Solutions & Notes of Homework 6

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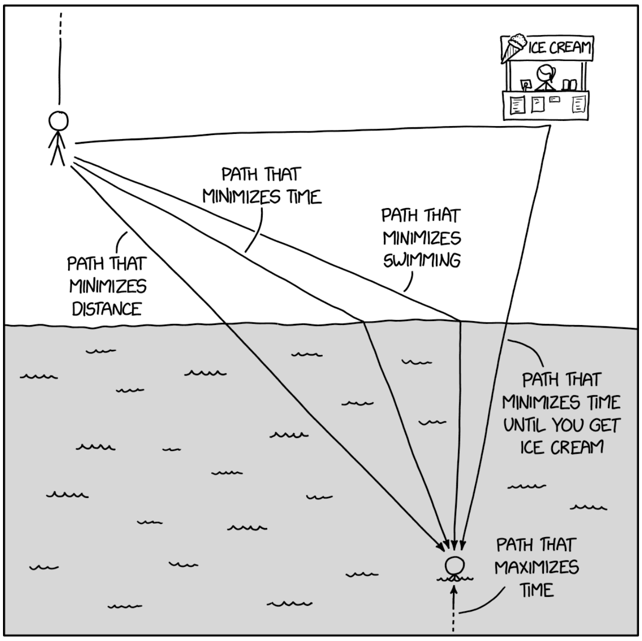

Figure 1: Picture from https://matthbeck.github.io/quotes.html, originally from https://xkcd.com/.

Problem 1

In $\mathbb{R}^4$, let

$$U = \operatorname{span}((1,1,0,0),(1,1,1,2)).$$

Find $u \in U$ such that $\|u-(1,2,3,4)\|$ is as small as possible.

Solution: $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}$, $\dim C(A) = 2$. Thus

$$u = A(A^TA)^{-1}A^T(1,2,3,4)^T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{7}{10} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix} (1,2,3,4)^T = \left(\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5}\right)^T.$$

Note 1.1: *You can also use Gram-Schmidt procedure to calculate the orthonormal basis*

$$e_1 = \frac{1}{\sqrt{2}}(1,1,0,0), \quad e_2 = \frac{1}{\sqrt{5}}(0,0,1,2),$$

and use equation 6.57 (i) to find the answer.

Problem 2

Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(0) = 0, p'(0) = 0$, and

$$\int_0^1 |2 + 3x - p(x)|^2 dx$$

is as small as possible.

Solution: $p(0) = 0, p'(0) = 0$ implies $p(x) = ax^3 + bx^2 (a, b \neq 0)$. We can easily verify that all $p(x)$'s form a subspace of $\mathcal{P}_3(\mathbb{R})$, and x^2, x^3 forms a basis of it. Apply Gram-Schmidt procedure to x^2, x^3 and we get an orthonormal basis e_1, e_2:

$$e_1 = \frac{x^2}{\sqrt{\int_0^1 (x^2)(x^2)dx}} = \sqrt{5}x^2,$$

$$f_2 = x^3 - \frac{\int_0^1 (x^2)(x^3)dx}{\int_0^1 (x^2)(x^2)dx} x^2 = x^3 - \frac{5}{6}x^2,$$

$$e_2 = \frac{f_2}{\sqrt{\int_0^1 f_2^2 dx}} = \frac{f_2}{\sqrt{\int_0^1 (x^6 + \frac{25}{36}x^4 - \frac{5}{3}x^5)dx}} = 6\sqrt{7}\left(x^3 - \frac{5}{6}x^2\right).$$

