Solutions & Notes of Homework 2

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October 1, 2025

In linear algebra, the *concepts* are as important as the *computations*. The simple numerical exercises only help you check your understanding of basic procedures. Later in your career, **computers will do the calculations**, but you will have to **choose** the calculations, know how to **interpret** the results, and then **explain** the results to other people.

You must avoid the temptation to look at answers before you have tried to write out the solution yourself. **Otherwise, you are likely to think you understand something when in fact you do not**.

—Linear Algebra and Its Applications, by David C. Lay et al.

Problem 1

Suppose
$$U = \{(x, x, y) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$$

Find a subspace W of $\mathbb{R}^3$ such that $\mathbb{R}^3 = U \oplus W$.

✔ **Solution (1):** *We can directly pick an W with our intuition, and give a basis-free verification.*

Take $W = \{(x, 0, 0) \in \mathbb{R}^3 : x \in \mathbb{R}\}$, it's easy to verify that W is a subspace of $\mathbb{R}^3$. The proof below shows that $\mathbb{R}^3 = U \oplus W$.

First we prove that $\mathbb{R}^3 = U + W$. Let $v = (x, y, z) \in \mathbb{R}^3$, then

$$v = (x - y, 0, 0) + (y, y, z).$$

Since $(y, y, z) \in U$ and $(x - y, 0, 0) \in W$, there exists $u \in U, w \in W$ such that $v = u + w$, which shows that $\mathbb{R}^3 \subset U + W$, and obviously $U + W \subset \mathbb{R}^3$, completing the proof that $\mathbb{R}^3 = U + W$.

Next, to show that $U \bigcap W = \{0\}$, suppose that $v = (x, y, z) \in U \bigcap W$. Then

$$\left.\begin{array}{l} v \in U \implies x = y \\ v \in W \implies y = 0, z = 0 \end{array}\right\} \implies x = 0 \implies v = 0,$$

thus $U \bigcap W = \{0\}$.

Now that $\mathbb{R}^3 = U + W$ and $U \bigcap W = \{0\}$, we can conclude that $\mathbb{R}^3 = U \oplus W$.

❗ **Note 1.1:** *In fact we only prove $U \bigcap W \subset \{0\}$, and the other direction is omitted here because it's straightforward.*

❗ **Note 1.2:** *We can also use the basis here, but the proof will become lengthy. In fact we use the basis for* **construction** *(see below) instead of* **verification**.

✔ **Solution (2):** *We can also use the procedure outlined in 2.30 and 2.32 of Linear Algebra Done Right (fourth edition), that is, to find a basis of U and extend it to a basis of $\mathbb{R}^3$. And then the new vector (or vectors) that we add forms the basis of W—see the proof 2.33.*

We know that $(1, 1, 0), (0, 0, 1)$ is a basis of U because it is linearly independent (not a scalar multiple of each other) and spans U $[(x, x, y) = x(1, 1, 0) + y(0, 0, 1)$, where x, y are scalars]. Let $u_1 = (1, 1, 0), u_2 = (0, 0, 1)$. Denote the jth standard basis of $\mathbb{R}^3$ by e_j. Following the procedure outlined in 2.30 and 2.32, we adjoin the three standard basis vectors in $\mathbb{R}^3$ to the basis of U to obtain the spanning set

$$u_1, u_2, e_1, e_2, e_3.$$

- e_1 does not belong to $\text{span}(u_1, u_2)$, so we do not delete it.

- $e_2 = u_1 - e_1$, so we delete e_2 from the list.

- $e_3 = u_2$, so we delete e_3 from the list.

We are left with the list u_1, u_2, e_1; as the proof of 2.32 shows, this must be a basis of $\mathbb{R}^3$. And as shown in the proof of 2.33, if we let

$$W = \text{span}(e_1) = \{(x, 0, 0) \in \mathbb{R}^3 : x \in \mathbb{R}\},$$

then $\mathbb{R}^3 = U \oplus W$.

