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CB1501012-062

Probability and Statistics

Lecture 10.2: Multiple random variables

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Agenda

1. Intro
2. Joint PDFs
3. Marginal PDFs
4. Joint CDFs
5. Independence

Introduction

X : age of a child in year

Y : discrete values of height of child in inches

X : the number rolled on the 1st dice

Y : the number rolled on the 2nd dice

X : the number rolled on the 1st dice

Y : sum of both dice

Introduction

X : age of a child in year

Y : discrete values of height of child in inches

X : the number rolled on the 1st dice

Y : the number rolled on the 2nd dice

X and Y are Discrete Random Variables

X : the number rolled on the 1st dice

Y : sum of both dice

What about when X and Y are Continuous Random Variables?

Introduction

X

Waiting time
before a call is picked up
[0 - 10 minutes]

2.4 minutes

1.5 minutes

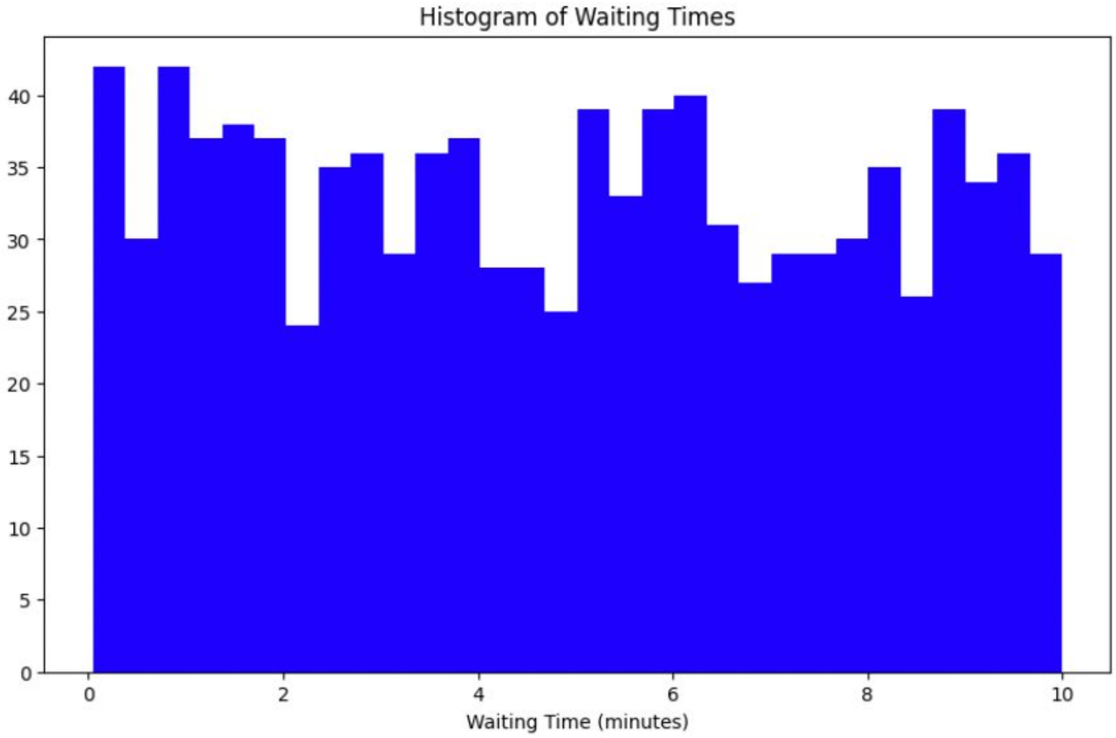
Introduction

X	Y
Waiting time before a call is picked up [0 - 10 minutes]	Customer satisfaction rating [0 - 10]
	
