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CB1501012-062

Probability and Statistics

Lecture 10.1: Multiple random variables

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Notice

Some feedbacks

- No further assignments on exam period
- Fully annotated pdf after classes

Further feedbacks

- Please provide any feedbacks you would like
- https://docs.google.com/forms/d/e/1FAIpQLScHgW58HT06Cj0NdE8OcpDEDRJEQgEgnE-vwx1cpzpoXr2nKw/viewform?usp=sharing

Agenda

1. Intro
2. Cartesian Products of Sets
3. Joint PMFs
4. Marginal PMFs
5. Independence

So far

- We've handled both discrete and continuous random variables

Discrete Distributions		
Distribution	**Parameters**	**Possible Description**
Uniform (disc)	$X \sim \text{Unif}(a, b)$ for $a, b \in \mathbb{Z}$ and $a \leq b$	Equally likely to be any *integer* in $[a, b]$
Bernoulli	$X \sim \text{Ber}(p)$ for $p \in [0, 1]$	Takes value 1 with prob p and 0 with prob $1 - p$
Binomial	$X \sim \text{Bin}(n, p)$ for $n \in \mathbb{N}$, and $p \in [0, 1]$	Sum of n iid Ber(p) rvs. # of heads in n independent coin flips with $P(\text{head}) = p$.
Poisson	$X \sim \text{Poi}(\lambda)$ for $\lambda > 0$	# of events that occur in **one** unit of time independently with rate λ per unit time
Geometric	$X \sim \text{Geo}(p)$ for $p \in [0, 1]$	# of independent Bernoulli trials with parameter p *up to and **including** first success*

Continuous Distributions		
Distribution	**Parameters**	**Possible Description**
Uniform	$X \sim \text{Unif}(a, b)$ for $a < b$	Equally likely to be any real number in $[a, b]$
Exponential	$X \sim \text{Exp}(\lambda)$ for $\lambda > 0$	Time until first event in Poisson process
Normal	$X \sim \mathcal{N}(\mu, \sigma^2)$ for $\mu \in \mathbb{R}$, and $\sigma^2 > 0$	Standard bell curve

Multiple random variables

- What happens if we want to model more than one at a time?
- How do we model and work with several RV simultaneously?

Multiple random variables

- What happens if we want to model more than one at a time?
- How do we model and work with several RV simultaneously?
- Let's say we have two normal distributions of MMR score

리그 오브 레전드	MMR 등급표
플래티넘 1 단계	2100
플래티넘 2 단계	2050
플래티넘 3 단계	2000
플래티넘 4 단계	1950
플래티넘 5 단계	1850
골 드 1 단계	1750
골 드 2 단계	1700
골 드 3 단계	1650
골 드 4 단계	1600
골 드 5 단계	1500

