Homework 4

April 22, 2022

1. Suppose $\mathbf{v}_1, ..., \mathbf{v}_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbb{R}^m, V)$ by
$$T(\mathbf{x}) = x_1\mathbf{v}_1 + ... + x_m\mathbf{v}_m,$$
for $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m$. (Injective or Surjective.)
   (a) What property of T corresponds to $\mathbf{v}_1, ..., \mathbf{v}_m$ spanning V? Why?
   (b) What property of T corresponds to $\mathbf{v}_1, ..., \mathbf{v}_m$ being linearly independent? Why?

2. (a) Suppose $T \in \mathcal{L}(V, W)$ is injective and $\mathbf{v}_1, ..., \mathbf{v}_n$ is linearly independent in V. Prove that $T(\mathbf{v}_1), ...T(\mathbf{v}_n)$ is linearly independent in W.
   (b) Suppose $\mathbf{v}_1, ..., \mathbf{v}_n$ spans V and $T \in \mathcal{L}(V, W)$. Prove that the list $T(\mathbf{v}_1), ...T(\mathbf{v}_n)$ spans $T(V)(\operatorname{range} T)$.
   (c) Suppose V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \operatorname{null} T = \{0\}$ and $T(V) = T(U)$. Find a basis.

3. (a) Suppose V and W are both finite-dimensional. Prove that there exists an injective linear transfomation from V to W if and only if $\dim V \leq \dim W$.
   (b) Suppose V and W are both finite-dimensional. Prove that there exists an surjective linear transfomation from V onto W if and only if $\dim V \geq \dim W$.
   (c) Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

4. Find the standard matrices of the following linear transformations.
   (a) $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ with $T(x) = \begin{bmatrix} x_1 - x_2 \\ -2x_1 + x_2 \\ x_1 \end{bmatrix}$ for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(b) $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ is a horizontal shear transformatin that leaves $\mathbf{e}_1$ unchanged and maps $\mathbf{e}_2$ into $\mathbf{e}_2 + 3\mathbf{e}_1$.

(c) $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ first performs a vertical shear transformatin that leaves $\mathbf{e}_2$ unchanged and maps $\mathbf{e}_1$ into $\mathbf{e}_1 - 2\mathbf{e}_2$ and then reflects points through the line $x_2 = -x_1$.

5. Let

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and let L be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ define by

$$L(\mathbf{x}) = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + (x_1 + x_2)\mathbf{b}_3,$$

find the matrix A representing L with respect to the ordered bases $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

6. Let

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{y}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and let $\mathcal{I}$ be the identity operator on $\mathbb{R}^3$.

(a) Find the coordinates of $\mathcal{I}(\mathbf{e}_1), \mathcal{I}(\mathbf{e}_2)$, and $\mathcal{I}(\mathbf{e}_3)$ with respect to $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.

(b) Find a matrix A such that $A\mathbf{x}$ is the coordinate vector of $\mathbf{x}$ with respect to $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.