Teacher: **Yanjie Li** — Assignment Number: **2**

Course: **Linear Algebra in Control Theory** — Disclosure date: May 19, 2025

Problem 1

Suppose $e_1, ..., e_m$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$$

if and only if $v \in \operatorname{span}(e_1, ..., e_m)$.

Problem 2

Suppose n is a positive integer. Prove that

$$\frac{1}{\sqrt{2}}, \cos x, \cos 2x, ..., \cos nx, \sin x, \sin 2x, ..., \sin nx$$

is an orthonormal list of vectors in $C\left[-\pi, \pi\right]$, the vector space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(x\right) g\left(x\right) dx.$$

Problem 3

On $\mathcal{P}_2\left(\mathbf{R}\right)$, consider the inner product given by

$$\langle p, q \rangle = \int_0^1 p\left(x\right) q\left(x\right) dx$$

Apply the Gram–Schmidt Procedure to the basis 1, x, x^2 to produce an orthonormal basis of $\mathcal{P}_2\left(\mathbf{R}\right)$.

Problem 4

For each of the following, use the Gram-Schmidt process find an orthonormal basis for $R(A)$:

$$1.A = \begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix}$$

$$2.A = \begin{bmatrix} 2 & 5 \\ 1 & 10 \end{bmatrix}$$

where $R(A)$ is the linear space spanned by the columns of A.

Problem 5

Given $\mathbf{x}_1 = \frac{1}{2}(1, 1, 1, -1)^T$ and $\mathbf{x}_2 = \frac{1}{6}(1, 1, 3, 5)^T$, verify that these vectors form an orthonormal set in $\mathbb{R}^4$. Extend this set to an orthonomal basis for $\mathbb{R}^4$ by finding an orthonomal basis for the null space of

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix}$$

[*Hint*: First find a basis for the null space and then use the Gram-Schmidt process.]

Problem 6

Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x)\, q(x)\, dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

Problem 7

Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$\int_0^1 p(x)(\cos \pi x)\, dx = \int_0^1 p(x)\, q(x)\, dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

Problem 8

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$$

(a) Use the Gram–Schmidt process to find an orthonormal basis for the column space of A.
(b) Factor A into a product QR, where Q has an orthonormal set of column vectors and R is upper triangular.
(c) Solve the least squares problem $A\mathbf{x} = \mathbf{b}$.

Problem 9

Suppose $v_1, ..., v_m \in V$. Prove that

$$\{v_1, ..., v_m\}^{\perp} = (\text{span}\,(v_1, ..., v_m))^{\perp}$$

Problem 10

Suppose U is the subspace of $\mathbf{R}^4$ defined by

$$U = \text{span}\,((1, 2, 3, -4)\,, (-5, 4, 3, 2))\,.$$

Find an orthonormal basis of U and an orthonormal basis of $U^{\perp}$.

Problem 11

Let U be an m-dimensional subspace of $\mathbb{R}^n$ and let V be a k-dimensional subspace of U, where $0 < k < m$.
(a) Show that any orthonormal basis

$$\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$$

for V can be expanded to form an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k, \mathbf{v}_{k+1}, ..., \mathbf{v}_m\}$ for U.
(b) Show that if $W = \text{Span}\{\mathbf{v}_{k+1}, ..., \mathbf{v}_m\}$, then $U = V \oplus W$.

Pay Attention

a) Mark your class number, name and student number on the homework.
b) Please hand in your homework to your TA before class next Thursday (May. 29).

References

[1] Axler, S. (1997). Linear algebra done right. Springer Science & Business Media.

[2] Lay, D. C. . Linear algebra and its applications. Academic Press.