Homework 2

April 2, 2022

1. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that
$$U_1 + W = U_2 + W,$$
then $U_1 = U_2$.

2. Suppose
$$U = \{(x, x, y) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$$
Find a subspace W of $\mathbb{R}^3$ such that $\mathbb{R}^3 = U \oplus W$.

3. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be a spanning set for a vector space V.

   (a) If we add another vector, $\mathbf{x}_{k+1}$, to the set, will we still have a spanning set? Explain.

   (b) If we delete one of the vectors,say, $\mathbf{x}_k$, from the set, will we still have a spanning set? Explain.

4. For each subspace in (a)-(d), (1) find a basis, and (2) state the dimension.

   (a)
$$\left\{ \begin{bmatrix} s - 2t \\ s + t \\ 3t \end{bmatrix} : s, t \text{ in } \mathbb{R} \right\}$$

   (b)
$$\left\{ \begin{bmatrix} 2c \\ a - b \\ b - 3c \\ a + 2b \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$$

   (c)
$$\left\{ \begin{bmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ -a + 2c \\ -3a + 7b + 6c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$$

(d)

$$\{(a, b, c, d) : a - 3b + c = 0\}$$

5. Let

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ a \end{bmatrix}$$

If $\dim(\mathrm{Span}(x_1, x_2, x_3)) = 2$, compute a.

6. V is a nonzero finite-dimensional vector space, and the vectors listed belong to V. Mark each statement True or False. Justify each answer (Prove it if True or give an anti-example if False).

   a. If there exists a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ that spans V, then $\dim V \leq p$.
   b. If there exists a linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V, then $\dim V \geq p$.
   c. If $\dim V = p$, then there exists a spanning set of $p + 1$ vectors in V.
   d. If there exists a linearly dependent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V, then $\dim V \leq p$.
   e. If every set of p elements in V fails to span V, then dim $V > p$.
   f. If $p \geq 2$ and $\dim V = p$, then every set of $p - 1$ nonzero vectors is linearly independent.