Optimal Estimation homework 3

Li Yude 24S053054

December 29, 2024

1 About the Kalman Filter

Estimating the position of a car. Figure 1 shows the estimation problem using a nonlinear measurement. The process model has been linearized and discretized as,

$$\begin{aligned} x_k &= f(x_{k-1}, u_{k-1}, w_{k-1}) \\ &= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} x_{k-1} + \begin{bmatrix} 0 \\ \Delta t \end{bmatrix} u_{k-1} + w_{k-1} \end{aligned}$$

The nonlinear measurement equation is,

$$y_k = \phi_k = h(p_k, v_k) = \arctan\left(\frac{S}{D - p_k}\right) + v_k$$

The process noise and measurement noise are assumed to be white noise, i.e.,

$$\begin{aligned} v_k &\sim N(0, 0.05) \\ w_k &\sim N(0, (0.1) \cdot I_{2\times 2}) \end{aligned}$$

The initial state is

$$x_0 \sim N\left(\begin{bmatrix} 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

The sample instant is $\Delta t = 0.5s$, the initial input is $u_0 = -2m/s^2$, the measurements available are

$$\begin{aligned} y_1 &= 20\text{deg}, \\ S &= 20m, \\ D &= 60m \end{aligned}$$

(1) Try to derive the Extended Kalman filter and Unscented Kalman filter estimate for $\hat{x}_1, P_1$, compare the 2 results.

Solution

Extended Kalman filter:

Motion model Jacobians:

$$F_k = \frac{\partial f}{\partial x_k}|_{\hat{x}_k, u_k, 0} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

$$L_k = \frac{\partial f_k}{\partial w_k}|_{\hat{x}_k, u_k, 0} = I_{2\times 2}$$

Measurement model Jacobians:

$$H_k = \frac{\partial h_k}{\partial x_k}|_{\check{x}_k,0} = \begin{bmatrix} \frac{S}{(D-\check{p}_k)^2+S^2} & 0 \end{bmatrix}$$

$$M_k = \frac{\partial h_k}{\partial v_k}|_{\check{x}_k,0} = 1$$

we have

$$F_0 = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}, L_0 = I_{2\times 2}$$

$$P_0 = \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix}, Q_0 = 0.1 \times I_{2\times 2}$$

So,

$$\begin{aligned}
\check{P}_1 =& F_0 \hat{P}_0 F_0^T + L_0 Q_0 L_0^T \\
=& \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}^T + 0.1 \times I_{2\times 2} \\
=& \begin{bmatrix} 0.36 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}
\end{aligned}$$

$$\check{x}_1 = f(x_0, u_0, 0) = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} \frac{20}{(60-2.5)^2+20^2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{16}{2965} & 0 \end{bmatrix}, M_1 = 1$$

$$\begin{aligned}
K_1 &= \check{P}_1 H_1^T (H_1 \check{P}_1 H_1^T + M_1 R_1 M_1^T)^{-1} \\
&= \begin{bmatrix} 0.36 & 0.5 \\ 0.5 & 1.1 \end{bmatrix} \begin{bmatrix} \frac{16}{2965} \\ 0 \end{bmatrix} (\begin{bmatrix} \frac{16}{2965} & 0 \end{bmatrix} \begin{bmatrix} 0.36 & 0.5 \\ 0.5 & 1.1 \end{bmatrix} \begin{bmatrix} \frac{16}{2965} \\ 0 \end{bmatrix} + 0.05)^{-1} \\
&= \begin{bmatrix} 0.03884 \\ 0.05395 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\hat{x}_1 &= \check{x}_1 + K_1 \left[y_1 - h_1(\check{x}_1, 0) \right] \\
&= \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} + \begin{bmatrix} 0.03884 \\ 0.05395 \end{bmatrix} (0.349 - 0.3347) \\
&= \begin{bmatrix} 2.50055 \\ 4.00077 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\hat{P}_1 &= (I - K_1 H_1) \check{P}_1 \\
&= (I - \begin{bmatrix} 0.03884 \\ 0.05395 \end{bmatrix} \begin{bmatrix} \frac{16}{2965} & 0 \end{bmatrix}) \begin{bmatrix} 0.36 & 0.5 \\ 0.5 & 1.1 \end{bmatrix} \\
&= \begin{bmatrix} 0.35992 & 0.49989 \\ 0.49989 & 1.09985 \end{bmatrix}
\end{aligned}$$

Unscented Kalman filter:

Prediction:
we have $n = 2$, we take $\lambda = 1$

$$\hat{L}_0 = \sqrt{\hat{P}_0} = \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\hat{X}_0^{(0)} = \hat{x}_0 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\hat{X}_0^{(1)} = \hat{x}_0 + \sqrt{3} \left[\sqrt{\hat{P}_0} \right]_1 = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + \sqrt{3} \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1732 \\ 5 \end{bmatrix}$$

$$\hat{X}_0^{(2)} = \hat{x}_0 + \sqrt{3} \left[\sqrt{\hat{P}_0} \right]_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + \sqrt{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6.7321 \end{bmatrix}$$

$$\hat{X}_0^{(3)} = \hat{x}_0 - \sqrt{3} \left[\sqrt{\hat{P}_0} \right]_1 = \begin{bmatrix} 0 \\ 5 \end{bmatrix} - \sqrt{3} \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.1732 \\ 5 \end{bmatrix}$$

$$\hat{X}_0^{(4)} = \hat{x}_0 - \sqrt{3} \left[\sqrt{\hat{P}_0} \right]_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix} - \sqrt{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3.2679 \end{bmatrix}$$

$$\check{X}_1^{(0)} = f(\hat{X}_0^{(0)}, u_0, 0) = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix}$$

