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Course: **Linear Algebra in Control Theory**

Assignment Number: **2**
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Problem 1

Suppose e_1, \dots, e_m is an orthonormal list of vectors in V . Let $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_m \rangle|^2$$

if and only if $v \in \text{span}(e_1, \dots, e_m)$.

Problem 2

Suppose n is a positive integer. Prove that

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots, \frac{\sin nx}{\sqrt{\pi}}$$

is an orthonormal list of vectors in $C[-\pi, \pi]$, the vector space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) dx.$$

Problem 3

On $\mathcal{P}_2(\mathbf{R})$, consider the inner product given by

$$\langle p, q \rangle = \int_0^1 p(x) q(x) dx$$

Apply the Gram–Schmidt Procedure to the basis $1, x, x^2$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$.

Problem 4

For each of the following, use the Gram-Schmidt process find an orthonormal basis for $R(A)$:

$$1. A = \begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix}$$

$$2. A = \begin{bmatrix} 2 & 5 \\ 1 & 10 \end{bmatrix}$$

where $R(A)$ is the linear space spanned by the columns of A .

Problem 5

Given $\mathbf{x}_1 = \frac{1}{2}(1, 1, 1, -1)^T$ and $\mathbf{x}_2 = \frac{1}{6}(1, 1, 3, 5)^T$, verify that these vectors form an orthonormal set in \mathbb{R}^4 . Extend this set to an orthonormal basis for \mathbb{R}^4 by finding an orthonormal basis for the null space of

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix}$$

[Hint: First find a basis for the null space and then use the Gram-Schmidt process.]

Problem 6

Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x) q(x) dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

Problem 7

Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$\int_0^1 p(x) (\cos \pi x) dx = \int_0^1 p(x) q(x) dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

Problem 8

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$$

- Use the Gram-Schmidt process to find an orthonormal basis for the column space of A .
- Factor A into a product QR , where Q has an orthonormal set of column vectors and R is upper triangular.
- Solve the least squares problem $A\mathbf{x} = \mathbf{b}$.

Problem 9

Suppose $v_1, \dots, v_m \in V$. Prove that

$$\{v_1, \dots, v_m\}^\perp = (\text{span}(v_1, \dots, v_m))^\perp$$

Problem 10

Suppose U is the subspace of \mathbf{R}^4 defined by

$$U = \text{span}((1, 2, 3, -4), (-5, 4, 3, 2)).$$

Find an orthonormal basis of U and an orthonormal basis of U^\perp .

Problem 11

Let U be an m -dimensional subspace of \mathbb{R}^n and let V be a k -dimensional subspace of U , where $0 < k < m$.

(a) Show that any orthonormal basis

$$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$$

for V can be expanded to form an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_m\}$ for U .

(b) Show that if $W = \text{Span}\{\mathbf{v}_{k+1}, \dots, \mathbf{v}_m\}$, then $U = V \oplus W$.

Pay Attention

- a) Mark your class number, name and student number on the homework.
- b) Please hand in your homework to your TA before class next Tuesday (Jun. 7).

References

- [1] Axler, S. (1997). Linear algebra done right. Springer Science & Business Media.
- [2] Lay, D. C. . Linear algebra and its applications. Academic Press.