

## Homework 5

May 4, 2022

1. Let  $E = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  and  $F = \{\mathbf{b}_1, \mathbf{b}_2\}$ , where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

and

$$\mathbf{b}_1 = (1, -1)^T, \quad \mathbf{b}_2 = (2, -1)^T$$

For each of the following linear transformations  $L$  from  $\mathbb{R}^3$  into  $\mathbb{R}^2$ , find the matrix representing  $L$  with respect to the ordered bases  $E$  and  $F$ :

(i)  $L(\mathbf{x}) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_3 \end{pmatrix}$

(ii)  $L(\mathbf{x}) = \begin{pmatrix} 2x_2 \\ -x_1 \end{pmatrix}$

2. Let  $D$  be the differentiation operator on  $\mathbb{P}_2(\mathbb{R})$ . Find the matrix  $B$  representing  $D$  with respect to  $[1, x, x^2]$ , the matrix  $A$  representing  $D$  with respect to  $[1, 2x, 4x^2 - 2]$ , and the nonsingular matrix  $S$  such that  $B = S^{-1}AS$ .
3. Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Prove that  $\dim \text{range } T = 1$  if and only if there exist a basis of  $V$  and a basis of  $W$  such that with respect to these bases, all entries of the matrix representation  $\mathcal{M}(T)$  equal 1.
4. For each of the following systems  $Ax = b$ , find all the least squares solutions:

(i)  $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix};$

(ii)  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$

5. Suppose  $T \in \mathcal{L}(U, V)$  and  $S \in \mathcal{L}(V, W)$  are both invertible linear transformations. Prove that  $ST \in \mathcal{L}(U, W)$  is invertible and that  $(ST)^{-1} = T^{-1}S^{-1}$ .
6. (a) Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ . Prove that the following statements are equivalent:
  - (i)  $T$  is invertible;
  - (ii)  $T$  is injective;
  - (iii)  $T$  is surjective.
- (b) Suppose  $V$  is finite-dimensional,  $U$  is a subspace of  $V$ , and  $S \in \mathcal{L}(U, V)$ . Prove there exists an invertible operator  $T \in \mathcal{L}(V)$  such that  $T(u) = S(u)$  for every  $u \in U$  if and only if  $S$  is injective.
- (c) Suppose  $V, W$  are finite-dimensional and  $T_1, T_2 \in \mathcal{L}(V, W)$ . Prove that  $\text{null } T_1 = \text{null } T_2$  if and only if there exists an invertible operator  $S \in \mathcal{L}(W)$  such that  $T_1 = ST_2$ .
- (d) Suppose  $V, W$  are finite-dimensional and  $T_1, T_2 \in \mathcal{L}(V, W)$ . Prove that  $\text{range } T_1 = \text{range } T_2$  if and only if there exists an invertible operator  $S \in \mathcal{L}(V)$  such that  $T_1 = T_2S$ .