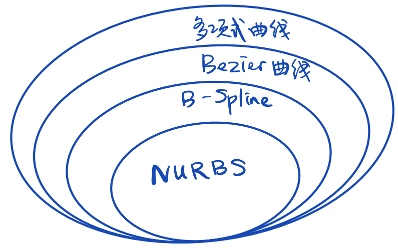
样条曲线的数学表达与样条拟合



1、 $C(u) = [a_0 \ a_1 \ \cdots \ a_n] \begin{bmatrix} 1 \\ u \\ \vdots \\ u^n \end{bmatrix} = [a_i]^T [u^i]$ 多项式

对于交互式设计是不自然的

2、 Bezier 曲线：

$C(u) = (1-u) P_0 + u P_1$ 两个点 即为直线

$C(u) = (1-u)^2 P_0 + 2u(1-u) P_1 + u^2 P_2$ 三个点 抛物线

$$\begin{cases} \frac{2!}{0!2!} (1-u)^2 = (1-u)^2 \\ \frac{2!}{1!1!} u(1-u) = 2u(1-u) \\ \frac{2!}{2!0!} u^2 = u^2 \end{cases}$$

$C(u) = (1-u)^3 P_0 + 3u(1-u)^2 P_1 + 3u^2(1-u) P_2 + u^3 P_3$ 四个点

$$\begin{cases} \frac{3!}{0!3!} u^0 (1-u)^3 \\ \frac{3!}{1!2!} u(1-u)^2 \quad (\to 6, \ \to 2) \\ \frac{3!}{2!1!} u^2(1-u) \\ \frac{3!}{3!0!} u^3 \end{cases}$$

通式

$$\boxed{\frac{n!}{i!(n-i)!} u^i (1-u)^{n-i}}$$

依此类推

Bezier 曲线具有凸包性，即曲线包含在其控制点的凸包中

3、 B-Spline （B样条曲线）

$N_{i,p}(u)$ p次B样条基函数 $C(u) = \sum_{i=0}^{n} N_{i,p}(u) P_i$ 控制点

$$N_{i,0}(u) = \begin{cases} 1 & u_i \le u < u_{i+1} \\ 0 & \text{其他} \end{cases}$$

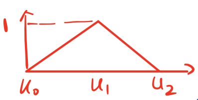
$$N_{i,p}(u) = \frac{u-u_i}{u_{i+p}-u_i} N_{i,p-1}(u) + \frac{u_{i+p+1}-u}{u_{i+p+1}-u_{i+1}} N_{i+1,p-1}(u)$$

控制点数

节点向量 $U = \{\underbrace{a, \cdots, a}_{p+1个}, \underbrace{u_{p+1}, \cdots, u_{m-p-1}}_{控制点数}, \underbrace{b, \cdots, b}_{p+1个}\}$

$u_0 \le u < u_1$ 时为1 $u_1 \le u < u_2$ 时为1

$$N_{\underset{p}{0,1}}(u) = \frac{u-u_0}{u_1-u_0} N_{\underset{i \ p-1}{0,0}}(u) + \frac{u_2-u}{u_2-u_1} N_{\underset{i+1 \ p-1}{1,0}}(u)$$



$u_0 \sim u_1$ $N_{0,0}$

$u_0 \sim u_2$ $N_{0,1}$ i+1

$u_1 \sim u_2$ $N_{1,0}$

$u_2 \sim u_3$ $N_{2,0}$

$u_1 \sim u_3$ $N_{1,1}$ i+1

$u_0 \sim u_3$ $u_0 \sim u_2$ $u_1 \sim u_3$

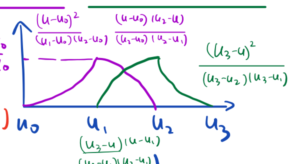
$$N_{0,2} = \frac{u-u_0}{u_2-u_0} N_{0,1}(u) + \frac{u_3-u}{u_3-u_1} N_{1,1}(u)$$

$\frac{(u-u_0)^2}{(u_1-u_0)(u_2-u_0)}$ $\frac{(u-u_0)(u_2-u)}{(u_1-u_0)(u_2-u_0)}$ $\frac{(u_3-u)(u-u_1)}{(u_3-u_1)(u_2-u_1)}$

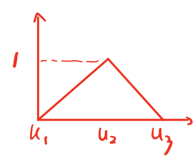
归一化线性组合

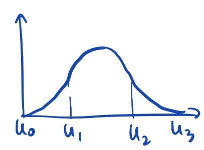
$$\frac{u-u_1}{u_2-u_1} N_{1,0}(u) + \frac{u_3-u}{u_3-u_2} N_{2,0}(u)$$

