Solutions & Notes of Homework 5

Yang He, Junda Wu

June 17, 2025

Tanquam ex ungue leonem.
One knows the lion by his claw.

—Johann Bernoulli

Problem 1

Suppose $e_1, ..., e_m$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

if and only if $v \in \text{span}(e_1, ..., e_m)$.

Proof:

Since $e_1, ..., e_m$ is an orthonormal list of vectors in V, it is linearly independent, thus it is an orthonormal basis of $\text{span}(e_1, ..., e_m)$.

If $v \in \text{span}\,(e_1, ..., e_m)$, we have $v = a_1 e_1 + \cdots + a_m e_m$, where $a_1, \ldots, a_m$ are scalars. Take inner product with e_k and we find that $a_k = \langle v, e_k\rangle$. Then

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m.$$

Thus,

$$\begin{aligned}
\|v\|^2 &= \langle \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m, \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m\rangle \\
&= \|\langle v, e_1\rangle e_1\|^2 + \cdots + \|\langle v, e_m\rangle e_m\|^2 \\
&= |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2 .
\end{aligned}$$

Conversely, if $\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$, we denote $u = v - (\langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m)$. Then we have

$$\langle u, e_i\rangle = \langle v, e_i\rangle - \langle v, e_i\rangle = 0, \quad i = 1, \ldots, m.$$

Thus, $u \perp \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m$. Since $v = u + (\langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m)$, it follows from Pythagorean theorem that

$$\|v\|^2 = \|u\|^2 + \|\langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m\|^2 .$$

Thus,

$$\begin{aligned}\|u\|^2 &= \|(\langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m)\|^2 - \|v\|^2 \\ &= \left(|\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2\right) - \left(|\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2\right) \\ &= 0,\end{aligned}$$

which implies $u = 0$. which implies that,

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m.$$

Hence, $v \in \text{span}\,(e_1, ..., e_m)$. ■

Problem 2

Suppose n is a positive integer. Prove that

$$\frac{1}{\sqrt{2}}, \cos x, \cos 2x, ..., \cos nx, \sin x, \sin 2x, ..., \sin nx$$

is an orthonormal list of vectors in $C\,[-\pi, \pi]$, the vector space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx.$$

Proof: First we show that the norm of every vector in the list is 1.

$$\left\|\frac{1}{\sqrt{2}}\right\| = \left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}\,dx\right)^{1/2} = \left(\frac{1}{\pi}\cdot\pi\right)^{1/2} = 1;$$

$$\|\cos kx\| = \left(\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 kx\,dx\right)^{1/2} = \left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos 2kx + 1}{2}\,dx\right)^{1/2} = 1 \quad (k \in \mathbb{N}^*).$$

$$\|\sin kx\| = \left(\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2 kx\,dx\right)^{1/2} = \left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1 - \cos 2kx}{2}\,dx\right)^{1/2} = 1 \quad (k \in \mathbb{N}^*).$$

$$\left(\text{using } \int_{-\pi}^{\pi}\frac{\cos 2kx}{2}\,dx = 0 \text{ and } \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}\,dx = \frac{1}{\pi}\cdot\pi = 1\right)$$

Next we show that every two different vectors in the list are orthogonal to each other.

For any pair of positive integers $i, j (i \neq j)$, we have

$$\begin{aligned}\langle \cos ix, \cos jx\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos ix \cdot \cos jx \mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\cos{(i+j)x} + \cos{(i-j)x}\right]\mathrm{d}x \\ &= \frac{1}{2\pi}\left[\frac{\sin{(i+j)x}}{i+j} + \frac{\sin{(i-j)x}}{i-j}\right]\Bigg|_{-\pi}^{\pi} \\ &= 0.\end{aligned}$$

Similarly, $\langle \sin ix, \sin jx\rangle = \langle \cos ix, \sin jx\rangle = 0$.

