Solutions & Notes of Homework 1

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If students struggle against an exercise for several hours (possibly working with other students), then they are likely to learn a lot.

—Sheldon Axler

Problem 1

Let V be a vector space and let $\mathbf{x}, \mathbf{y} \in V$. Show that

(a) $\beta\mathbf{0} = \mathbf{0}$ for each scalar β.

(b) $\mathbf{x} + \mathbf{y} = \mathbf{0}$ implies that $\mathbf{y} = -\mathbf{x}$, i.e., the additive inverse of $\mathbf{x}$ is unique.

✔ **Proof:** (a) Using the distributive property, we have

$$\beta\mathbf{0} = \beta(\mathbf{0} + \mathbf{0}) = \beta\mathbf{0} + \beta\mathbf{0}.$$

Add the additive inverse of $\beta\mathbf{0}$ to both sides of the equation above, and we have $\mathbf{0} = \beta\mathbf{0}$, as desired.

(b) **Note:** *We need only to prove* $\mathbf{y} = -\mathbf{x}$*, and it indicates the uniqueness.*

$$\begin{aligned}
-\mathbf{x} &= -\mathbf{x} + \mathbf{0} && \text{(additive identity)}\\
&= -\mathbf{x} + (\mathbf{x} + \mathbf{y})\\
&= (-\mathbf{x} + \mathbf{x}) + \mathbf{y} && \text{(associativity)}\\
&= \mathbf{0} + \mathbf{y} && \text{(additive identity)}\\
&= \mathbf{y} + \mathbf{0} && \text{(commutativity)}\\
&= \mathbf{y}. && \text{(additive identity)}
\end{aligned}$$

Therefore, $\mathbf{x} + \mathbf{y} = \mathbf{0}$ implies that $\mathbf{y} = -\mathbf{x}$, as desired.

An alternative proof: add $-\mathbf{x}$ to both sides of $\mathbf{x} + \mathbf{y} = \mathbf{0}$ and we have

$$\begin{aligned}
\mathbf{0} + (-\mathbf{x}) &= \mathbf{x} + \mathbf{y} + (-\mathbf{x})\\
-\mathbf{x} + \mathbf{0} &= \mathbf{y} + \mathbf{x} + (-\mathbf{x}) && \text{(commutativity)}\\
-\mathbf{x} + \mathbf{0} &= \mathbf{y} + (\mathbf{x} + (-\mathbf{x})) && \text{(associativity)}\\
-\mathbf{x} &= \mathbf{y} + \mathbf{0} && \text{(additive identity)}\\
-\mathbf{x} &= \mathbf{y}. && \text{(additive identity)}
\end{aligned}$$

Therefore, $\mathbf{x} + \mathbf{y} = \mathbf{0}$ implies that $\mathbf{y} = -\mathbf{x}$, as desired. ■

Note for typesetting: *Use displayed equations (equations in a single line) to enhance readability. Consider the displayed equation as a part of the sentence and add an appropriate punctuation to it.*

Problem 2

Let V be the set of all ordered pairs of real numbers with addition defined by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)\,,$$

and scalar multiplication defined by

$$\alpha \circ (x_1, x_2) = (\alpha x_1, x_2)\,,$$

Scalar multiplication for this system is defined in an unusual way, and consequently we use the symbol $\circ$ to avoid confusion with the ordinary scalar multiplication of row vectors. Is V a vector space with these operations? Justify your answer.

✔ **Solution (1):** No. We have

$$(a + b) \circ (x_1, y_1) = ((a + b) \times x_1, y_1) = (ax_1 + bx_1, y_1),$$

while

$$a \circ (x_1, y_1) + b \circ (x_1, y_1) = (ax_1, y_1) + (bx_1, y_1) = (ax_1 + bx_1, 2y_1).$$

Therefore, $(a + b) \circ (x_1, y_1) \neq a \circ (x_1, y_1) + b \circ (x_1, y_1)$, which implies distributive property does not hold, and V is not a vector space.

✔ **Solution (2):** No. Suppose V is a vector space. According to the distributive property, we have

$$(x_1, y_1) + (x_1, y_1) = 1 \circ (x_1, y_1) + 1 \circ (x_1, y_1) = (1 + 1) \circ (x_1, y_1) = (2x_1, y_1). \tag{1}$$

However, according to the definition of addition, we have

$$(x_1, y_1) + (x_1, y_1) = (2x_1, 2y_1),$$

which is contradictory to (1). Therefore, V is not a vector space.