Use equation 6.57 (i), we find that

$$\begin{aligned}
p(x) &= \langle 2 + 3x, \sqrt{5}x^2 \rangle \cdot \sqrt{5}x^2 + \left\langle 2 + 3x, 6\sqrt{7}\left(x^3 - \frac{5}{6}x^2\right) \right\rangle \cdot 6\sqrt{7}\left(x^3 - \frac{5}{6}x^2\right) \\
&= \int_0^1 (2x^2 + 3x^3)dx \cdot 5x^2 + \int_0^1 \left(2x^3 + 3x^4 - \frac{5}{3}x^2 - \frac{5}{2}x^3\right) dx \cdot 252 \left(x^3 - \frac{5}{6}x^2\right) \\
&= \left(\frac{2}{3} + \frac{3}{4}\right) \cdot 5x^2 + 252\left(\frac{2}{4} + \frac{3}{5} - \frac{5}{9} - \frac{5}{8}\right)\left(x^3 - \frac{5}{6}x^2\right) \\
&= \frac{85}{12}x^2 - \frac{203}{10}x^3 + \frac{203}{12}x^2 \\
&= -20.3x^3 + 24x^2.
\end{aligned}$$

Problem 3

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V.

(a) Prove that if $U \subset \operatorname{null} T$, then U is invariant under T.

(b) Prove that if $\operatorname{range} T \subset U$, then U is invariant under T.

Proof: (a) If $U \subset \operatorname{null} T$, then for every $u \in U$, we have $T(u) = 0$. $0 \in U$, so U is invariant under T.

(b) For every $u \in U$, we have $T(u) = v$, where $v \in \operatorname{range} T$. Because $\operatorname{range} T \subset U$, then $v \in U$, so U is invariant under T. ■

Problem 4

Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that range S is invariant under T.

Proof: For every $u \in$ range S, we can find a $v \in V$ such that $S(v) = u$. $T(u) = T(S(v)) = TS(v) = ST(v) = S(T(v))$, thus $T(u) \in$ range S. Thus range S is invariant under T. ∎

Problem 5

Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that null S is invariant under T.

Proof: For every $u \in$ null S, we have $S(u) = 0$. $ST(u) = TS(u) = T(S(u)) = T(0) = 0$, thus $T(u) \in$ null S. Thus null S is invariant under T. ∎

Problem 6

Define $T \in \mathcal{L}\left(\mathbb{F}^3\right)$ by

$$T\left(z_1, z_2, z_3\right) = \left(2z_2, 0, 5z_3\right)$$

Find all eigenvalues and eigenvectors of T.

Solution: With respect to the standard basis, the matrix of T is $\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. This matrix is upper triangular. Thus by 5.41, the eigenvalues of T are 0, 0 and 5. For the eigenvalue 0, let $Tx = 0$ where $x \in \mathbb{F}^3$, and we get a linearly independent vector $(1, 0, 0)$; let $(T - 5I)x = 0$ where $x \in \mathbb{F}^3$ and I is the standard matrix of identity operator, and we get a linearly independent vector $(0, 0, 1)$. So for $\lambda = 0$ we have the eigenvector $k(1, 0, 0)(k \in \mathbb{F}\backslash\{0\})$; for $\lambda = 5$ we have the eigenvector $k(0, 0, 1)(k \in \mathbb{F}\backslash\{0\})$.

Problem 7

Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

Solution: $(x^n)' = nx^{n-1}$. Suppose λ is an eigenvalue of T with an eigenvector q, then $q' = Tq = \lambda q$. If $\lambda \neq 0$, then $\deg \lambda q > \deg q'$, we get a contradiction. If $\lambda = 0$, then we can take $q = c$ for nonzero $c \in \mathbb{R}$. Hence the only eigenvalue of T is zero with nonzero constant polynomials as eigenvectors.

Problem 8

Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.
(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.
(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

Proof: (a) First, suppose $\lambda \in \mathbb{F}$ is an eigenvalue of T. Then there exists an eigenvector $x \in V$ such that $(T - \lambda I)x = 0$. Since $S \in \mathcal{L}(V)$ is invertible, $I = SS^{-1}$, and $T - \lambda I$ can be written as $T - \lambda SS^{-1}$. Then $S(S^{-1}TS - \lambda I)S^{-1}x = 0$. Because S is invertible, S is injective, which implies $(S^{-1}TS - \lambda I)S^{-1}x = 0$.

Because x is nonzero and S^{-1} is injective, $S^{-1}x \neq 0$, which implies $S^{-1}TS - \lambda I$ maps a nonzero vector into 0. Thus $S^{-1}TS - \lambda I$ is not injective, so λ is an eigenvalue of $S^{-1}TS$.

