Homework 1

March 13, 2025

1. Let V be a vector space and let $\mathbf{x}, \mathbf{y} \in V$. Show that

   (a) $\beta\mathbf{0} = \mathbf{0}$ for each scalar β.

   (b) $\mathbf{x} + \mathbf{y} = \mathbf{0}$ implies that $\mathbf{y} = -\mathbf{x}$, $i.e.$, the additive inverse of $\mathbf{x}$ is unique.

2. Let V be the set of all ordered pairs of real numbers with addition defined by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

   and scalar multiplication defined by

$$\alpha \circ (x_1, x_2) = (\alpha x_1, x_2)$$

   Scalar multiplication for this system is defined in an unusual way, and consequently we use the symbol o to avoid confusion with the ordinary scalar multiplication of row vectors. Is V a vector space with these operations? Justify your answer.

3. Suppose V is a real vector space.

   - The *complexification* of V, denoted by $V_{\mathbb{C}}$, equals $V \times V$. An element of $V_{\mathbb{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
   - Addition on $V_{\mathbb{C}}$ is defined by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

     for all $u_1, v_1, u_2, v_2 \in V$.
   - Complex scalar multiplication on $V_{\mathbb{C}}$ is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

     for all $a, b \in \mathbb{R}$ and all $u, v \in V$.

   Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbb{C}}$ is a complex vector space.

4. Let A be a fixed vector in $\mathbb{R}^{n\times n}$ and let S be the set of all matrices that commute with A, that is,

$$S = \{B \mid AB = BA\}$$

Show that S is a subspace of $\mathbb{R}^{n\times n}$.

5. Verify the following statements.

   (a) Is $\mathbb{R}^3$ a subspace of the complex vector space $\mathbb{C}^3$?

   (b) Is $\{(x, y, z) \in \mathbb{R}^3 : x^3 = y^3\}$ a subspace of $\mathbb{R}^3$?

   (c) Is $\{(x, y, z) \in \mathbb{C}^3 : x^3 = y^3\}$ a subspace of $\mathbb{C}^3$?

6. Suppose U_1 and U_2 are subspaces of V.

   (a) Is the intersection $U_1 \bigcap U_2$ a subspace of V? Prove or give a counterexample.

   (b) Is the union $U_1 \bigcup U_2$ a subspace of V? Prove or give a counterexample.

7. Suppose $v_1, \dots, v_m$ is a list of vectors in V. For $k \in \{1, \dots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that span$(v_1, \dots, v_m) =$ span$(w_1, \dots, w_m)$.

8. Suppose $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \dots, v_m + w$ is linearly dependent, then $w \in$ span$(v_1, \dots, v_m)$.