❗ **Note 1.3:** *Another W is $W = \{(0, x, 0) \in \mathbb{R}^3 : x \in \mathbb{R}\}$. Actually, W can be any line in $\mathbb{R}^3$ which contains the origin but doesn't lie on the plane U.*

Problem 2

For each subspace in (a)-(c), (1) find a basis, and (2) state the dimension.

(a)
$$\left\{ \begin{bmatrix} x + 2y \\ 2x - 3y \\ -x \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\}$$

(b)
$$\left\{ \begin{bmatrix} x + 3y - z \\ 4x + 5y + 3z \\ 3x + 6z \\ -x + 7y - 9z \end{bmatrix} : x, y, z \text{ in } \mathbb{R} \right\}$$

(c)
$$\{(x, y, z, w) : x - 4y + 3w = 0\}$$

❗ **Note 2.1:** *For convenience, we denote* $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ *as* $(x_1, \ldots, x_n)$.

✔ **Solution:** (a) for all $x, y \in \mathbb{R}$, $(x + 2y, 2x - 3y, -x) = x(1, 2, -1) + y(2, -3, 0)$, which shows that $(1, 2, -1), (2, -3, 0)$ is a spanning set. None of the two vectors is the scalar multiple of another vector, which indicates their linear independence. Therefore, we find a basis $(1, 2, -1), (2, -3, 0)$, and the dimension is 2.

(b) for all $x, y, z \in \mathbb{R}$,

$$
\begin{aligned}
(x + 3y - z, 4x + 5y + 3z, 3x + 6z, -x + 7y - 9z) &= x(1, 4, 3, -1) + y(3, 5, 0, 7) + z(-1, 3, 6, -9) \\
&= x(1, 4, 3, -1) + y(3, 5, 0, 7) \\
&\quad + z(2(1, 4, 3, -1) + (-1)(3, 5, 0, 7)) \\
&= (x + 2z)(1, 4, 3, -1) + (y - z)(3, 5, 0, 7),
\end{aligned}
$$

which shows that $(1, 4, 3, -1), (3, 5, 0, 7)$ is a spanning set, and we know that it is linearly independent (similar to (a)). Therefore, we find a basis $(1, 4, 3, -1), (3, 5, 0, 7)$, and the dimension is 2.

! **Note 2.2:** *We can utilize matrix and row reduction to help discover the linear dependence relation. For $x_1, \ldots, x_m \in \mathbb{F}$ and $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{F}^n$ ($\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$), the following* **equivalence among vector equation, matrix equation and system of linear equations** *holds:*

$$
\mathbf{a}_1 x_1 + \cdots + \mathbf{a}_m x_m = \mathbf{0} \iff
\begin{cases}
a_{11} x_1 + \cdots + a_{1m} x_m = 0 \\
\cdots \\
a_{n1} x_1 + \cdots + a_{nm} x_m = 0
\end{cases}
\iff
\begin{bmatrix}
a_{11} & \cdots & a_{1m} \\
\vdots & & \vdots \\
a_{n1} & \cdots & a_{nm}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ \vdots \\ x_m
\end{bmatrix} = \mathbf{0}.
$$

Therefore, linear dependence $\iff$ the homogeneous system has nonzero (nontrivial) solutions $\iff$ the coefficient matrix has free variables (less than m pivot columns). Important fact: the pivot columns correspond to **basic variables and linearly independent vectors***, while nonpivot columns correspond to* **free variables and vectors that can be written as linear combinations of other vectors**. [1]

For (b), row reduce the augmented matrix and we get

$$
\begin{bmatrix}
1 & 3 & -1 & 0 \\
4 & 5 & 3 & 0 \\
3 & 0 & 6 & 0 \\
-1 & 7 & -9 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & -1 & 0 \\
0 & -7 & 7 & 0 \\
0 & -9 & 9 & 0 \\
0 & 10 & -10 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
$$

where we find that the first two columns are pivot columns, and the corresponding two vectors $(1, 4, 3, -1)$, $(3, 5, 0, 7)$ are linearly independent. Assign the free variable x_3 as 1 and we find $x_1 = -2, x_2 = 1$, which means $-2(1, 4, 3, -1) + (3, 5, 0, 7) + (-1, 3, 6, -9) = 0$, and these coefficients are used in the solution above.