2.4 minutes	0.0
1.5 minutes	5.7

Introduction

X variable: Waiting time (mins)
0 - 10 mins

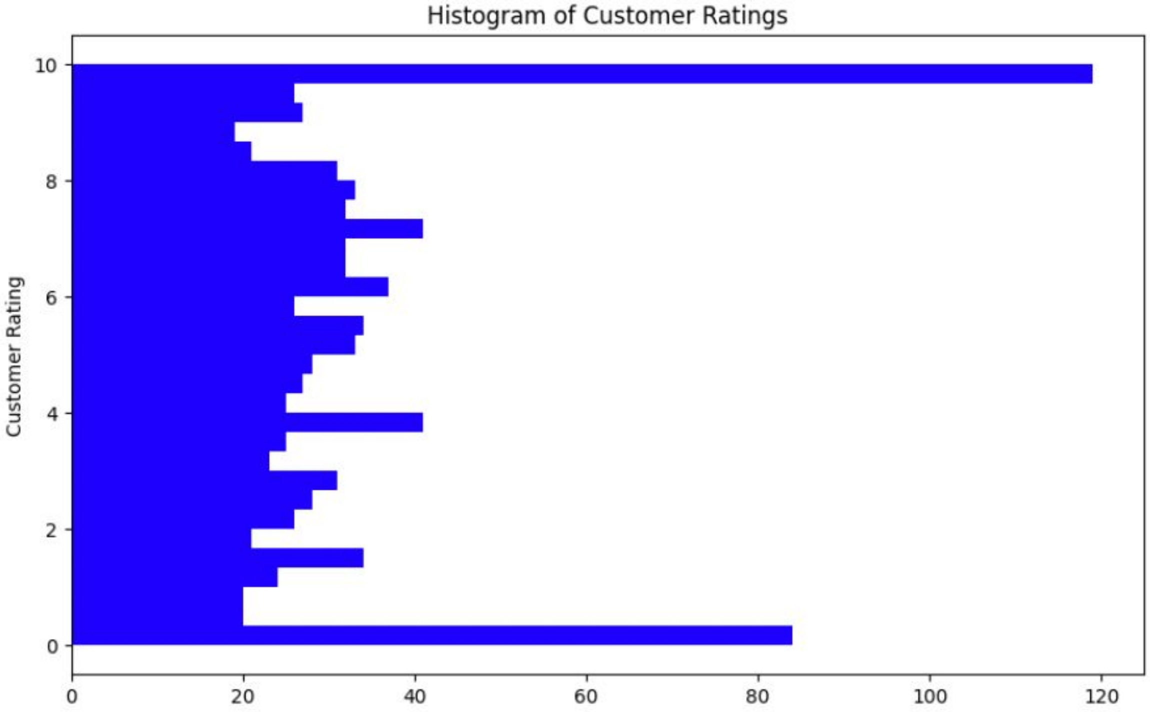
1000 customers



Introduction

Y variable: Satisfaction rating 0 - 10

1000 customers



Introduction

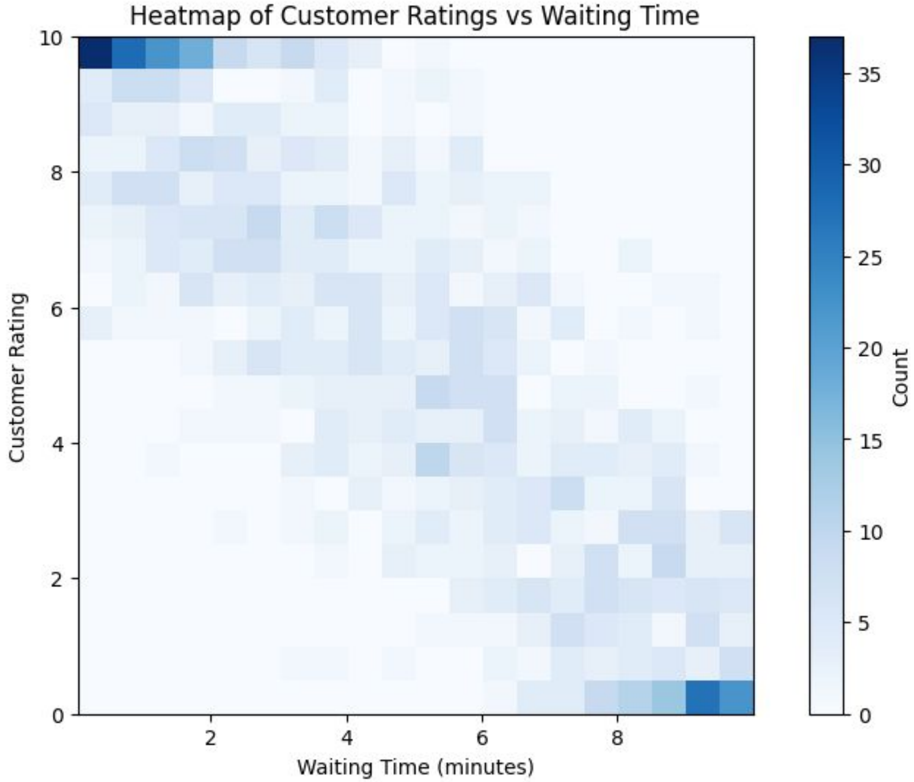
X variable: Waiting time (mins)
0 - 10 mins