$u_1 \le u < u_2$ 为1 $u_2 \le u < u_3$ 时为1



$N_{1,1}$



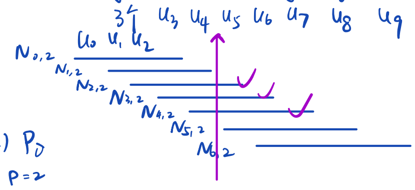


例 $p=2$, $U = \{0,0,0,1,2,3,4,5,5,5\}$

求 $C(2.5) = ?$

$p=2$, 2次 $U = \{\underbrace{0,0,0}_{3个}, \overbrace{1,2,3,4,}^{控制点数} \overbrace{5,5,5}^{次数+1}\}$

$u_0 \ u_1 \ u_2 \ u_3 \ u_4 \ u_5 \ u_6 \ u_7 \ u_8 \ u_9$



$\sum N_{i,\underline{p}}(u) P_i \quad p=2$

$C(2.5) = N_{2,2}(\frac{5}{2}) P_2 + N_{3,2}(\frac{5}{2}) P_3 + N_{4,2}(\frac{5}{2}) P_4$

例：写出基函数 $N_{0,3}$ ？

$N_{0,0}$ $N_{1,0}$ $N_{2,0}$ $N_{3,0}$ $N_{4,0}$

$N_{0,1}$ $N_{1,1}$ $N_{2,1}$ $N_{3,1}$

$N_{0,2}$ $N_{1,2}$ $N_{2,2}$

$N_{0,3}$ $N_{1,3}$

$$N_{0,3} = \frac{u-u_0}{u_{\underset{0+2+1}{3}}-u_0} N_{0,2}(u) + \frac{u_4-u}{u_4-u_{\underset{1+2+1}{1}}} N_{1,2}(u) \quad p=3$$

4、 NURBS：

$$C(u) = \frac{\sum_{i=0}^{n} N_{i,p}(u) w_i P_i}{\sum_{i=0}^{n} N_{i,p}(u) w_i}$$

（Non Uniform Rational B-Spline）

5、分段三次样条　PCS

目标：在给定待拟合点时，确定方程组并求解系数

$$f_i(u) = D_i + C_i(u-u_i) + B_i(u-u_i)^2 + A_i(u-u_i)^3 \qquad u \in [u_i, u_{i+1}]$$

已知：待拟合各点：(u_i, t_i)

则有 ① $f_i(u_i) = t_i = D_i$

② $f_i''(u_i) = 2B_i$　记 $f_i''(u_i) = m_i$　则 $B_i = \frac{1}{2} m_i$

③ $$\underbrace{\underbrace{f_i(u_{i+1})}_{t_{i+1}} = \underbrace{D_i}_{t_i}}_{t_{i+1} - t_i} + C_i \underbrace{(u_{i+1}-u_i)}_{\text{记为}h_i} + B_i \underbrace{(u_{i+1}-u_i)^2}_{h_i^2} + A_i \underbrace{(u_{i+1}-u_i)^3}_{h_i^3}$$

则 $$\underbrace{\frac{t_{i+1}-t_i}{h_i}}_{\text{记为 } d_i} = C_i + B_i h_i + A_i h_i^2$$

则 $$C_i = d_i - \frac{h_i(m_{i+1}-m_i) + 3m_i h_i}{6} = d_i - \frac{h_i(m_{i+1}+2m_i)}{6}$$

④ 又有 $$\underbrace{f_i''(u_{i+1})}_{m_{i+1}} = \underbrace{2B_i}_{m_i} + 6A_i \underbrace{(u_{i+1}-u_i)}_{h_i}$$

$$\Rightarrow A_i = \frac{m_{i+1}-m_i}{6h_i}$$

由 $f_i'(u_i) = C_i \overset{\text{连续}}{=} f_{i-1}'(u_i) = C_{i-1} + 2B_{i-1}(u_i - u_{i-1}) + 3A_i(u_i - u_{i-1})^2$

$$\Rightarrow d_i - \frac{h_i(m_{i+1}+2m_i)}{6} = d_{i-1} - \frac{h_{i-1}(m_i+2m_{i-1})}{6} + m_{i-1}h_{i-1} + \frac{1}{2}h_{i-1}(m_i - m_{i-1})$$

$$\Rightarrow d_i - d_{i-1} = \frac{1}{6}h_i m_{i+1} + \frac{1}{6}\left[2m_i h_i + 2m_{i-1} h_i\right] + \frac{1}{6}h_{i-1} m_{i-1}$$

$$\Rightarrow \underbrace{6(d_i - d_{i-1})}_{w_i} = h_i m_{i+1} + 2h_i(m_i + m_{i-1}) + h_{i-1} m_{i-1}$$