$$\check{X}_1^{(1)} = f(\hat{X}_0^{(0)}, u_0, 0) = \begin{bmatrix} 2.6732 \\ 4 \end{bmatrix}$$

$$\check{X}_1^{(2)} = f(\hat{X}_0^{(0)}, u_0, 0) = \begin{bmatrix} 3.366 \\ 5.7321 \end{bmatrix}$$

$$\check{X}_1^{(3)} = f(\hat{X}_0^{(0)}, u_0, 0) = \begin{bmatrix} 2.3268 \\ 4 \end{bmatrix}$$

$$\check{X}_1^{(3)} = f(\hat{X}_0^{(0)}, u_0, 0) = \begin{bmatrix} 1.634 \\ 2.2679 \end{bmatrix}$$

$$W_0^{(m)} = W_0^{(c)} = \frac{1}{3}, W_i^{(m)} = W_i^{(c)} = \frac{1}{6}$$

$$\check{x}_1 = \sum_i W_i^{(m)} \check{X}_1^{(i)} = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix}$$

$$\check{P}_1 = \sum_i W_i^{(c)} (\check{X}_1^{(i)} - \check{x}_1)(\check{X}_1^{(i)} - \check{x}_1)^T + Q_0 = \begin{bmatrix} 0.36 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}$$

Correction:

$$\check{L}_1 \check{L}_1^T = \check{P}_1$$

$$\check{L}_1 = \sqrt{\check{P}_1} = \begin{bmatrix} 0.6 & 0.8333 \\ 0 & 0.6368 \end{bmatrix}$$

$$\check{X}_1^{(0)} = \check{x}_1 = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix}$$

$$\check{X}_1^{(1)} = \check{x}_1 + \sqrt{3} \left[\sqrt{\check{P}_1} \right]_1 = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} + \sqrt{3} \begin{bmatrix} 0.6 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5392 \\ 4 \end{bmatrix}$$

$$\check{X}_1^{(2)} = \check{x}_1 + \sqrt{3}\left[\sqrt{\check{P}_1}\right]_2 = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} + \sqrt{3}\begin{bmatrix} 0.8333 \\ 0.6368 \end{bmatrix} = \begin{bmatrix} 3.9433 \\ 5.103 \end{bmatrix}$$

$$\check{X}_1^{(3)} = \check{x}_1 - \sqrt{3}\left[\sqrt{\check{P}_1}\right]_1 = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} - \sqrt{3}\begin{bmatrix} 0.6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.4608 \\ 4 \end{bmatrix}$$

$$\check{X}_1^{(4)} = \check{x}_1 - \sqrt{3}\left[\sqrt{\check{P}_1}\right]_2 = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} - \sqrt{3}\begin{bmatrix} 0.8333 \\ 0.6368 \end{bmatrix} = \begin{bmatrix} 1.0567 \\ 2.897 \end{bmatrix}$$

$$\hat{Y}_1^{(0)} = h(\check{X}_1^{(0)}, 0) = \arctan(\frac{20}{60 - 2.5}) = 0.3347\text{rad}$$

$$\hat{Y}_1^{(1)} = 0.3404\text{rad}$$

$$\hat{Y}_1^{(2)} = 0.3427\text{rad}$$

$$\hat{Y}_1^{(3)} = 0.3292\text{rad}$$

$$\hat{Y}_1^{(4)} = 0.3271\text{rad}$$

$$\mu_1 = \sum_i W_i^{(m)} \hat{Y}_1^{(i)} = 0.3348\text{rad}$$

$$S_1 = \sum_i W_i^{(c)} (\hat{Y}_1^{(i)} - \mu_1)(\hat{Y}_1^{(i)} - \mu_1)^T + R_1 = 0.05$$

$$C_1 = \sum_i W_i^{(m)} (\check{X}_1^{(i)} - \check{x}_1)(\hat{Y}_1^{(i)} - \mu_1)^T = \begin{bmatrix} 0.0019 \\ 0.0026 \end{bmatrix}$$

$$K_1 = C_1 S_1^{-1} = \begin{bmatrix} 0.038 \\ 0.052 \end{bmatrix}$$

$$\hat{x}_1 = \check{x}_1 = K_1(y_1 - \mu_1) = \begin{bmatrix} 2.50053 \\ 4.00073 \end{bmatrix}$$

$$\hat{P}_1 = \check{P}_1 - K_1 S_1 K_1^T = \begin{bmatrix} 0.3599 & 0.4999 \\ 0.4999 & 1.0999 \end{bmatrix}$$

□

(2) If the position observation is given as

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k$$

discuss the existence of a steady - state discrete - time Kalman filter.
Solution

$$F = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}; Q = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}; R = 0.05; H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

consider if (F, H) is detectable:

$$\begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0.5 \end{bmatrix}$$

so rank $\begin{bmatrix} H \\ HF \end{bmatrix} = 2$, it's detectable.
here we have

$$J = \begin{bmatrix} \frac{\sqrt{10}}{10} & 0 \\ 0 & \frac{\sqrt{10}}{10} \end{bmatrix}$$

that $JJ^T = Q$.
Consider if (F, J) is stabilizable: rank $\begin{bmatrix} J \\ JF \end{bmatrix} = 2$, so it's stabilizable.
So The DARE has a unique positive semidefinite solution.

$$P = FPF^T - FPH^T(HPH^T + R)^{-1}HPF^T + Q = \begin{bmatrix} 0.2414 & 0.1707 \\ 0.1707 & 0.3828 \end{bmatrix}$$

$$K = PH^T(HPH^T + R)^{-1} = \begin{bmatrix} 0.8284 \\ 0.5858 \end{bmatrix}$$

$\square$