Note: *If the scalar multiplication in the problem above is redefined as* $\alpha \circ (x_1, x_2) = (\alpha x_1, 0)$, *we will find that the sets given in both this example and the problem above include* 0 *and are closed under addition and scalar multiplication. However, they are not vector spaces. What makes this happen?*

Hint: *The conditions for* **subspaces** *and those for* **vector spaces** *are different!*

Problem 3

Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbb{C}}$, equals $V \times V$. An element of $V_{\mathbb{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbb{C}}$ is defined by

  $$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2),$$

  for all $u_1, v_1, u_2, v_2 \in V$.

- Complex scalar multiplication on $V_{\mathbb{C}}$ is defined by

$$(a+bi)(u+iv) = (au-bv)+i(av+bu),$$

for all $a,b \in \mathbb{R}$ and all $u,v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbb{C}}$ is a complex vector space.

Proof: We need to verify each condition in the definition of vector space.

(1) **commutativity:** for all $u_1, v_1, u_2, v_2 \in V$, we have

$$\begin{aligned}(u_1+iv_1)+(u_2+iv_2) &= (u_1+u_2)+i(v_1+v_2)\\ &= (u_2+u_1)+i(v_2+v_1)\\ &= (u_2+iv_2)+(u_1+iv_1),\end{aligned}$$

Thus commutativity holds.

(2) **associativity:** for all $u_1, v_1, u_2, v_2, u_3, v_3 \in V$, we have

$$\begin{aligned}\big((u_1+iv_1)+(u_2+iv_2)\big)+(u_3+iv_3) &= (u_1+u_2)+i(v_1+v_2)+(u_3+iv_3)\\ &= (u_1+u_2+u_3)+i(v_1+v_2+v_3)\\ &= (u_1+iv_1)+(u_2+u_3)+i(v_2+v_3)\\ &= (u_1+iv_1)+\big((u_2+iv_2)+(u_3+iv_3)\big),\end{aligned}$$

and for all $a,b,c,d \in \mathbb{R}$ and all $u,v \in V$, we have

$$\begin{aligned}\big((a+bi)(c+di)\big)(u+iv) &= \big((ac-bd)+i(ad+bc)\big)(u+iv)\\ &= \big((ac-bd)u-(ad+bc)v\big)+i\big((ac-bd)v+(ad+bc)u\big)\\ &= \big((cu-dv)a-(cv+du)b\big)+i\big((cv+du)a+(cu-dv)b\big)\\ &= (a+bi)\big((cu-dv)+i(cv+du)\big)\\ &= (a+bi)\big((c+di)(u+iv)\big).\end{aligned}$$

Thus associativity holds.

(3) **additive identity:** for all $u,v \in V$, we have

$$(u+iv)+(0+i0) = (u+0)+i(v+0) = (u+iv),$$

thus $0+i0$ is an (in fact, the unique) additive identity of $V_{\mathbb{C}}$.

(4) **additive inverse:** for all $u,v \in V$, we have

$$(u+iv)+(-u+i(-v)) = (u+(-u))+i(v+(-v)) = 0+i0,$$

thus for every $(u,v) \in V_{\mathbb{C}}$, there exists an additive inverse $(-u,-v)$.

(5) **multiplicative identity:** for all $u,v \in V$, we have

$$(1+i0)(u+iv) = (1u-0v)+i(1v+0u) = u+iv,$$

thus $1 + i0$ is a (in fact, the unique) multiplicative identity of $V_{\mathbb{C}}$.