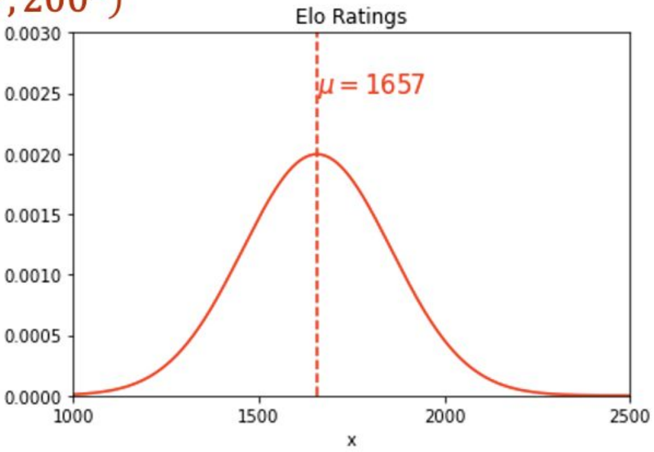
심해

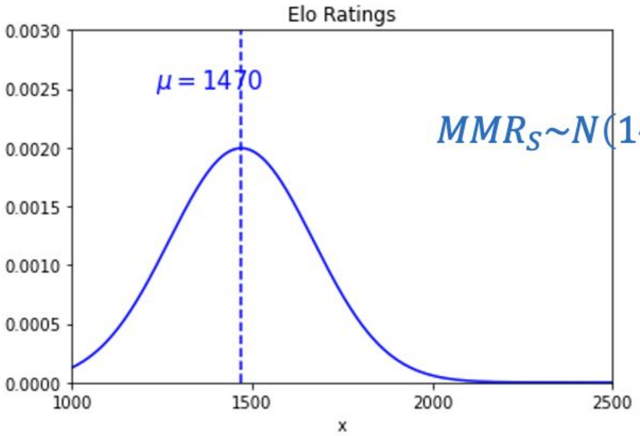
실 버 1 단계	1400
실 버 2 단계	1350
실 버 3 단계	1300
실 버 4 단계	1250
실 버 5 단계	1150
브론즈 1 단계	1050
브론즈 2 단계	1000
브론즈 3 단계	950
브론즈 4 단계	900
브론즈 5 단계	800

Multiple random variables

- What happens if we want to model more than one at a time?
- How do we model and work with several RV simultaneously?
- Let's say we have two normal distributions of MMR score
- What is the probability gold tier gamer win the match?

$MMR_G \sim N(1657, 200^2)$





$MMR_S \sim N(1470, 200^2)$

Multiple random variables

$$P(Gold\ tier\ win) = P(MMR_G > MMR_T)$$

```python
import scipy.stats as st

Gold_MMR = 1657
Silver_MMR = 1470

STDEV = 200
NTRIALS = 10000
nSuccess = 0

for i in range(NTRIALS):
    g = st.norm.rvs(Gold_MMR, STDEV)
    t = st.norm.rvs(Silver_MMR, STDEV)
    if g > t:
        nSuccess += 1

print("Win fraction", float(nSuccess) / NTRIALS)
```

≈0.7468 calculated by sampling

Cartesian product of sets

Definition: Cartesian Product of Sets

Let $\boldsymbol{A}$,$\boldsymbol{B}$ be sets. The Cartesian product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is denoted as

$$\mathbf{A}\times\boldsymbol{B} = \{(\boldsymbol{a},\boldsymbol{b})\colon \boldsymbol{a}\in\boldsymbol{A}, \boldsymbol{b}\in\boldsymbol{B}\}$$

If $\boldsymbol{A}$,$\boldsymbol{B}$ are finite sets, then $|A\times B|=|A|\cdot|B|$ by the product rule of counting

Example

- $\{1,2,3\}\times\{4,5\}$?

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If $\boldsymbol{A}$,$\boldsymbol{B}$ are finite sets, then $|A\times B|=|A|\cdot|B|$ by the product rule of counting

Example

- $\{1,2,3\}\times\{4,5\}=\{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$

Joint PMFs

Suppose we roll two fair 4-sided die independently, one blue and one red. Let X be the value of the blue die and Y be the value of the red die.

$$\Omega_X = \{1,2,3,4\} \qquad \Omega_Y = \{1,2,3,4\}$$

What would be joint range of X & Y and corresponding pmfs?

$$\Omega_{X\times Y} = \Omega_X \times \Omega_Y$$

Joint PMFs

Suppose we roll two fair 4-sided die independently, one blue and one red. Let X be the value of the blue die and Y be the value of the red die.

If we want to write it as a formula

$$p_{X,Y}(x,y) = P(X = x, Y = y)$$

for $x, y \in \Omega_{X,Y}$

$$p_{X,Y}(x,y) = \begin{cases} \frac{1}{16}, & x, y \in \Omega_{X,Y} \\ 0, & otherwise \end{cases}$$

X\Y	1	2	3	4
1	1/16	1/16	1/16	1/16
2	1/16	1/16	1/16	1/16
3	1/16	1/16	1/16	1/16
4	1/16	1/16	1/16	1/16

Joint PMFs

Definition: Joint PMF (결합확률질량함수)

Let X,Y be discrete random variable. The joint PMF of X and Y is

$$p_{X,Y}(x,y) = P(X = x, Y = y)$$

The joint range is

$$\Omega_{X\times Y} = \{(c,d): p_{X,Y}(c,d) > 0\} \subseteq \Omega_X \times \Omega_Y$$

