Homework 3

April 17, 2025

1. Suppose $\mathbf{v}_1, ..., \mathbf{v}_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbb{R}^m, V)$ by

$$T(\mathbf{x}) = x_1\mathbf{v}_1 + ... + x_m\mathbf{v}_m,$$

for $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m$. (Injective or Surjective.)

   (a) What property of T corresponds to $\mathbf{v}_1, ..., \mathbf{v}_m$ spanning V? Why?

   (b) What property of T corresponds to $\mathbf{v}_1, ..., \mathbf{v}_m$ being linearly independent? Why?

2. (a) Suppose $T \in \mathcal{L}(V, W)$ is injective and $\mathbf{v}_1, ..., \mathbf{v}_n$ is linearly independent in V. Prove that $T(\mathbf{v}_1), ...T(\mathbf{v}_n)$ is linearly independent in W.

   (b) Suppose V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \operatorname{null} T = \{0\}$ and $T(V) = T(U)$. Find a basis.

3. (a) Suppose V and W are both finite-dimensional. Prove that there exists an surjective linear transfomation from V onto W if and only if $\dim V \geq \dim W$.

   (b) Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$

4. Find the standard matrices of the following linear transformations.

   (a) $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ with $T(x) = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + x_2 \\ x_2 \end{bmatrix}$ for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

   (b) $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ is a vertical shear transformatin that leaves $\mathbf{e}_2$ unchanged and maps $\mathbf{e}_1$ into $2\mathbf{e}_2 + \mathbf{e}_1$.

   (c) $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ first performs a horizontal shear transformatin that leaves $\mathbf{e}_1$ unchanged and maps $\mathbf{e}_2$ into $-2\mathbf{e}_1 + \mathbf{e}_2$ and then reflects points through the line $x_2 = -x_1$.

5. Let
$$\mathbf{b}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$$
and let L be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ define by
$$L(\mathbf{x}) = (x_1 - x_2)\mathbf{b}_1 + x_2\mathbf{b}_2 + (x_1 + x_2)\mathbf{b}_3,$$
find the matrix A representing L with respect to the ordered bases $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

6. Let D be the differentiation operator on $\mathbb{P}_2(\mathbb{R})$. Find the matrix B representing D with respect to $\left[1, 2x, x^2\right]$, the matrix A representing D with respect to $\left[2, 4x, 4x^2 - 4\right]$, and the nonsingular matrix S such that $B = S^{-1}AS$.

7. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of the matrix representation $\mathcal{M}(T)$ equal 1.

8. (a) Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove there exists an invertible operator $T \in \mathcal{L}(V)$ such that $T(u) = S(u)$ for every $u \in U$ if and only if S is injective.

   (b) Suppose V, W are finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} T_1 = \operatorname{null} T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$.

   (c) Suppose V, W are finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$ if and only if there exists an invertible operators $R \in \mathcal{L}(V)$ and $S \in \mathcal{L}(W)$ such that $T_1 = ST_2R$.

9. Suppose $V_1, \ldots, V_m$ and $W_1, \ldots, W_m$ are vector spaces.

   (a) Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

   (b) Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are isomorphic vector spaces.

10. Suppose that v, x are vectors in V and that U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.