Solution Cont. (c) for all $x, y, z, w \in \mathbb{F}$,

$$
(x, y, z, w) = (4y - 3w, y, z, w) = y(4, 1, 0, 0) + z(0, 0, 1, 0) + w(-3, 0, 0, 1),
$$

which shows $(4, 1, 0, 0), (0, 0, 1, 0), (-3, 0, 0, 1)$ is a spanning set, and for scalars $x, y, z \in \mathbb{R}$,

$$
x(4, 1, 0, 0) + y(0, 0, 1, 0) + z(-3, 0, 0, 1) = 0 \iff y = 0, x = 0, z = 0,
$$

which shows that the above list of vector is linearly independent. Therefore, we find a basis $(4, 1, 0, 0)$, $(0, 0, 1, 0), (-3, 0, 0, 1)$, and the dimension is 3.

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[1]Hint: The row reduction does not change the solution of the equations, thus it does not change the linear dependence relation between columns. We can observe from the reduced row echelon form that the pivot columns are linearly independent (cannot be represented by the preceding vectors), and variables corresponding to them (basic variables) can be defined by others (free variables, by setting each of them to 1). Read *Linear Algebra and Its Applications* for details.

Problem 3

V is a nonzero finite-dimensional vector space, and the vectors listed belong to V. Mark each statement True or False. Justify each answer (Prove it if True or give an anti-example if False).

a. If there exists a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ that spans V, then $\dim V \leq p$.

b. If there exists a linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V, then $\dim V \geq p$.

c. If $\dim V = p$, then there exists a spanning set of $p + 1$ vectors in V.

d. If there exists a linearly dependent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V, then $\dim V \leq p$.

e. If every set of p elements in V fails to span V, then $\dim V > p$.

f. If $p \geq 2$ and $\dim V = p$, then every set of $p - 1$ nonzero vectors is linearly independent.

✔ **Solution:**

a. True. In a finite-dimensional vector space, a spanning set can always be reduced (pared down) to a basis, so the number of vectors in a basis is no greater than that of a spanning set. So $\dim V \leq p$.

b. True. In a finite-dimensional vector space, every linearly independent list of vectors can be extended to a basis, so the basis will contain more vectors than $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. So $\dim V \geq p$.

c. True. Suppose that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ is a basis of V and $\mathbf{x}_{p+1} \in V$. For every vector $v \in V$, we have

$$v = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_p\mathbf{x}_p = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_p\mathbf{x}_p + 0\mathbf{x}_{p+1},$$

which shows that v can be written as a linear combination of $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p, \mathbf{x}_{p+1}\}$, hence we have a list of $p + 1$ vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p, \mathbf{x}_{p+1}\}$ spanning V, completing the proof.

d. False. Counterexample: the set $\{(0,0,1), (0,0,2)\}$ is a linearly dependent set in $\mathbb{R}^3$, but $\dim \mathbb{R}^3 = 3 > 2$.

e. True. $\dim V \neq p$ because we can't find a linearly independent set of p elements to span V. If $\dim V < p$, then using (c.) repeatedly (we can take a scalar multiple of any vector in the basis of V) and we can find a spanning list of p vectors, which contradicts with our assumption. Therefore $\dim V > p$.

f. False. Counterexample: $p = \dim \mathbb{R}^3 = 3$, the set $\{(0,0,1), (0,0,2)\}$ is a set of $2 (= p - 1)$ vectors in $\mathbb{R}^3$, but this set is linearly dependent.

Problem 4

Prove that If V_1 and V_2 are subspaces of a finite-dimensional vector space, then $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$.

✔ **Proof:** Please refer to 2.43 of *Linear Algebra Done Right* (fourth edition). The key is to construct the basis of $V_1 + V_2$.