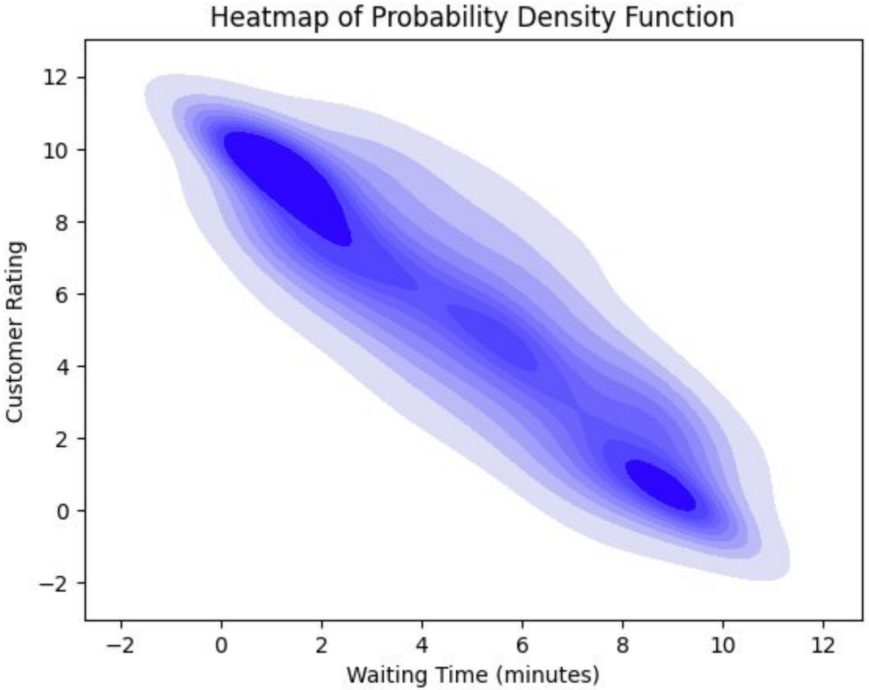
Y variable: Satisfaction rating
0 - 10

1000 customers

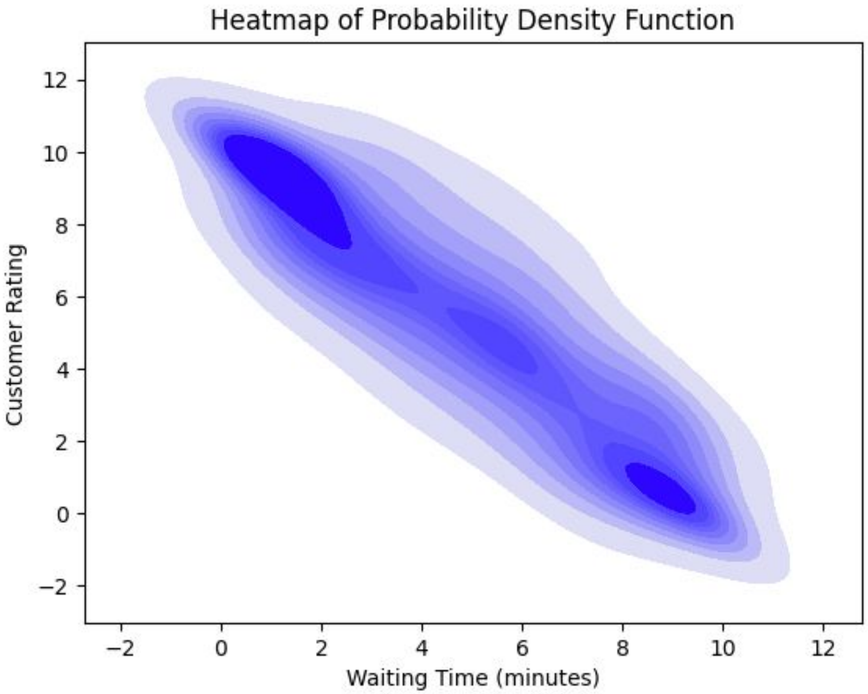
Introduction

Introduction





Introduction



Joint PDFs

Definition: Continuous Joint Probability Density Functions

Let X and Y be continuous random variables. The joint PDF of X and Y is

$$f_{X,Y}(a,b)$$

The joint range is

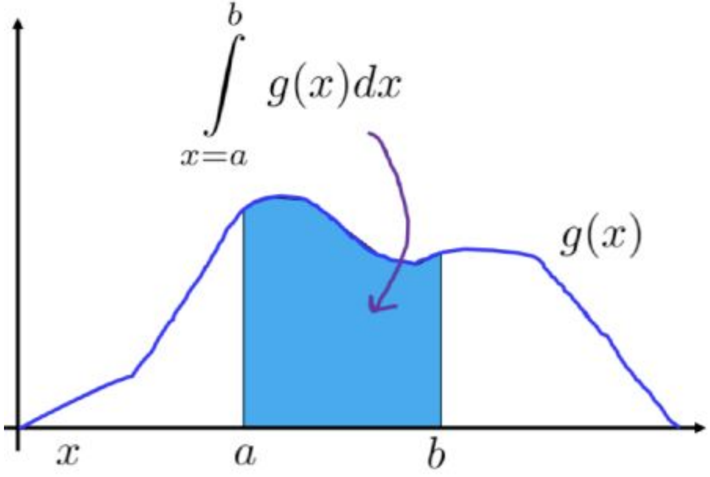
$$\Omega_{X,Y} = \{(c,d): f_{X,Y}(c,d) > 0\} \subseteq \Omega_X \times \Omega_Y$$

- If $g : \mathbb{R}^2 \to \mathbb{R}$ is a function, then

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(s,t) f_{X,Y}(s,t)\,ds\,dt$$