$i = 1, 2, \cdots, N-1$　　$N-1$ 个方程，$N+1$ 个未知数

增加两个边界条件：

- 自然样条　$m_0 = m_N = 0$
- 或
- 夹持样条　$f'(u_0) = C_0 = d_0 - \frac{h_0(m_1+2m_0)}{6}$

$f'(u_0)$ ↓ 指定值

$$m_0 = \frac{6(d_0 - f'(u_0))}{2h_0} - \frac{m_1}{2} = \frac{3}{h_0}\left[d_0 - f'(u_0)\right] - \frac{m_1}{2}$$

类似地，指定终端条件 可约束 m_N

可写为线性方程组 $AM=V$，其中对于夹持样条，A 为三对角矩阵，可用追赶法（Thomas 算法）求解。

6、NUPCS (Nearly Unit Parameterized Cubic Splines)

$P_i(u) = A_i u^3 + B_i u^2 + C_i u + D_i \quad u \in [0,1]$ $\qquad AM = V$ 中的 A 是固定的

$M = A^{-1} B \qquad A^{-1}$ 行中有些项很小可舍去，进一步降低计算量

7. 给定一系列点，尝试用 p次 B样条来近似。

$Q_k = C(\bar{u}_k) = \sum_{i=0}^{n} N_{i,p}(\bar{u}_k) P_i \qquad$ 设 $\forall k,\ \bar{u}_k \in [0,1]$ $\qquad$ 现要为每个拟合点分配 $\bar{u}_k$ 值

↓

一个控制点 P_k 对应一个拟合点 Q_k

$\bar{u}_k$ 的选取有以下三种规则：

① 均匀：$\bar{u}_0 = 0, \quad \bar{u}_k = \frac{k}{n} \quad k = 1, \cdots, n-1 \quad \bar{u}_n = 1$ $\qquad$ 不推荐

② 长度（弦长）：$\quad d = \sum_{k=1}^{n} |Q_k - Q_{k-1}|$

$\bar{u}_0 = 0, \quad \bar{u}_n = 1, \quad \bar{u}_k = \bar{u}_{k-1} + \frac{|Q_k - Q_{k-1}|}{d}$

③ Centripetal（向心）：$\quad d = \sum_{k=1}^{n} \sqrt{|Q_k - Q_{k-1}|}$

$\bar{u}_0 = 0, \quad \bar{u}_n = 1, \quad \bar{u}_k = \bar{u}_{k-1} + \frac{\sqrt{|Q_k - Q_{k-1}|}}{d}$

再用公式 $u_{j+p} = \frac{1}{p} \sum_{i=j}^{j+p-1} \bar{u}_i$ 进行平均化

作为节点向量

也可直接取平均 $u_{j+p} = \frac{j}{n-p+1}$（不推荐）

例 令 $Q_k = \{(0,0), (3,4), (-1,4), (-4,0), (-4,-3)\}$ $\qquad$ Interpolate it with a cubic curve.

$\qquad\qquad 5 \qquad 4 \qquad 5 \qquad 3$

$\begin{array}{ccccc} P_0 & P_1 & P_2 & P_3 & P_4 \\ Q_0 & Q_1 & Q_2 & Q_3 & Q_4 \end{array}$ 对应

用上述规则②

总长为 17，则 $\bar{u}_0 = 0, \ \bar{u}_1 = \frac{5}{17}, \ \bar{u}_2 = \frac{9}{17}, \ \bar{u}_3 = \frac{14}{17}, \ \bar{u}_4 = 1$

$N_{i,3}$

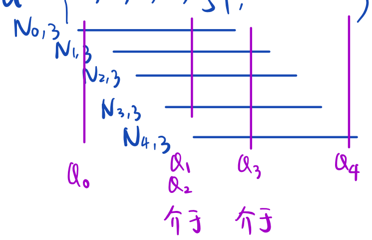
↓

u 形如 $\{\underbrace{a,a,a,a,u_4}_{\text{控制点数}}, \underbrace{b,b,b,b}_{\text{次数}+1}\}$

u_{1+3}

则 $u_4 = \frac{1}{3} \sum_{i=1}^{3} \bar{u}_i = \frac{1}{3}\left(\frac{5}{17} + \frac{9}{17} + \frac{14}{17}\right) = \frac{28}{51}$

