Homework 4

$$[B \,\vdots\, T(u_1) \cdots T(u_n)]$$

April 29, 2021

$$[B \,\vdots\, L(u_1)\ L(u_2)\ L(u_3)]$$

1. Let $E = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $F = \{\mathbf{b}_1, \mathbf{b}_2\}$, where

2. $D(1)=0=[1\ x\ x^2]\begin{bmatrix}0\\0\\0\end{bmatrix}$

$D(x)=1=[1\ x\ x^2]\begin{bmatrix}1\\0\\0\end{bmatrix}$

$D(x^2)=2x=[1\ x\ x^2]\begin{bmatrix}0\\2\\0\end{bmatrix}$

V:
$$\mathbf{u}_1 = \begin{pmatrix}1\\0\\-1\end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix}1\\2\\1\end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$B=\begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix}$

and

W:
$$\mathbf{b}_1 = (1, -1)^T, \quad \mathbf{b}_2 = (2, -1)^T$$

$D(1)=0=[1\ 2x\ 4x^2-2]\begin{bmatrix}0\\0\\0\end{bmatrix}$

$D(2x)=2=[1\ 2x\ 4x^2-2]\begin{bmatrix}2\\0\\0\end{bmatrix}$

$D(4x^2-2)=8x=[1\ 2x\ 4x^2-2]\begin{bmatrix}0\\4\\0\end{bmatrix}$

For each of the following linear transformations L from $\mathbb{R}^3$ into $\mathbb{R}^2$, find the matrix representing L with respect to the ordered bases E and F:

(i) $L(\mathbf{x}) = \begin{pmatrix}x_3\\x_1\end{pmatrix}$ $\quad \begin{bmatrix}1&2&-1&1&1\\-1&-1&1&1&-1\end{bmatrix} \to \begin{bmatrix}1&2&-1&1&1\\0&1&0&2&0\end{bmatrix} \to \begin{bmatrix}1&0&-1&-3&1\\0&1&0&2&0\end{bmatrix}$

A

(ii) $L(\mathbf{x}) = \begin{pmatrix}x_1+x_2\\x_1-x_3\end{pmatrix}$ $\quad \begin{bmatrix}-5&-3&4\\3&3&-2\end{bmatrix}$

$A=\begin{bmatrix}0&2&0\\0&0&4\\0&0&0\end{bmatrix}$

(iii) $L(\mathbf{x}) = \begin{pmatrix}2x_2\\-x_1\end{pmatrix}$ $\quad \begin{bmatrix}2&-2&-4\\-1&3&3\end{bmatrix}$

$S^{-1}AS \to [1, 2x, 4x^2-2]$

$B \to [1, x, x^2]$

2. Let D be the differentiation operator on $P_2(R)$. Find the matrix B representing D with respect to $[1, x, x^2]$, the matrix A representing D with respect to $[1, 2x, 4x^2 - 2]$, and the nonsingular matrix S such that $B = S^{-1}AS$.

$A=S^{-1}BS \Leftarrow B=SAS^{-1} \quad A=SBS^{-1}$

$[1,x,x^2]=[1,2x,4x^2-2]\,S$

$=[1,2x,4x^2-2]\begin{bmatrix}1&0&\frac{1}{2}\\0&\frac{1}{2}&0\\0&0&\frac{1}{4}\end{bmatrix}$

3. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim T(V) = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of the matrix representation $\mathcal{M}(T)$ equal 1.

4. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear transformation. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

S^{-1}

$[1,2x,4x^2-2]=[1,x,x^2]\,S^{-1}$

$=[1,x,x^2]\begin{bmatrix}1&0&-2\\0&2&0\\0&0&4\end{bmatrix}$

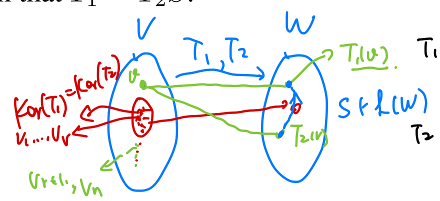
5. (a) Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following statements are equivalent:
(i) T is invertible;
(ii) T is injective;
(iii) T is surjective.

(b) Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove there exists an invertible operator $T \in \mathcal{L}(V)$ such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

(c) Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\ker(T_1) = \ker(T_2)$ if and only if there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$. $T_1(v) = S\,T_2(v)$

(d) Suppose V is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $T_1(V) = T_2(V)$ if and only if there exists an invertible operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2S$.

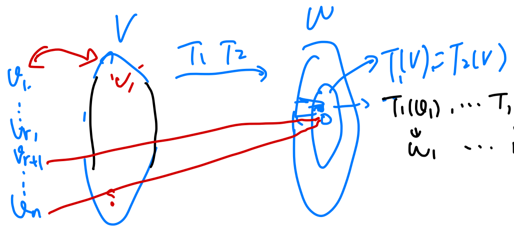
(c) idea



$T_1(v_{r+1}), \dots, T_1(v_n),\ w_{n-r+1}, \dots, w_m,$
$\updownarrow S \quad \updownarrow S \quad \uparrow S \quad \uparrow S$
$T_2(v_{r+1}), \dots, T_2(v_n),\ w_{n-r+1}, \dots, w'_m.$

$\{v_1, \dots, v_r\}$ is a basis of $\ker(T_1)$ and $\ker(T_2)$
$\downarrow$
$\{v_1, \dots, v_r, v_{r+1}, \dots, v_n\}$. is a basis of V.

(d).



$T_1(v_1), \dots T_1(v_r)$
$\downarrow \qquad \downarrow$
$w_1 \ \dots \ w_r.$