哈尔滨工业大学（深圳）2025 年春季学期

控制理论中的代数基础试题（A）

一、Justify each answer: draw “✓” if True and “×” if False. (30 points)

1. The subset $S = \{(w_1, w_2) \in \mathbb{R}^2 | w_1^2 = w_2^2\}$ is a subspace of $\mathbb{R}^2$.
2. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^5$. ( )
3. If U is a subspace of a finite-dimensional vector space V, then $\dim(U) \leqslant \dim(V)$. ( )
4. V is a finite-dimensional vector space. If there exists a linearly dependent set $\{v_1, \cdots, v_p\}$ in V, then $\dim V \leqslant p$. ( )
5. Suppose that you have 2025 vectors $v_1, v_2, \cdots, v_{2025} \in \mathbb{R}^{2024}$. Each v_i is a combination of the other 2024 vectors. ( )
6. Every finite-dimensional vector space has a unique basis. ( )
7. Let $V = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\2\end{bmatrix}\right)$ and $W = \operatorname{span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}\right)$. The matrix representing a linear transformation from V to W is a 2×3 matrix. ( )
8. A linear transformation to a larger dimensional space cannot be surjective. ( )
9. Suppose V is a real inner product space. If $u, v \in V$ have the same norm, then $u + v$ is orthogonal to $u - v$. ( )
10. Suppose $T \in \mathcal{L}(V)$ and $U \subset \operatorname{null} T$, then U is invariant under T. ( )
11. If A is a matrix with $m \times n$ dimension, then $A^T A$ and AA^T have the same nonzero eigenvalues. ( )
12. If W is a subspace of $\mathbb{R}^n$, then W and $W^\perp$ have no vectors in common. ( )
13. If a 5×5 matrix A has fewer than 5 distinct eigenvalues, then A is not diagonalizable. ( )
14. The forward shift operator $T \in \mathcal{L}(\mathbb{F}^\infty)$ defined by $T(z_1, z_2, \cdots) = (0, z_1, z_2, \cdots)$ has no eigenvalues. ( )
15. If W is a linear space and $\{v_1, v_2, v_3\}$ is an orthogonal set in W, then $\{v_1, v_2, v_3\}$ is a basis for W. ( )

二、**(4 points)**

Suppose $v_1, \cdots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 - w, \cdots, v_p - w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \cdots, v_p)$.

三、**(6 points)**

$S = \{(x_1, x_2, x_3, x_4) : x_1 + 2x_2 - x_4 = 0, x_2 - x_3 = 0\}$ is a subspace of $\mathbb{R}^4$.

(a) (3 points) Write a basis of S.

(b) (3 points) Write a basis of a subspace W of $\mathbb{R}^4$ satisfying $\mathbb{R}^4 = S \oplus W$.

四、**(8 points)**

Let $A \in \mathbb{R}^{3\times 4}$ with reduced row echelon form give by $R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The first and fourth columns of A are $a_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $a_4 = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$.

(a) (4 points) Find the bases for the four fundamental subspaces of A.

(b) (4 points) Write down the fundamental theorem of linear algebra: Part 1 and Part 2, and verify them by the above four fundamental subspaces.

五、**(10 points)**

Compute.

(a) (3 points) $T \in \mathcal{L}(\mathbb{R}^2)$ first performs a transformation that maps $\mathbf{e}_1$ into $2\mathbf{e}_2$ and maps $\mathbf{e}_2$ into $\mathbf{e}_1 - 3\mathbf{e}_2$ and then reflects points through the line $x_1 = 0$. Compute the standard matrix of T.

(b) (3 points) For the linear transformation $T(x) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix}$ with $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ from $\mathbb{R}^2$ to $\mathbb{R}^3$, find the matrix representing T with respect to the ordered basis $E = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$ and $F = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \right\}$.

(c) (4 points) Let U be the subspace of $C[a, b]$ spanned by e^x, xe^x, x^2e^x. Let D be the differentiation operator on U. Find the matrix A representing D with respect to $[(2 - x)e^x, -e^x, (x^2 - 2x)e^x]$.

六、**(10 points)**

Suppose V and W are both finite-dimensional vector spaces and $T_1, T_2 \in \mathcal{L}(V, W)$.

(a) (4 points) For $T \in \mathcal{L}(V, W)$, write down the fundamental theorem of linear transformation.

(b) (3 points) Use the above theorem to show that if $T \in \mathcal{L}(V, W)$ is injective, then $\dim V \leqslant \dim W$.

(c) (3 points) Prove that $\operatorname{range} T_1 = \operatorname{range} T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2 S$.

七、**(8 points)**

Find U, Σ, V in the Singular Value Decomposition $A = U\Sigma V^T$:

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 2 \\ 2 & -1 \end{bmatrix}.$$

八、**(8 points)**

Define $T \in \mathcal{L}(\mathbb{R}^2)$ by $T(x, y) = (41x + 7y, -20x + 74y)$.

(a) (3 points) Please describe the matrix respect to the standard basis of $\mathbb{R}^2$.

(b) (5 points) Find a basis of $\mathbb{R}^2$ with respect to which T has a diagonal matrix.

九、(8 points)

Please solve the following 2th-order systems of linear differential equations

$$\ddot{x}_1 = -2x_1 + x_2$$

$$\ddot{x}_2 = x_1 - 2x_2$$

with initial condition $x_1(0) = x_2(0) = 0$ and $x_1'(0) = x_2'(0) = 2$.

十、**(8 points)**

Find $p(x) \in P_2(\mathbb{R})$ such that $p'(0) = 0$ and

$$\int_0^1 |x - p(x)|^2 \, \mathrm{d}x$$

is as small as possible.