Homework 2

March 26, 2025

1. Suppose
$$U = \{(x, x, y) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$$
Find a subspace W of $\mathbb{R}^3$ such that $\mathbb{R}^3 = U \oplus W$.

2. For each subspace in (a)-(d), (1) find a basis, and (2) state the dimension.

   (a)
$$\left\{ \begin{bmatrix} x + 2y \\ 2x - 3y \\ -x \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\}$$

   (b)
$$\left\{ \begin{bmatrix} x + 3y - z \\ 4x + 5y + 3z \\ 3x + 6z \\ -x + 7y - 9z \end{bmatrix} : x, y, z \text{ in } \mathbb{R} \right\}$$

   (c)
$$\{(x, y, z, w) : x - 4y + 3w = 0\}$$

3. V is a nonzero finite-dimensional vector spaces, and the vectors listed belong to V. Mark each statement True or False.

   a. If there exists a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ that spans V, then dim $V \leq p$.
   b. If there exists a linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V, then then dim $V \geq p$.
   c. If dim $V = p$, then there exists a spanning set of $p + 1$ vectors in V.
   d. If there exists a linearly dependent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V, then dim $V \leq p$.
   e. If every set of p elements in V fails to span V, then dim $V > p$.
   f. If $p \geq 2$ and dim $V = p$, then every set of $p - 1$ nonzero vectors is linearly independent.

4. Prove that If V_1 and V_2 are subspaces of a finite-dimensional vector space, then $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$.

5. Consider two ordered bases $E = \{v_1, v_2, v_3\}$ and $F = \{w_1, w_2, w_3\}$ for a vector space V, and suppose

$$v_1 = -w_1 + w_2 + w_3, \quad v_2 = w_2 + 3w_3, \quad v_3 = 4w_1 - 2w_2$$

   (a) Find the transition matrix S from E to F.

   (b) Compute the coordinate vector $[v]_F$ for $v = 2v_1 + 1v_2 - v_3$.

6. Consider two ordered bases $E = \{v_1, v_2, v_3\}$ and $F = \{w_1, w_2, w_3\}$ for $\mathbb{R}^3$, where

$$v_1 = \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

   (a) Find the transition matrix S_1 from E to F.

   (b) Find the transition matrix S_2 from F to E.

   (c) Verify that $S_1S_2 = S_2S_1 = I_3$.

   (d) If $[v]_E = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$, compute $[v]_F$ and use S_1 or S_2 (decide by yourself) to verify your answer.

7. Let $A \in \mathbb{R}^{4\times 5}$ be

$$A = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 2 & 0 & -2 \\ 0 & 1 & 3 & 1 & 4 \\ 1 & 2 & 13 & 5 & 5 \end{bmatrix}.$$

   (a) Find the four subspaces of the matrix ($C(A)$, $C(A^T)$, $N(A)$, and $N(A^T)$, determine their bases and dimensions).

   (b) Write down the fundamental theorem of linear algebra: Part 1 and Part 2. And verify them by the answers in (a).

8. Let $A \in \mathbb{R}^{4\times 5}$ and let R be the reduced row echelon form of A. If the first and fourth columns of A are

$$a_1 = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \quad a_4 = \begin{bmatrix} -3 \\ -3 \\ -1 \\ -5 \end{bmatrix}$$

and

$$R = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

(a) find a basis for $N(A)$.

(b) given that x_0 is a solution to $Ax = b$, where

$$b = \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

determine the remaining column vectors of A.

9. If P is the plane of vectors in $\mathbb{R}^4$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$, write a basis for $P^\perp$. Construct a matrix that has P as its nullspace.

10. Let x and y be linearly independent vectors in $\mathbb{R}^n$ and let $S = \mathrm{Span}(x, y)$. We can use x and y to define a matrix A be setting

$$A = xy^T + yx^T$$

(a) Show that $N(A) = S^\perp$.

(b) Show that $\dim C(A) = 2$ (the rank of A must be 2).