Lecture 2: Vector Spaces

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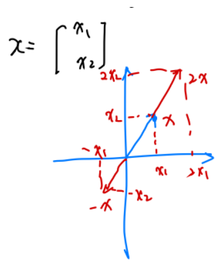
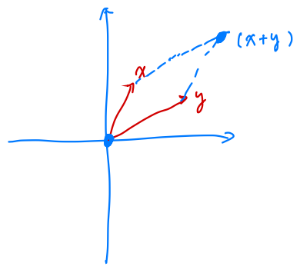
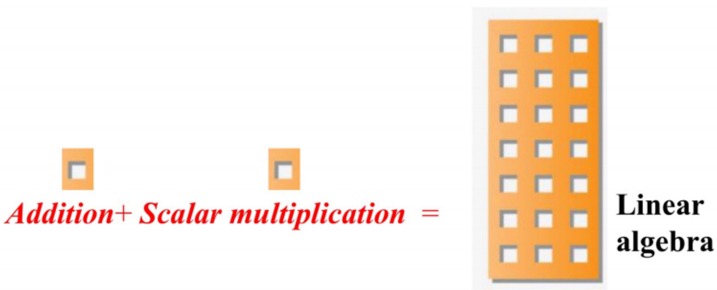
Review: Euclidean Vector Space $\mathbb{R}^n$: $\quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Let us consider first $\mathbb{R}^2$.





Note: We can add any vectors, and we can multiple any vector by a scalar.



Q: Do we have vector spaces other than $\mathbb{R}^n$?

- The vector space of all real 2×2 matrices.
- The vector space of all continuous functions.
- The vector space that consists only of a zero vector.

2.1 Vector Space Axioms

Definition 2.1.1: *The set V together with the operations of addition and scalar multiplication over a field $\mathbb{F}$, V is said to form a vector space if the following eight axioms are satisfied:*

A1. ***commutative*(交换律): $x + y = y + x$ for any x and y in V.**

A2. ***associative*(结合律): $(x + y) + z = x + (y + z)$ for any $x, y,$ and z in V.**

A3. ***associative:* $(ab)x = a(bx)$ for any scalars a and b in $\mathbb{F}$ and any x in V.**

A4. ***distributive*(分配律): $a(x + y) = ax + ay$ for each scalar a in $\mathbb{F}$ and any x and y in V.**

A5. ***distributive:* $(a + b)x = ax + bx$ for any scalars a and b in $\mathbb{F}$ and any x in V.**

A6. ***zero*(零元): There exists an element $\mathbf{0}$ in V such that $x + \mathbf{0} = x$ for each x in V.**

A7. ***additive inverse*(加法逆元): For each x in V, there exists an element $-x$ in V such that $x + (-x) = \mathbf{0}$.**

A8. ***multiplicative identity*(乘法单位元): $1x = x$ for all x in V.**

An important component of the definition is the closure properties of the two operations:

C1: If $x \in V$ and α is a scalar, then $\alpha x \in V$
C2: If $x, y \in V$, then $x + y \in V$

Additional Properties of Vector Space.

Theorem 2.1.1: *If V is a vector space and x is a element of V, then*

(i) $0x = 0$.

(ii) $x + y = 0$ *implies that* $y = -x$, *i.e., the additive inverse of* x *is unique.*

Proof: (i)

$$\begin{aligned} x &= 1 \cdot x \quad (A8) \\ &= (1 + 0)x \\ &= 1 \cdot x + 0 \cdot x \quad (A5) \\ &= x + 0x \quad (A8) \end{aligned}$$

$$\begin{array}{rcl} -x + x = -x + (x + 0x) = & (-x + x) + 0x & (A2) \\ \downarrow (A1)(A7) & \downarrow (A1)(A7) & \\ 0 = & 0 + 0x = 0x(A6) & \end{array}$$

(ii)

$$
\begin{aligned}
-x+(x+y) &= -x+0 \\
-x+(x+y) &\overset{A2}{=} (-x+x)+y \overset{A1}{=} (x+(-x))+y \overset{A7}{=} 0+y \overset{A1}{=} y+0 \overset{A6}{=} y \\
-x+0 &\overset{A6}{=} -x
\end{aligned}
$$

■

Example 2.1.1:

(i) $\mathbb{R}^n$: $\quad \alpha \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix} \quad \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{bmatrix}$

(ii) The set of all real 2×2 *matrices:*

$M_1 = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \quad M_2 = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$, x_{ij}, y_{ij} *are real numbers.*

$$\alpha M_1 = \begin{bmatrix} \alpha x_{11} & \alpha x_{12} \\ \alpha x_{21} & \alpha x_{22} \end{bmatrix} \quad M_1 + M_2 = \begin{bmatrix} x_{11}+y_{11} & x_{12}+y_{12} \\ x_{21}+y_{21} & x_{22}+y_{22} \end{bmatrix}.$$

(iii) $\mathbb{R}_+$*: the set of all positive real numbers.* $\quad \mathbb{F} = \mathbb{R}$.

Scalar multiplication: $\alpha \circ x \triangleq x^\alpha \in \mathbb{R}_+$, *for* $\alpha \in \mathbb{R}$ *and* $x \in \mathbb{R}_+$.

Addition: $x \oplus y \triangleq xy \in \mathbb{R}_+$.

2.2 Subspace

Given a vector space V, it is often possible to form another vector space by taking a subset U of V and using the operations of V.

Example 2.2.1: *Let* $U = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_2 = 2x_1 \right\}$. *$U$ is a subset of $\mathbb{R}^2$.*

$$\begin{aligned} \alpha x =& \alpha \begin{bmatrix} c \\ 2c \end{bmatrix} = \begin{bmatrix} \alpha c \\ 2\alpha c \end{bmatrix} \in U \\ \textit{for} \quad y =& \begin{bmatrix} b \\ 2b \end{bmatrix}, \quad x + y = \begin{bmatrix} c \\ 2c \end{bmatrix} + \begin{bmatrix} b \\ 2b \end{bmatrix} = \begin{bmatrix} c+b \\ 2(c+b) \end{bmatrix} \in U \end{aligned}$$

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It is easily seen that the mathematical system consisting of the set U, together with the operations from $\mathbb{R}^2$, is itself a vector space.

Definition 2.2.1: *If U is a nonempty subset of a vector space V, and U satisfies the following conditions*

(i) $\mathbf{0} \in U$;

(ii) $\alpha x \in U$ *wherever* $x \in U$ *for any scalar* α;

(iii) $x + y \in U$ *wherever* $x \in U$ *and* $y \in U$.

Note 2.2.1:

(i) U is closed under scalar multiplication and addition.

(ii) Every subspace is a vector space.

Example 2.2.2:

1. *The set consisting of only the zero vector in a vector space V is a subspace of V, called the zero subspace and written as $\{0\}$.*

   $\{0\}$ and V are subspaces of V.

   All other subspaces are referred to as proper subspaces.