For any positive integer k, we have

$$\left\langle \frac{1}{\sqrt{2}}, \cos kx \right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \cdot \cos kx\mathrm{d}x = \frac{1}{\sqrt{2}\pi}\int_{-\pi}^{\pi} \cos kx\mathrm{d}x = \frac{1}{\sqrt{2}\pi}\left[\frac{1}{k}\sin kx\right]\Bigg|_{-\pi}^{\pi} = 0,$$

$$\left\langle \frac{1}{\sqrt{2}}, \sin kx \right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \cdot \sin kx\mathrm{d}x = \frac{1}{\sqrt{2}\pi}\int_{-\pi}^{\pi} \sin kx\mathrm{d}x = \frac{1}{\sqrt{2}\pi}\left[-\frac{1}{k}\cos kx\right]\Bigg|_{-\pi}^{\pi} = 0,$$

and

$$\langle \cos kx, \sin kx \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin kx \cos kx\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin 2kx\mathrm{d}x = \frac{1}{2\pi}\left[-\frac{1}{2k}\cos 2kx\right]\Bigg|_{-\pi}^{\pi} = 0.$$

Thus, every two different vectors in the list are orthogonal to each other. Hence, the list is an orthonormal list. ■

Problem 3

On $\mathcal{P}_2(\mathbb{R})$, consider the inner product given by

$$\langle p, q \rangle = \int_0^1 p(x)\, q(x)\, dx$$

Apply the Gram-Schmidt Procedure to the basis 1, x, x^2 to produce an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

✔ **Solution:** Let $\alpha_1 = 1$. Then

$$\|\alpha_1\| = \sqrt{\int_0^1 1\, dx} = 1,$$

so the first vector of the orthonormal basis is $\beta_1 = 1$.

Now, let $\alpha_2 = x$ and compute:

$$\beta_2 = \frac{\alpha_2 - \langle\alpha_2, \beta_1\rangle\beta_1}{\|\alpha_2 - \langle\alpha_2, \beta_1\rangle\beta_1\|}, \quad \langle\alpha_2, \beta_1\rangle = \int_0^1 x\, dx = \frac{1}{2}, \quad \alpha_2 - \langle\alpha_2, \beta_1\rangle\beta_1 = x - \frac{1}{2},$$

$$\left\|x - \frac{1}{2}\right\| = \sqrt{\int_0^1 \left(x - \frac{1}{2}\right)^2 dx} = \sqrt{\int_0^1 \left(x^2 - x + \frac{1}{4}\right) dx} = \sqrt{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}} = \frac{1}{2\sqrt{3}},$$

$$\Rightarrow \beta_2 = 2\sqrt{3}x - \sqrt{3}.$$

Let $\alpha_3 = x^2$. Then,

$$\beta_3 = \frac{\alpha_3 - \langle\alpha_3, \beta_1\rangle\beta_1 - \langle\alpha_3, \beta_2\rangle\beta_2}{\|\alpha_3 - \langle\alpha_3, \beta_1\rangle\beta_1 - \langle\alpha_3, \beta_2\rangle\beta_2\|}.$$

We compute:

$$\langle\alpha_3, \beta_1\rangle = \int_0^1 x^2\, dx = \frac{1}{3}, \quad \langle\alpha_3, \beta_2\rangle = \int_0^1 (2\sqrt{3}x^3 - \sqrt{3}x^2)\, dx = \frac{\sqrt{3}}{6},$$

$$\Rightarrow \alpha_3 - \langle\alpha_3, \beta_1\rangle\beta_1 - \langle\alpha_3, \beta_2\rangle\beta_2 = x^2 - x + \frac{1}{6}.$$

Now compute the norm:

$$
\begin{aligned}
\left\|x^2 - x + \frac{1}{6}\right\| &= \sqrt{\int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx} \\
&= \sqrt{\int_0^1 \left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}\right) dx} \\
&= \sqrt{\frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}} = \frac{1}{6\sqrt{5}}, \\
&\Rightarrow \beta_3 = 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}.
\end{aligned}
$$

Therefore, the orthonormal basis consists of:

$$1, \quad 2\sqrt{3}x - \sqrt{3}, \quad 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}.$$

Problem 4

For each of the following, use the Gram-Schmidt process find an orthonormal basis for $R(A)$:

$$1. A = \begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix}$$

$$2. A = \begin{bmatrix} 2 & 5 \\ 1 & 10 \end{bmatrix}$$

where $R(A)$ is the linear space spanned by the columns of A.