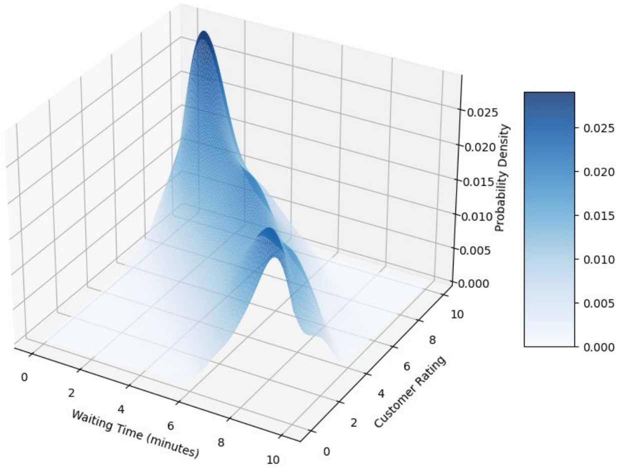
Joint PDFs

Single continuous RV X



$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

Jointly continuous RV X and Y



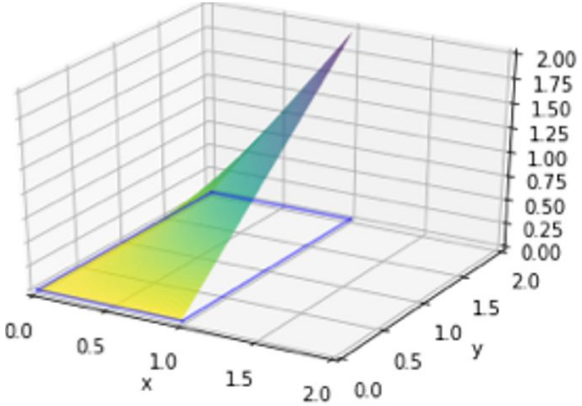
$$P(a_1 \leq X \leq a_2, b_1 \leq Y \leq b_2)$$

$$= \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x, y)\, dy\, dx$$

Joint PDFs

Let X and Y be two continuous random variables defined within

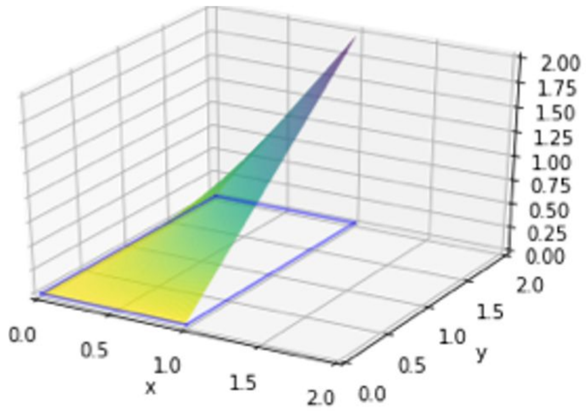
$$\mathbf{0 \leq X \leq 1}, \ \ \mathbf{0 \leq Y \leq 2}$$



Joint PDFs

Let X and Y be two continuous random variables defined within

$0 \le X \le 1,\ \ 0 \le Y \le 2$



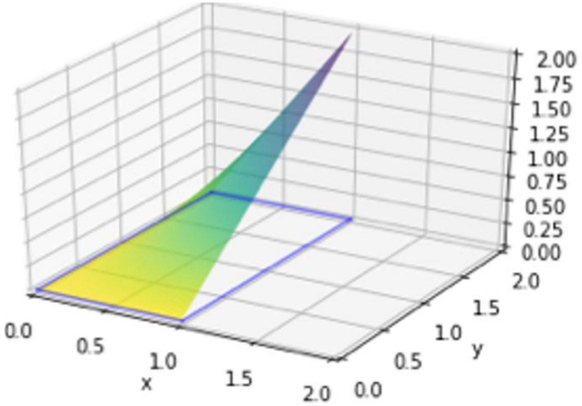
Is $g(x, y) = xy$ a valid joint PDF over X and Y?

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy\ dx = 1$$

Joint PDFs

Let X and Y be two continuous random variables defined within

$\mathbf{0 \leq X \leq 1}$, $\mathbf{0 \leq Y \leq 2}$



Is $g(x, y) = xy$ a valid joint PDF over X and Y?

$$\int_{y=0}^{2} \int_{x=0}^{1} xy \, dx \, dy = 1$$

$$\int_{y=0}^{2} \left(\int_{x=0}^{1} xy \, dx \right) dy \quad = \int_{y=0}^{2} y \left(\int_{x=0}^{1} x \, dx \right) dy$$