2. *Let $\mathbb{P}$ be the set of all polynomials with real coefficients. Then $\mathbb{P}$ is a subspace of the space of all real-valued functions defined on $\mathbb{R}$. Also, for each $n \geqslant 0, \mathbb{P}_n$ (the set of all polynomials with degree at most n) is a subspace of $\mathbb{P}$, because the sum of two polynomials in $\mathbb{P}_n$ is also in $\mathbb{P}_n$, and a scalar multiple of a polynomial in $\mathbb{P}_n$ is also in $\mathbb{P}_n$.*

3. *Is the vector space $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?*

   No! $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$, since the vectors in $\mathbb{R}^3$ all have three components, whereas the vectors in $\mathbb{R}^2$ have only two.

$$\boldsymbol{H} = \left\{ \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ 0 \end{bmatrix} : \boldsymbol{x}_1 \textit{ and } \boldsymbol{x}_2 \textit{ are real} \right\} \textit{ is a subset of } \mathbb{R}^3.$$

4. *The set $\mathbb{S}$ of all functions f in $C^2[a,b]$ such that*

$$\ddot{f}(x) + f(x) = 0$$

   for all x in $[a,b]$.

$$\begin{aligned} & \forall f, g \in \mathbb{S}, \alpha \in \mathbb{R} \\ & \alpha f \quad \alpha\ddot{f} + \alpha f = 0 \Rightarrow \alpha(\ddot{f} + f) = 0 \Rightarrow \alpha f \in \mathbb{S} \\ & f + g \quad (\ddot{f} + \ddot{g}) + f + g = 0 \Rightarrow (\ddot{f} + f) + (\ddot{g} + g) = 0. \quad f + g \in \mathbb{S} \end{aligned}$$

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5. *Lines through the origin of $\mathbb{R}^3$.*
6. $W = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$.

2.3 Sum of Subspaces

When dealing with vector spaces, we are usually interested only in subspaces, as opposed to arbitrary subsets. The notion of the sum of subspaces will be useful.

Definition 2.3.1: *Suppose $U_1, \ldots, U_m$ are subsets of V. The sum of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$. More precisely,*

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\}.$$

Example 2.3.1:

1. $U = \{(x, 0, 0) \in \mathbb{F}^3 : x \in \mathbb{F}\}$ *and* $W = \{(0, y, 0) \in \mathbb{F}^3 : y \in \mathbb{F}\}$.

$$U + W = \{(x, y, 0) : x, y \in \mathbb{F}\}$$

2. $U = \{(x, 0, 0) \in \mathbb{F}^3 : x \in \mathbb{F}\}$ *and* $W = \{(0, x, 0) \in \mathbb{F}^3 : x \in \mathbb{F}\}$.

$$U + W = \{(x, y, 0) : x, y \in \mathbb{F}\}$$

3. $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$ *and* $W = \{(x, x, x, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$.

$$U + W = \{(x, x, y, z) : x, y, z \in \mathbb{F}\}$$

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Theorem 2.3.1: *Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.*

Proof: We first verify the three conditions in Definition 2.2.1. It is easy to see that $0 \in U_1 + \cdots + U_m$. For any $u, v \in U_1 + \cdots + U_m$, there exist $u_1, v_1 \in U_1, \ldots, u_m, v_m \in U_m$ such that

$$u = u_1 + \cdots + u_m, \quad v = v_1 + \cdots + v_m.$$

Since $U_1, \dots, U_m$ are subspaces, for any scalar α, $\alpha u_1 \in U_1, \dots, \alpha u_m \in U_m$. And $u_1 + v_1 \in U_1, \dots, u_m + v_m \in U_m$. Therefore,

$$\alpha u = \alpha u_1 + \cdots + \alpha u_m \in U_1 + \cdots + U_m$$

and

$$u + v = (u_1 + v_1) + \cdots + (u_m + v_m) \in U_1 + \cdots + U_m.$$

Thus from Definition 2.2.1 one can conclude that $U_1 + \cdots + U_m$ is a subspace of V.

By considering sums $u_1 + \cdots + u_m$ where all except one of the us are 0, it is clearly that $U_1, \dots, U_m$ are all contained in $U_1 + \cdots + U_m$. Conversely, every subspace of V containing $U_1, \dots, U_m$ contains $U_1 + \cdots + U_m$ because subspaces must contain all finite sums of their elements. Thus $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \dots, U_m$. ■

Definition 2.3.2: *Suppose $U_1, \dots, U_m$ are subspaces of V.*

- *The sum $U_1 + \cdots + U_m$ is called a* **direct sum** *if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j.*
- *If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.*

Example 2.3.2:

1. $U = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$ *and* $W = \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\}$.
2. $U_1 = \{(x, 0, \dots, 0) \in \mathbb{F}^n : x \in \mathbb{F}\}$, $U_2 = \{(0, x, \dots, 0) \in \mathbb{F}^n : x \in \mathbb{F}\}$, ..., $U_n = \{(0, 0, \dots, x) \in \mathbb{F}^n : x \in \mathbb{F}\}$.
3. $U_1 = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$, $U_2 = \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\}$, $U_3 = \{(0, y, y) \in \mathbb{F}^3 : y \in \mathbb{F}\}$.

■

Theorem 2.3.2: *Suppose $U_1, \dots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j, is by taking each u_j equal to 0.*

Proof: ($\Longrightarrow$) The necessity can be easily verified by the definition of the direct sum.

For the sufficiency, suppose that the only way to write 0 as a sum $u_1+\cdots+u_m$, where each u_j is in U_j, is by taking each u_j equal to 0. To show that $U_1+\cdots+U_m$ is a direct sum, let $v \in U_1+\cdots+U_m$. We can write

$$v = u_1 + \cdots + u_m$$

for some $u_1 \in U_1, \ldots, u_m \in U_m$. To show that this representation is unique, suppose we also have

$$v = v_1 + \cdots + v_m$$

where $v_1 \in U_1, \ldots, v_m \in U_m$. Subtracting these two equations, we have

$$0 = (u_1 - v_1) + \cdots + (u_m - v_m)$$

Because $u_1 - v_1 \in U_1, \ldots, u_m - v_m \in U_m$, the equation above implies that each $u_j - v_j = 0$. Thus $u_1 = v_1, \ldots, u_m = v_m$, as desired. ■

Theorem 2.3.3: *Suppose U and W are subspaces of V. Then $U+W$ is a direct sum if and only if $U \bigcap W = \{0\}$.*

Proof: ($\Longrightarrow$) First suppose that $U+W$ is a direct sum. If $v \in U \bigcap W$, then $0 = v + (-v)$, where $v \in U$ and $-v \in W$. By the unique representation of 0 as the sum of a vector in U and a vector in W, we have $v = 0$. Thus $U \bigcap W = \{0\}$.

($\Longleftarrow$) Now suppose that $U \bigcap W = \{0\}$. To prove that $U+W$ is a direct sum, suppose $u \in U, w \in W$, and

$$0 = u + w$$

To complete the proof, we need only show that $u = 0$ and $w = 0$. The equation above implies that $u = -w \in W$. Thus $u \in U \bigcap W$. Hence $u = 0$, which by the equation above implies that $w = 0$. ■

2.4 The Span of a Set of Vectors

Definition 2.4.1: *Let $v_1, v_2, \cdots, v_n$ be vectors in a vector space V.*

A sum of the form $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$, with α_1, α_2, ..., α_n being scalars, is called a **linear combination** *of $v_1, v_2, \cdots, v_n$.*

The set of all linear combinations of $v_1, v_2, \cdots, v_n$ is called the **span** *of v_1, v_2, ..., v_n, denoted by* **Span** $(v_1, v_2, \cdots, v_n)$.