Solution:

1. To get started, we have $\left\|\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\| = \sqrt{2}$, and hence $e_1 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

   Now the numerator in the expression for e_2 is

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} - \left\langle \begin{pmatrix} 3 \\ 5 \end{pmatrix}, e_1 \right\rangle e_1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \left(3 \cdot \frac{-1}{\sqrt{2}} + 5 \cdot \frac{1}{\sqrt{2}}\right) \cdot \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

   We have $\left\|\begin{pmatrix} 4 \\ 4 \end{pmatrix}\right\| = 4\sqrt{2}$, and hence $e_2 = \frac{1}{4\sqrt{2}} \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

   Thus

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

   is an orthonormal list of length 2 in $R(A)$, and hence an orthonormal basis of $R(A)$.

2. To get started, we have $\left\|\begin{pmatrix}2\\1\end{pmatrix}\right\| = \sqrt{5}$, and hence $e_1 = \begin{pmatrix}\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{pmatrix}$.

   Now the numerator in the expression for e_2 is

$$\begin{pmatrix}5\\10\end{pmatrix} - \left\langle \begin{pmatrix}5\\10\end{pmatrix}, e_1 \right\rangle e_1 = \begin{pmatrix}5\\10\end{pmatrix} - \left(5 \cdot \frac{2}{\sqrt{5}} + 10 \cdot \frac{1}{\sqrt{5}}\right) \cdot \begin{pmatrix}\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{pmatrix} = \begin{pmatrix}-3\\6\end{pmatrix}.$$

   We have $\left\|\begin{pmatrix}-3\\6\end{pmatrix}\right\| = 3\sqrt{5}$, and hence $e_2 = \frac{1}{3\sqrt{5}}\begin{pmatrix}-3\\6\end{pmatrix} = \begin{pmatrix}-\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{pmatrix}$.

   Thus

$$\begin{pmatrix}\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{pmatrix}, \begin{pmatrix}-\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{pmatrix}$$

   is an orthonormal list of length 2 in $R(A)$, and hence an orthonormal basis of $R(A)$.

Problem 5

Given $\mathbf{x}_1 = \frac{1}{2}(1,1,1,-1)^T$ and $\mathbf{x}_2 = \frac{1}{6}(1,1,3,5)^T$, verify that these vectors form an orthonormal set in $\mathbb{R}^4$. Extend this set to an orthonormal basis for $\mathbb{R}^4$ by finding an orthonormal basis for the null space of

$$\begin{bmatrix}1 & 1 & 1 & -1\\ 1 & 1 & 3 & 5\end{bmatrix}$$

[*Hint*: First find a basis for the null space and then use the Gram-Schmidt process.]

Solution:

We have $\|\mathbf{x}_1\| = \sqrt{\frac{1}{4}\left(1^2+1^2+1^2(-1)^2\right)} = 1$, $\|\mathbf{x}_2\| = \sqrt{\frac{1}{36}(1^2+1^2+3^2+5^2)} = 1$, and

$$\langle \mathbf{x}_1, \mathbf{x}_2\rangle = \frac{1}{2}\cdot\frac{1}{6} + \frac{1}{2}\cdot\frac{1}{6} + \frac{1}{2}\cdot\frac{3}{6} + \frac{-1}{2}\cdot\frac{5}{6} = 0.$$

Thus, these vectors form an orthonormal set in $\mathbb{R}^4$.

We have

$$\begin{bmatrix}1 & 1 & 1 & -1\\ 1 & 1 & 3 & 5\end{bmatrix} \to \begin{bmatrix}1 & 1 & 1 & -1\\ 0 & 0 & 2 & 6\end{bmatrix} \to \begin{bmatrix}1 & 1 & 0 & -4\\ 0 & 0 & 1 & 3\end{bmatrix}.$$

Thus, we have a basis of the null space:

$$\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}4\\0\\-3\\1\end{pmatrix}.$$

From the Fundamental Theorem of Linear Algebra, the null space is orthogonal to $\mathrm{span}(\mathbf{x}_1, \mathbf{x}_2)$. So we only need to apply the Gram-Schmidt Procedure to the basis of the null space.