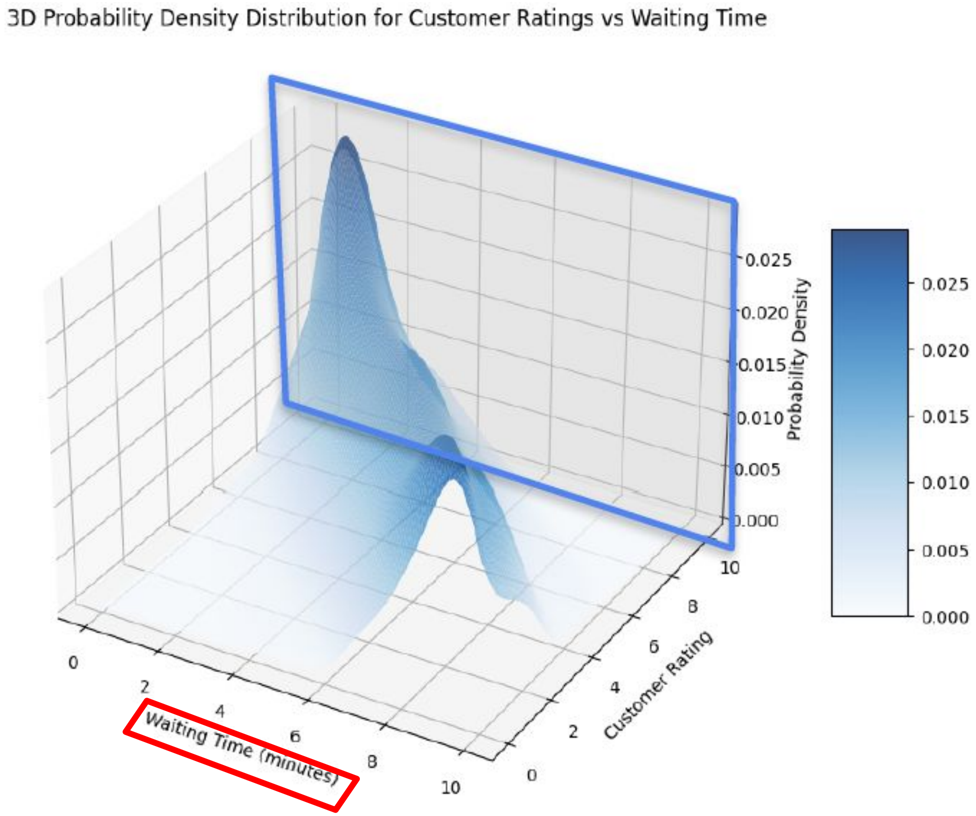
$$= \int_{y=0}^{2} y \left[\frac{1}{2} x^2 \right]_0^1 dy \quad = \int_{y=0}^{2} y \frac{1}{2} dy$$