Example 2.4.1: *In $\mathbb{R}^3$, the span of $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.*

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix}$$

■

Theorem 2.4.1: *If $v_1, v_2, \cdots, v_n$ are elements of a vector space V, then Span$(v_1$, ..., $v_n)$ is a subspace of V.*

Proof: Let $v \in \text{Span}\,(v_1, \cdots, v_n)$ with $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$.
For a scalar β,

$$\beta v = (\beta \alpha_1)\, v_1 + (\beta \alpha_\alpha)\, v_2 + \ldots + (\beta \alpha_n)\, v_n.$$

It follows $\beta v \in \text{Span}\,(v_1, \ldots, v_n)$. Let $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$ and $w = \beta_1 v_1 + \ldots + \beta_n v_n$.

$$v + w = (\alpha_1 + \beta_1) v_1 + \ldots + (\alpha_n + \beta_n) v_n \quad \in \text{Span}(v_1, \ldots, v_n)$$

Therefore, $\text{Span}\,(v_1, \ldots, v_n)$ is a subspace of V. ■

Note 2.4.1: *It may happen that* $\text{Span}\,(v_1, \ldots, v_n) = V$.

Definition 2.4.2: *Let $v_1, v_2, \cdots, v_n$ be vectors in a vector space V. The set $\{v_1, \ldots, v_n\}$ is a* **spanning set** *for V if and only if every vector in V can be written as a linear combination of $v_1, v_2, \cdots, v_n$.*

Example 2.4.2:

1. *Which of the following are spanning sets for $\mathbb{R}^3$?*

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} \alpha_1 = x_3 \\ \alpha_2 = x_2 - x_3 \\ \alpha_3 = x_1 - x_2 \end{array}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_1 \end{bmatrix}$$

2. *Can the set* $\{1 - x^2, x + 2, x^2\}$ span $\mathbb{P}_2$ *?*

$$\begin{aligned} ax^2 + bx + c &= \alpha_1 \left(1 - x^2\right) + \alpha_2(x + 2) + \alpha_3 x^2 \\ &= (\alpha_3 - \alpha_1)\, x^2 + \alpha_2 x + 2\alpha_2 + \alpha_1 \end{aligned}$$

$$\begin{array}{r} \alpha_3 - \alpha_1 = a \\ \alpha_2 = b \\ 2\alpha_2 + \alpha_1 = c \end{array} \Rightarrow \begin{cases} \alpha_3 = a + c - 2b \\ \alpha_2 = b \\ \alpha_1 = c - 2b \end{cases}$$

■

In the next section, we will consider the problem of finding minimal spanning sets for a vector space V (i.e., spanning sets that contain the smallest possible number of vectors.)

2.5 Linear Independence

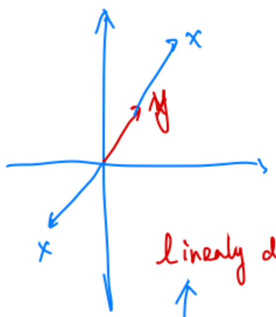
Definition 2.5.1: *The vectors $v_1, v_2, \ldots, v_n$ in a vector space V are said to be* **linearly independent** *if*

$$c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$$

implies that all scalars $c_1, \ldots, c_n$ must equal 0.

Definition 2.5.2: *The vectors $v_1, v_2, \ldots, v_n$ in a vector space V are said to be* **linearly dependent** *if there exist scalars $c_1, c_2, \cdots, c_n$, not all zero, such that*

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0.$$

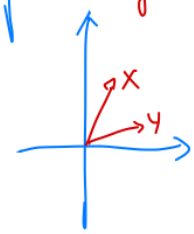


$\forall\ x, y \in \mathbb{R}^2$.

$c_1 x + c_2 y = 0$, where c_1, c_2 are not both 0.

$x = -\frac{c_2}{c_1} y$. (for example, $c_1 \neq 0$)

linearly dependent



linearly independent.

Example 2.5.1:

1. *$\{P_1, P_2, P_3\}$ with $P_1(x) = 1, P_2(x) = x, P_3(x) = 4 - x$.*

   $P_3 = 4P_1 - P_2 \Rightarrow 4P_1 - P_2 - P_3 = 0.$ *linearly dependent.*

2. *The set $\{\sin t, \cos t\}$ in $C[0, 1]$, the space of all continuous functions on $0 \leqslant t \leqslant 1$.*

   $c_1 \sin t + c_2 \cos t = 0, \quad c_1 = 0 \quad c_2 = 0.$ *linearly independent.*

3. *The set $\{\sin t \cos t, \sin 2t\}$ in $C[0, 1]$.*

   $\sin 2t = 2 \sin t \cos t.$ *linearly dependent.*

■

Next, we consider a very important property of linearly independent vectors: **linear combinations of linearly independent vectors are unique.**

Theorem 2.5.1: *Let $v_1, \ldots, v_n$ be vectors in a vector space V. A vector $v \in \mathrm{Span}(v_1, \ldots, v_n)$ can be written uniquely as a linear combination of $v_1, v_2, \ldots, v_n$ if and only if $v_1, \ldots, v_n$ are linearly independent.*

Proof: If $v \in \mathrm{Span}\,\{v_1, \ldots, v_n\}$, then v can be written as a linear combination

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \tag{2.1}$$

Suppose that v can also be expressed as a linear combination

$$v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n \tag{2.2}$$

$\Leftarrow$ (Sufficiency). If $v_1, \cdots, v_n$ are linearly independent, then substracting (2.2) from (2.1) yields

$$0 = (\alpha_1 - \beta_1)\, v_1 + (\alpha_2 - \beta_2)\, v_2 + \cdots + (\alpha_n - \beta_n)\, v_n \tag{2.3}$$

By the linear independence of $v_1, \cdots, v_n$, the coefficients of (2.3) must all be 0. Hence,

$$\alpha_1 = \beta_1, \alpha_2 = \beta_2, \cdots, \alpha_n = \beta_n.$$

$\Rightarrow$ (Necessity). If $v_1, \cdots, v_n$ are linearly dependent, there exist $c_1, \ldots, c_n$, not all 0, such that

$$0 = c_1 v_1 + \cdots + c_n v_n \tag{2.4}$$

Now if we set

$$\beta_1 = \alpha_1 + c_1, \cdots, \beta_n = \alpha_n + c_n.$$

Adding (2.4) and (2.1), we get

$$\begin{aligned} v &= (\alpha_1 + c_1)\, v_1 + \ldots + (\alpha_n + c_n)\, v_n \\ &= \beta_1 v_1 + \ldots + \beta_n v_n \end{aligned}$$

Since the c_i are not all $0, \beta_i \neq \alpha_i$ for at least one value of i. Thus, the linear combination of $v_1, \ldots, v_n$ is not unique. ■

Theorem 2.5.2: *Suppose $v_1, \ldots, v_n$ is a linearly dependent list in V. Then there exists $j \in \{1, \ldots, n\}$ such that the following conditions hold:*

(1) $v_j \in \mathrm{Span}(v_1, \ldots, v_{j-1})$.

(2) if the jth term is removed from $v_1, \ldots, v_n$, the span of the remaining list equals $\mathrm{Span}(v_1, \ldots, v_n)$.

Proof: Since $v_1, \ldots, v_n$ are linearly dependent, there exist scalars $c_1, \ldots, c_n$, not all 0, such that

$$c_1 v_1 + \cdots + c_n v_n = 0$$

Let j be the largest number element of $\{1, \ldots, n\}$ such that $c_j \neq 0$. Then,

$$v_j = \frac{c_1}{c_j} v_1 + \cdots + \frac{c_{j-1}}{c_j} v_{j-1}, \tag{2.5}$$

proving (1).

To prove (2), for any $v \in \mathrm{Span}(v_1, \ldots, v_n)$, there exist scalars $b_1, \ldots, b_n$ such that

$$u = b_1 v_1 + \cdots + b_n v_n$$

In the equation above, we can replace v_j with the right side of (2.5), which shows that u is in the span of the list obtained by removing the jth term from $v_1, \ldots, v_n$. Thus (2) holds. ■

2.6 Basis and Dimension

Example 2.6.1: *Two vectors can't span all of $\mathbb{R}^3$, even if they are independent. Four vectors can't be independent, even if they span $\mathbb{R}^3$.*

We want enough independent vectors to span the vector space (and not more, *i.e.*, a minimal spanning set).