To get started, we have $\left\| \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\| = \sqrt{2}$, and hence $\mathbf{x}_3 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$.

Now the numerator in the expression for $\mathbf{x}_4$ is

$$\begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix} - \left\langle \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix}, \mathbf{x}_3 \right\rangle \mathbf{x}_3 = \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix} - (-2\sqrt{2}) \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -3 \\ 1 \end{pmatrix}.$$

We have $\left\| \begin{pmatrix} 2 \\ 2 \\ -3 \\ 1 \end{pmatrix} \right\| = 3\sqrt{2}$, and hence $\mathbf{x}_4 = \left(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{6} \right)^T$.

Thus $\mathbf{x}_3, \mathbf{x}_4$ is an orthonormal basis of the null space.

Hence, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$, *i.e.*,

$$\frac{1}{2}(1,1,1,-1)^T, \frac{1}{6}(1,1,3,5)^T, \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right)^T, \left(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{6} \right)^T$$

is an orthonormal basis of $\mathbb{R}^4$.

Problem 6

Find a polynomial $q \in \mathcal{P}_2(\mathbb{R})$ such that

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x)\, q(x)\, dx$$

for every $p \in \mathcal{P}_2(\mathbb{R})$.

Solution1: *We recommend that you use this solution.*

Let $\varphi(p) = p\left(\frac{1}{2}\right)$. From the proof of the Riesz representation theorem, using the orthonormal basis from Problem 3, we have

$$\begin{aligned} q(x) &= 1 \cdot 1 + \left(2\sqrt{3} \cdot \frac{1}{2} - \sqrt{3}\right) \cdot \left(2\sqrt{3}x - \sqrt{3}\right) \\ &\quad + \left(6\sqrt{5} \cdot \left(\frac{1}{2}\right)^2 - 6\sqrt{5} \cdot \frac{1}{2} + \sqrt{5}\right)\left(6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}\right) \\ &= 1 + 0 - \frac{\sqrt{5}}{2}\left(6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}\right) \\ &= -15x^2 + 15x - \frac{3}{2}. \end{aligned}$$

Solution2:

Suppose $q(x) = a_0 + a_1x + a_2x^2$ (a_0, a_1, a_2 are constants), and $p(x) = k_0 + k_1x + k_2x^2,\ k_0, k_1, k_2 \in \mathbb{R}$. Thus, $p\left(\frac{1}{2}\right) = k_0 + \frac{1}{2}k_1 + \frac{1}{4}k_2$.

We have

$$\begin{aligned}
&\int_0^1 p(x)\,q(x)\,\mathrm{d}x = \int_0^1 \left[\left(k_0 + k_1x + k_2x^2\right)\cdot\left(a_0 + a_1x + a_2x^2\right)\right]\mathrm{d}x \\
&= \int_0^1 \left[a_0k_0 + (a_1k_0 + a_0k_1)x + (a_2k_0 + a_1k_1 + a_0k_2)x^2 + (a_2k_1 + a_1k_2)x^3 + a_2k_2x^4\right]\mathrm{d}x \\
&= \left[a_0k_0x + \frac{1}{2}(a_1k_0 + a_0k_1)x^2 + \frac{1}{3}(a_2k_0 + a_1k_1 + a_0k_2)x^3 + \frac{1}{4}(a_2k_1 + a_1k_2)x^4 + \frac{1}{5}a_2k_2x^5\right]_0^1 \\
&= (a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2)k_0 + (\frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2)k_1 + (\frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2)k_2 \\
&= k_0 + \frac{1}{2}k_1 + \frac{1}{4}k_2, \quad \text{for any } k_0, k_1, k_2 \in \mathbb{R}.
\end{aligned}$$

Compare the corresponding coefficients and we have

$$\begin{cases} a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 = 1 \\ \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 = \frac{1}{2} \\ \frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2 = \frac{1}{4} \end{cases} \Longrightarrow \begin{cases} a_0 = -\frac{3}{2} \\ a_1 = 15 \\ a_2 = -15 \end{cases}$$

Hence, $q(x) = -\dfrac{3}{2} + 15x - 15x^2$.