(6) **distributive properties:** for all $a, b \in \mathbb{R}$ and all $u_1, v_1, u_2, v_2 \in V$, we have

$$\begin{aligned}(a+bi)\big((u_1+iv_1)+(u_2+iv_2)\big) &= (a+bi)\big((u_1+u_2)+i(v_1+v_2)\big)\\ &= \big((u_1+u_2)a-(v_1+v_2)b\big)+i\big((v_1+v_2)a+(u_1+u_2)b\big)\\ &= au_1-bv_1+i(av_1+bv_1)+au_2-bv_2+i(av_2+bv_2)\\ &= (a+bi)(u_1+iv_1)+(a+bi)(u_2+iv_2),\end{aligned}$$

and for all $a, b, c, d \in \mathbb{R}$ and all $u, v \in V$, we have

$$\begin{aligned}\big((a+bi)+(c+di)\big)(u+iv) &= \big((a+c)+i(b+d)\big)(u+iv)\\ &= \big((a+c)u-(b+d)v\big)+i\big((a+c)v+(b+d)u\big)\\ &= au-bv+i(av+bu)+cu-dv+i(cv+du)\\ &= (a+bi)(u+iv)+(c+di)(u+iv).\end{aligned}$$

Thus distributive properties hold.

Therefore, $V_{\mathbb{C}}$ is a complex vector space. ■

Note (1): *This problem illustrates the important role that* **definitions** *play in proof. All of the equalities above come from the definition of addition and scalar multiplication on $V_{\mathbb{C}}$ or V.*

Note (2): *The closure of addition and scalar multiplication is obvious and it's all right to omit the verification here. The following is listed for your reference.*

Closure under addition: for all $u_1, v_1, u_2, v_2 \in V$, we have $(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$. Since $u_1 + u_2 \in V$ and $v_1 + v_2 \in V$, it follows that $(u_1 + iv_1) + (u_2 + iv_2) \in V_{\mathbb{C}}$.

Closure under scalar multiplication: for all $u, v \in V$ and all $a, b \in \mathbb{R}$, we have $(a + bi)(u + iv) = (au - bv) + i(av + bu)$. Since $au - bv \in V$ and $av + bu \in V$, we have $(a + bi)(u + iv) \in V_{\mathbb{C}}$.

Problem 4

Let A be a fixed matrix in $\mathbb{R}^{n\times n}$ and let S be the set of all matrices that commute with A, that is,

$$S = \{B \mid AB = BA\}$$

Show that S is a subspace of $\mathbb{R}^{n\times n}$.

✔ **Proof:** Clearly $0_{n\times n} \in S$ because $A0 = 0A = 0$.

Also, S is closed under addition, because for $B_1, B_2 \in S$ we have

$$A(B_1 + B_2) = AB_1 + AB_2 = B_1A + B_2A = (B_1 + B_2)A.$$

Furthermore, S is closed under scalar multiplication, because for $B \in S, k \in \mathbb{R}$ we have

$$A(kB) = k(AB) = k(BA) = (kB)A.$$

Hence S is a subspace of $\mathbb{R}^{n\times n}$. ■

Problem 5

Verify the following statements.

(a) Is $\mathbb{R}^3$ a subspace of the complex vector space $\mathbb{C}^3$?

(b) Is $\{(x,y,z) \in \mathbb{R}^3 : x^3 = y^3\}$ a subspace of $\mathbb{R}^3$?

(c) Is $\{(x,y,z) \in \mathbb{C}^3 : x^3 = y^3\}$ a subspace of $\mathbb{C}^3$?

Solution:

(a) **No**. By mentioning $\mathbb{C}^3$ as a complex vector space, we are taking scalars from $\mathbb{C}$, i.e., $\mathbb{F} = \mathbb{C}$. A real number multiplied by a complex number does not always give a real number, thus the result of scalar multiplication may not be in $\mathbb{R}^3$, so $\mathbb{R}^3$ is not a subspace of $\mathbb{C}^3$.

(b) **Yes**. For convenience, we denote $S = \{(x,y,z) \in \mathbb{R}^3 : x^3 = y^3\}$. It is easy to see that $(0,0,0) \in S$.

For $v_1 = (x_1, y_1, z_1), v_2 = (x_2, y_2, z_2) \in S$, we have $v_1 + v_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$. We can derive from $x^3 = y^3$ that $x = y$ (*Hint:* $(x-y)(x^2+xy+y^2) = 0$), hence $x_1 = y_1, x_2 = y_2$, and thus $(x_1 + x_2)^3 = (y_1 + y_2)^3$, which implies that S is closed under addition.

For $v_3 = (x_3, y_3, z_3) \in S, k \in \mathbb{R}$, we have $kv_3 = (kx_3, ky_3, kz_3)$. Because $(kx_3)^3 = k^3x_3^3 = k^3y_3^3 = (ky_3)^3$, we have S is closed under scalar multiplication.

