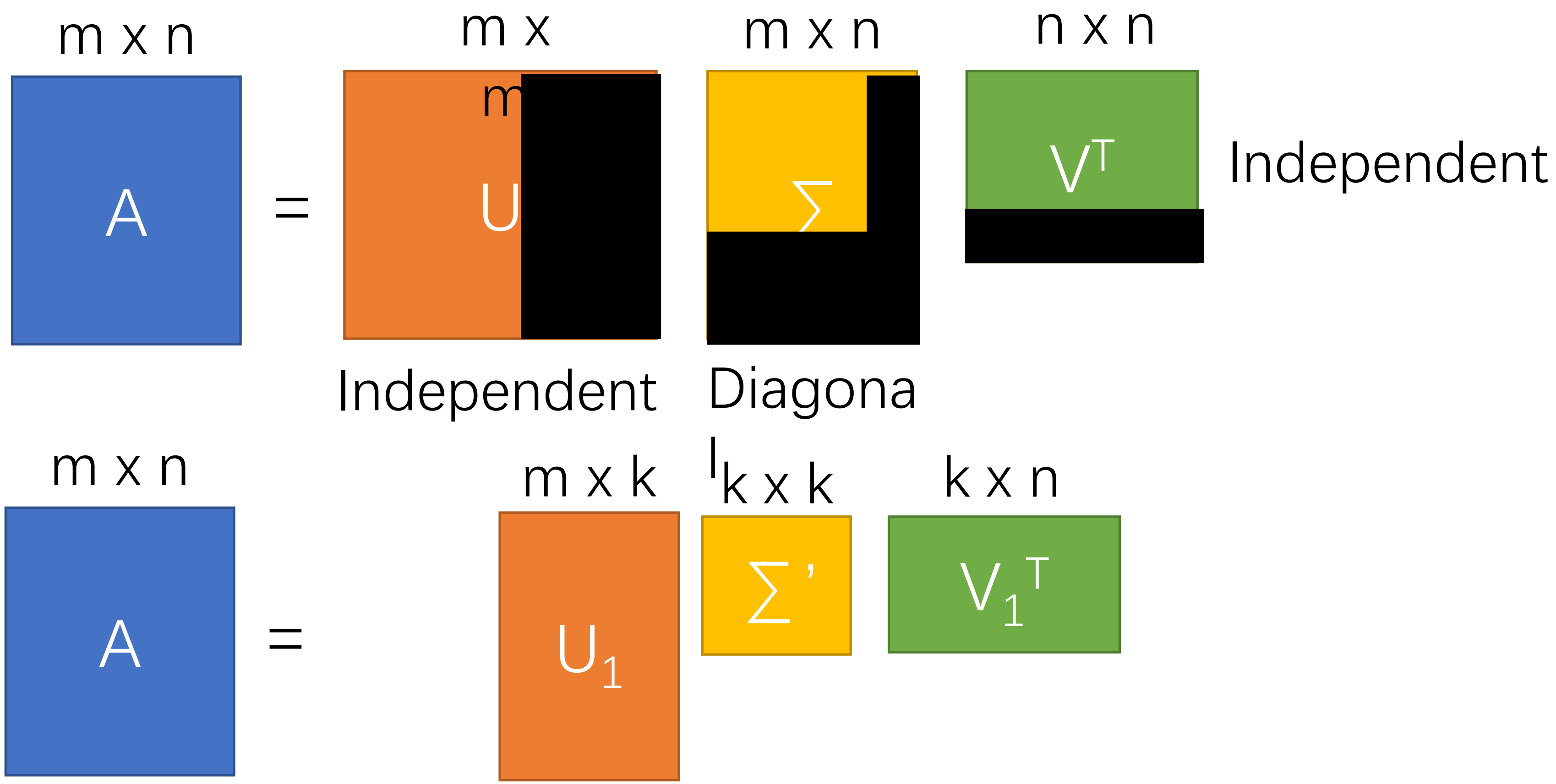
If B is a matrix of rank n, then $Rank(AB) = Rank(A)$

If A is a matrix of rank n, then $Rank(AB) = Rank(B)$

SVD

- Any m x n matrix A





SVD

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$$

σ_k is deleted

$$\begin{bmatrix} \sigma_1 & 0 & \cdots \\ 0 & \sigma_2 & \cdots \\ \vdots & \vdots & \sigma_{k-1} \end{bmatrix}$$ [redacted]