Problem 5

Consider two ordered bases $E = \{v_1, v_2, v_3\}$ and $F = \{w_1, w_2, w_3\}$ for a vector space V, and suppose

$$v_1 = -w_1 + w_2 + w_3, \quad v_2 = w_2 + 3w_3, \quad v_3 = 4w_1 - 2w_2$$

(a) Find the transition matrix S from E to F.

(b) Compute the coordinate vector $[v]_F$ for $v = 2v_1 + 1v_2 - v_3$.

✔ **Solution:** (a) By definition,

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} S = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & -2 \\ 1 & 3 & 0 \end{bmatrix}.$$

Therefore, the transition matrix $S = \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & -2 \\ 1 & 3 & 0 \end{bmatrix}$.

(b) By definition,

$$[v]_F = S[v]_E = \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & -2 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \\ 5 \end{bmatrix}.$$

⚠ **Attention:** *You should distinguish between* **old bases** *and* **new bases** *and carefully read the discussion above Example 2.7.3! Pay attention that in* **your first year course***, the transition matrix is defined as* $F = ES$ *(E is the old basis, F is the new one, S is the transition matrix) where the starting point is* **basis***, and the corresponding coordinate transition is* $[v]_F = S^{-1}[v]_E$*, while* <u>*in* **the course of this semester** *the definition has* **coordination** *as its starting point, which reads* $[v]_F = S[v]_E$ *and* $E = FS$</u>. **Don't mix them up!**

Problem 6

Consider two ordered bases $E = \{v_1, v_2, v_3\}$ and $F = \{w_1, w_2, w_3\}$ for $\mathbb{R}^3$, where

$$v_1 = \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

(a) Find the transition matrix S_1 from E to F.

(b) Find the transition matrix S_2 from F to E.

(c) Verify that $S_1 S_2 = S_2 S_1 = I_3$.

(d) If $[v]_E = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$, compute $[v]_F$ and use S_1 or S_2 (decide by yourself) to verify your answer.

Solution: (a) $[v_1 \quad v_2 \quad v_3] = [w_1 \quad w_2 \quad w_3]S_1 \implies S_1 = [w_1 \quad w_2 \quad w_3]^{-1}[v_1 \quad v_2 \quad v_3]$, thus with row reduction we have (denote $[v_1 \quad v_2 \quad v_3] = V$, $[w_1 \quad w_2 \quad w_3] = W$)

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}}_{W} \vdots \underbrace{\begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix}}_{V} \rightarrow \begin{bmatrix} 1 & 1 & 2 & \vdots & 4 & 0 & 0 \\ 0 & 1 & 1 & \vdots & 2 & 1 & 1 \\ 0 & 1 & 2 & \vdots & 3 & 1 & 2 \end{bmatrix} \begin{matrix} \\ r_2 \leftarrow r_2 - r_1 \\ r_3 \leftarrow r_3 - r_1 \end{matrix} \qquad \text{(from top to bottom)}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 2 & \vdots & 4 & 0 & 0 \\ 0 & 1 & 1 & \vdots & 2 & 1 & 1 \\ 0 & 0 & 1 & \vdots & 1 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ r_3 \leftarrow r_3 - r_2 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 & \vdots & 2 & 0 & -2 \\ 0 & 1 & 0 & \vdots & 1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 1 & 0 & 1 \end{bmatrix} \begin{matrix} r_1 \leftarrow r_1 - 2r_3 \\ r_2 \leftarrow r_2 - r_3 \\ \\ \end{matrix} \qquad \text{(from bottom back to top)}$$

$$\rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{W^{-1}W=I} \vdots \underbrace{\begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{W^{-1}V=S_1} \begin{matrix} r_1 \leftarrow r_1 - r_2 \\ \\ \\ \end{matrix} \implies S_1 = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(b) Similarly, $[w_1 \quad w_2 \quad w_3] = [v_1 \quad v_2 \quad v_3]S_2 \implies S_2 = [v_1 \quad v_2 \quad v_3]^{-1}[w_1 \quad w_2 \quad w_3]$, thus with row reduction we have