$$= \left[\frac{1}{4} y^2 \right]_0^2 \quad = 1 - 0 = 1$$

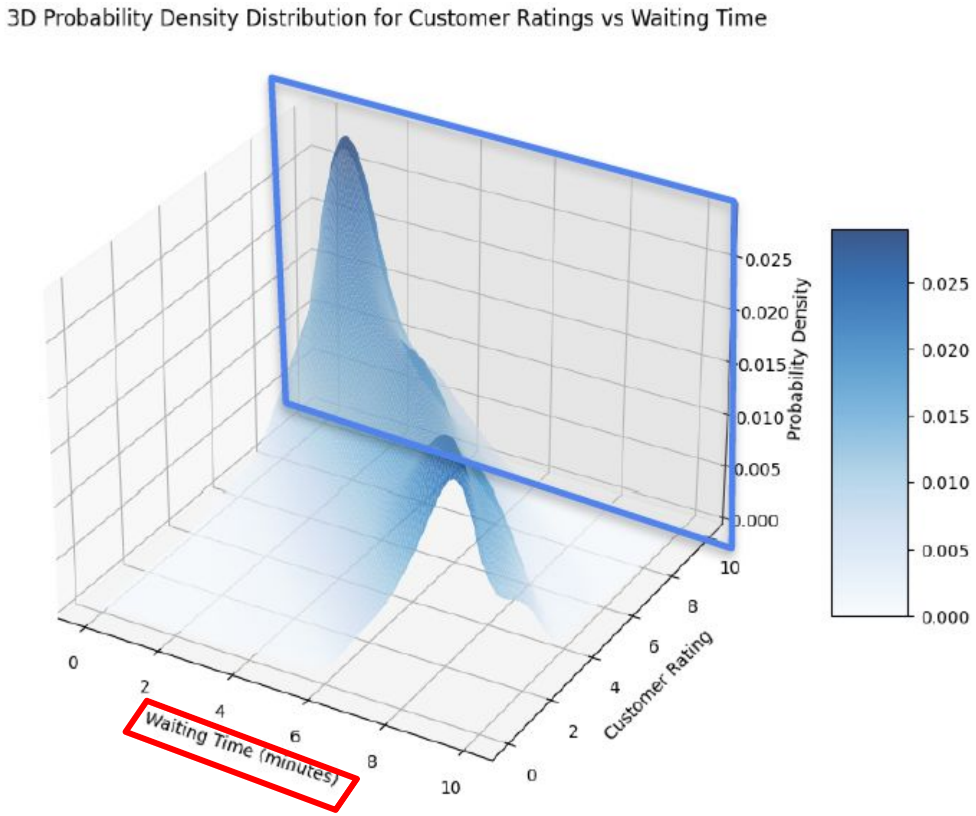
Marginal PDFs

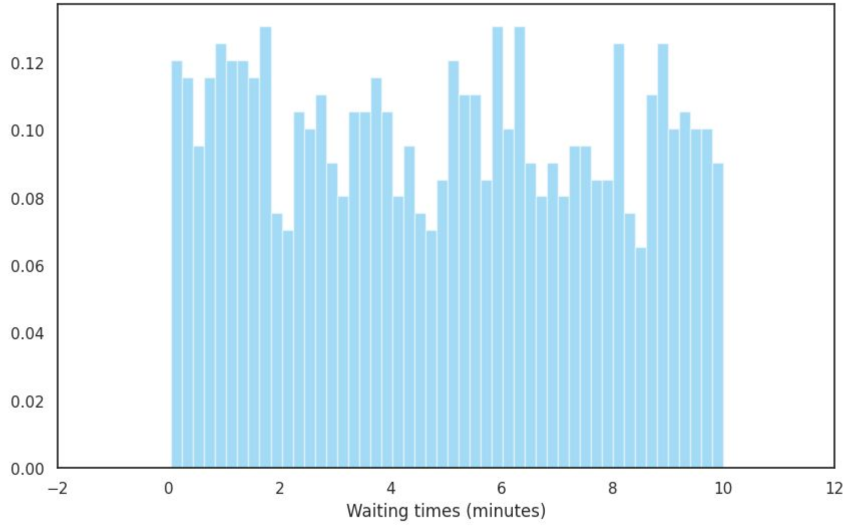
Marginal PDFs

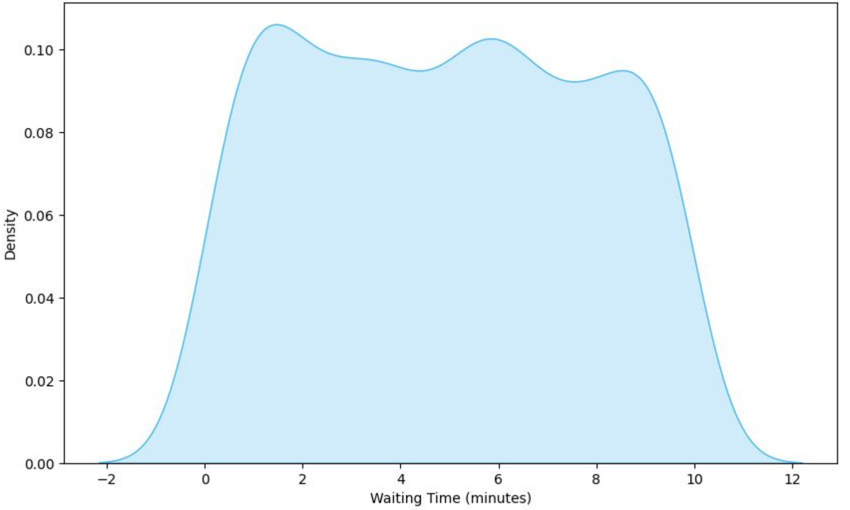
Marginal PDFs



Marginal PDFs







Marginal PDFs

Definition: Continuous Marginal Probability Density Functions

Let X and Y be continuous random variables. The Marginal PDFs are

$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y)dy \qquad f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x, b)dx$$

Find the marginal PDFs given the joint PDF

$$f_{X,Y}(x, y) = \begin{cases} x + \frac{3}{2}y^2 & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Marginal PDFs

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\
&= \int_0^1 \left(x + \frac{3}{2}y^2\right) dy \\
&= \left[xy + \frac{1}{2}y^3\right]_{y=0}^{y=1} \\
&= x + \frac{1}{2}
\end{aligned}$$

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\
&= \int_0^1 \left(x + \frac{3}{2}y^2\right) dx \\
&= \left[\frac{1}{2}x^2 + \frac{3}{2}y^2x\right]_{x=0}^{x=1} \\
&= \frac{3}{2}y^2 + \frac{1}{2}
\end{aligned}$$

$$f_Y(y) = \begin{cases} \frac{3}{2}y^2 + \frac{1}{2} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Marginal PDFs

Compute $\mathrm{E}[X]$ given the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} x + \frac{3}{2}y^2 & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x \left(x + \frac{1}{2} \right) dx$$

Taking $g(X,Y) = X$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y}(x,y) dx dy = \int_0^1 \int_0^1 x \left(x + \frac{3}{2}y^2 \right) dx dy$$

