# Homework 3

April 30, 2021

1. Suppose $\mathbf{v}_1, ..., \mathbf{v}_m$ is a list of vectors in $V$. Define $T \in \mathcal{L}(\mathbb{R}^m, V)$ by

$$T(\mathbf{x}) = x_1\mathbf{v}_1 + ... + x_m\mathbf{v}_m,$$

$T(\mathbb{R}^m) = \text{span}(v_1, \cdots v_m) = V.$

for $\mathbf{x} = \left\{ \begin{matrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{matrix} \right\} \in \mathbb{R}^m$.

Surjective. $T(\mathbb{R}^m) = V$.

   (a) What property of $T$ corresponds to $\mathbf{v}_1, ..., \mathbf{v}_m$ spanning $V$? Why?

   (b) What property of $T$ corresponds to $\mathbf{v}_1, ..., \mathbf{v}_m$ being linearly independent? Why? Injective. If $T(x)=0$, $x_1 v_1 + \cdots + x_m v_m = 0$. Since $v_1, \cdots, v_m$ are linearly independent, $x_1 = \cdots = x_m = 0$. $\Rightarrow$ $x = \vec{0}$ $\Rightarrow \ker(T) = \{0\}$. $\Rightarrow$ $T$ is injective.

2. (a) Suppose $T \in \mathcal{L}(V, W)$ is injective and $\mathbf{v}_1, ..., \mathbf{v}_n$ is linearly independent in $V$. Prove that $T(\mathbf{v}_1), ...T(\mathbf{v}_n)$ is linearly independent in $W$.

   (b) Suppose $\mathbf{v}_1, ..., \mathbf{v}_n$ spans $V$ and $T \in \mathcal{L}(V, W)$. Prove that the list $T(\mathbf{v}_1), ...T(\mathbf{v}_n)$ spans $T(V)$.

   (c) Suppose $V$ is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace $U$ of $V$ such that $U \cap$ Ker$(T)$ $= \{0\}$ and $T(V) = T(U)$. Find a basis.

3. (a) Suppose $V$ and $W$ are both finite-dimensional. Prove that there exists an injective linear transfomation from $V$ to $W$ if and only if dim$V$ $\leq$dim $W$.

   (b) Suppose $V$ and $W$ are both finite-dimensional. Prove that there exists an surjective linear transfomation from $V$ onto $W$ if and only if dim$V$ $\geq$dim $W$.

   (c) Suppose $V$ and $W$ are finite-dimensional and that $U$ is a subspace of $V$. Prove that there exists $T \in \mathcal{L}(V, W)$ such that Ker$(T) = U$ if and only if dim$U$ $\geq$dim$V -$dim$W$.

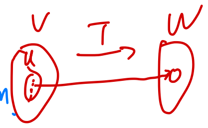
$r \geqslant n - m \Leftrightarrow m \geqslant n - r$

Idea:

$\dim V = n$

Let $\dim U = r$.

$\dim W = m$




$T \in \mathcal{L}(V, W)$

$V: \{v_1, \cdots, v_n\}$.

$T(v_1) \in W \quad \cdots \quad T(v_n) \in W$.

$\forall v \in V, \quad v = c_1 v_1 + \cdots + c_n v_n$

$T(v) = c_1 T(v_1) + \cdots + c_n T(v_n)$

a basis of $U$, $\{v_1, \cdots, v_r\} \to$ a basis of $V$ $\{v_1, \cdots, v_r, v_{r+1}, \cdots, v_n\}$

$T(v_1) = 0 \quad T(v_r) = 0 \quad T(v_{r+1}) = w_1 \quad \vdots \quad T(v_n) = w_{n-r}$

Let $\{w_1, \cdots, w_m\}$ be a basis of $W$.

1) $T(U) = \{0\}$. $U \subset \ker(T)$

2) For $v \in V$, if $T(v) = 0$, $v = c_1 v_1 + \cdots + c_r v_r + c_{r+1} v_{r+1} + \cdots + c_n v_n$

$0 = T(v) = c_1 T(v_1) + \cdots + c_r T(v_r) + c_{r+1} T(v_{r+1}) + \cdots + c_n T(v_n)$

$m \geqslant n - r$

$= C_{r+1} w_1 + \cdots + C_n w_{n-r} = 0$

$w_1, \cdots, w_{n-r}$ linearly independent $C_{r+1} = \cdots = C_n = 0$

$v = c_1 v_1 + \cdots + c_r v_r \in U$.

$\ker(T) \subset U$.

$A = \left[ [L(e_1)]_{\{b_1,b_2,b_3\}} \quad [L(e_2)]_{\{b_1,b_2,b_3\}} \right]$

$L(e_1) = (b_1\ b_2\ b_3)\begin{pmatrix}1\\0\\1\end{pmatrix}$ $\quad L(e_2) = (b_1\ b_2\ b_3)\begin{pmatrix}0\\1\\1\end{pmatrix}$ $\quad A = \begin{pmatrix}1 & 0\\0 & 1\\1 & 1\end{pmatrix}$

$x_1 = 1,\ x_2 = 0$ $\quad x_1 = 0\ \ x_2 = 1$

4. Let

$$\mathbf{b}_1 = \begin{Bmatrix}1\\1\\0\end{Bmatrix}, \mathbf{b}_2 = \begin{Bmatrix}1\\0\\1\end{Bmatrix}, \mathbf{b}_3 = \begin{Bmatrix}0\\1\\1\end{Bmatrix}$$

and let $L$ be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ define by

$$L(\mathbf{x}) = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + (x_1 + x_2)\mathbf{b}_3, \qquad x = \begin{bmatrix}x_1\\x_2\end{bmatrix}$$

find the matrix $A$ representing $L$ with respect to the ordered bases $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

$\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}$

$e_1$ $\quad x_1 = 1$, $x_2 = 0$

5. Let

$$\mathbf{y}_1 = \begin{Bmatrix}1\\1\\1\end{Bmatrix}, \mathbf{y}_2 = \begin{Bmatrix}1\\1\\0\end{Bmatrix}, \mathbf{y}_3 = \begin{Bmatrix}1\\0\\0\end{Bmatrix}$$

and let $\mathcal{I}$ be the identity operator on $\mathbb{R}^3$.

(a) Find the coordinates of $\mathcal{I}(\mathbf{e}_1)$, $\mathcal{I}(\mathbf{e}_2)$, and $\mathcal{I}(\mathbf{e}_3)$ with respect to $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.

(b) Find a matrix $A$ such that $A\mathbf{x}$ is the coordinate vector of $\mathbf{x}$ with respect to $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.

$y = Ax$ ; $T$ $\quad T(x) = Ix$

$[T(x)]_{\{y_1, y_2, y_3\}}$ $\quad \begin{bmatrix}x_1 + x_2\\x_3\\x_1\end{bmatrix}$

(a) $\mathcal{I}(e_1) = e_1 = (y_1\ y_2\ y_3)\begin{pmatrix}0\\0\\1\end{pmatrix}$

$\mathcal{I}(e_2) = e_2 = (y_1, y_2\ y_3)\begin{pmatrix}0\\1\\-1\end{pmatrix}$ $\quad x \xrightarrow{T} Ax$

$\mathcal{I}(e_3) = e_3 = (y_1\ y_2\ y_3)\begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\quad (T(x))$

(b). $A = \begin{pmatrix}0 & 0 & 1\\0 & 1 & -1\\1 & -1 & 0\end{pmatrix}$ $\quad T(e_1) = \begin{bmatrix}1\\0\\1\end{bmatrix} = (y_1, y_2, y_3)\left(\quad\right)$