$$\underbrace{\begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix}}_{V} \vdots \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}}_{W} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 6 & 1 & 1 & \vdots & 1 & 2 & 3 \\ 7 & 1 & 2 & \vdots & 1 & 2 & 4 \end{bmatrix} \begin{matrix} r_1 \leftarrow \frac{1}{4}r_1 \\ \\ \\ \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & 1 & \vdots & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 2 & \vdots & -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{matrix} \\ r_2 \leftarrow r_2 - 6r_1 \\ r_3 \leftarrow r_3 - 7r_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & 1 & \vdots & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \vdots & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{matrix} \\ \\ r_3 \leftarrow r_3 - r_2 \end{matrix}$$

$$\rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{V^{-1}V=I} \vdots \underbrace{\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}}_{V^{-1}W=S_2} \begin{matrix} \\ r_2 \leftarrow r_2 - r_3 \\ \\ \end{matrix} \implies S_2 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

(c) (Routine computation.)

(d) To compute $[v]_F$, consider the following equation:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} [v]_F = v = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 9 \end{bmatrix}.$$

Row reduce the augmented matrix and we find

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 1 & 2 & 3 & 11 \\ 1 & 2 & 4 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix} \implies [v]_F = \begin{bmatrix} 7 \\ 5 \\ -2 \end{bmatrix}.$$

We can also find that

$$[v]_F = S_1[v]_E = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ -2 \end{bmatrix},$$

completing the verification.

Problem 7

Let $A \in \mathbb{R}^{4 \times 5}$ be

$$A = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 2 & 0 & -2 \\ 0 & 1 & 3 & 1 & 4 \\ 1 & 2 & 13 & 5 & 5 \end{bmatrix}.$$

(a) Find the four subspaces of the matrix $C(A)$, $C(A^{\mathrm{T}})$, $N(A)$, and $N(A^{\mathrm{T}})$. Determine their bases and dimensions.

(b) Write down the fundamental theorem of linear algebra: Part 1 and Part 2. And verify them by the answers in (a).

✔ **Solution:** (a) Transform A to the reduced row echelon form:

$$A \to \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 & 4 \\ 0 & 4 & 12 & 4 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & 3 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So the basis of the column space of A consists of the 1$^{\text{st}}$, 2$^{\text{nd}}$ and 5$^{\text{th}}$ column of A:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \\ 5 \end{bmatrix},$$

and $\dim C(A) = 3$.

Consider the equation $Ax = 0$, where $x = (x_1, x_2, x_3, x_4, x_5)^{\mathrm{T}} \in \mathbb{R}^5$. Then the free variables are x_3 and x_4. Assign one free variable to 1 and others to 0, we obtain the basis of the nullspace of A:

$$\begin{bmatrix} -7 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

and $\dim N(A) = 2$.

According to the basis of $C(A)$ and the reduced row echelon form, we have the CR factorization of A:

$$A = CR = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 3 & -2 \\ 0 & 1 & 4 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 7 & 3 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and the basis of the row space of A corresponds to the three rows of R:

$$\begin{bmatrix} 1 \\ 0 \\ 7 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and $\dim C(A^{\mathrm{T}}) = 3$.

To find the basis of $N(A^{\mathrm{T}})$, we transform A^{T} to the reduced row echelon form:

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ -2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 13 \\ 1 & 0 & 1 & 5 \\ 2 & -2 & 4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 3 & 3 & 12 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{17}{4} \\ 0 & 1 & 0 & \frac{13}{4} \\ 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Consider the equation $A^{\mathrm{T}} x = 0$, where $x = (x_1, x_2, x_3, x_4)^{\mathrm{T}} \in \mathbb{R}^4$. Then the free variable is x_4. Assign this free variable to 1, we obtain the basis of the nullspace of A^{T}:

$$\begin{bmatrix} -\frac{17}{4} \\ -\frac{13}{4} \\ -\frac{3}{4} \\ 1 \end{bmatrix},$$

and $\dim N(A^{\mathrm{T}}) = 1$.