Joint CDFs

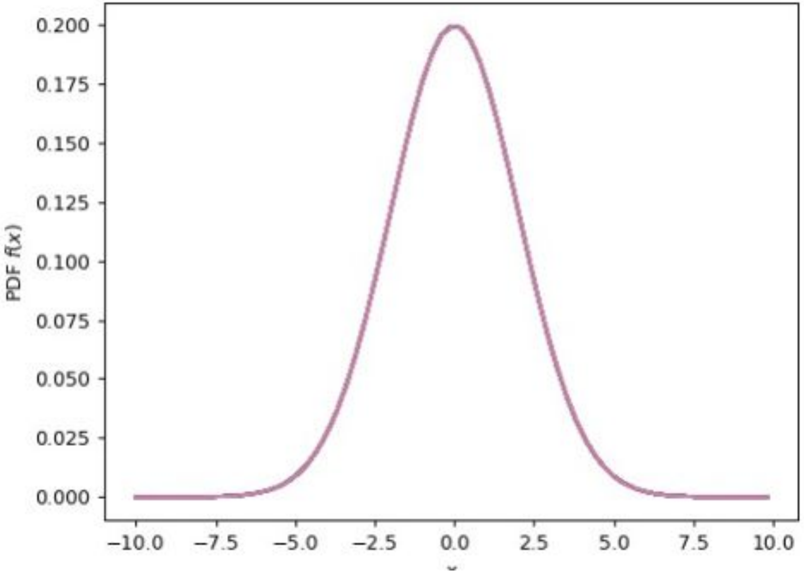
For a continuous random variable X with PDF f, the CDF (cumulative distribution function) is

$$P(X \leq a) = F(a) = \int_{-\infty}^{a} f(x)dx$$

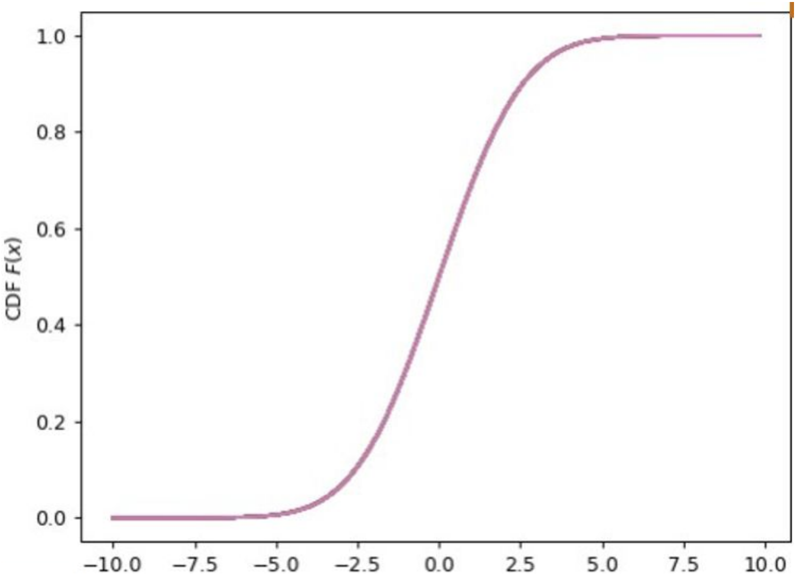
The density f is therefore the derivative of the CDF, F:

$$f(a) = \frac{d}{da}F(a)$$

Joint CDFs



$f_X(x)$



$F_X(x) = P(X \leq x)$

Joint CDFs

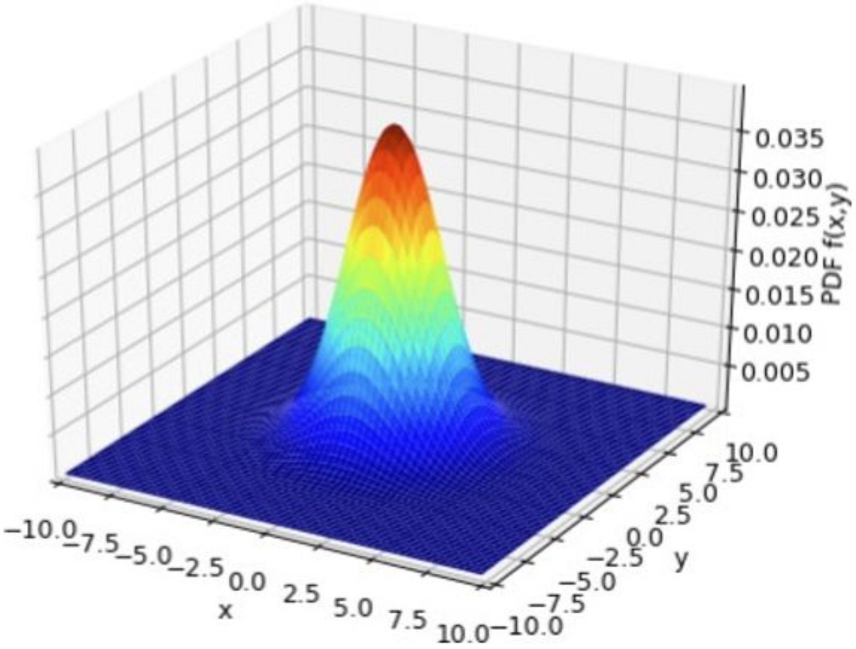
For two random variables X and Y, there can be a joint cumulative distribution function

