Homework 3

April 12, 2022

1. Consider two ordered bases $E = \{v_1, v_2, v_3\}$ and $F = \{w_1, w_2, w_3\}$ for a vector space V, and suppose

$$v_1 = 4w_1 - w_2, \quad v_2 = -w_1 + w_2 + w_3, \quad v_3 = w_2 - 2w_3$$

   (a) Find the transition matrix S from E to F.

   (b) Compute the coordinate vector $[v]_F$ for $v = 3v_1 + 4v_2 + v_3$.

2. Consider two ordered bases $E = \{v_1, v_2, v_3\}$ and $F = \{w_1, w_2, w_3\}$ for $\mathbb{R}^3$, where

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$
$$w_1 = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

   (a) Find the transition matrix S_1 from F to E.

   (b) Find the transition matrix S_2 from E to F.

   (c) Verify that $S_1 S_2 = S_2 S_1 = I_3$.

   (d) Compute the coordinate vector $[v]_E$, where $v = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$, and use S_1 or S_2 (decide by yourself) to compute $[v]_F$.

   (e) Check your work by computing $[v]_F$ directly.

3. Let $A \in \mathbb{R}^{4\times 6}$ be

$$A = \begin{bmatrix} 1 & 2 & 4 & 2 & 3 & 1 \\ 2 & 5 & 9 & 4 & 5 & 4 \\ 3 & 7 & 9 & 6 & 8 & 5 \\ 1 & 3 & 5 & 2 & 2 & 3 \end{bmatrix}.$$

(a) Find the four subspaces of the matrix ($C(A)$, $C(A^T)$, $N(A)$, and $N(A^T)$, determine their bases and dimensions).

(b) Write down the fundamental theorem of linear algebra: Part 1 and Part 2. And verify them by the answers in (a).

4. Let $A \in \mathbb{R}^{4\times 5}$ and let R be the reduced row echelon form of A. If the first and second columns of A are

$$a_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ -2 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

and

$$R = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

(a) find a basis for $N(A)$.

(b) given that x_0 is a solution to $Ax = b$, where

$$b = \begin{bmatrix} 0 \\ 5 \\ 3 \\ 4 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix},$$

determine the remaining column vectors of A.

5. Let x and y be linearly independent vectors in $\mathbb{R}^n$ and let $S = \mathrm{Span}(x, y)$. We can use x and y to define a matrix A be setting

$$A = xy^T + yx^T$$

(a) Show that $N(A) = S^{\perp}$.

(b) Show that $\dim C(A) = 2$ (the rank of A must be 2).