❗ **Note 7.1:** *To obtain the CR factorization, you need to row reduce A to the **reduced** row echelon form, not just general row echelon form. The row reduction procedure can be expressed as*

$$EA = R_0 \implies A = E^{-1} R_0 = \begin{bmatrix} C & * \end{bmatrix} \underbrace{\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}}_{R_0} P,$$

where E denotes the elementary matrix, R_0 denotes the reduced row echelon form, and P is the permutation matrix which puts the columns of $I_{r \times r}$ into their correct positions. Delete the zero rows from R_0 and corresponding columns from E^{-1}, and we obtain

$$A = C \underbrace{\begin{bmatrix} I & F \end{bmatrix}}_{R} P = \begin{bmatrix} C & CF \end{bmatrix} P,$$

where C and CF possess respectively the linear independent and dependent columns of A.

(b) **Fundamental Theorem of Linear Algebra: Part 1**

$\dim C(A) = \dim C(A^{\mathrm{T}}) = r, \dim N(A) = n - r, \dim N(A^{\mathrm{T}}) = m - r$

Fundamental Theorem of Linear Algebra: Part 2

$N(A) = C(A^{\mathrm{T}})^{\perp}, N(A^{\mathrm{T}}) = C(A)^{\perp}$

Verification of part 1:

$\dim C(A) = \dim C(A^{\mathrm{T}}) = \mathbf{3} = \mathbf{r}, \dim N(A) = \mathbf{2} = \mathbf{n} - \mathbf{r}, \dim N(A^{\mathrm{T}}) = \mathbf{1} = \mathbf{m} - \mathbf{r};$

Verification of part 2: (The routine calculations are omitted here. You can do the verification either by showing the corresponding bases are orthogonal or writing the general expression of vectors in corresponding subspaces and then prove their orthogonality.)

Problem 8

Let $A \in \mathbb{R}^{4 \times 5}$ and let R be the reduced row echelon form of A. If the first and fourth columns of A are

$$
a_1 = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \quad a_4 = \begin{bmatrix} -3 \\ -3 \\ -1 \\ -5 \end{bmatrix}
$$

and

$$
R = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},
$$

(a) find a basis for $N(A)$.

(b) given that x_0 is a solution to $Ax = b$, where

$$
b = \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \\ 0 \end{bmatrix},
$$

determine the remaining column vectors of A.

✔ **Solution:** (a) Consider the equation $Ax = 0$, where $x = (x_1, x_2, x_3, x_4, x_5)^{\mathrm{T}} \in \mathbb{R}^5$. Then the free variables are x_3 and x_5. Assign one free variable to 1 and others to 0, and we know that there are 2 vectors in the basis for $N(A)$, which are

$$
\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}.
$$

(b) *view the matrix equation as a linear combination of columns!*

From $Ax_0 = b$ we have

$$0a_1 + 3a_2 + 1a_3 + (-1)a_4 + 0a_5 = b \implies a_3 = -3a_2 + a_4 + b \quad \text{①}$$

From the basis obtained in (a), we know that

$$1a_1 + 2a_2 + 1a_3 + 0a_4 + 0a_5 = 0 \implies a_3 = -a_1 - 2a_2 \quad \text{②}$$
$$(-1)a_1 + 0a_2 + 0a_3 + (-2)a_4 + 1a_5 = 0 \implies a_5 = a_1 + 2a_4 \quad \text{③}$$

Substitute ② into ① and we find $a_2 = a_1 + a_4 + b$, and then with ① and ③ we can obtain a_3 and a_5. The results are shown below.

$$a_2 = \begin{bmatrix} -1 \\ -5 \\ -3 \\ -4 \end{bmatrix}, a_3 = \begin{bmatrix} -2 \\ 9 \\ 8 \\ 8 \end{bmatrix}, a_5 = \begin{bmatrix} -2 \\ -5 \\ -4 \\ -10 \end{bmatrix}.$$

Problem 9

If P is the plane of vectors in $\mathbb{R}^4$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$, write a basis for $P^\perp$. Construct a matrix that has P as its nullspace.