$$F_{X,Y}(a,b) = P(X \leq a, Y \leq b)$$

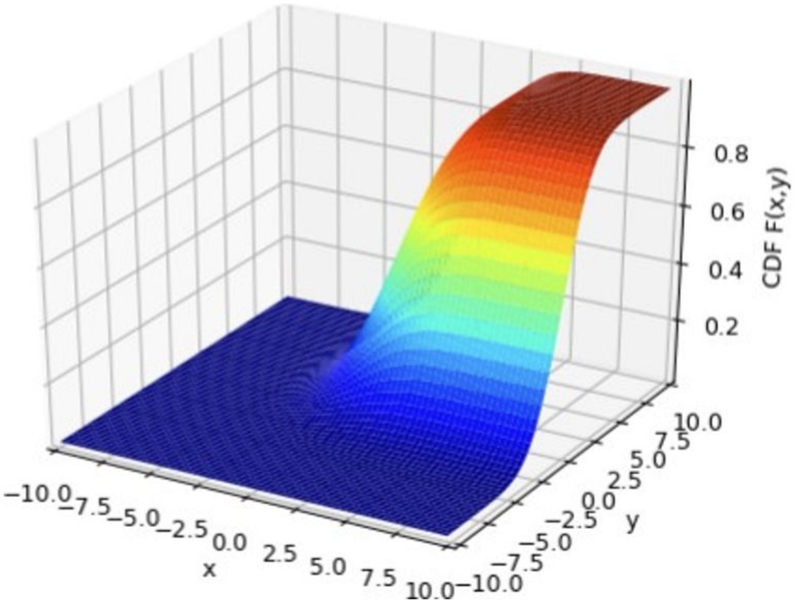
$$= \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x,y) dy\, dx$$

$$f_{X,Y}(a,b) = \frac{\partial^2}{\partial \mathrm{a} \partial b} F_{X,Y}(a,b)$$

Joint CDFs



$$f_{X,Y}(x,y)$$



$$F_{X,Y}(a,b) = P(X \leq x, Y \leq y)$$

Independence

Definition: Independent continuous RVs

Two continuous random variables X and Y are independent if:

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y) \quad \forall x, y$$

$$F_{X,Y}(x, y) = F_X(x)F_Y(y) \quad \forall x, y$$
$$f_{X,Y}(x, y) = f_X(x)f_Y(y) \quad \forall x, y$$

More generally, X and Y are independent if joint density factors separately

$$\boldsymbol{f_{X,Y}(x, y) = g(x)h(y)}, \text{ where } -\infty < \boldsymbol{x, y} < \infty$$

Independence

$$F_{X,Y}(x,y) = F_X(x)F_Y(y) \quad \forall x, y$$
$$f_{X,Y}(x,y) = f_X(x)f_Y(y) \quad \forall x, y$$

Proof)

$$f_{X,Y}(x,y) \quad = \frac{\partial^2}{\partial \mathrm{x} \partial y} F_{X,Y}(x,y) \quad = \frac{\partial^2}{\partial \mathrm{x} \partial y} F_X(x)F_Y(y)$$

$$= \frac{\partial}{\partial \mathrm{x}} \frac{\partial}{\partial y} F_X(x)F_Y(y) \quad = \frac{\partial}{\partial \mathrm{x}} F_X(x) \frac{\partial}{\partial y} F_Y(y)$$

$$= f_X(x)f_Y(y)$$

Independence

You have a disk surface, a circle of radius R. Suppose you have a single point imperfection uniformly distributed on the disk.

1. What is the joint PDF of X and Y?
2. What are the marginal distributions of X and Y?
3. Are X and Y independent?

Independence

You have a disk surface, a circle of radius R. Suppose you have a single point imperfection uniformly distributed on the disk.

1. What is the joint PDF of X and Y?

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\pi R^2} & x^2 + y^2 \leq R^2 \\ 0 & otherwise \end{cases}$$

Independence

You have a disk surface, a circle of radius R. Suppose you have a single point imperfection uniformly distributed on the disk.

2. What are the marginal distributions of X and Y?

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(a, y)dy \qquad = \frac{1}{\pi R^2}\int_{x^2+y^2\leq R^2} dy \qquad \text{where } -\mathrm{R} \leq x \leq R$$

$$= \frac{1}{\pi R^2}\int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \qquad = \frac{2\sqrt{R^2 - x^2}}{\pi R^2}$$

$$f_Y(y) = \frac{2\sqrt{R^2 - y^2}}{\pi R^2}$$

3. Are X and Y independent?

$$f_{X,Y}(x, y) \neq f_X(x)\ f_Y(y)$$

Summary

	Joint discrete PMF	**Joint continuous PDF**
Marginal distributions	$p_X(a) = \sum_y p_{X,Y}(a,y)$	$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a,y)dy$
Independent RVs	$p_{X,Y}(x,y) = p_X(x)p_Y(y)$	$f_{X,Y}(x,y) = f_X(x)f_Y(y)$
Expectation	$E[g(X,Y)] = \sum_x \sum_y g(x,y)p_{X,Y}(x,y)$	$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)\,dy\,dx$