Definition 2.6.1: *The vectors $v_1, \ldots, v_n$ form a* **basis** *for a vector space V if and only if*

(i) $v_1, \ldots, v_n$ *are linearly independent.*

(ii) $\operatorname{Span}(v_1, \ldots, v_n) = V$.

Example 2.6.2:

1. $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$

   The set $\{e_1, e_2, \ldots, e_n\}$ is called the standard basis for $\mathbb{R}^n$.

2. $S = \{1, x, x^2, \ldots, x^n\}$ *is a basis for $\mathbb{P}_n$.*

3. 2×2 *real matrix*

$$S = \{E_{11}, E_{12}, E_{21}, E_{22}\}$$

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

■

Note 2.6.1:

- *Since $\operatorname{Span}(v_1, \ldots, v_n) = V$, every vector v in the space is a linear combination of the basis vectors.*
- *The combination that produce v is unique, because the basis vectors are linearly independent.*

Theorem 2.6.1: *Let $v_1, v_2, \cdots, v_n$ be vectors in a vector space V. If $\{v_1, v_2, \ldots, v_n\}$ is any spanning set for V, then the number of any linearly independent vectors is less or equal to n.*

Proof: Suppose $u_1, \ldots, u_m$ are linearly independent. We need to prove $m \leq n$. We do so through the multi-step process described below; note that in each step we add one of the us and remove one of the vs.

Step 1: Since $\{v_1, v_2, \ldots, v_n\}$ is a spanning set of V, the list of vectors

$$u_1, v_1, \ldots, v_n \tag{2.6}$$

is linear dependent. From Theorem 2.5.2, we can remove some v_j from (2.6) to form a new list B with n vectors consisting of u_1 and the remaining v's which can span V.

Step i: The new list B with n vectors from the $i-1$th step can span V. Thus adjoining any vector to this list produces a linearly dependent list. In particular, the list of length $n+1$ obtained by adjoining u_i to B, placing it just after $u_i, \ldots, u_{i-1}$, is linearly dependent. By Theorem 2.5.2, one of the vectors in this list is in the span of the previous ones, and because $u_i, \ldots, u_{i-1}$ is linearly independent, this vector is one of the v's, not one of the us. We can remove that w from B so that the new list B (of length n) consisting of $u_i, \ldots, u_i$ and the remaining vs spans V.

After step m, we have added all the u's and the process stops. At each step as we add a u to B, Theorem 2.5.2 implies that there is some v to remove. Thus there are at least as many v's as u's. ■

Corollary 2.6.1: *If $\{v_1, v_2, \ldots, v_n\}$ is a spanning set for a vector space V, then any collection of m vectors in V, where $m > n$, is linearly dependent.*

Example 2.6.3:

1. *Are the vectors $(1, 2, 3)$, $(-1, 3, 4)$, $(2, 5, 7)$, $(9, -1, 2)$ linear dependent in $\mathbb{R}^3$?*
2. *Can the vectors $(1, 2, 3, 5)$, $(-1, 3, 4, 8)$, $(2, 5, 7, 9)$ span $\mathbb{R}^4$?*

Corollary 2.6.2: *If both $\{v_1, \ldots, v_n\}$ and $\{u_1, \ldots, u_m\}$ are bases for a vector space V, then $n = m$.*

Note 2.6.2: *The number of basis vectors depends on the space - not a particular basis. It is the same for every basis.*

Definition 2.6.2: *Let V be a vector space. If V has a basis consisting of n vectors, we say that V has* **dimension** *n.* $(\dim V = n)$

The subspace $\{0\}$ of V is said to have **dimension** *0.*

V is said to be **finite dimensional** *if there is a finite set of vectors spans V; otherwise, we say that V is* **infinite dimensional**.

Example 2.6.4:

1. *$\mathbb{R}^n$.*
2. *The space $\mathbb{P}_n$ of the polynomials with order less or equal to n.*
3. *The space $\mathbb{P}$ of all polynomials.*
4. *The subspaces of $\mathbb{R}^3$.*
   - *the zero subspace: 0-dimensional subspace.*
   - *a line: 1-dimensional subspace.*
     (Any subspace spanned by a nonzero vector)
   - *a plane: 2-dimensional subspace.*
     (Any subspace spanned by two linearly independent vectors)
   - *$\mathbb{R}^3$: 3-dimensional subspace.*
     (Any three linearly independent vectors in $\mathbb{R}^3$ span all of $\mathbb{R}^3$)

■

Theorem 2.6.2: *If V is a vector space of dimension $n > 0$, then*

(i) any set of n linearly independent vectors span V.

(ii) no set of fewer than n vectors can span V.

(iii) any n vectors that span V are linearly independent.

(iv) any subset of fewer than n linearly independent vectors can be extended to form a basis for V.

(v) any spanning set containing more than n vectors can be pared down to form a basis for V.

Proof:

(i) Suppose that $v_1, \ldots, v_n$ are linearly independent and v is any other vector in V. Since $\dim V = n$, it has a basis consisting of n vectors which span V. Then the $n+1$ vectors $v_1, \ldots, v_n, v$ are linearly dependent. Thus there exist scalars $c_1, \ldots, c_n, c_{n+1}$, not all zero such that

$$c_1 v_1 + \cdots + c_n v_n + c_{n+1} v = 0$$

c_{n+1} cannot be zero, otherwise $c_1 = \cdots = c_n = 0$ since $v_1, \ldots, v_n$ are linearly independent. Then $v = -\frac{c_1}{c_{n+1}} v_1 - \cdots - \frac{c_n}{c_{n+1}} v_n$ is a linear combination of $v_1, \ldots, v_n$. Thus, $V = \text{Span}\{v_1, \cdots, v_n\}$.

(ii) If there are $k < n$ vectors which can span V, then any collection of $m > k$ vectors are linearly dependent, which is a contradiction since $\dim V = n$.

(iii) Suppose that $v_1, \cdots, v_n$ span V. If $v_1, \cdots, v_n$ are linearly dependent, then one of the v_i' s, say v_n, can be written as a linear combination of the others. It follows that $V = \text{Span}\,(v_1, \ldots, v_{n-1})$, which is a contradiction with (ii).

(iv) Suppose there are $k < n$ vectors $v_1, \ldots, v_k$ which are linearly independent. From (ii), $\text{Span}\,(v_1, \ldots, v_k)$ is a proper subspace of V and hence there exists a vector v_{k+1} that is in V but not in $\text{Span}\,(v_1, \ldots, v_k)$. It then follows that $v_1, \ldots, v_k, v_{k+1}$ must be linearly independent. If $k+1 < n$, then, in the same manner, $\{v_1, \cdots, v_k, v_{k+1}\}$ can be extended to a set of $k+2$ linearly independent vectors.

This extension process can be continued until a set $\{v_1, \cdots, v_n\}$ of n linearly independent vectors is obtained.

(v) Suppose that $V = \text{Span}\,(v_1, \ldots, v_m)$ and $m > n$. Since $\dim V = n$, $v_1, \ldots,$ v_m must be linearly dependent. It follows that one of the vectors, say v_j, can be written as a linear combination of the others. Hence, if v_j is eliminated from the set, the remaining $m-1$ vectors still span V. If $m-1 > n$, we can continue to eliminate vectors in this manner until we arrive at a spanning set containing n vectors.