- Any m x n matrix A

m x n		m x m	m x n	n x n	
A	=	[redacted]	[redacted]	V^T [redacted]	Independent
		Independent	Diagonal		

What is the rank of A'?

m x n		m x (k-1)	(k-1) x (k-1)	(k-1) x n
A'	=	U_1	Σ'	V_1^T

k-1

A' is the rank k-1 matrix minimizing $||A - A'||$

$$AA^T = U\Sigma V^T V \Sigma^T U^T = U\Sigma\Sigma^T U^T$$

$$A^T A = V\Sigma^T U^T U \Sigma V^T = V\Sigma^T \Sigma V^T$$

$$A^T A v = \sigma^2 v \qquad AA^T u = \sigma^2 u$$

假设A是一个$m \times n$的矩阵，记A的转置为A^T。
首先证明$r(AA^T) = r(A^T A) = r(A) = r(A^T)$.

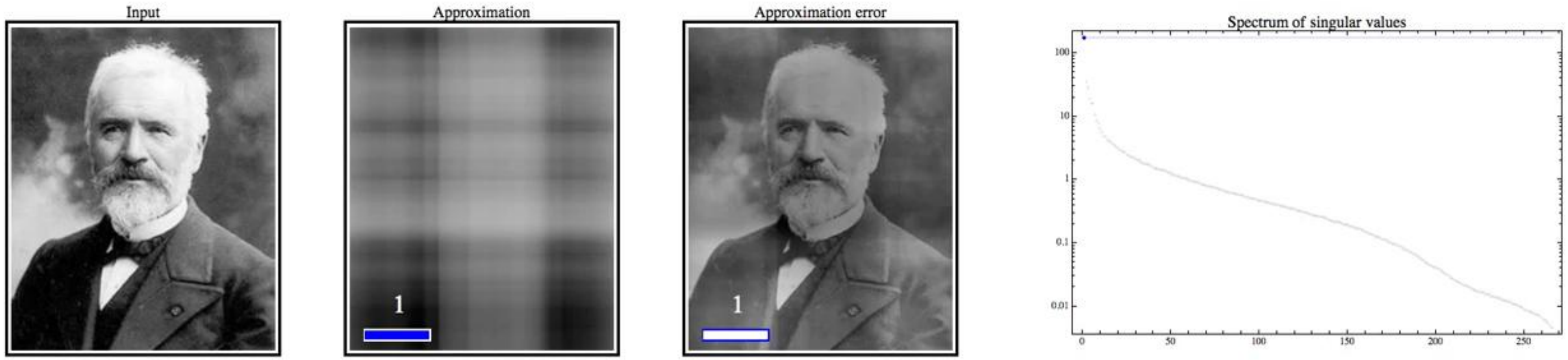
假设线程方程组为 $Ax = 0$ 和 $A^T Ax = 0$。
如果$Ax = 0$,则$A^T(Ax) = 0$,所以$Ax = 0$的解为$A^T Ax = 0$的解。

对于$A^T Ax = 0$,两边同时乘以 x^T,得到$x^T A^T Ax = x^T * 0 = 0$.
则有$(Ax)^T(Ax) = 0$,. 即，$||Ax|| = 0$。所以得到$Ax = 0$.所以，$A^T(Ax) = 0$的解都为$Ax = 0$的解。

所以$Ax = 0$和$A^T Ax = 0$有相同的解空间，所以$r(A) = r(A^T A)$。同理，$r(A^T) = r(AA^T)$。所以
$r(AA^T) = r(A^T A) = r(A) = r(A^T)$.

下面证明 特征值相同。
假设x是$A^T A$的输入特征值λ的特征向量。$A^T Ax = \lambda x$.
两边同乘以A,得到$AA^T Ax = \lambda Ax$, 则有$AA^T(Ax) = \lambda(Ax)$。
所以$A^T A$和AA^T有相同的非零特征值。
同理可得，AB 和BA有相同的非零特征值。

Low rank approximation using the singular value decomposition



https://www.youtube.com/watch?v=pAiVb7gWUrM

谢　谢