Conversely, suppose $\lambda \in \mathbb{F}$ is an eigenvalue of $S^{-1}TS$, then there exists an eigenvector $x \in V$ such that $(S^{-1}TS - \lambda I)x = 0$. Then $(S^{-1}TS - \lambda S^{-1}S)x = 0$, which further implies $S^{-1}(T - \lambda I)Sx = 0$. Because S is invertible, S^{-1} is injective, which implies $(T - \lambda I)Sx = 0$. Because x is nonzero and S is injective, $Sx \neq 0$, which implies $T - \lambda I$ maps a nonzero vector into 0. Thus $T - \lambda I$ is not injective, so λ is an eigenvalue of T. This completes the proof.

(b) Let $\lambda \in \mathbb{F}$ be an eigenvalue of T. As we showed in part (a), this is the case if and only if λ is an eigenvalue of $S^{-1}TS$. Define

$$E(\lambda, T) = \{v \in V : v \neq 0, Tv = \lambda v\} \quad \text{and} \quad E(\lambda, S^{-1}TS) = \{u \in V : u \neq 0, (S^{-1}TS)(u) = \lambda u\}.$$

That is, $E(\lambda, T)$ is the collection of eigenvectors of T corresponding to the eigenvalue λ and $E(\lambda, S^{-1}TS)$ is the collection of eigenvectors of $S^{-1}TS$ corresponding to the eigenvalue λ. Our calculations in part (a) show that

$$E(\lambda, T) = \{Su : u \in E(\lambda, S^{-1}TS)\} \quad \text{and} \quad E(\lambda, S^{-1}TS) = \{S^{-1}v : v \in E(\lambda, T)\}. \qquad \blacksquare$$

Problem 9

Find all eigenvalues and eigenvectors of the backward shift operator $T \in \mathcal{L}(\mathbb{F}^\infty)$ defined by

$$T(z_1, z_2, z_3, \ldots) = (z_2, z_3, \ldots).$$

✓ **Solution:** Suppose λ is an eigenvalue of T. For this particular operator, the eigenvalue-eigenvector equation $Tz = \lambda z$ becomes the system of equations $z_2 = \lambda z_1, z_3 = \lambda z_2, \ldots$. From this we see that we can choose z_1 arbitrarily and then solve for the other coordinates: $z_2 = \lambda z_1, z_3 = \lambda z_2 = \lambda^2 z_1, \ldots$. Thus each $\lambda \in \mathbb{F}$ is an eigenvalue of T and the set of corresponding eigenvector is $\{(w, \lambda w, \lambda^2 w, \ldots) : w \in \mathbb{F}\backslash\{0\}\}$.

Problem 10

If A is a matrix with $m \times n$ dimension, please show that A^TA and AA^T have the same nonzero eigenvalues.

✓ **Proof:** Suppose x is the eigenvector for the eigenvalue λ of A^TA. Then $A^TAx = \lambda x$. Multiply both sides of the equation with A, then $AA^TAx = \lambda Ax$, so $AA^T(Ax) = \lambda(Ax)$. If $Ax = 0$, then $A^TAx = 0 = \lambda x$, which contradicts with the fact that $\lambda \neq 0$ and $x \neq 0$. Hence $Ax \neq 0$, which means λ is also an eigenvalue of AA^T with Ax as the corresponding eigenvector.

Conversely, suppose x is the eigenvector for the eigenvalue λ of AA^T. Then $AA^Tx = \lambda x$. Multiply both sides of the equation with A^T, then $A^TAA^Tx = \lambda A^Tx$, so $A^TA(A^Tx) = \lambda(A^Tx)$. If $A^Tx = 0$, then $AA^Tx = 0 = \lambda x$, which contradicts with the fact that $\lambda \neq 0$ and $x \neq 0$. Hence $A^Tx \neq 0$, which means λ is also an eigenvalue of A^TA with A^Tx as the corresponding eigenvector. This completes the proof. $\blacksquare$