Therefore, $S = \{(x,y,z) \in \mathbb{R}^3 : x^3 = y^3\}$ is a subspace of $\mathbb{R}^3$.

(c) **No**. Consider two vectors in $\{(x,y,z) \in \mathbb{C}^3 : x^3 = y^3\}$:

$$u_1 = (1,1,1), u_2 = (-1+\sqrt{3}i, -1-\sqrt{3}i, 1),$$

Both of them satisfy the condition $x^3 = y^3$. (In fact, $1^3 = 1^3 = 1$; $(-1+\sqrt{3}i)^3 = (-1-\sqrt{3}i)^3 = 8$) However, their sum is

$$u_3 = (\sqrt{3}i, -\sqrt{3}i, 2),$$

where the cube of the first coordinate is $(\sqrt{3}i)^3 = -3\sqrt{3}i$, while that of the second coordinate is $(-\sqrt{3}i)^3 = 3\sqrt{3}i$. Hence $u_3 \notin \{(x,y,z) \in \mathbb{C}^3 : x^3 = y^3\}$. Therefore, $\{(x,y,z) \in \mathbb{C}^3 : x^3 = y^3\}$ is not closed under addition, which implies it is not a subspace of $\mathbb{C}^3$.

Note: *We can also take* $u_1 = \left(\frac{-1+\sqrt{3}i}{2}, 1, 0\right)$ *and* $u_2 = \left(\frac{-1-\sqrt{3}i}{2}, 1, 0\right)$.

Problem 6

Suppose U_1 and U_2 are subspaces of V.

(a) Is the intersection $U_1 \bigcap U_2$ a subspace of V? Prove or give a counterexample.

(b) Is the union $U_1 \bigcup U_2$ a subspace of V? Prove or give a counterexample.

Solution:

(a) **Yes**. Clearly $\mathbf{0} \in U_1 \bigcap U_2$ because $\mathbf{0} \in U_1$ and $\mathbf{0} \in U_2$.

Consider two vectors $w_1, w_2 \in U_1 \bigcap U_2$. Since $w_1, w_2 \in U_1$ and U_1 is closed under addition (because it is a subspace of V), we have $w_1 + w_2 \in U_1$. Similarly, $w_1 + w_2 \in U_2$. Hence $w_1 + w_2 \in U_1 \bigcap U_2$. Therefore, $U_1 \bigcap U_2$ is closed under addition.

Consider a vector $w_3 \in U_1 \bigcap U_2$, and a scalar $\alpha \in \mathbb{F}$. Since $w_3 \in U_1$ and U_1 is closed under scalar multiplication (because it is a subspace of V), we have $\alpha w_3 \in U_1$. Similarly, $\alpha w_3 \in U_2$. Hence $\alpha w_3 \in U_1 \bigcap U_2$. Therefore, $U_1 \bigcap U_2$ is closed under scalar multiplication.

Now we conclude that the intersection $U_1 \bigcap U_2$ is a subspace of V.

(b) **No**. Consider two subspaces of $\mathbb{R}^3$:

$$U_1 = \{(x, 0, 0), x \in \mathbb{R}\}, U_2 = \{(0, y, 0), y \in \mathbb{R}\}.$$

U_1 includes all vectors parallel to x-axis while U_2 includes all vectors parallel to y-axis. Their union includes all vectors parallel to either x-axis or y-axis. However, consider the following two vectors in $U_1 \bigcup U_2$:

$$u_1 = (1, 0, 0), \quad u_2 = (0, 1, 0),$$

add them together and it turns out $u_1 + u_2 = (1, 1, 0)$, which is parallel to neither the x-axis nor the y-axis, and thus is not in $U_1 \bigcup U_2$.

Therefore, $U_1 \bigcup U_2$ is not closed under addition, which implies it is not a subspace of V.

Note: *The union of subspaces is usually not a subspace, which is the reason for us to consider the sum of subspaces.*

Problem 7

Suppose $v_1, \ldots, v_m$ is a list of vectors in V. For $k \in \{1, \ldots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that $\text{span}(v_1, \ldots, v_m) = \text{span}(w_1, \ldots, w_m)$.

