(Fall Semester of 2025) Final Examination of *Optimal Estimation*

December 9, 2025

P.S.: This paper was memorized and typeset after the examination ended, and there's no cheating behaviour during the examination. Since the exam was a long time ago, my memory may not be entirely accurate, but I can guarantee that the knowledge covered is correct.

1. (20 marks) Suppose X and Y are two random variables that are jointly normal. Let $Z = [X, Y]^T$ be the vector of these variables. The mean vector of Z is $\boldsymbol{\mu} = [\mu_X, \mu_Y]^T$, and the covariance matrix is

$$\mathbf{C}_Z = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix},$$

where $\sigma_X > 0$, $\sigma_Y > 0$, and $|\rho| < 1$ (ensuring the covariance matrix is positive definite). Answer the following questions:

   (a) Write the joint probability density function of X and Y. (5 marks)

   (b) Write the conditional probability density function $f_{X|Y}(x|y)$. (10 marks)

   (c) Under what condition are the two random variables X and Y independent? (5 marks)

2. (20 marks) (About KF. The first question requires writing the formulas for the Kalman gain, prior estimate, prior covariance, posterior estimate, and updated covariance, and performing two iterative steps using the data provided in the problem. The second question asks you to discuss the existence of a steady-state discrete-time Kalman filter.)

3. (20 marks) Let $\{x_k\}$ be a discrete-time white noise process with zero mean and variance σ^2, i.e.,

$$\mathbb{E}[x_k] = 0, \quad \mathbb{E}[x_k^2] = \sigma^2,$$

and x_i is independent of x_j for $i \neq j$. Consider the process $\{y_k\}$ obtained by applying the following FIR (Finite Impulse Response) filter to $\{x_k\}$:

$$y_k = \frac{1}{4}x_{k-2} + \frac{1}{2}x_{k-1} + \frac{1}{4}x_k.$$

(a) Compute the mean of y_k, i.e., $\mathbb{E}[y_k]$, and the autocorrelation function of y_k, defined as

$$R_y(m) = \mathbb{E}[y_n y_{n+m}].$$

(b) Is the process $\{y_k\}$ wide-sense stationary (WSS)? Justify your answer.

4. (20 marks) (About EKF. Given a nonlinear system, you are asked to derive its linearized observation equation and provide the formulas for the Kalman gain, prior estimate, prior covariance, posterior estimate, and updated covariance. No need for explicit calculations.)

5. (20 marks, 5 marks each) Suppose that $\boldsymbol{z} = \boldsymbol{s} + \boldsymbol{v}$, where $\boldsymbol{s}$ and $\boldsymbol{v}$ are independent, jointly distributed RVs with $\boldsymbol{s} \sim \mathcal{N}(1, 4)$ and $\boldsymbol{v} \sim \mathcal{N}(0, 1)$. Given measurement z:

   (a) Derive the maximum likelihood estimate for $\boldsymbol{s}$;

   (b) Derive the maximum a posteriori estimate for $\boldsymbol{s}$;

   (c) Derive the minimum mean square estimate for $\boldsymbol{s}$;

   (d) Derive the linear minimum mean square estimate for $\boldsymbol{s}$.