- Note that the probabilities in the table must sum to 1 $\sum_{(s,t)\in\Omega_{X\times Y}} p_{X,Y}(s,t) = 1$
- If $g : \mathbb{R}^2 \to \mathbb{R}$ is a function, then

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) \cdot p_{X,Y}(x,y)$$

Joint PMFs

The joint range $\Omega_{X,Y}$ was always a subset of $\Omega_X \times \Omega_Y$

, but they are not necessarily equal.

$$\Omega_{U \times V} = \{(u, v) \in \Omega_U \times \Omega_V\} \neq \Omega_U \times \Omega_V$$

Joint PMFs

Suppose I roll two 4-sided die independently. Let U=min{X,Y} (the smaller of the two die rolls) and V=max{X,Y} (the larger of the two die rolls).

$$\Omega_U = \{1,2,3,4\} \qquad \Omega_V = \{1,2,3,4\}$$

What would be joint range of X & Y and corresponding pmfs?

→ **Any pair (u,v) with u>v has zero probability**

Joint PMFs

Suppose I roll two 4-sided die independently. Let U=min{X,Y} (the smaller of the two die rolls) and V=max{X,Y} (the larger of the two die rolls).

$$p_{U,V}(u,v) = P(U = u, V = v)$$

for $u, v \in \Omega_{U,V}$

$$p_{U,V}(u,v) = \begin{cases} \frac{2}{16}, & u,v \in \Omega_U \times \Omega_V, \quad v > u \\ \frac{1}{16}, & u,v \in \Omega_U \times \Omega_V, \quad v = u \\ 0, & \text{otherwise} \end{cases}$$

U\V	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

Marginal PMFs

Suppose I roll two 4-sided die independently. Let U=min{X,Y} (the smaller of the two die rolls) and V=max{X,Y} (the larger of the two die rolls).

What if we didn't care about both U and V, just U (the minimum value)?

→ What is the PMF $p_U(u) = \mathbb{P}(U = u) \text{ for } u \in \Omega_U$?

Marginal PMFs

Suppose I roll two 4-sided die independently. Let U=min{X,Y} (the smaller of the two die rolls) and V=max{X,Y} (the larger of the two die rolls). $p_U(u) = \mathbb{P}(U = u)$ for $u \in \Omega_U$?

$$\mathbb{P}(U=1) = \mathbb{P}(U=1, V=1) + \mathbb{P}(U=1, V=2) + \mathbb{P}(U=1, V=3) + \mathbb{P}(U=1, V=4)$$

$$= \frac{1}{16} + \frac{2}{16} + \frac{2}{16} + \frac{2}{16} = \frac{7}{16}$$

U\V	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

Marginal PMFs

Suppose I roll two 4-sided die independently. Let U=min{X,Y} (the smaller of the two die rolls) and V=max{X,Y} (the larger of the two die rolls). $p_U(u) = \mathbb{P}(U = u)$ for $u \in \Omega_U$?

$$p_U(u) = \begin{cases} \frac{7}{16}, & u = 1 \\ \frac{5}{16}, & u = 2 \\ \frac{3}{16}, & u = 3 \\ \frac{1}{16}, & u = 4 \end{cases}$$

U\V	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

Marginal PMFs

Definition: Marginal PMF (주변확률질량함수)

Let X, Y be discrete random variable. The marginal PMF of X is

$$p_X(a) = \sum_{b \in \Omega_Y} p_{X,Y}(a, b)$$

Similarly, the marginal PMF of Y is

$$p_Y(d) = \sum_{c \in \Omega_X} p_{X,Y}(c, d)$$

- Use marginal distributions to get a 1-D RV from a joint PMF.

Example

Suppose X, Y are jointly distributed with joint PMF:

$X \setminus Y$	6	9	Row Total
0	3/12	5/12	?
2	1/12	2/12	?
3	0	1/12	?
Col Total	?	?	1

1. Find the marginal probability mass functions $p_X(x)$ and $p_Y(y)$.
2. Find $\mathbb{E}[Y]$.
3. Are X and Y independent?
4. Find $\mathbb{E}\left[X^Y\right]$.