✔ **Proof:** First suppose $v = a_1 v_1 + \cdots + a_m v_m \in \text{span}(v_1, \ldots, v_m)$, where $a_1, \ldots, a_m$ are scalars. Note that for every $k = 2, \ldots, m$, we have $v_k = w_k - w_{k-1}$, and that $v_1 = w_1$. Then we have

$$\begin{aligned} v &= a_1 v_1 + \cdots + a_m v_m \\ &= a_1 w_1 + a_2(w_2 - w_1) + \cdots + a_m(w_m - w_{m-1}) \\ &= (a_1 - a_2) w_1 + (a_2 - a_3) w_2 + \cdots + (a_{m-1} - a_m) w_{m-1} + a_m w_m, \end{aligned}$$

thus $v \in \text{span}(w_1, \ldots, w_m)$, which implies that $\text{span}(v_1, \ldots, v_m) \subset \text{span}(w_1, \ldots, w_m)$.

Conversely, suppose $w = a_1 w_1 + \cdots + a_m w_m \in \text{span}(w_1, \ldots, w_m)$, where $a_1, \ldots, a_m$ are scalars. Then we have

$$\begin{aligned} w &= a_1 w_1 + \cdots + a_m w_m \\ &= a_1 v_1 + a_2(v_1 + v_2) + \cdots + a_m(v_1 + \cdots + v_m) \\ &= (a_1 + \cdots + a_m) v_1 + (a_2 + \cdots + a_m) v_2 + \cdots + a_m v_m, \end{aligned}$$

thus $w \in \text{span}(v_1, \ldots, v_m)$, which implies that $\text{span}(w_1, \ldots, w_m) \subset \text{span}(v_1, \ldots, v_m)$.

Therefore, $\text{span}(v_1, \ldots, v_m) = \text{span}(w_1, \ldots, w_m)$. ■

Problem 8

Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_m)$.

Proof (1): Because $v_1 + w, \ldots, v_m + w$ is linearly dependent, we can conclude from linear dependence lemma (2.19 of the textbook, or Theorem 2.5.2 of the lecture notes) that there is a $j \in \{1, \ldots, m\}$ such that $v_j + w \in \operatorname{span}(v_1 + w, \ldots, v_{j-1} + w)$.

If $j = 1$, then $v_1 + w \in \operatorname{span}(\,)$, which implies that $v_1 + w = 0$, i.e., $w = -v_1$. Thus $w \in \operatorname{span}(v_1, \ldots, v_m)$.

If $j \geq 2$, then there are scalars $a_1, \ldots, a_{j-1}$ such that

$$v_j + w = a_1(v_1 + w) + \cdots + a_{j-1}(v_{j-1} + w) \iff v_j + \lambda w = a_1 v_1 + \cdots + a_{j-1} v_{j-1},$$

where $\lambda = 1 - (a_1 + \cdots + a_{j-1})$. Note that λ must be nonzero, otherwise v_j would lie in the span of $v_1, \ldots, v_{j-1}$, which cannot happen since the list $v_1, \ldots, v_j$ is linearly independent. It follows that

$$w = \frac{1}{\lambda}(a_1 v_1 + \cdots + a_{j-1} v_{j-1} - v_j),$$

so that $w \in \operatorname{span}(v_1, \ldots, v_m)$. ■

Proof (2): *We can also construct the proof with the definition of linear independence and dependence.*

Since the vectors $v_1 + w, \ldots, v_m + w$ are linearly dependent, there exist scalars $a_1, a_2, \ldots, a_m$, not all zero, such that

$$a_1(v_1 + w) + a_2(v_2 + w) + \cdots + a_m(v_m + w) = 0.$$

Expand the left side of the equation and we have

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m + (a_1 + a_2 + \cdots + a_m) w = 0. \tag{2}$$

Let $c = a_1 + a_2 + \cdots + a_m$, and we can rewrite the equation as

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m + c w = 0.$$

If $c = 0$, then the equation becomes $a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0$. Since $v_1, \ldots, v_m$ are linearly independent, we must have $a_1 = a_2 = \cdots = a_m = 0$. But this contradicts with the fact that $a_1, a_2, \ldots, a_m$ are not all zero. Therefore, $c \neq 0$. We can then solve the equation (2) for w:

$$w = -\frac{1}{c}(a_1 v_1 + a_2 v_2 + \cdots + a_m v_m),$$

which shows that w can be written as a linear combination of $v_1, \ldots, v_m$. This completes the proof. ■