$\Rightarrow u = \{\overset{u_0}{0}, 0, 0, 0, \overset{u_4}{\frac{28}{51}}, 1, 1, 1, \overset{u_8}{1}\}$



则 $\sum N_{i,3}(u_k) P_k = Q_k \Rightarrow$

$$\begin{cases} k=0: & N_{0,3}(\overset{\bar{u}_0}{0}) P_0 = Q_0 \\ k=1: & \sum_{i=0}^{3} N_{i,3}(\bar{u}_1) P_1 = Q_1 \\ k=2: & \sum_{i=0}^{3} N_{i,3}(\bar{u}_2) P_2 = Q_2 \\ k=3: & \sum_{i=1}^{4} N_{i,3}(\bar{u}_3) P_3 = Q_3 \\ k=4: & N_{4,3}(\underset{\bar{u}_4}{1}) P_4 = Q_4 \end{cases}$$

长线前后分开插值；急转前后分开插值。

8、最小二乘曲线逼近 (Least Squares Curve Approximation, LSCA)

与上面的区别：控制点数少于待拟合点数　　假设有 n 个控制点，k 个待拟合点，且 $n<k$，

步骤：

(1) 指定 P_0, P_N，使 $P_0=Q_0$，$P_n=Q_K$

(2) 选取 $\bar{u}_k$　$k=0,1,\cdots,K$　规则与上页所述相同（一般用规则2或3）

(3) 确定节点向量　$u_{p+i}=\frac{1}{p}\sum_{j=\alpha}^{\alpha+p-1}u_j$　注意 α 取为 $\mathrm{int}\left(\frac{i\cdot k}{n}\right)$ 而不像之前那样直接取为 i

比如 $k=9$　$n=5$　$p=3$　则 $U=\{\underbrace{0,0,0,0,u_4}_{\text{点数}},\underbrace{1,1,1,1}_{p+1}\}$

$$\underset{3+1}{u_4}=\frac{1}{3}\sum_{j=2}^{2+3-1}\bar{u}_j=\frac{1}{3}(\bar{u}_2+\bar{u}_3+\bar{u}_4)$$

$$\alpha=\mathrm{int}\left(1\times\frac{9}{5}\right)=2$$

(4) 构建平方误差：

$$f=\sum_{k=1}^{K-1}\left|Q_k-C(\bar{u}_k)\right|^2=\sum_{k=1}^{K-1}\left|Q_k-\sum_{i=0}^{n}N_{i,p}(\bar{u}_k)P_i\right|^2=\sum_{k=1}^{K-1}\left|R_k-\sum_{i=1}^{n-1}N_{i,p}(\bar{u}_k)P_i\right|^2$$

$k=0$, N 完美拟合，无需考虑

$R_k = Q_k-N_{0,p}Q_0-N_{n,p}(\bar{u}_k)\underset{(P_n)}{Q_K}$,

则精确匹配时应满足　$NP=R$

其中 $N=\begin{bmatrix}N_{1,p}(\bar{u}_1) & N_{2,p}(\bar{u}_1) & \cdots & N_{n-1,p}(\bar{u}_1)\\ \vdots & & & \\ N_{1,p}(\bar{u}_{k-1}) & N_{2,p}(\bar{u}_{k-1}) & \cdots & N_{n-1,p}(\bar{u}_{k-1})\end{bmatrix}_{(k-1)\times(n-1)}$，　$P=\begin{bmatrix}P_1\\P_2\\\vdots\\P_{n-1}\end{bmatrix}$，　$R=\begin{bmatrix}R_1\\\vdots\\R_{k-1}\end{bmatrix}$

则相应的法方程（最小二乘时满足的方程）应为 $\boxed{N^TNP=N^TR}$ → 由此获得最小二乘时 P

(5) 检验误差

拟合方法常需迭代进行，主要有两种策略：

1、一开始就拟合一整段，如果不符合要求就往回（缩小范围），直至拟合成功，再从末尾点往前走；

2、一开始只拟合一小段，若符合就扩大范围，直至不符合要求，则退一步并从末尾点往前走。

But, the best results are obtained with new files made with the right CAM process in mind. You can use fitting method with old files but the results may not be as optimal.

还有变控制点数的方法（见 ppt）