Problem 7

Find a polynomial $q \in \mathcal{P}_2(\mathbb{R})$ such that

$$\int_0^1 p(x)(\cos \pi x)\,dx = \int_0^1 p(x)\,q(x)\,dx$$

for every $p \in \mathcal{P}_2(\mathbb{R})$.

✔ **Solution1:** *We recommend that you use this solution.*

Let $\varphi(p) = \int_0^1 p(x)(\cos \pi x)\mathrm{d}x$. Using the orthonormal basis from Problem 3, we have

$$\begin{aligned}
q(x) = &\left(\int_0^1 1(\cos \pi x)\mathrm{d}x\right)\cdot 1 + \left(\int_0^1 \left(2\sqrt{3}x - \sqrt{3}\right)(\cos \pi x)\mathrm{d}x\right)\cdot\left(2\sqrt{3}x - \sqrt{3}\right) \\
&+ \left(\int_0^1 \left(6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}\right)(\cos \pi x)\mathrm{d}x\right)\cdot\left(6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}\right),
\end{aligned}$$

where

$$\int_0^1 x\cos \pi x\mathrm{d}x = \frac{1}{\pi}\int_0^1 x\mathrm{d}(\sin \pi x) = \frac{1}{\pi}\left(x\sin \pi x\Big|_0^1 - \int_0^1 \sin \pi x\mathrm{d}x\right) = -\frac{2}{\pi^2}$$

and

$$\int_0^1 x^2 \cos \pi x \mathrm{d}x = \frac{1}{\pi} \int_0^1 x^2 \mathrm{d}(\sin \pi x) = \frac{1}{\pi} \left(x^2 \sin \pi x|_0^1 - 2 \int_0^1 x \sin \pi x \mathrm{d}x \right)$$
$$= \frac{2}{\pi^2} \left(x \cos \pi x|_0^1 - \int_0^1 \cos \pi x \mathrm{d}x \right) = -\frac{2}{\pi^2}$$

Thus, we have

$$q(x) = -\frac{4\sqrt{3}}{\pi^2}(2\sqrt{3}x - \sqrt{3}) = -\frac{12}{\pi^2}(2x - 1).$$

Solution2:

Suppose $q(x) = a_0 + a_1 + a_2x^2$ (a_0, a_1, a_2 are constants), and $p(x) = k_0 + k_1x + k_2x^2, k_0, k_1, k_2 \in \mathbb{R}$. Thus, we have

$$\int_0^1 p(x)(\cos \pi x)\,\mathrm{d}x = \int_0^1 \left(k_0 + k_1x + k_2x^2\right)(\cos \pi x)\,\mathrm{d}x$$
$$= k_0 \int_0^1 \cos \pi x \mathrm{d}x + k_1 \int_0^1 x \cos \pi x \mathrm{d}x + k_2 \int_0^1 x^2 \cos \pi x \mathrm{d}x$$
$$= 0 \cdot k_0 + \left(-\frac{2}{\pi^2}\right) \cdot k_1 + \left(-\frac{2}{\pi^2}\right) \cdot k_2.$$

From **Solution2** of Problem 6, for any $k_0, k_1, k_2 \in \mathbb{R}$, we have

$$\int_0^1 p(x)\,q(x)\,\mathrm{d}x = (a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2)k_0 + (\frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2)k_1 + (\frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2)k_2.$$

Compare the corresponding coefficients and we have

$$\begin{cases} a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 = 0 \\ \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 = -\frac{2}{\pi^2} \\ \frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2 = -\frac{2}{\pi^2} \end{cases} \Longrightarrow \begin{cases} a_0 = \frac{12}{\pi^2} \\ a_1 = -\frac{24}{\pi^2} \\ a_2 = 0 \end{cases}$$

Hence, $q(x) = \dfrac{12}{\pi^2} - \dfrac{24}{\pi^2}x$.

Problem 8

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$$

(a) Use the Gram-Schmidt process to find an orthonormal basis for the column space of A.

(b) Factor A into a product QR, where Q has an orthonormal set of column vectors and R is upper triangular.

(c) Solve the least squares problem $A\mathbf{x} = \mathbf{b}$.