■

Theorem 2.6.3: *Suppose U is a subspace of a finite-dimensional vector space V. Then there exists a subspace W of V such that $V = U \oplus W$.*

Proof: Let $u_1, \ldots, u_m$ be a basis of U. Clearly, $u_1, \ldots, u_m$ are linearly independent in V. Therefore, it can be extended to a basis of V, say, $u_1, \ldots, u_m, w_1, \ldots, w_n$. Let $W = \text{Span}(w_1, \ldots, w_n)$. Clearly, W is a subspace of V.

To prove $V = U \oplus W$, from Theorem 2.3.3, we only need to show that

$$V = U + W \quad \text{and} \quad U \cap W = \{0\}.$$

For any $v \in V$, since $\{u_1, \ldots, u_m, w_1, \ldots, w_n\}$ is a basis of V, there exists scalars $b_1, \ldots, b_m, c_1, \ldots, c_n$ such that

$$v = \underbrace{b_1 u_1 + \cdots + b_m u_m}_{\in U} + \underbrace{c_1 w_1 + \cdots + c_m w_m}_{\in W}$$

Thus $v \in U + W$, which implies that $V = U + W$.

To show that $U \cap W = \{0\}$, for any $v \in U \cap W$, there exists scalars $b_1, \ldots, b_m, c_1, \ldots, c_n$ such that

$$v = b_1 u_1 + \cdots + b_m u_m = c_1 w_1 + \cdots + c_m w_m$$

Thus

$$b_1 u_1 + \cdots + b_m u_m - c_1 w_1 - \cdots - c_m w_m = 0$$

Since $u_1, \ldots, u_m, w_1, \ldots, w_n$ are linearly independent, $b_1 = \cdots = b_m = c_1 = \cdots = c_n = 0$. Then $v = 0$, which implies that $U \cap W = \{0\}$. ■

2.7 Change of Basis

Definition 2.7.1: *Let V be a vector space and let $E = \{v_1, v_2, \cdots v_n\}$ be an ordered basis for V. For any $v \in V$, it can be written in the form*

$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

where $c_1, c_2, \cdots, c_n$ are scalars. Thus, we can associate with each vector v a unique vector $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ in $\mathbb{F}^n$. The vector c defined in this way is called the **coordinate vector** *of v with respect to the ordered basis E and is denoted $[v]_E$. The c_i's are called the* **coordinates** *of v relative to E.*

Example 2.7.1: *Consider a basis $E = \{v_1, v_2\}$ for $\mathbb{R}^2$, where $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Suppose an x in $\mathbb{R}^2$ has the coordinate vector $[x]_E = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find x.*

$$x = (-2)v_1 + 3v_2 = (-2)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

The entries in the vector $\begin{bmatrix} 1 \\ 6 \end{bmatrix}$ are the coordinates of x relative to the standard basis $\{e_1, e_2\}$, since

$$\begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \cdot e_1 + 6 \cdot e_2$$

■

Example 2.7.2: *Consider two bases $E = \{v_1, v_2\}$ and $F = \{w_1, w_2\}$ for a vector space V, such that*

$$v_1 = 4w_1 + w_2, \quad v_2 = -6w_1 + w_2. \tag{2.7}$$

Suppose

$$x = 3v_1 + v_2 \tag{2.8}$$

That is, suppose $[x]_E = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[x]_F$.

Note that

$$\begin{aligned}
[x]_F &= [3v_1 + v_2]_F \\
&= 3[v_1]_F + [v_2]_F \\
&= \left[\begin{array}{cc} [v_1]_F & [v_2]_F \end{array} \right] \left[\begin{array}{c} 3 \\ 1 \end{array} \right]
\end{aligned} \tag{2.9}$$

This formula gives $[x]_F$, once we know the columns of the matrix. From (2.7),

$$[v_1]_F = \left[\begin{array}{c} 4 \\ 1 \end{array} \right], \quad [v_2]_F = \left[\begin{array}{c} -6 \\ 1 \end{array} \right]$$

Thus (2.9) provides the solution:

$$[x]_F = \left[\begin{array}{cc} 4 & -6 \\ 1 & 1 \end{array} \right] \left[\begin{array}{c} 3 \\ 1 \end{array} \right] = \left[\begin{array}{c} 6 \\ 4 \end{array} \right]$$

The matrix $\left[\begin{array}{cc} 4 & -6 \\ 1 & 1 \end{array} \right]$ is called the **transition matrix** from the ordered basis $\{v_1, v_2\}$ to the ordered basis $\{w_1, w_2\}$.

■

If V is any $n-$dimensional vector space, it is possible to change from one basis to another by means of an $n \times n$ transition matrix. Let $E = [v_1, \ldots, v_n]$ and $F = [w_1, \ldots, w_n]$ be two ordered bases of V. The key step is to express each basis vector v_j as a linear combination of the w_i's.

$$v_1 = \left[\begin{array}{cccc} w_1 & w_2 & \cdots & w_n \end{array} \right] \left[\begin{array}{c} s_{11} \\ s_{21} \\ \vdots \\ s_{n1} \end{array} \right]$$

$$v_2 = \left[\begin{array}{cccc} w_1 & w_2 & \cdots & w_n \end{array} \right] \left[\begin{array}{c} s_{12} \\ s_{22} \\ \vdots \\ s_{n2} \end{array} \right]$$

$$\vdots$$

$$v_n = \left[\begin{array}{cccc} w_1 & w_2 & \cdots & w_n \end{array} \right] \left[\begin{array}{c} s_{1n} \\ s_{2n} \\ \vdots \\ s_{nn} \end{array} \right]$$

For a vector $v \in V$, let $x = [v]_E$. It follows that

$$
\begin{aligned}
v &= x_1 v_1 + \cdots + x_n v_n \\
&= \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \begin{bmatrix} x_1 s_{11} \\ x_1 s_{21} \\ \vdots \\ x_1 s_{n1} \end{bmatrix} \\
&\quad + \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \begin{bmatrix} x_2 s_{12} \\ x_2 s_{22} \\ \vdots \\ x_2 s_{2n} \end{bmatrix} \\
&\quad + \cdots \\
&\quad + \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \begin{bmatrix} x_n s_{1n} \\ x_n s_{2n} \\ \vdots \\ x_n s_{nn} \end{bmatrix} \\
&= \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \begin{bmatrix} x_1 s_{11} + x_2 s_{12} + \cdots + x_n s_{1n} \\ x_1 s_{21} + x_2 s_{22} + \cdots + x_n s_{2n} \\ \vdots \\ x_1 s_{n1} + x_2 s_{2n} + \cdots + x_n s_{nn} \end{bmatrix} \\
&= \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \underbrace{\begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ s_{n1} & s_{2n} & \cdots & s_{nn} \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x=[v]_E}
\end{aligned}
$$

Let $y = [v]_F$. Then we have

$$y = Sx$$

The $n \times n$ matrix S is called the **transition matrix** from the ordered basis $E = [v_1, \ldots, v_n]$ to the ordered basis $F = [w_1, \ldots, w_n]$.

And the matrix S^{-1} is the transition matrix from the ordered basis $F = [w_1, \ldots, w_n]$ to the ordered basis $E = [v_1, \ldots, v_n]$.