✔ **Solution (1):** For any vector $x = (x_1, x_2, x_3, x_4)^\mathrm{T} \in P$, we have $x_1 + x_2 + x_3 + x_4 = 0$, that is,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} x = 0.$$

Let A denote the 1 by 4 matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ and let v denote the vector $(1, 1, 1, 1)^\mathrm{T}$. Then we have $P = N(A)$ and $C(A^\mathrm{T}) = \mathrm{span}(v)$. Since v is not a zero vector, it is linear independent and thus a basis of $C(A^\mathrm{T})$. From the fundamental theorem of linear algebra (Part II), we have $C(A^\mathrm{T}) = (N(A))^\perp = P^\perp$. Hence $v = (1, 1, 1, 1)^\mathrm{T}$ is a basis of $P^\perp$. Since $P = N(A)$, the matrix A has P as its nullspace, which is required to construct.

✔ **Solution (2):** First we describe P. The equation $x_1 + x_2 + x_3 + x_4 = 0$ can be represented as

$$x_1 = -x_2 - x_3 - x_4$$
$$x_2 = x_2$$
$$x_3 = x_3$$
$$x_4 = x_4$$

Hence a basis of P is

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

Let

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $P = C(A)$. From the fundamental theorem of linear algebra (Part II), we have $P^{\perp} = N(A^{\mathrm{T}})$. Solve $A^{\mathrm{T}}x = 0$ and we find that a basis of $P^{\perp}$ is $(1, 1, 1, 1)^{\mathrm{T}}$.

Next we construct a matrix B such that P is its null space. Since $P = N(B)$, we have $P^{\perp} = (N(B))^{\perp} = C(B^{\mathrm{T}})$. Then B can be the 1 by 4 matrix $[1 \quad 1 \quad 1 \quad 1]$, or any rank-1 matrix whose rows are nonzero scalar multiples of $[1 \quad 1 \quad 1 \quad 1]$.

Problem 10

Let x and y be linearly independent vectors in $\mathbb{R}^n$ and let $S = \mathrm{Span}(x, y)$. We can use x and y to define a matrix A be setting

$$A = xy^{\mathrm{T}} + yx^{\mathrm{T}}$$

(a) Show that $N(A) = S^{\perp}$.
(b) Show that $\dim C(A) = 2$ (the rank of A must be 2).

✔ **Proof:** (a) Let $x_1 \in N(A)$, then $0 = Ax_1 = (xy^{\mathrm{T}} + yx^{\mathrm{T}})x_1 = x(y^{\mathrm{T}}x_1) + y(x^{\mathrm{T}}x_1)$. Since x, y are linearly independent, $y^{\mathrm{T}}x_1 = x^{\mathrm{T}}x_1 = 0$. Let $s \in S$ and express it as $s = ax + by, a, b \in \mathbb{F}$, then we have $s^{\mathrm{T}}x_1 = ax^{\mathrm{T}}x_1 + by^{\mathrm{T}}x_1 \equiv 0$, which implies that $x_1 \in S^{\perp}$, completing the proof for $N(A) \subset S^{\perp}$.

On the other hand, let $z \in S^{\perp}$, then $x^{\mathrm{T}}z = 0, y^{\mathrm{T}}z = 0$. Thus $Az = (xy^{\mathrm{T}} + yx^{\mathrm{T}})z = xy^{\mathrm{T}}z + yx^{\mathrm{T}}z = 0$, which implies that $z \in N(A)$, completing the proof for $S^{\perp} \subset N(A)$. Therefore, $N(A) = S^{\perp}$.

(b) We have

$$\begin{aligned} N(A) = S^{\perp} &\implies N(A)^{\perp} = (S^{\perp})^{\perp} = S \\ N(A) = C(A^{\mathrm{T}})^{\perp} &\implies N(A)^{\perp} = (C(A^{\mathrm{T}})^{\perp})^{\perp} = C(A^{\mathrm{T}}) \end{aligned} \implies S = C(A^{\mathrm{T}}).$$

Since x, y are linearly independent and span S, $\dim S = 2$. Therefore, $\dim C(A) = \dim C(A^{\mathrm{T}}) = 2$. ∎