# Solutions & Notes of Homework 4

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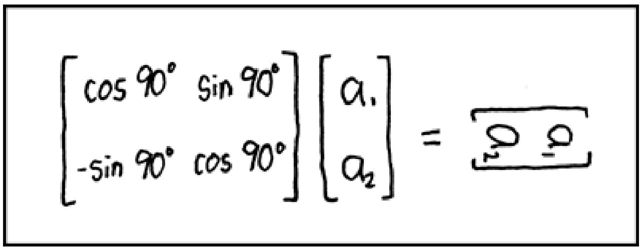



Figure 1: Picture from https://matthbeck.github.io/325.html, originally from https://xkcd.com/.

**Problem 1**

Show that the function that takes $((x_1, x_2), (y_1, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$ to $|x_1y_1| + |x_2y_2|$ is not an inner product on $\mathbb{R}^2$.

**Proof:** Denote the function as $(x_1, x_2) \circ (y_1, y_2)$. Let $v_1 = (1, 2), v_2 = (-1, -2), v_3 = (1, 1)$. Then

$$\left.\begin{aligned} v_1 \circ v_3 + v_2 \circ v_3 = 3 + 3 = 6, \\ (v_1 + v_2) \circ v_3 = (0, 0) \circ (1, 1) = 0, \end{aligned}\right\} \implies (v_1 + v_2) \circ v_3 \neq v_1 \circ v_3 + v_2 \circ v_3.$$

Therefore, the additivity in the first slot doesn't hold for the function, and this function is not an inner product on $\mathbb{R}^2$. ■

**Problem 2**

Suppose $V$ is a real inner product space, show that:

a) the inner product $\langle u + v, u - v \rangle = \|u\|^2 - \|v\|^2$ for every $u, v \in V$.

b) if $u, v \in V$ have the same norm, then $u + v$ is orthogonal to $u - v$.

c) use part(b) to show that the diagonals of a rhombus are perpendicular to each other.

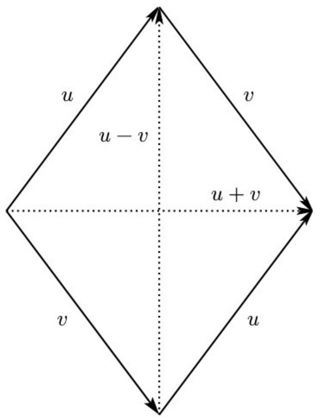
**Note 2.1:** *The inner product on a* **real** *inner product space satisfies additivity in both slots (note that the conjugate of a real number remains unchanged):*

$$\langle u, v + w \rangle = \overline{\langle v + w, u \rangle} = \langle v + w, u \rangle = \langle v, u \rangle + \langle w, u \rangle = \overline{\langle u, v \rangle} + \overline{\langle u, w \rangle} = \langle u, v \rangle + \langle u, w \rangle.$$

**Proof:** a) $\langle u+v, u-v\rangle = \langle u,u\rangle - \langle u,v\rangle + \langle v,u\rangle - \langle v,v\rangle = \langle u,u\rangle - \langle u,v\rangle + \langle u,v\rangle - \langle v,v\rangle = \|u\|^2 - \|v\|^2$.

b) This is immediate from part (a).

c) Suppose two adjacent sides of a rhombus are represented by two vectors $u, v \in \mathbb{R}^2$. Then $u$ and $v$ have the same norm. The two diagonals of the rhombus can be written as $u+v$ and $u-v$ respectively. From part (b), we know that $u+v$ is orthogonal to $u-v$, so the diagonals of a rhombus are perpendicular to each other. ■




**Problem 3**

Suppose $u, v \in V$, prove that the inner product $\langle u, v\rangle = 0$ if and only if $\|u\| \leqslant \|u + av\|$ for all $a \in \mathbb{F}$.

**Proof:** If $v$ or $u$ equals $0$, then it's easy to draw the conclusion. The proof below assume that $v \neq 0, u \neq 0$. First we assume that $\langle u, v\rangle = 0$. Then $\langle u, av\rangle = \overline{a}\langle u, v\rangle = 0$, hence $u$ and $av$ are orthogonal. Then from Pythagorean theorem, we have

$$\|u+av\|^2 = \|u\|^2 + \|av\|^2 = \|u\|^2 + |a|^2\|v\|^2 \geq \|u\|^2 .$$

Conversely, assume that $\|u\| \leqslant \|u + av\|$. We write the orthogonal decomposition of $u$ from 6.13:

$$u = cv + w, \langle w, v\rangle = 0, c \in \mathbb{F}.$$

Then we have

$$\|cv + w\| \leqslant \|cv + w + av\| .$$

From the arbitrariness of $a$, take $a = -c$ and we have

$$\|cv + w\| \leqslant \|w\| . \tag{1}$$

Since $\langle cv, w\rangle = c\langle v, w\rangle = 0$, $cv$ and $w$ are orthogonal. Then from Pythagorean theorem, we have

$$\|cv + w\| = \|cv\| + \|w\| . \tag{2}$$

It follows from (1) and (2) that

$$\|cv\| \leqslant 0 \implies \|cv\| = 0 \implies c = 0 \implies u = w.$$

Thus $\langle u, v\rangle = 0$, completing the proof. ■

! **Note 3.1:** *Draw a picture in $\mathbb{R}^2$ may help. If $u$ has nonzero "component" along $v$, then an appropriate selection of $a$ can "cancel" this component and leads to a smaller norm.*

**Problem 4**

Suppose $u, v \in V$, prove that $\|au + bv\| = \|bu + av\|$ for all $a, b \in \mathbb{R}$ if and only if $\|u\| = \|v\|$.