Solution: (a) To get started, we have $\left\| \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\| = 3$, hence $e_1 = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)^T$.

Now the numerator in the expression for e_2 is

$$\begin{bmatrix}1\\1\\1\end{bmatrix} - \left\langle \begin{bmatrix}1\\1\\1\end{bmatrix}, e_1 \right\rangle e_1 = \left(-\frac{1}{9}, \frac{4}{9}, -\frac{1}{9}\right)^T.$$

We have $\left\| \left(-\frac{1}{9}, \frac{4}{9}, -\frac{1}{9}\right)^T \right\| = \frac{\sqrt{2}}{3}$, and hence $e_2 = \left(-\frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}, -\frac{\sqrt{2}}{6}\right)^T$.

Thus

$$\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)^T, \left(-\frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}, -\frac{\sqrt{2}}{6}\right)^T$$

is an orthonormal basis of the column space of A.

(b) From (a), we have

$$Q = \begin{bmatrix} \frac{2}{3} & -\frac{\sqrt{2}}{6} \\ \frac{1}{3} & \frac{2\sqrt{2}}{3} \\ \frac{2}{3} & -\frac{\sqrt{2}}{6} \end{bmatrix}.$$

Since e_1, e_2 is an orthonormal set of column vectors,

$$R = \begin{bmatrix} \left\langle \begin{bmatrix}2\\1\\2\end{bmatrix}, e_1 \right\rangle & \left\langle \begin{bmatrix}1\\1\\1\end{bmatrix}, e_1 \right\rangle \\ \left\langle \begin{bmatrix}2\\1\\2\end{bmatrix}, e_2 \right\rangle & \left\langle \begin{bmatrix}1\\1\\1\end{bmatrix}, e_2 \right\rangle \end{bmatrix} = \begin{bmatrix} 3 & \frac{5}{3} \\ 0 & \frac{\sqrt{2}}{3} \end{bmatrix}.$$

Thus, $A = QR$.

(c) The unique least squares solution is

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b} = \left(\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 5 \\ 5 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -\frac{5}{2} & \frac{9}{2} \end{bmatrix} \begin{bmatrix} 66 \\ 36 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}.$$

Problem 9

Suppose $v_1, ..., v_m \in V$. Prove that

$$\{v_1, ..., v_m\}^\perp = (\text{span}\,(v_1, ..., v_m))^\perp$$

✔ **Proof:** Let $u \in \{v_1, ..., v_m\}^\perp$, then we have

$$u \perp v_i, \quad i = 1, \ldots, m,$$

which implies that $\langle v_1, u\rangle = \cdots = \langle v_m, u\rangle = 0$. Thus, for any $v \in \text{span}(v_1, ..., v_m)$, which can be written as $v = a_1 v_1 + \cdots + a_m v_m$, where $a_1, \ldots, a_m \in \mathbb{F}$, we have

$$\begin{aligned}\langle v, u\rangle &= \langle a_1 v_1 + \cdots + a_m v_m, u\rangle \\ &= a_1 \langle v_1, u\rangle + \cdots + a_m \langle v_m, u\rangle \\ &= a_1 \cdot 0 + \cdots + a_m \cdot 0 \\ &= 0,\end{aligned}$$

which implies that $u \in (\text{span}(v_1, ..., v_m))^{\perp}$. Hence, $\{v_1, ..., v_m\}^{\perp} \subseteq (\text{span}(v_1, ..., v_m))^{\perp}$.

Conversely, if $u \in (\text{span}(v_1, ..., v_m))^{\perp}$, since $v_i \in \text{span}(v_1, ..., v_m)$, $i = 1, \ldots, m$, we have

$$u \perp v_i, \quad i = 1, \ldots, m.$$

Thus, $u \in \{v_1, ..., v_m\}^{\perp}$. Hence, $(\text{span}(v_1, ..., v_m))^{\perp} \subseteq \{v_1, ..., v_m\}^{\perp}$.
Hence,

$$\{v_1, ..., v_m\}^{\perp} = (\text{span}(v_1, ..., v_m))^{\perp}. \qquad \blacksquare$$

Problem 10

Suppose U is the subspace of $\mathbb{R}^4$ defined by

$$U = \text{span}((1, 2, 3, -4), (-5, 4, 3, 2)).$$

Find an orthonormal basis of U and an orthonormal basis of $U^{\perp}$.

