Homework 1

March 31, 2022

1. Let V be a vector space and let $\mathbf{x}, \mathbf{y} \in V$. Show that

   (a) $\beta \mathbf{0} = \mathbf{0}$ for each scalar β.

   (b) $\mathbf{x} + \mathbf{y} = \mathbf{0}$ implies that $\mathbf{y} = -\mathbf{x}$, *i.e.*, the additive inverse of $\mathbf{x}$ is unique.

2. Let V be the set of all ordered pairs of real numbers with addition defined by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

   and scalar multiplication defined by

$$\alpha \circ (x_1, x_2) = (\alpha x_1, x_2)$$

   Scalar multiplication for this system is defined in an unusual way, and consequently we use the symbol o to avoid confusion with the ordinary scalar multiplication of row vectors. Is V a vector space with these operations? Justify your answer.

3. Let R^+ denote the set of positive real numbers. Define the operation of scalar multiplication, denoted o, by

$$\alpha \circ x = x^\alpha$$

   for each $x \in R^+$ and for any real number α. Define the operation of addition, denoted $\oplus$, by

$$x \oplus y = x \cdot y \quad \text{for all} \quad x, y \in R^+$$

   Thus, for this system, the scalar product of -3 times $\frac{1}{2}$ is given by

$$-3 \circ \frac{1}{2} = \left(\frac{1}{2}\right)^{-3} = 8$$

   and the sum of 2 and 5 is given by

$$2 \oplus 5 = 2 \cdot 5 = 10$$

   Is R^+ a vector space with these operations? Prove your answer.

4. Let A be a fixed vector in $\mathbb{R}^{n \times n}$ and let S be the set of all matrices that commute with A, that is,

$$S = \{B \mid AB = BA\}$$

Show that S is a subspace of $\mathbb{R}^{n \times n}$.

5. Verify the following statements.

   (a) Is $\mathbb{R}^3$ a subspace of the complex vector space $\mathbb{C}^3$?
   (b) Is $\{(x, y, z) \in \mathbb{R}^3 : x^3 = y^3\}$ a subspace of $\mathbb{R}^3$?
   (c) Is $\{(x, y, z) \in \mathbb{C}^3 : x^3 = y^3\}$ a subspace of $\mathbb{C}^3$?

6. Suppose U_1 and U_2 are subspaces of V.

   (a) Is the intersection $U_1 \cap U_2$ a subspace of V? Prove or give a counterexample.
   (b) Is the union $U_1 \cup U_2$ a subspace of V? Prove or give a counterexample.