**Proof:** Note that $\|au+bv\| = \|bu+av\| \iff \langle au+bv, au+bv\rangle = \langle bu+av, bu+av\rangle$. We have

$$\begin{aligned}\langle au+bv, au+bv\rangle &= a\langle u, au+bv\rangle + b\langle v, au+bv\rangle \\ &= a\overline{\langle au+bv, u\rangle} + b\overline{\langle au+bv, v\rangle} \\ &= a\overline{a\langle u,u\rangle + b\langle v,u\rangle} + b\overline{a\langle u,v\rangle + b\langle v,v\rangle} \\ &= a^2\langle u,u\rangle + ab\langle u,v\rangle + b^2\langle v,v\rangle + ab\langle v,u\rangle. \end{aligned} \tag{3}$$

Similarly we have

$$\langle bu+av, bu+av\rangle = b^2\langle u,u\rangle + ab\langle u,v\rangle + a^2\langle v,v\rangle + ab\langle v,u\rangle. \tag{4}$$

It follows from (3) and (4) that

$$\langle au+bv, au+bv\rangle = \langle bu+av, bu+av\rangle \iff (a^2-b^2)(\langle u,u\rangle - \langle v,v\rangle) = 0.$$

Given this, it will suffice to show that $(a^2-b^2)(\langle u,u\rangle - \langle v,v\rangle) = 0$ for all $a, b \in \mathbb{R}$ if and only if $\|u\| = \|v\|$. The reverse implication is clear; for the forward implication, simply assign $a = 1, b = 0$. ■

**Problem 5**

Suppose $u, v \in V$, $\|u\| = \|v\| = 1$ and $\langle u, v\rangle = 1$, prove that $u = v$.

**Proof (1):** $|\langle u,v\rangle| = 1 = \|u\|\,\|v\|$, then from Cauchy-Schwarz inequality, $u$ must be a scalar multiple of $v$, i.e., $u = av, a \in \mathbb{F}$. Then $\langle u,v\rangle = \langle av, v\rangle = a\langle v,v\rangle = a = 1$. So $u = v$, as desired. ■

**Proof (2):** Observe that $\langle u-v, u-v\rangle = \langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle = \|u\|^2 - 2\operatorname{Re}\langle u,v\rangle + \|v\|^2 = 0$. It follows from definiteness that $u - v = 0$. ■

**Problem 6**

Find vectors $u, v \in \mathbb{R}^2$ such that $u$ is a scalar multiple of $(1,3)$, $v$ is orthogonal to $(1,3)$, and $(1,2) = u + v$.

**Solution:** Let $u = k(1,3) = (k, 3k) (k \in \mathbb{R})$, then $v = (1-k, 2-3k)$. Because $v$ is orthogonal to $(1,3)$, then $\langle v, (1,3)\rangle = 0$, so $1 - k + 6 - 9k = 0$, $k = \frac{7}{10}$. So $u = (\frac{7}{10}, \frac{21}{10})$, $v = (\frac{3}{10}, -\frac{1}{10})$.

**Note 6.1:** *Please refer to 6.13 of Linear Algebra Done Right (fourth edition).*

**Problem 7**

Prove that $(x_1 + \cdots + x_n)^2 \leqslant n\left(x_1^2 + \cdots + x_n^2\right)$ for all positive integers $n$ and all real numbers $x_1, \ldots, x_n$.

**Proof:** Let $u = (x_1, x_2, \ldots, x_n)$, $v = (1, 1, \ldots, 1)$. Then from Cauchy-Schwarz inequality, we find

$$|\langle u,v\rangle| = |x_1 + x_2 + \cdots + x_n| \leqslant \|u\|\,\|v\| = \sqrt{n}\sqrt{x_1^2 + \cdots + x_n^2}.$$

Take the square of both sides of the equation and we complete the proof. ■

**Problem 8**

Suppose $V$ is a real inner product space, prove that

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}$$

for all $u, v \in V$.

**Proof:**

$$\begin{aligned}
\|u+v\|^2 - \|u-v\|^2 &= \langle u+v, u+v\rangle - \langle u-v, u-v\rangle \\
&= (\langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle) - (\langle u, u\rangle - \langle u, v\rangle - \langle v, u\rangle + \langle v, v\rangle) \\
&= (\langle u, u\rangle + 2\langle u, v\rangle + \langle v, v\rangle) - (\langle u, u\rangle - 2\langle u, v\rangle + \langle v, v\rangle) \\
&= 4\langle u, v\rangle .
\end{aligned}$$

■

**Note 8.1:** *Suppose that $V$ is a complex inner product space. Then for all $u, v \in V$,*

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+\mathrm{i}v\|^2\mathrm{i} - \|u-\mathrm{i}v\|^2\mathrm{i}}{4}.$$

*Try to prove it by yourself.*