Optimal Estimation homework 1

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1. Gaussian RV: uncorrelatedness implies independence

   - Show that if two random variables X and Y are jointly normal and uncorrelated, then they are independent.
   - Hint: Two random variables X, Y are jointly normal if $pdf(X,Y) = f_{X,Y}(x,y) = \frac{1}{2\pi|\det(C_Z)|^{1/2}} \exp\left[-\frac{1}{2}(Z-E(Z))^T C_Z^{-1}(Z-E(Z))\right]$, in which $Z = [X,Y]^T$.

   Proof: Since X and Y are uncorrelated, we have $Cov(X,Y) = E[(X - E(X))(Y - E(Y))] = 0$. So we can get

   $$C_Z = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}$$

   then we can get

   $$|\det(C_Z)|^{1/2} = \sigma_x \sigma_y$$
   $$C_Z^{-1} = \begin{bmatrix} 1/\sigma_x^2 & 0 \\ 0 & 1/\sigma_y^2 \end{bmatrix}$$

   Since X and Y are jointly normal,

   $$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y} \exp\left[-\frac{(x-E(X))^2}{2\sigma_X^2} - \frac{(y-E(Y))^2}{2\sigma_Y^2}\right]$$

   marginal distribution can be determined

   $$\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} f_{XY}(x,y)dy \\ &= \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left[-\frac{(x-E(X))^2}{2\sigma_X^2}\right] \\ &\quad \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left[-\frac{(y-E(Y))^2}{2\sigma_Y^2}\right] dy \\ &= \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left[-\frac{(x-E(X))^2}{2\sigma_X^2}\right] \end{aligned}$$

similarly,

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left[-\frac{(x - E(Y))^2}{2\sigma_Y^2}\right]$$

notice that $f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$, we can concluse that X and Y are independent.

2. Wide-sense stationary

- Consider the signal plus noise $z(n) = s + v(n)$, where s is a RV with $E[s] = 1$, $E[s^2] = 2$, and for each value of n, $v(n) \sim N(0, 1)$. It is known that $E[sv(i)] = 1$ for all i, and $v(i)$ is independent of $v(j)$ for all $i \neq j$.
  (a) Compute the autocorrelation function $R_z(i, j)$ of $z(n)$ for all integers i, j.
  (b) Is $z(n)$ WSS (Wide sense stationary)? If so, derive a mathematical expression for $R_z(k)$.

Solution:
(1)

$$\begin{aligned} R_z(i, j) &= E(z(i)z(j)) \\ &= E(s^2 + s \cdot (v(i) + v(j)) + v(i) \cdot v(j)) \\ &= E(s^2) + E(s \cdot v(i)) + E(s \cdot v(j)) + E(v(i) \cdot v(j)) \end{aligned}$$

when $i = j$

$$E((v(i))^2) = E((v(i) - E(v(i)))^2) + E(v(i))^2 = 1$$

$$R_z(i, j) = 2 + 1 + 1 + 1 = 5$$

when $i \neq j$

$$E(v(i) \cdot v(j)) = 0$$

$$R_z(i, j) = 2 + 1 + 1 + 0 = 4$$

(2)

$$E(z(n)) = E(s) + E(v(n)) = 1$$

$$R_z(k) = \begin{cases} 4 & k \neq 0 \\ 5 & k = 0 \end{cases}$$

so $E(z(n))$is a constant and $R_z(k)$ is independent of time, which imply $z(n)$ IS wide sense stationary.