Example

$X \setminus Y$	6	9	Row Total $p_X(x)$
0	$3/12$	$5/12$	$8/12$
2	$1/12$	$2/12$	$3/12$
3	0	$1/12$	$1/12$
Col Total $p_Y(y)$	$4/12$	$8/12$	1

1. Marginal probability

$$p_X(x) = \begin{cases} 8/12 & x = 0 \\ 3/12 & x = 2 \\ 1/12 & x = 3 \end{cases} \qquad p_Y(y) = \begin{cases} 4/12 & y = 6 \\ 8/12 & y = 9 \end{cases}$$

Example

2. Expectation of Y

$$\mathbb{E}[Y] = \sum_y y p_Y(y) = 6 \cdot \frac{4}{12} + 9 \cdot \frac{8}{12} = 8$$

3. Independence → Check whether we have $p_{X,Y}(x,y) = p_X(x) p_Y(y)$

$$p_{X,Y}(3,6) = 0 \text{ but } p_X(3) > 0 \text{ and } p_Y(6) > 0$$

4. $$\mathbb{E}[X^Y] = \sum_x \sum_y x^y p_{X,Y}(x,y) = 0^6 \cdot \frac{3}{12} + 0^9 \cdot \frac{5}{12} + 2^6 \cdot \frac{1}{12} + 2^9 \cdot \frac{2}{12} + 3^6 \cdot 0 + 3^9 \cdot \frac{1}{12}$$

Independence

- Redefine independence of RVs in terms of the joint PMF, which is completely the same as the earlier definition but with new notation

Definition: Independence

Discrete random variables ***X***,***Y*** are independent, $X \perp Y$, if for all $x \in \Omega_X \text{ and } y \in \Omega_Y$

$$p_{X,Y}(x,y) = p_X(x)p_Y(y)$$

Independence

Let's prove that if X and Y are independent random variables, we have

$$Var(X+Y) = Var(X) + Var(Y)$$

Which leads to

$$Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y)$$

Independence

$$
\begin{aligned}
Var(X+Y) &= E[(X+Y)^2] - (E[X+Y])^2 \\
&= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2 \\
&= E[X^2] + 2E[XY] + E[Y^2] - (E[X])^2 - 2E[X]E[Y] - (E[Y])^2
\end{aligned}
$$

$$
= E[X^2] - (E[X])^2 \quad + \quad E[Y^2] - (E[Y])^2 \quad + 2E[XY] - 2E[X]E[Y]
$$

Independence

$$
\begin{aligned}
E[XY] &= \sum_x \sum_y xyp_{X,Y}(x,y) \\
&= \sum_x \sum_y xyp_X(x)p_Y(y) \\
&= \sum_x xp_X(x) \sum_y yp_Y(y) \\
&= E[X]E[Y]
\end{aligned}
$$

$$
\begin{aligned}
Var(X+Y) &= E[(X+Y)^2] - (E[X+Y])^2 \\
&= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2 \\
&= E[X^2] + 2E[XY] + E[Y^2] - (E[X])^2 - 2E[X]E[Y] - (E[Y])^2
\end{aligned}
$$

$$
= E[X^2] - (E[X])^2 \quad + \quad E[Y^2] - (E[Y])^2 \quad + 2E[XY] - 2E[X]E[Y]
$$

Independence

$$
\begin{aligned}
Var(X+Y) &= E[(X+Y)^2] - (E[X+Y])^2 \\
&= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2 \\
&= E[X^2] + 2E[XY] + E[Y^2] - (E[X])^2 - 2E[X]E[Y] - (E[Y])^2
\end{aligned}
$$

$$= E[X^2] - (E[X])^2 \quad + \quad E[Y^2] - (E[Y])^2 \quad + 2E[XY] - 2E[X]E[Y]$$

$$= Var(X) + Var(Y) \qquad\qquad = 0 \text{ since } X \perp Y$$