Example 2.7.3: *Consider two ordered bases in $\mathbb{P}_2$: $[1, x, x^2]$ and $[1, 2x, 4x^2 - 2]$. Find the transition matrix S_1 from $[1, 2x, 4x^2 - 2]$ to $[1, x, x^2]$ and the transition matrix S_2 from $[1, x, x^2]$ to $[1, 2x, 4x^2 - 2]$.*

Note

$$1 = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$2x = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$4x^2 - 2 = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$$

The transition matrix S_1 *from* $[1, 2x, 4x^2-2]$ *to* $[1, x, x^2]$ *is* $S_1 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

$$1 = \begin{bmatrix} 1 & 2x & 4x^2 - 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 & 2x & 4x^2 - 2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

$$x^2 = \begin{bmatrix} 1 & 2x & 4x^2 - 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{4} \end{bmatrix}$$

The transition matrix S_2 *from* $[1, x, x^2]$ *to* $[1, 2x, 4x^2-2]$ *is* $S_2 = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$.

Note that $S_1 S_2 = S_2 S_1 = I_3$. ■

Q: How to compute the transition matrix S in a general case?

Consider two ordered bases $E = [v_1, \ldots, v_n]$ and $F = [w_1, \ldots, w_n]$ for $\mathbb{R}^n$. Then $v_j, w_j \in \mathbb{R}^n$. For a vector $v \in \mathbb{R}^n$, let $x = [v]_E$ and $y = [v]_F$.

$$v = \underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{y}$$

$$y = \underbrace{W^{-1}V}_{S}x$$

Clearly, S can be computed by considering the reduced row echelon form of $\left[\begin{array}{cc} W & V \end{array}\right]$.

Example 2.7.4: *Consider two ordered bases $E = [v_1, v_2]$ and $F = [w_1, w_2]$ in $\mathbb{R}^2$, with*

$$v_1 = \left[\begin{array}{c} 1 \\ -3 \end{array}\right], v_2 = \left[\begin{array}{c} -2 \\ 4 \end{array}\right], w_1 = \left[\begin{array}{c} -7 \\ 9 \end{array}\right], w_2 = \left[\begin{array}{c} -5 \\ 7 \end{array}\right]$$

(1) Find the transition matrix S_1 from E to F.

(2) Find the transition matrix S_2 from F to E.

(1) Compute

$$\begin{aligned}
\left[\begin{array}{cc:cc} w_1 & w_2 & v_1 & v_2 \end{array}\right] &= \left[\begin{array}{cc:cc} -7 & -5 & 1 & -2 \\ 9 & 7 & -3 & 4 \end{array}\right] \\
&\sim \left[\begin{array}{cc:cc} 1 & 5/7 & -1/7 & 2/7 \\ 9 & 7 & -3 & 4 \end{array}\right] \\
&\sim \left[\begin{array}{cc:cc} 1 & 5/7 & -1/7 & 2/7 \\ 0 & 4/7 & -12/7 & 10/7 \end{array}\right] \\
&\sim \left[\begin{array}{cc:cc} 1 & 5/7 & -1/7 & 2/7 \\ 0 & 1 & -3 & 5/2 \end{array}\right] \\
&\sim \left[\begin{array}{cc:cc} 1 & 0 & 2 & -3/2 \\ 0 & 1 & -3 & 5/2 \end{array}\right]
\end{aligned}$$

Thus, the transition matrix S_1 from E to F is $\left[\begin{array}{cc} 2 & -3/2 \\ -3 & 5/2 \end{array}\right]$.

(2) Compute

$$\begin{aligned}
\left[\begin{array}{cc:cc} v_1 & v_2 & w_1 & w_2 \end{array}\right] &= \left[\begin{array}{cc:cc} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right] \\
&\sim \left[\begin{array}{cc:cc} 1 & -2 & -7 & -5 \\ 0 & -2 & -12 & -8 \end{array}\right] \\
&\sim \left[\begin{array}{cc:cc} 1 & 0 & 5 & 3 \\ 0 & -2 & -12 & -8 \end{array}\right] \\
&\sim \left[\begin{array}{cc:cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{array}\right]
\end{aligned}$$

Thus, the transition matrix S_2 from F to E is $\left[\begin{array}{cc} 5 & 3 \\ 6 & 4 \end{array}\right]$.

Note that $S_1S_2 = S_2S_1 = I_2$.

■

2.8 Four Subspaces of a Matrix A

1. The column space of a matrix

Review the problem of solving linear equations.

$$Ax = b$$

$$\downarrow$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\downarrow$$

$$x_1 \underbrace{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}}_{a_1} + x_2 \underbrace{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}}_{a_2} + \cdots + x_n \underbrace{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{a_n} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b}$$

$$\downarrow$$

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b$$

Linear Combination of the Columns of Matrix A

Definition 2.8.1: *The* **column space** *of a matrix A consists of all linear combinations of the columns, denoted by $C(A)$.*

$C(A) = \text{Span}\,(a_1, \cdots .a_n)$ is a subspace of $\mathbb{R}^m$.

Note 2.8.1:

(1) To solve $Ax = b$ is to express b as a linear combination of the columns.

(2) The equation $Ax = b$ is solvable if and only if $b \in C(A)$.

(3) $Ax = b$ *is solvable for every* $b \in \mathbb{R}^m$ *if and only if* $C(A) = \mathbb{R}^m$.

(4) $Ax = b$ *has at most one solution for every* $b \in \mathbb{R}^m$ *if and only if the column vectors of* A *are linearly independent.*

(5) *Let* A *be an* $m \times n$ *matrix.*

If $C(A) = \mathbb{R}^m$, *then* $n \geqslant m$, *since no set of fewer than* m *vectors could span* $\mathbb{R}^m$.

If the column vectors of A *are linearly independent, then* $n \leqslant m$, *since every set of more than* m *vectors in* R^m *is linearly dependent.*

If the column vectors of A *form a basis for* $\mathbb{R}^m$, *(span* $\mathbb{R}^m$ *and linearly independent),* $n = m$.

Example 2.8.1:

$$\boldsymbol{Ax} = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix}$$

Q: The column space of this example is a line, a plane or $\mathbb{R}^3$ *?*

$$\text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix} \right) = \text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right) \text{ is a plane.}$$

■

2. The nullspace of a matrix

Definition 2.8.2: *The* **nullspace** *of an* $m \times n$ *matrix* A *is the set of all solutions of the homogeneous equation* $Ax = 0$, *denoted by* $N(A)$.

$N(A) = \{x \mid Ax = 0\}$ is a subspace of $\mathbb{R}^n$.

Example 2.8.2:

1. *Find the nullspace of* $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

$$N(A) = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$$

2. *Find the nullspace of* $A = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$.

$$N(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)$$

■

3. The row space of a matrix

Definition 2.8.3: *The* **row space** *of an $m \times n$ matrix A is defined as the column space of A^T, denoted by $C\left(A^T\right)$.*

$C(A^T)$ is a subspace of $\mathbb{R}^n$.

4. The left nullspace of a matrix

Definition 2.8.4: *The* **left nullspace** *of an $m \times n$ matrix A is defined as the nullspace of A^T, denoted by $N\left(A^T\right)$.*

$N\left(A^T\right)$ is the set of all solutions of the homogeneous equation $A^T y = 0$.

$N\left(A^T\right)$ is a subspace of $\mathbb{R}^m$.