✔ **Solution:** To get started, we have $\alpha_1 = (1, 2, 3, -4)$, $\alpha_2 = ((-5, 4, 3, 2))$. Then we get

$$\beta_1 = \frac{1}{\|\alpha_1\|}\alpha_1 = \frac{1}{\sqrt{30}}\alpha_1 = \left(\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{3}{\sqrt{30}}, -\frac{4}{\sqrt{30}}\right).$$

To get β_2, we compute it step-by-step:

$$\begin{aligned}\langle \alpha_2, \beta_1\rangle = \frac{4}{\sqrt{30}}, \langle \alpha_2, \beta_1\rangle \beta_1 &= \left(\frac{2}{15}, \frac{4}{15}, \frac{2}{5}, -\frac{8}{15}\right), \\ \alpha_2 - \langle \alpha_2, \beta_1\rangle \beta_1 &= \left(-\frac{77}{15}, \frac{56}{15}, \frac{13}{5}, \frac{38}{15}\right), \\ \beta_2 = \frac{\alpha_2 - \langle \alpha_2, \beta_1\rangle\beta_1}{\|\alpha_2 - \langle \alpha_2, \beta_1\rangle\beta_1\|} &= \frac{15}{\sqrt{12030}}\left(-\frac{77}{15}, \frac{56}{15}, \frac{13}{5}, \frac{38}{15}\right).\end{aligned}$$

Now we get an orthonormal basis of U:

$$\beta_1 = \frac{1}{\sqrt{30}}(1, 2, 3, -4), \beta_2 = \frac{1}{\sqrt{12030}}(-77, 56, 39, 38).$$

To find an orthonormal basis of $U^{\perp}$, we let $A = \begin{pmatrix} 1 & 2 & 3 & -4 \\ -5 & 4 & 3 & 2 \end{pmatrix}$. Solve $Ax = 0$, and then we can get two linearly independent vectors that span the null space of A, which is also $U^{\perp}$. Such two vectors form a basis of $U^{\perp}$.

We have

$$\begin{pmatrix} 1 & 2 & 3 & -4 \\ -5 & 4 & 3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & -4 \\ 0 & 14 & 18 & -18 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & -4 \\ 0 & 1 & \frac{9}{7} & -\frac{9}{7} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{3}{7} & -\frac{10}{7} \\ 0 & 1 & \frac{9}{7} & -\frac{9}{7} \end{pmatrix}$$

Thus, we have a basis of $U^{\perp}$:

$$x_1 = \left(-\frac{3}{7}, -\frac{9}{7}, 1, 0\right), x_2 = \left(\frac{10}{7}, \frac{9}{7}, 0, 1\right).$$

Then we apply the Gram-Schmidt process to these two vectors. The step-by-step computation is shown as follows.

$$\gamma_1 = \frac{1}{\|x_1\|} x_1 = \frac{7}{\sqrt{139}} \left(-\frac{3}{7}, -\frac{9}{7}, 1, 0\right)$$

$$\langle x_2, \gamma_1 \rangle = -\frac{7}{\sqrt{139}} \cdot \frac{111}{49} = -\frac{111}{7\sqrt{139}}$$

$$\langle x_2, \gamma_1 \rangle \gamma_1 = -\frac{111}{7\sqrt{139}} \frac{7}{\sqrt{139}} \left(-\frac{3}{7}, -\frac{9}{7}, 1, 0\right) = -\frac{111}{139} \left(-\frac{3}{7}, -\frac{9}{7}, 1, 0\right) = \left(\frac{333}{973}, \frac{999}{973}, -\frac{111}{139}, 0\right)$$

$$x_2 - \langle x_2, \gamma_1 \rangle \gamma_1 = \left(\frac{151}{139}, \frac{36}{139}, \frac{111}{139}, 1\right)$$

$$\gamma_2 = \frac{x_2 - \langle x_2, \gamma_1 \rangle \gamma_1}{\|x_2 - \langle x_2, \gamma_1 \rangle \gamma_1\|} = \frac{139}{\sqrt{55739}} \left(\frac{151}{139}, \frac{36}{139}, \frac{111}{139}, 1\right)$$

Now we get an orthonormal basis of $U^{\perp}$:

$$\gamma_1 = \frac{1}{\sqrt{139}} (-3, -9, 7, 0), \gamma_2 = \frac{1}{\sqrt{55739}} (151, 36, 111, 139).$$

! **Note:** *We can also add the standard basis of $\mathbb{R}^4$ to the basis of U and apply Gram-Schmidt process to them, and we get* $\frac{1}{\sqrt{76190}}(190, 117, 60, 151)$, $\frac{1}{\sqrt{190}}(0, 9, -10, -3)$ *as an orthonormal basis of $U^{\perp}$.*

Problem 11

Let U be an m-dimensional subspace of $\mathbb{R}^n$ and let V be a k-dimensional subspace of U, where $0 < k < m$.

(a) Show that any orthonormal basis

$$\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$$

for V can be expanded to form an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k, \mathbf{v}_{k+1}, ..., \mathbf{v}_m\}$ for U.

(b) Show that if W = Span$\{\mathbf{v}_{k+1}, ..., \mathbf{v}_m\}$, then $U = V \oplus W$.

✔ **Proof:** (a) Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthonormal list in U, it is linearly independent. Thus we can extend it to a basis of U:

$$\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k, \mathbf{u}_{k+1}, ..., \mathbf{u}_m\}$$

And then apply the Gram-Schmidt procedure to it and get an orthonormal basis of U:

$$\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k, \mathbf{v}_{k+1}, ..., \mathbf{v}_m\}.$$

Here the Gram-Schmidt procedure leaves the first m vectors unchanged because they are already orthonormal. Gram-Schmidt does not change the span of the list of vectors, thus we find the desired basis.

(b) There are at least two different proofs.

Proof1: From the Gram-Schmidt procedure, it's seen that

$$\mathbf{v}_i \perp \mathbf{v}_j, \qquad i = k+1, \dots, m, \quad j = 1, \dots, k,$$

which implies that

$$\mathbf{v}_i \in \{\mathbf{v}_1, ..., \mathbf{v}_k\}^{\perp}, \quad i = k+1, \dots, m.$$

From Problem 9, we have

$$\{\mathbf{v}_1, ..., \mathbf{v}_k\}^{\perp} = (\text{span}(\mathbf{v}_1, ..., \mathbf{v}_k))^{\perp},$$

Thus, $W \subseteq V^{\perp}$. Since $\{\mathbf{v}_{k+1}, ..., \mathbf{v}_m\}$ is a basis of W, we have $\dim W = m-k$. Since $\dim U = m$, $\dim V = k$, and $U = V \oplus V^{\perp}$, we have $\dim V^{\perp} = m - k = \dim W$. Thus, $W = V^{\perp}$. Hence, $U = V \oplus W$.

Proof2: First we show that $U = V + W$. From (a), we have $v_1, \dots, v_k, v_{k+1}, \dots, v_m$ is an orthonormal basis of U, then every vector $u \in U$ can be written as

$$u = \underbrace{a_1 v_1 + \cdots + a_k v_k}_{v} + \underbrace{a_{k+1} v_{k+1} + \cdots + a_m v_m}_{w},$$

where $v \in V$ and $w \in W$. Thus, $U = V + W$.

To show that $U = V \oplus W$, we now need only show that $V \cap W = \{0\}$. Suppose $u \in V \cap W$. Then we have $u = a_1 v_1 + \cdots + a_k v_k = b_1 v_{k+1} + \cdots + b_{m-k} v_m$, which implies that

$$a_1 v_1 + \cdots + a_k v_k - (b_1 v_{k+1} + \cdots + b_{m-k} v_m) = 0.$$

Since $v_1, \dots, v_k, v_{k+1}, \dots, v_m$ are linearly independent, the equation above holds if and only if $a_1 = \cdots = a_k = b_1 = \cdots = b_{m-k} = 0$, which implies $u = 0$. Thus, $V \cap W = \{0\}$. Hence, $U = V \oplus W$. ■