Example 2.8.3: $m = 3, \quad n = 5.$

$$Ax = \begin{bmatrix} 1 & 4 & 5 & -3 & 6 \\ 3 & 2 & 5 & 1 & 8 \\ 2 & 1 & 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 6 \\ 8 \\ 5 \end{bmatrix}$$

The reduced row echelon form of A:

$$\begin{bmatrix} 1 & 4 & 5 & -3 & 6 \\ 3 & 2 & 5 & 1 & 8 \\ 2 & 1 & 3 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 5 & -3 & 6 \\ 0 & -10 & -10 & 10 & -10 \\ 0 & -7 & -7 & 7 & -7 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 4 & 5 & -3 & 6 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 & 5 & -3 & 6 \\ 3 & 2 & 5 & 1 & 8 \\ 2 & 1 & 3 & 1 & 5 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 2 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 1 \end{bmatrix}}_{R}$$

The column space $C(A) = \operatorname{Span}(a_1, a_2, a_3, a_4, a_5) = \operatorname{Span}(a_1, a_2)$.

A basis of $C(A)$ *is* $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right\}$, $\quad \dim C(A) = 2$.

$$A^T = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 2 & 1 \\ 5 & 5 & 3 \\ -3 & 1 & 1 \\ 6 & 8 & 5 \end{bmatrix} = R^T C^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 4 & 2 & 1 \end{bmatrix}$$

A basis of $C\left(A^T\right)$ *is* $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$, $\quad \dim C\left(A^T\right) = 2$.

$\dim C(A) = \dim C(A^T) = \mathrm{Rank}$ **of A.**
The columns of C form a basis for the column space.
The rows of R form a basis for the row space.

$$N(A): \quad Ax = 0 \quad \Leftrightarrow \quad Rx = 0 \quad \Leftrightarrow \quad \begin{array}{l} x_1 + x_3 + x_4 + 2x_5 = 0 \\ x_2 + x_3 - x_4 + x_5 = 0 \end{array}$$

The general solution is

$$\begin{aligned} x_1 &= -x_3 - x_4 - 2x_5 \\ x_2 &= -x_3 + x_4 - x_5 \\ x_3 &= x_3 \\ x_4 &= \phantom{-x_3 - {}} x_4 \\ x_5 &= \phantom{-x_3 - x_4 - 2{}} x_5 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A basis of $N(A)$ *is* $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, $\quad \dim N(A) = 3.$

$N\left(A^T\right)$:

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 2 & 1 \\ 5 & 5 & 3 \\ -3 & 1 & 1 \\ 6 & 8 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 \\ 0 & -10 & -7 \\ 0 & -10 & -7 \\ 0 & 10 & 7 \\ 0 & -10 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0.7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -0.1 \\ 0 & 1 & 0.7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis of $N\left(A^T\right)$ *is* $\left\{ \begin{bmatrix} 0.1 \\ -0.7 \\ 1 \end{bmatrix} \right\}$, $\quad \dim N\left(A^T\right) = 1.$

■

Summary:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 5 & -3 & 6 \\ 3 & 2 & 5 & 1 & 8 \\ 2 & 1 & 3 & 1 & 5 \end{bmatrix}$$

$\mathrm{m} = 3, \mathrm{n} = 5$

A basis of $\boldsymbol{C}(\boldsymbol{A})$ is $\left\{\begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}4\\2\\1\end{bmatrix}\right\}$, $\quad \dim \boldsymbol{C}(\boldsymbol{A}) = 2.$

A basis of $\boldsymbol{C}\left(\boldsymbol{A}^T\right)$ is $\left\{\begin{bmatrix}1\\0\\1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\\-1\\1\end{bmatrix}\right\}$, $\quad \dim \boldsymbol{C}\left(\boldsymbol{A}^{\mathrm{T}}\right) = 2.$

A basis of $\boldsymbol{N}(\boldsymbol{A})$ is $\left\{\begin{bmatrix}-1\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\-1\\0\\0\\1\end{bmatrix}\right\}$, $\quad \dim \boldsymbol{N}(\boldsymbol{A}) = 3.$

A basis of $\boldsymbol{N}\left(\boldsymbol{A}^T\right)$ is $\left\{\begin{bmatrix}0.1\\-0.7\\1\end{bmatrix}\right\}$, $\quad \dim \boldsymbol{N}\left(\boldsymbol{A}^T\right) = 1.$

Fundamental Theorem of Linear Algebra:Part 1

$$\dim C(A) = \dim C\left(A^T\right) = r$$
$$\dim N\left(A^T\right) = m - r, \dim N(A) = n - r$$

Proof: For $A \in \mathbb{R}^{m\times n}$, $\dim C(A) = r$, a basis of $C(A)$ is $\{c_1, \ldots, c_r\}$. We have

$$A = \begin{bmatrix} c_1 & c_2 & \cdots & c_r \end{bmatrix}_{m\times r} \begin{bmatrix} r_1^\top \\ r_2^\top \\ \vdots \\ r_r^\top \end{bmatrix}_{r\times n}$$

$$A^T = \begin{bmatrix} r_1 & r_2 & \cdots & r_r \end{bmatrix} \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_r^T \end{bmatrix} \Rightarrow C\left(A^T\right) \subset \operatorname{Span}\left(r_1, r_2, \cdots, r_r\right)$$

We want to prove that $\dim C(A^T) = r$. Note $\dim C(A^T) \leq r$.

Assume that $\dim C(A^T) = k < r$. Let $r'_1, \ldots, r'_k$ be a basis of $C(A^T)$. Then

$$\mathrm{Span}\,(r'_1, r'_2, \ldots, r'_k) = C(A^T)$$

$$A^T = \begin{bmatrix} r'_1 & r'_2 & \cdots & r'_k \end{bmatrix} \begin{bmatrix} c_1'^{\top} \\ c_2'^{\top} \\ \vdots \\ c_k'^{\top} \end{bmatrix}_{k\times m}$$

$$A = \begin{bmatrix} c'_1 & c'_2 & \cdots & c'_k \end{bmatrix} \begin{bmatrix} r_1'^{\top} \\ r_2'^{\top} \\ \vdots \\ r_k'^{\top} \end{bmatrix}$$

which implies that $C(A) \subset \mathrm{Span}(c'_1, \ldots, c'_k)$, which is a contradiction with $\dim C(A) = r$.

Let R be the reduced row echelon form of A. The equation $Ax = 0$ is equivalent to the equation $Rx = 0$. If $\dim C(A) = r$, then R will have r nonzero rows, and consequently the equation $Rx = 0$ will involve r pivots and $n - r$ free variables. And $\dim N(A)$ will equal the number of free variables. ■

2.9 Orthogonality of the Four Subspaces

Definition 2.9.1: *Two vectors x and y in $\mathbb{R}^n$ are said to be* **orthogonal** *if $x^T y = 0$.*

Definition 2.9.2: *Two subspaces X and Y in $\mathbb{R}^n$ are said to be* **orthogonal** *if $x^T y = 0$ for every $x \in X$ and every $y \in Y$. If X and Y are orthogonal, we write $X \perp Y$.*

Example 2.9.1: *The floor of our room is a subspace X in $\mathbb{R}^3$. The line where two walls meet is a subspace Y in $\mathbb{R}^3$. $X \perp Y$.*

Two walls look perpendicular but these two subspaces are not orthogonal! ■

Definition 2.9.3: *The* **orthogonal complement** *of a subspace Y in $\mathbb{R}^n$ is the set of all vectors that are orthogonal to every vector in Y, denoted $Y^\perp$. Thus*

$$Y^\perp = \left\{ x \in \mathbb{R}^n \mid x^\top y = 0 \textit{ for every } y \in Y \right\}$$

Example 2.9.2: *In $\mathbb{R}^3$, $X = \text{Span}\,(e_1)\,, Y = \text{Span}\,(e_2)$.*

$$X \perp Y? \longrightarrow X^\perp = \text{Span}\,(e_2, e_3) \quad Y^\perp = \text{Span}\,(e_1, e_3)$$

■

Theorem 2.9.1: *If X and Y are orthogonal subspaces of $\mathbb{R}^n$, then $X \cap Y = \{0\}$.*

Proof: If $x \in X \cap Y$ and $X \perp Y, x^\top x = 0 \Rightarrow x = 0$. ■

Theorem 2.9.2: *If Y is a subspace of $\mathbb{R}^n$, then $Y^\perp$ is also a subspace of $\mathbb{R}^n$.*

Proof: $0 \in Y^\perp$.

If $x_1, x_2 \in Y^\perp$, for any $y \in Y, x_1^\top y = 0, x_2^\top y = 0$. Then we have

$$(x_1 + x_2)^\top y = x_1^\top y + x_2^\top y = 0.$$

For a scalar α,

$$(\alpha x_1)^\top y = \alpha x_1^\top y = 0.$$

■

Four Fundamental Subspaces of $A \in \mathbb{R}^{m\times n}$:

$C(A), N(A^T)$: subspaces of $\mathbb{R}^m$; $\quad C(A^T), N(A)$: subspaces of $\mathbb{R}^n$.

$$\dim C(A) = \dim C(A^T) = r$$
$$\dim N(A) = n - r, \quad \dim N(A^T) = m - r.$$

Fundamental Theorem of Linear Algebra: Part 2

$$N(A) = C\left(A^T\right)^{\perp} \text{ and } N\left(A^T\right) = C(A)^{\perp}$$

$N(A)$ is the orthogonal complement of the row space $C\left(A^T\right)$ (in $\mathbb{R}^n$).
$N\left(A^T\right)$ is the orthogonal complement of the column space $C(A)$ (in $\mathbb{R}^m$).

Proof: For any $x \in N(A)$, $Ax = 0$.

$$\begin{bmatrix} \text{row 1 of} & A \\ \vdots & \\ \text{row m of} & A \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$$

$N(A) \perp C\left(A^T\right) \Rightarrow N(A) \subset C\left(A^T\right)^{\perp}$.
For any $x \in C(A^T)^{\perp}$, x is orthogonal to each of the column vectors of A^T, and consequently $Ax = 0$. Thus $x \in N(A)$ and then $C\left(A^T\right)^{\perp} \subset N(A)$. ■

Theorem 2.9.3: *If S is a subspace of $\mathbb{R}^n$, then $\dim S + \dim S^{\perp} = n$. Furthermore, if $\{x_1, \cdots, x_r\}$ is a basis for S and $\{x_{r+1}, \cdots, x_n\}$ is a basis for $S^{\perp}$, then $\{x_1, \cdots, x_r, x_{r+1}, \cdots, x_n\}$ is a basis for $\mathbb{R}^n$.*

Proof: If $S = \{0\}$, then $S^{\perp} = \mathbb{R}^n$ and

$$\dim S + \dim S^{\perp} = 0 + n = n.$$

If $S \neq \{0\}$, then let $\{x_1, \cdots, x_r\}$ be a basis for S. Define

$$A = \begin{bmatrix} x_1^T \\ \vdots \\ x_r^T \end{bmatrix} \in \mathbb{R}^{r\times n}, \quad A^T = \begin{bmatrix} x_1 & \cdots & x_r \end{bmatrix} \in R^{n\times r}.$$

Clearly, $C\left(A^T\right) = \text{Span}\left(x_1, \cdots, x_r\right)$. Since $x_1, \cdots, x_r$ are linearly independent, $\{x_1, \cdots, x_r\}$ is a basis of $C\left(A^T\right)$. Then $C\left(A^T\right) = S$, and $\dim S = \dim C\left(A^T\right) = r$.

$$S^{\perp} = C\left(A^{\top}\right)^{\perp} = N(A) \Rightarrow \dim S^{\perp} = \dim N(A) = n - r.$$

We have $\dim S + \dim S^{\perp} = n$.

Suppose that $\underbrace{c_1 x_1 + \cdots + c_r x_r}_{\in S} + \underbrace{c_{r+1} x_{r+1} + \cdots + c_n x_n}_{\in S^{\perp}} = 0$. Let $y = c_1 x_1 + \cdots + c_r x_r$ and $z = c_{r+1} x_{r+1} + \cdots + c_n x_n$. We then have

$$y + z = 0$$

Since $y \in S$ and $z \in S^{\perp}, y^T z = 0$. Therefore, $(-z)^T z = 0 \Rightarrow z = 0$ and $y = 0$.

Since $x_1, \cdots, x_r$ are linearly independent and $x_{r+1}, \cdots, x_n$ are linearly independent, $c_1 = \cdots = c_r = 0$ and $c_{r+1} = \cdots = c_n = 0$. So $x_1, \cdots, x_n$ are linearly independent and thus form a basis for $\mathbb{R}^n$ since $\dim \mathbb{R}^n = n$. ■

Theorem 2.9.4: *If S is a subspace of $\mathbb{R}^n$, then*

$$\mathbb{R}^n = S \oplus S^{\perp}.$$

Proof: When $S = \{0\}$ or $S = \mathbb{R}^n$, the result is trivial.

Assume that $\dim S = r, 0 < r < n$. Let $\{x_1, \cdots, x_r\}$ be a basis for S and $\{x_{r+1}, \cdots, x_n\}$ a basis for $S^{\perp}$. Then for any $x \in \mathbb{R}^n$,

$$x = \underbrace{c_1 x_1 + \cdots + c_r x_r}_{\in S} + \underbrace{c_{r+1} x_{r+1} + \cdots + c_n x_n}_{\in S^{\perp}}.$$

Let $y = c_1 x_1 + \cdots + c_r x_r$ and $z = c_{r+1} x_{r+1} + \cdots + c_n x_n$. Then $x = y + z, y \in S$, and $z \in S^{\perp}$. Thus $\mathbb{R}^n = S + S^{\perp}$.

Note that $S \cap S^{\perp} = \{0\}$. We can conclude from Theorem 2.3.3 that $\mathbb{R}^n = S \oplus S^{\perp}$.

Theorem 2.9.5: *If S is a subspace of R^n, then $\left(S^{\perp}\right)^{\perp} = S$.*

Proof: On the one hand, if $x \in S$, then x is orthogonal to each y in $S^{\perp}$. Therefore, $x \in \left(S^{\perp}\right)^{\perp}$ and $S \subset \left(S^{\perp}\right)^{\perp}$.

On the other hand, for any $z \in \left(S^{\perp}\right)^{\perp}$, since $z \in \mathbb{R}^n$, $z = u + v$, where $u \in S$ and $v \in S^{\perp}$.

Note $v^T u = 0$ and $v^T z = 0$.

$$0 = v^{\top} z = v^{\top}(u + v) = v^{\top} u + v^{\top} v = v^{\top} v \Rightarrow v = 0.$$

Therefore $z = u \in S$, and hence $\left(S^{\perp}\right)^{\perp} \subset S$. ■

Note 2.9.1: *If T is the orthogonal complement of a subspace S, then S is the orthogonal complement of T, and we may say simply that S and T are orthogonal complements.*

Note 2.9.2: *Two pairs of orthogonal subspaces.*

$$N(A) \perp C\left(A^{\top}\right) \text{ (dimensions: } n-r \perp r\text{)}$$
$$N\left(A^{\top}\right) \perp C(A) \text{ (dimensions: } m-r \perp r\text{)}$$
$$\mathbb{R}^n = N(A) \oplus C(A^T), \quad \mathbb{R}^m = N\left(A^T\right) \oplus C(A)$$

Big Picture of Linear Algebra: