Digital Design & Computer Architecture

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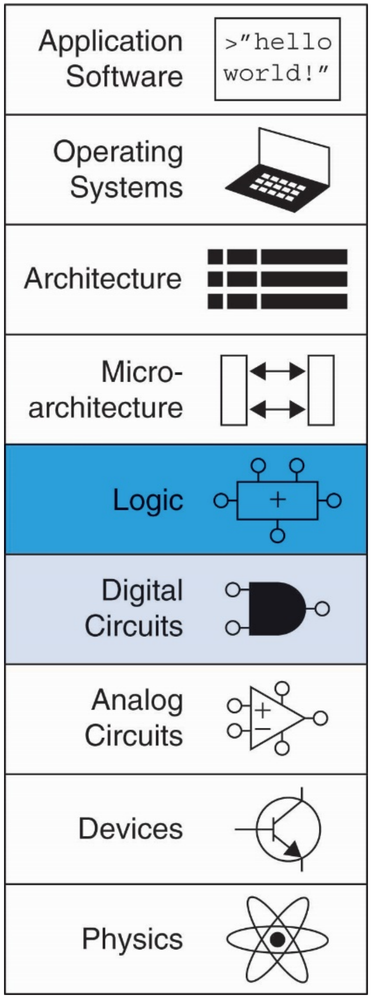
Chapter 2: Combinational Logic Design

Modified by Younghwan Yoo, 2023

Chapter 2 :: Topics

- **Combinational Circuits**
- **Boolean Equations**
- **Boolean Algebra**
- **From Logic to Gates**
- **X’s and Z’s, Oh My**
- **Karnaugh Maps**
- **Combinational Building Blocks**
- **Timing**

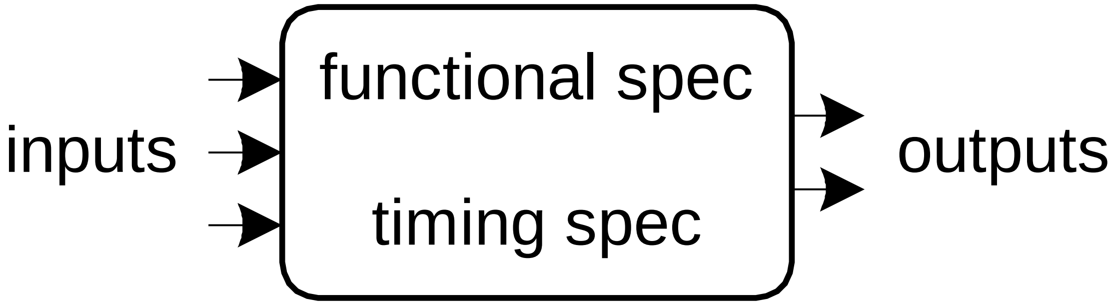


Combinational Circuits

Introduction

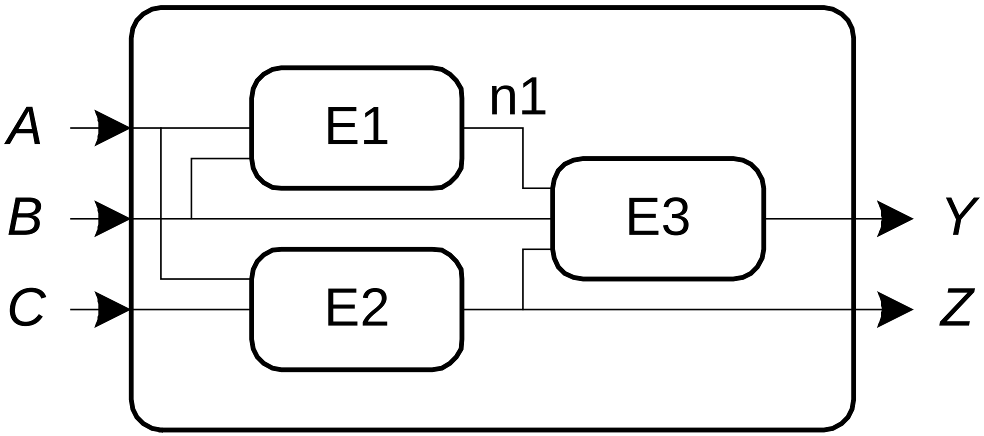
A logic circuit is composed of:

- Inputs
- Outputs
- Functional specification
- Timing specification



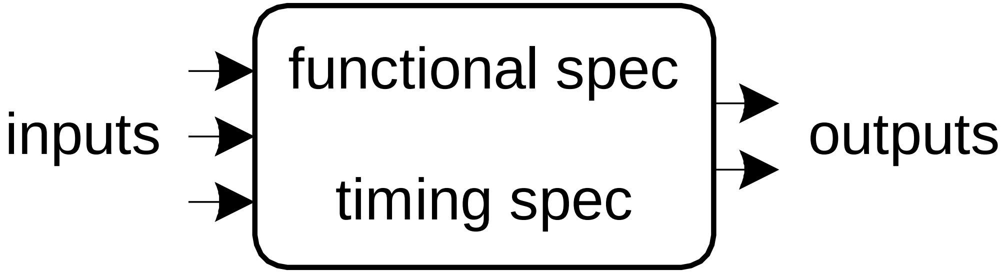
Circuits

- **Nodes**
  - Inputs: *A*, *B*, *C*
  - Outputs: *Y*, *Z*
  - Internal: n1
- **Circuit elements**
  - E1, E2, E3
  - Each itself a circuit



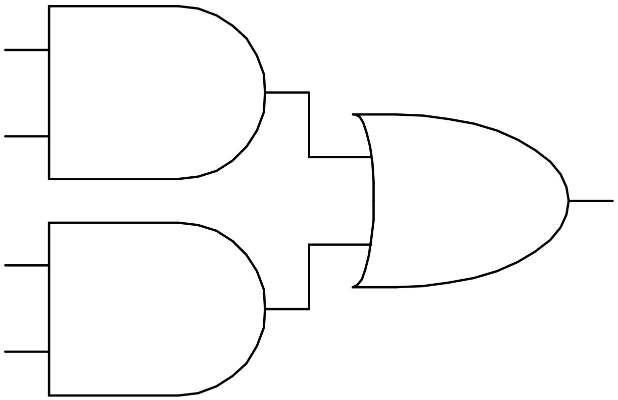
Types of Logic Circuits

- **Combinational logic**
  - Memory*less*
  - Outputs determined by current values of inputs
- **Sequential logic**
  - Has memory
  - Outputs determined by previous and current values of inputs



Rules of Combinational Composition

- Every element is combinational
- Every node is either an input or connects to *exactly one* output
- The circuit contains no cyclic paths
- **Example:**

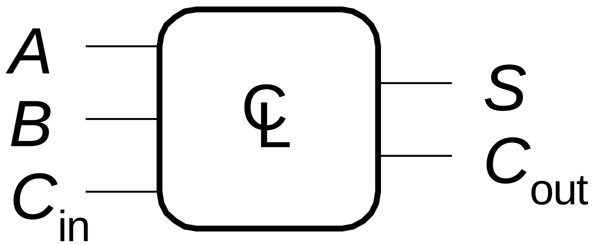


Boolean Equations

Boolean Equations

- Functional specification of outputs in terms of inputs
- **Example:** $S = F(A, B, C_{in})$

  $C_{out} = F(A, B, C_{in})$



$$S = A \oplus B \oplus C_{in}$$
$$C_{out} = AB + AC_{in} + BC_{in}$$

Some Definitions

- **Complement**: variable with a bar over it
  $\bar{A}, \bar{B}, \bar{C}$
- **Literal**: variable or its complement
  $A, \bar{A}, B, \bar{B}, C, \bar{C}$
- **Implicant**: product of literals
  $AB\bar{C}, \bar{A}C, BC$
- **Minterm**: product that includes all input variables
  $AB\bar{C}, A\bar{B}\bar{C}, ABC$
- **Maxterm**: sum that includes all input variables
  $(A+\bar{B}+C), (\bar{A}+B+\bar{C}), (\bar{A}+\bar{B}+C)$

Sum-of-Products (SOP) Form

- All Boolean equations can be written in SOP form
- Each row has a **minterm**
- A minterm is a **product** (AND) of literals
- Each minterm is **TRUE** for that row (and only that row)
- Form function by **ORing minterms** where output is **1**
- Thus, a **sum** (OR) of **products** (AND terms)

A	***B***	***Y***	**minterm**	**minterm name**
0	0	0	$\overline{A}\ \overline{B}$	m_0
0	1	1	$\overline{A}\ B$	m_1
1	0	0	$A\ \overline{B}$	m_2
1	1	1	$A\ B$	m_3

$$Y = \mathrm{F}(A, B) =$$

Sum-of-Products (SOP) Form

- All Boolean equations can be written in SOP form
- Each row has a **minterm**
- A minterm is a **product** (AND) of literals
- Each minterm is **TRUE** for that row (and only that row)
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- Thus, a **sum** (OR) of **products** (AND terms)

A	***B***	***Y***	**minterm**	**minterm name**
0	0	0	$\overline{A}\,\overline{B}$	m_0
0	1	1	$\overline{A}\,B$	m_1
1	0	0	$A\,\overline{B}$	m_2
1	1	1	$A\,B$	m_3

$$Y = F(A, B) = \overline{A}B + AB = \Sigma(m_1, m_3) = \Sigma(1, 3)$$

Long-hand **Short-hand**

Product-of-Sums (POS) Form

- All Boolean equations can be written in POS form
- Each row has a **maxterm**
- A maxterm is a **sum** (OR) of literals
- Each maxterm is **FALSE** for that row (and only that row)
- Form function by **ANDing maxterms** where output is **0**
- Thus, a **product** (AND) of **sums** (OR terms)

A	*B*	*Y*	**maxterm**	**maxterm name**
0	0	**0**	$A + B$	M_0
0	1	1	$A + \overline{B}$	M_1
1	0	**0**	$\overline{A} + B$	M_2
1	1	1	$\overline{A} + \overline{B}$	M_3

$$Y = \mathrm{F}(A, B) = (A + B) \bullet (\overline{A} + B) = \Pi(0, 2)$$

Long-hand Short-hand

Boolean Equations Example

- You are going to the cafeteria for lunch
  - You won't eat lunch (E = 0)
    - If it's not clean (C = 0) or
    - If they only serve meatloaf (M = 1)
- Write a truth table for determining if you will eat lunch (E).

C	M	E
0	0	
0	1	
1	0	
1	1	

SOP & POS Form

SOP – sum-of-products

C	M	E	minterm
0	0	0	$\overline{C}\ \overline{M}$
0	1	0	$\overline{C}\ M$
1	0	1	$C\ \overline{M}$
1	1	0	$C\ M$

POS – product-of-sums

C	M	E	maxterm
0	0	0	$C + M$
0	1	0	$C + \overline{M}$
1	0	1	$\overline{C} + M$
1	1	0	$\overline{C} + \overline{M}$

SOP & POS Form

SOP – sum-of-products

C	M	E	minterm
0	0	0	$\overline{C}\ \overline{M}$
0	1	0	$\overline{C}\ M$
1	0	1	$C\ \overline{M}$
1	1	0	$C\ M$

$$E = C\overline{M}$$
$$= \Sigma(2)$$

POS – product-of-sums

C	M	E	maxterm
0	0	0	$C + M$
0	1	0	$C + \overline{M}$
1	0	1	$\overline{C} + M$
1	1	0	$\overline{C} + \overline{M}$

$$E = (C + M)(C + \overline{M})(\overline{C} + \overline{M})$$
$$= \Pi(0, 1, 3)$$

Forming Boolean Expressions

Example 1:

We will go to the Park (***P*** is the output) if it's not Raining ($\overline{\boldsymbol{R}}$) and we have Sandwiches (***S***).

Boolean Equation:

Forming Boolean Expressions

Example 2:

You will be considered a Winner (**W** is the output) if we send you a Million dollars (**M**) or a small Notepad (**N**).

Boolean Equation:

Forming Boolean Expressions

Example 3:

You can Eat delicious food (***E*** is the output) if you Make it yourself (***M***) or you have a personal Chef (***C***) and she/he is talented (***T***) but not eXpensive ($\overline{X}$).

Boolean Equation:

Forming Boolean Expressions

Example 4:

You can Enter the building if you have a Hat and Shoes on or if you have a Hat on.

Boolean Equation:

Forming Boolean Expressions

Example 5:

You can Enter the building if you have a Hat and Shoes on or if you have a Hat and no Shoes on.

Boolean Equation:

Boolean Algebra: Axioms

Boolean Algebra

- Axioms and theorems to **simplify** Boolean equations
- Like regular algebra, but simpler: variables have only two values (1 or 0)
- **Duality** in axioms and theorems:
  - ANDs and ORs, 0's and 1's interchanged

Boolean Axioms

Number	Axiom	Name
A1	$B = 0$ if $B \neq 1$	Binary Field
A2	$\overline{0} = 1$	NOT
A3	$0 \bullet 0 = 0$	AND/OR
A4	$1 \bullet 1 = 1$	AND/OR
A5	$0 \bullet 1 = 1 \bullet 0 = 0$	AND/OR

Boolean Axioms

Number	Axiom	Dual	Name
A1	$B = 0$ if $B \neq 1$	$B = 1$ if $B \neq 0$	Binary Field
A2	$\overline{0} = 1$	$\overline{1} = 0$	NOT
A3	$0 \bullet 0 = 0$	$1 + 1 = 1$	AND/OR
A4	$1 \bullet 1 = 1$	$0 + 0 = 0$	AND/OR
A5	$0 \bullet 1 = 1 \bullet 0 = 0$	$1 + 0 = 0 + 1 = 1$	AND/OR

Dual: Replace: $\bullet$ with $+$
0 with 1
1 with 0

Boolean Algebra: Theorems of One Variable

Boolean Theorems of One Variable

Number	Theorem	Name
T1	$B \bullet 1 = B$	Identity
T2	$B \bullet 0 = 0$	Null Element
T3	$B \bullet B = B$	Idempotency
T4	$\overline{\overline{B}} = B$	Involution
T5	$B \bullet \overline{B} = 0$	Complements

Dual: Replace: $\bullet$ with +
0 with 1
1 with 0

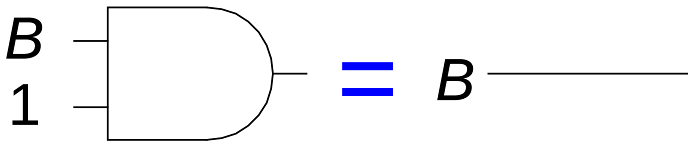
Boolean Theorems of One Variable

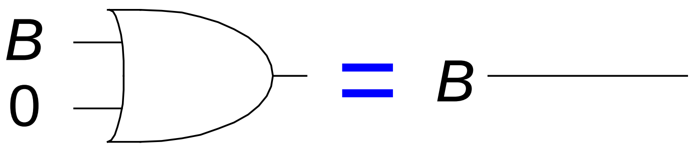
Number	Theorem	Dual	Name
T1	$B \bullet 1 = B$	$B + 0 = B$	Identity
T2	$B \bullet 0 = 0$	$B + 1 = 1$	Null Element
T3	$B \bullet B = B$	$B + B = B$	Idempotency
T4	$\overline{\overline{B}} = B$		Involution
T5	$B \bullet \overline{B} = 0$	$B + \overline{B} = 1$	Complements

Dual: Replace: • with +
0 with 1
1 with 0

T1: Identity Theorem

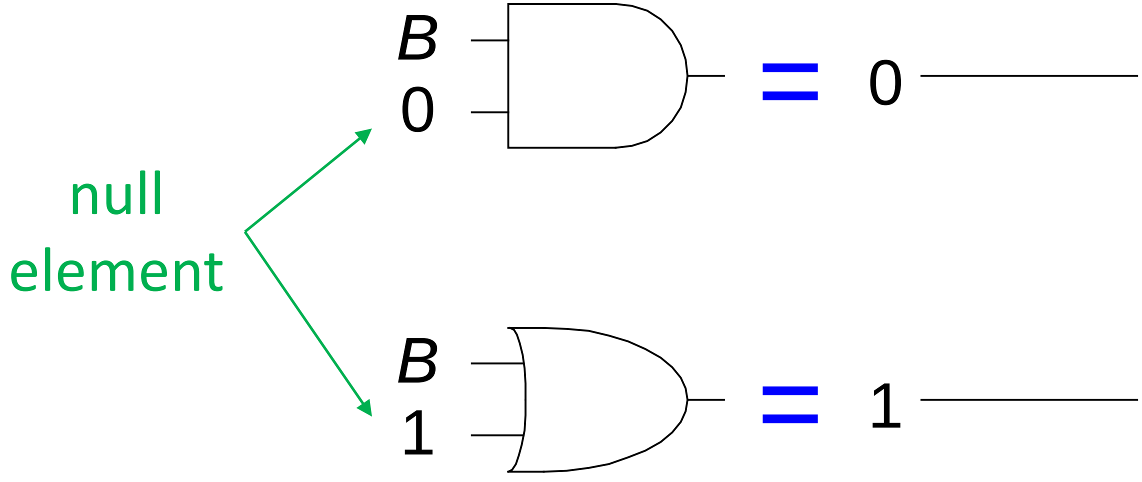
- $B \cdot 1 = B$
- $B + 0 = B$





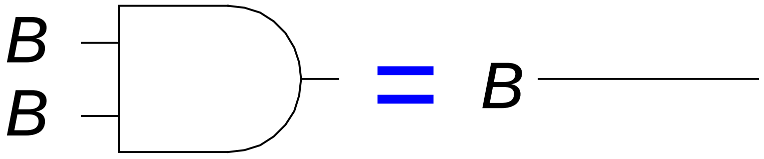
T2: Null Element Theorem

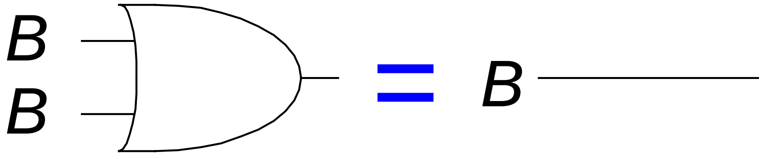
- $B \cdot 0 = 0$
- $B + 1 = 1$



T3: Idempotency Theorem

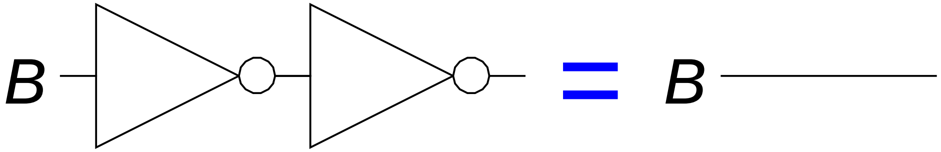
- $B \cdot B = B$
- $B + B = B$





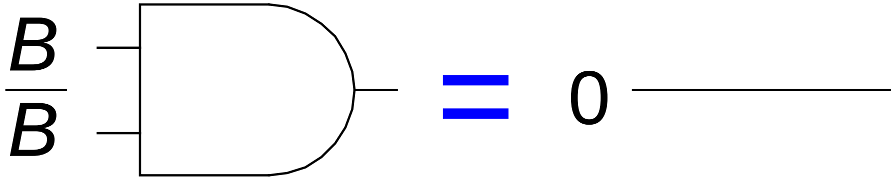
T4: Identity Theorem

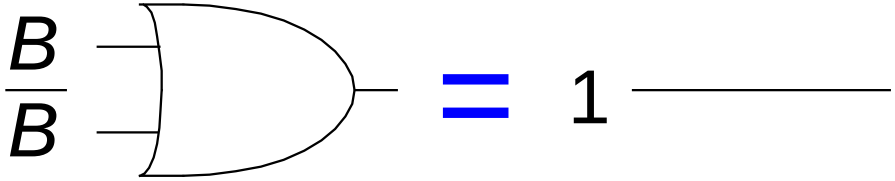
- $\overline{\overline{B}} = B$



T5: Complement Theorem

- $B \bullet \overline{B} = 0$
- $B + \overline{B} = 1$





Recap: Basic Boolean Theorems

Number	Theorem	Dual	Name
T1	$B \bullet 1 = B$	$B + 0 = B$	Identity
T2	$B \bullet 0 = 0$	$B + 1 = 1$	Null Element
T3	$B \bullet B = B$	$B + B = B$	Idempotency
T4	$\overline{\overline{B}} = B$		Involution
T5	$B \bullet \overline{B} = 0$	$B + \overline{B} = 1$	Complements

Boolean Algebra: Theorems of Several Variables

Boolean Theorems of Several Vars

#	Theorem	Dual	Name
T6	B•C = C•B	B+C = C+B	Commutativity
T7	(B•C) • D = B • (C•D)	(B + C) + D = B + (C + D)	Associativity
T8	B • (C + D) = (B•C) + (B•D)	B + (C•D) = (B+C) (B+D)	Distributivity
T9	B • (B+C) = B	B + (B•C) = B	Covering
T10	(B•C) + (B•$\overline{C}$) = B	(B+C) • (B+$\overline{C}$) = B	Combining
T11	(B•C) + ($\overline{B}$•D) + (C•D) = (B•C) + ($\overline{B}$•D)	(B+C) • ($\overline{B}$+D) • (C+D) = (B+C) • ($\overline{B}$+D)	Consensus

Warning: T8’ differs from traditional algebra:
OR (+) distributes over AND (•)

How to Prove

- **Method 1:** Perfect induction
- **Method 2:** Use other theorems and axioms to simplify the equation
  - Make one side of the equation look like the other

Proof by Perfect Induction

- Also called: **proof by exhaustion**
- Check every possible input value
- If the two expressions produce the same value for every possible input combination, the expressions are equal

T9: Covering

Number	Theorem	Name
T9	B• (B+C) = B	Covering

Prove true by:

- **Method 1:** Perfect induction
- **Method 2:** Using other theorems and axioms

T9: Covering

Number	Theorem	Name
T9	B• (B+C) = B	Covering

Method 1: Perfect Induction

B	*C*	*(B+C)*	*B(B+C)*
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Covering

Method 2: Prove true using other axioms and theorems.

$B \bullet (B+C)$	$= B \bullet B + B \bullet C$	T8: Distributivity
	$= \mathbf{B} + B \bullet C$	T3: Idempotency
	$= \mathbf{B \bullet 1} + B \bullet C$	T1: Identity
	$= B \bullet (1 + C)$	T8: Distributivity
	$= B \bullet (\mathbf{1})$	T2: Null element
	$= B$	T1: Identity

T10: Combining

Number	Theorem	Name
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	Combining

Prove true using other axioms and theorems:

$B \bullet C + B \bullet \overline{C}$	$= B \bullet (C + \overline{C})$	T8: Distributivity
	$= B \bullet (1)$	T5': Complements
	$= B$	T1: Identity

De Morgan's Theorem: Dual

#	Theorem	Dual	Name
T12	$\overline{B \bullet C \bullet D \ldots} = \overline{B} + \overline{C} + \overline{D} \ldots$	$\overline{B + C + D \ldots} = \overline{B} \bullet \overline{C} \bullet \overline{D} \ldots$	De Morgan's Theorem

The **complement** of the **product** is the **sum** of the **complements**.

Dual:
The **complement** of the **sum** is the **product** of the **complements**.

Recap: Theorems of Several Vars

#	Theorem	Dual	Name
T6	B•C = C•B	B+C = C+B	Commutativity
T7	(B•C) • D = B • (C•D)	(B + C) + D = B + (C + D)	Associativity
T8	B • (C + D) = (B•C) + (B•D)	B + (C•D) = (B+C) (B+D)	Distributivity
T9	B • (B+C) = B	B + (B•C) = B	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	$(B+C) \bullet (B+\overline{C}) = B$	Combining
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D) =$ $(B \bullet C) + (\overline{B} \bullet D)$	$(B+C) \bullet (\overline{B}+D) \bullet (C+D) =$ $(B+C) \bullet (\overline{B}+D)$	Consensus
T12	$\overline{B \bullet C \bullet D \ldots} = \overline{B}+\overline{C}+\overline{D}\ldots$	$\overline{B+C+D\ldots} = \overline{B} \bullet \overline{C} \bullet \overline{D}\ldots$	De Morgan’s

Boolean Algebra: Simplifying Equations

Simplifying an Equation

Simplifying may mean minimal sum of products form:

- SOP form that has the **fewest number of implicants**, where each implicant has the **fewest literals**
  - **Implicant:** product of literals
    $A\overline{B}C$, $A\overline{C}$, $B\overline{C}$
  - **Literal:** variable or its complement
    A, $\overline{A}$, B, $\overline{B}$, C, $\overline{C}$

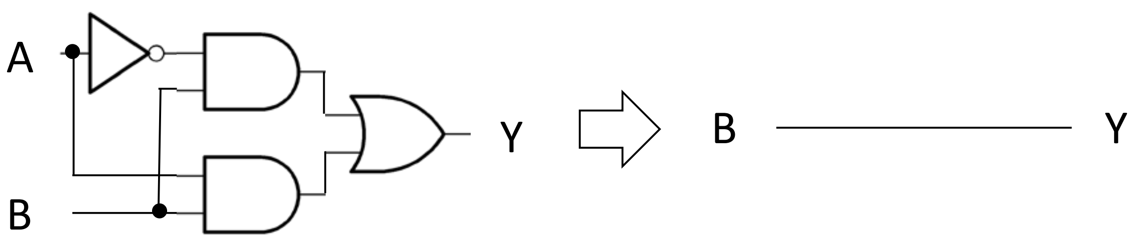
Simplifying could also mean fewest number of gates, lowest cost, lowest power, etc. For example, Y = A XOR B is likely simpler than minimal Sum of Products $Y = A\overline{B} + \overline{A}B$. These depend on details of the technology.

Simplifying Boolean Equations

Example 1:

$Y = \overline{A}B + AB$

Y = B T10: Combining



or

$Y = B(A + \overline{A})$ T8: Distributivity

$= B(1)$ T5': Complements

$= B$ T1: Identity

Simplifying Boolean Equations

Example 2:

$Y = A\overline{B}C + ABC + \overline{A}BC$

$= A\overline{B}C + ABC + ABC + \overline{A}BC$ T3': Idempotency

$= (A\overline{B}C+ABC) + (ABC+\overline{A}BC)$ T7': Associativity

$= AC + BC$ T10: Combining

Chapter 2: Combinational Logic

Extra Examples

Boolean Algebra: Simplifying Equations

Simplification Methods

- **Distributivity (T8, T8')** $B(C+D) = BC + BD$

  $B + CD = (B+C)(B+D)$

- **Covering (T9')** $A + AP = A$

- **Combining (T10)** $P\overline{A} + PA = P$

- **Expansion** $P = P\overline{A} + PA$

  $A = A + AP$

- **Idempotency (duplication)** $A = A + A$

- **"Simplification" theorem** $A + \overline{A}P = A + P$

  $\overline{A} + AP = \overline{A} + P$

Proving the "Simplification" Theorem

"Simplification" theorem

$$A + \overline{A}P = A + P$$

Method 1: $A + \overline{A}P = \mathbf{A + AP} + \overline{A}P$ — **T9' Covering**

$= A + \mathbf{(AP + \overline{A}P)}$ — **T7 Associativity**

$= A + \mathbf{P}$ — **T10 Combining**

Method 2: $A + \overline{A}P = (A + \overline{A})\,(A + P)$ — **T8' Distributivity**

$= \mathbf{1 \bullet}\;(A + P)$ — **T5' Complements**

$= \mathbf{A + P}$ — **T1 Identity**

T11: Consensus

Number	Theorem	Name
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D) = (B \bullet C) + (\overline{B} \bullet D)$	Consensus

Prove using other theorems and axioms:

$$\mathbf{B \bullet C + \overline{B} \bullet D + C \bullet D}$$

$= BC + \overline{B}D + (\mathbf{CD}B + \mathbf{CD}\overline{B})$ — **T10: Combining**

$= BC + \overline{B}D + \mathbf{BCD} + \mathbf{\overline{B}CD}$ — **T6: Commutativity**

$= \mathbf{BC + BCD} + \mathbf{\overline{B}D + \overline{B}CD}$ — **T6: Commutativity**

$= \mathbf{BC(1 + D)} + \mathbf{\overline{B}D(1 + C)}$ — **T8: Distributivity**

$= \mathbf{BC} + \mathbf{\overline{B}D}$ — **T2': Null Element**

Simplification Methods

- **Distributivity (T8, T8')** $B(C+D) = BC + BD$
  $B + CD = (B+C)(B+D)$
- **Covering (T9')** $A + AP = A$
- **Combining (T10)** $P\overline{A} + PA = P$
- **Expansion** $P = P\overline{A} + PA$
  $A = A + AP$
- **Idempotency (duplication)** $A = A + A$
- **"Simplification" theorem** $A + \overline{A}P = A + P$
  $\overline{A} + AP = \overline{A} + P$

Simplification Methods

- **Distributivity (T8, T8')** **$B(C+D) = BC + BD$**
  $B + CD = (B+C)(B+D)$
- **Covering (T9')** $A + AP = A$
- Combining (T10) $P\overline{A} + PA = P$
- **Expansion** $P = P\overline{A} + PA$
  $A = A + AP$
- **Idempotency (duplication)** $A = A + A$
- **"Simplification" theorem** $A + \overline{A}P = A + P$
  $\overline{A} + AP = \overline{A} + P$

Simplifying Boolean Equations

Example 3:

$Y = A(AB + ABC)$

$= A(AB(1 + C))$	T8: Distributivity
$= A(AB(1))$	T2': Null Element
$= A(AB)$	T1: Identity
$= (AA)B$	T7: Associativity
$= AB$	T3: Idempotency

Simplification Methods

- **Distributivity (T8, T8')** $B(C+D) = BC + BD$
  $B + CD = (B+C)(B+D)$
- **Covering (T9')** $A + AP = A$
- **Combining (T10)** $P\overline{A} + PA = P$
- **Expansion** $P = P\overline{A} + PA$
  $A = A + AP$
- **Idempotency (duplication)** $A = A + A$
- **"Simplification" theorem** $A + \overline{A}P = A + P$
  $\overline{A} + AP = \overline{A} + P$

Simplifying Boolean Equations

Example 4:

$Y = A'BC + A'$	**Recall:** $A' = \bar{A}$
$= A'$	T9' Covering: $X + XY = X$
or	
$= A'(BC + 1)$	T8: Distributivity
$= A'(\mathbf{1})$	T2': Null Element
$= \mathbf{A'}$	T1: Identity

Multiplying Out: SOP Form

An expression is in **sum-of-products (SOP)** form when all products contain literals only.

- **SOP form:** Y = AB + BC' + DE
- **NOT SOP form:** Y = DF + E(A'+B)
- **SOP form:** Z = A + BC + DE'F

Multiplying Out: SOP Form

Example 5:

$$Y = (A + C + D + E)(A + B)$$

Apply T8' first when possible: W+XZ = (W+X)(W+Z)

Make: X = (C+D+E), Z = B and rewrite equation

Y	= (A+**X**)(A+**Z**)	substitution (**X**=(C+D+E), **Z**=B)
	= A + **XZ**	T8': Distributivity
	= A + **(C+D+E)B**	substitution
	= A + BC + BD + BE	T8: Distributivity

or

Y	= AA+AB+AC+BC+AD+BD+AE+BE	T8: Distributivity
	= **A**+AB+AC+AD+AE+BC+BD+BE	T3: Idempotency
	= **A** + BC+BD+BE	T9': Covering

This method is called ***multiplying out*.**

Simplifying Boolean Equations

Example 6:

$Y = AB + BC + B'D' + AC'D'$

Method 1:

$Y = AB + BC + B'D' + \mathbf{(ABC'D' + AB'C'D')}$	T10: Combining
$= (AB + \mathbf{ABC'D'}) + BC + (B'D' + \mathbf{AB'C'D'})$	T6: Commutat.
	T7: Associativity
$= \mathbf{AB} + BC + \mathbf{B'D'}$	T9: Covering

Method 2:

$Y = \mathbf{AB} + BC + \mathbf{B'D}' + AC'D' + \mathbf{AD'}$	T11: Consensus
$= AB + BC + B'D' + \mathbf{AD'}$	T9: Covering
$= AB + BC + B'D'$	T11: Consensus

Literal and implicant ordering

- Variables within an implicant should be in alphabetical order.
- The order of implicants doesn't matter.

Examples:

- **Correct:** $Y = AB + BC + \overline{B}\,\overline{D}$
- **Correct:** $Y = BC + \overline{B}\,\overline{D} + AB$
- **Incorrect:** $Y = \mathbf{CB} + \overline{B}\,\overline{D} + \mathbf{BA}$
- **Incorrect:** $Y = AB + BC + \mathbf{\overline{D}\,\overline{B}}$

Simplifying Boolean Equations

Example 7:

$Y = (A + BC)(A + DE)$

Apply T8' first when possible: $W+XZ = (W+X)(W+Z)$

Make: $X = BC$, $Z = DE$ and rewrite equation

Y	$= (A+\mathbf{X})(A+\mathbf{Z})$	substitution (**X**=BC, **Z**=DE)
	$= A + \mathbf{XZ}$	T8': Distributivity
	$= A + \mathbf{BCDE}$	substitution

or

Y	$= AA + ADE + ABC + BCDE$	T8: Distributivity
	$= A + ADE + ABC + BCDE$	T3: Idempotency
	$= \mathbf{A + A}DE + \mathbf{A}BC + BCDE$	
	$= \mathbf{A} \qquad + BCDE$	T9': Covering
	$= A + BCDE$	T9': Covering

Review: Canonical SOP & POS Forms

SOP – sum-of-products

$$E = \boxed{C\overline{M}}$$

C	M	E	minterm
0	0	0	$\overline{C}\ \overline{M}$
0	1	0	$\overline{C}\ M$
1	0	1	$C\ \overline{M}$
1	1	0	$C\ M$

same

POS – product-of-sums

C	M	E	maxterm
0	0	0	$C + M$
0	1	0	$C + \overline{M}$
1	0	1	$\overline{C} + M$
1	1	0	$\overline{C} + \overline{M}$

$$
\begin{aligned}
E &= (C + M)(C + \overline{M})(\overline{C} + \overline{M}) \\
&= (C + M\overline{M}) * (\overline{C} + \overline{M}) \\
&= (C + 0) * (\overline{C} + \overline{M}) \\
&= C * (\overline{C} + \overline{M}) \\
&= C\overline{C} + C\overline{M} \\
&= 0 + C\overline{M} \\
&= \boxed{C\overline{M}}
\end{aligned}
$$

Factoring: POS Form

An expression is in **product-of-sums (POS)** form when all sums contain literals only.

- **POS form:** $Y = (A+B)(C+D)(E'+F)$
- **NOT POS form:** $Y = (D+E)(F'+GH)$
- **POS form:** $Z = A(B+C)(D+E')$

Factoring: POS Form

Example 8:

$Y = (A + B'CDE)$

Apply T8' first when possible: $W+XZ = (W+X)(W+Z)$

Make: $X = B'C$, $Z = DE$ and rewrite equation

Y	$= (A+XZ)$	substitution ($X=B'C$, $Z=DE$)
	$= (A+B'C)(A+DE)$	T8': Distributivity
	$= (A+B')(A+C)(A+D)(A+E)$	T8': Distributivity

Factoring: POS Form

Example 9:

$Y = AB + C'DE + F$

Apply T8' first when possible: $W+XZ = (W+X)(W+Z)$

Make: $W = AB$, $X = C'$, $Z = DE$ and rewrite equation

Y	$= (W+XZ) + F$	substitution $W = AB$, $X = C'$, $Z = DE$
	$= (W+X)(W+Z) + F$	T8': Distributivity
	$= (AB+C')(AB+DE)+F$	substitution
	$= (A+C')(B+C')(AB+D)(AB+E)+F$	T8': Distributivity
	$= (A+C')(B+C')(A+D)(B+D)(A+E)(B+E)+F$	T8': Distributivity
	$= (A+C'+F)(B+C'+F)(A+D+F)(B+D+F)(A+E+F)(B+E+F)$	T8': Distr.

De Morgan's Theorem

Example 10:

$$Y = \overline{(A+\overline{BD})\overline{C}}$$

$$= \overline{(A+\overline{BD})} + \overline{\overline{C}}$$

$$= (\overline{A}\bullet(\overline{\overline{BD}})) + C$$

$$= (\overline{A}\bullet(BD)) + C$$

$$= \overline{A}BD + C$$

- Work from the **outside in** (i.e., top bar, then down)
- Use **involution** when possible

De Morgan's Theorem

Example 11:

$$Y = \overline{(\overline{AC}E+\overline{D}) + B}$$

$$= \overline{(\overline{AC}E+\overline{D})} \bullet \overline{B}$$

$$= (\overline{\overline{AC}E} \bullet \overline{\overline{D}}) \bullet \overline{B}$$

$$= ((\overline{\overline{AC}}+\overline{E}) \bullet D) \bullet \overline{B}$$

$$= ((AC+\overline{E}) \bullet D) \bullet \overline{B}$$

$$= (ACD + D\overline{E}) \bullet \overline{B}$$

$$= A\overline{B}CD + \overline{B}D\overline{E}$$

Common Errors

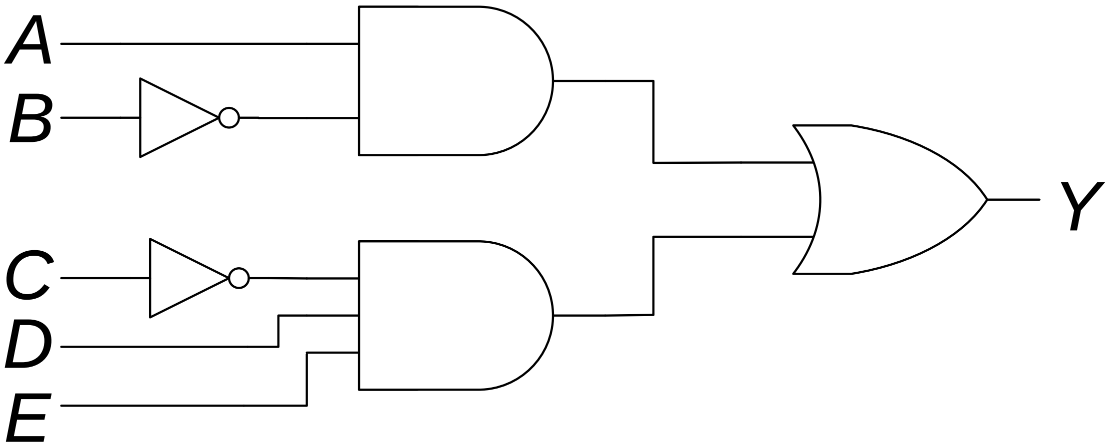
- **Losing bars** (alignment will help you avoid this)
- **Losing terms** (alignment will help you avoid this)
- **Trying to do multiple steps at once** – this is prone to errors!
- **Applying theorems incorrectly**, for example:
  - **Wrong:** $ABC + \bar{A}B\bar{C} = B$ **Correct:** $ABC + A\bar{B}C = AC$. Products may only **differ in a single term** when using the combining theorem.
  - **Wrong:** $(A + \bar{A}) = 0$ **Correct:** $A + \bar{A} = 1$
  - **Wrong:** $(A \bullet \bar{A}) = 1$ **Correct:** $A \bullet \bar{A} = 0$
  - **Wrong:** ABC = B **Correct:** **B** + ABC = B. In order to use the covering theorem, you must have a term that covers the other terms.
  - **Wrong:** $\overline{AC} = \bar{A}\bar{C}$ **Correct:** $\overline{AC} = \bar{A} + \bar{C}$ (De Morgan's)
  - **Wrong:** $\overline{A + C} = \bar{A} + \bar{C}$ **Correct:** $\overline{A + C} = \bar{A}\bar{C}$ (De Morgan's)

From Logic to Gates

From Logic to Gates

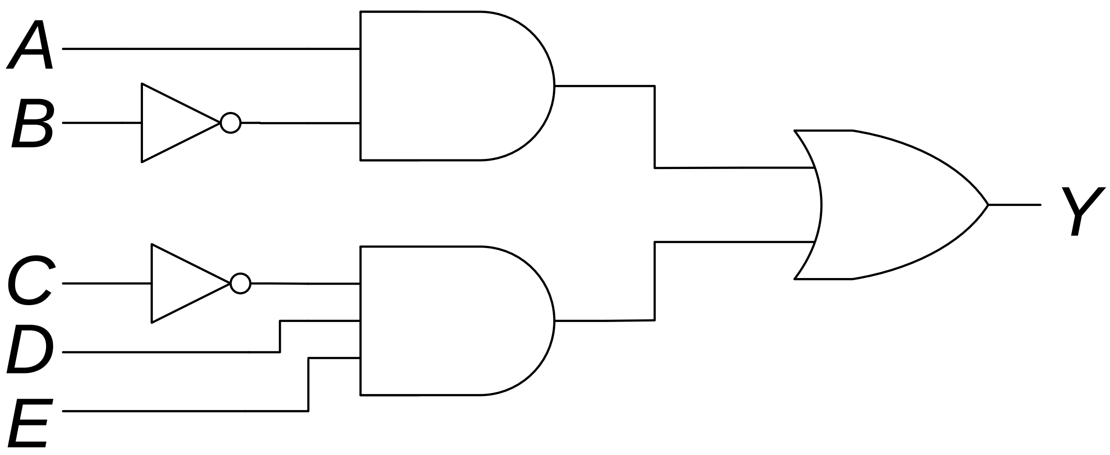
Build the following equation using logic gates:

$$Y = A\overline{B} + \overline{C}DE$$



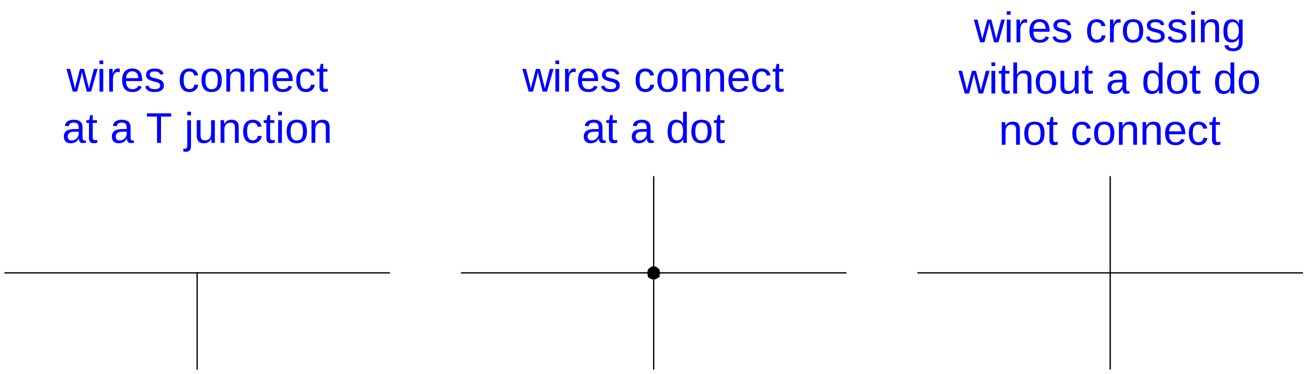
Circuit Schematics Rules

- Inputs on the left (or top)
- Outputs on right (or bottom)
- Gates flow from left to right
- Straight wires are best



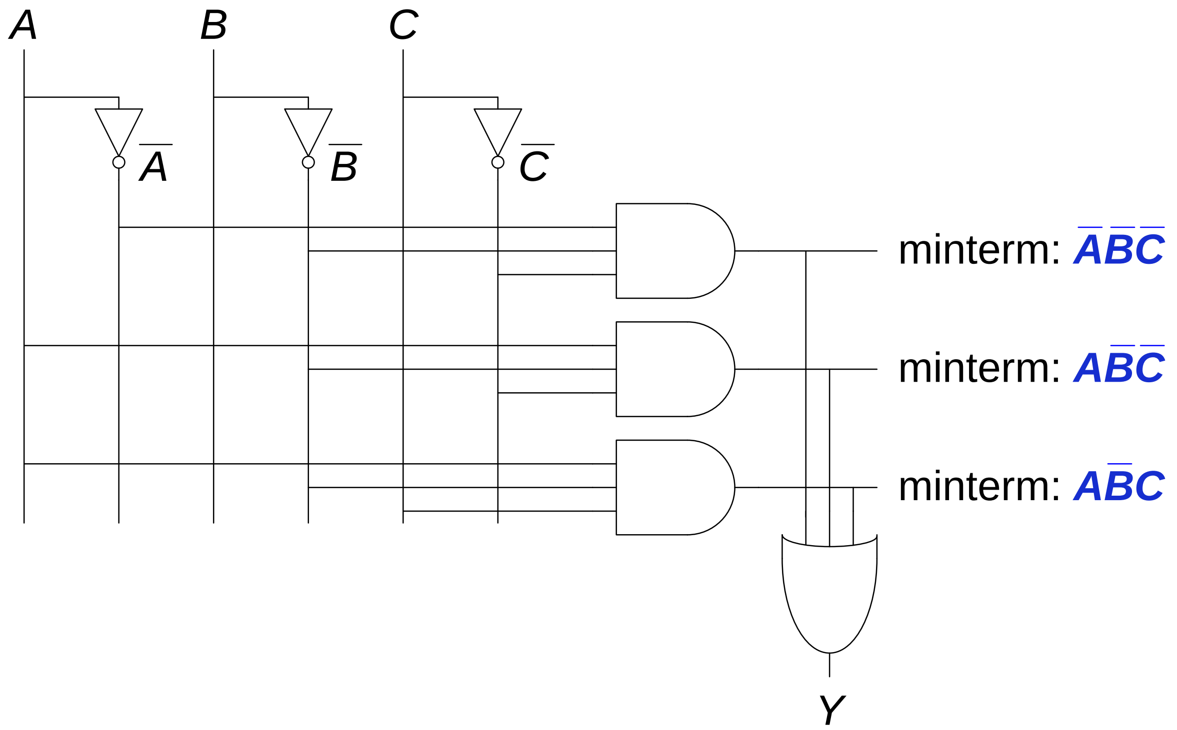
Circuit Schematic Rules (cont.)

- Wires always connect at a T junction
- A dot where wires cross indicates a connection between the wires
- Wires crossing *without* a dot make no connection



Two-Level Logic

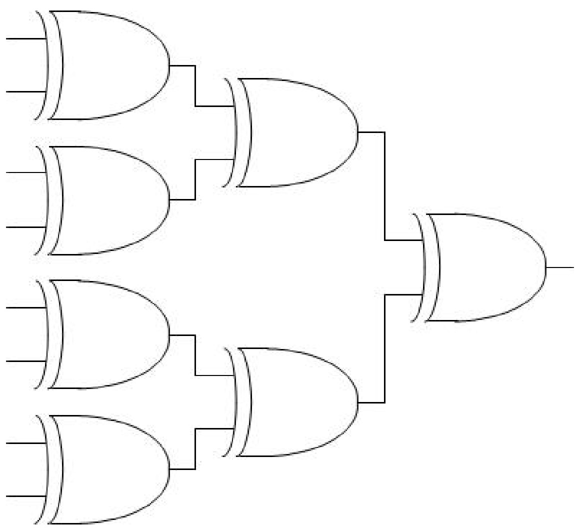
- Two-level logic: **ANDs** followed by **ORs**
- Example: $Y = \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C} + A\overline{B}C$

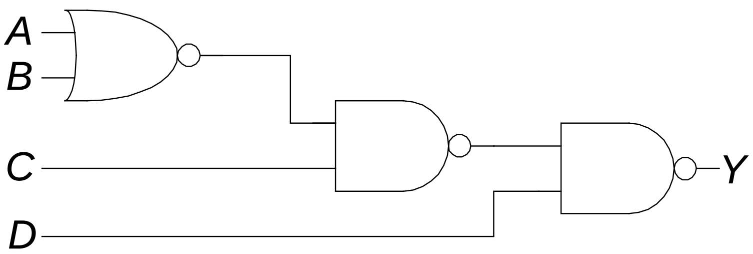


Implements functions in **SOP form**

Multilevel Logic

- Complex logic is often built from many stages of simpler gates.

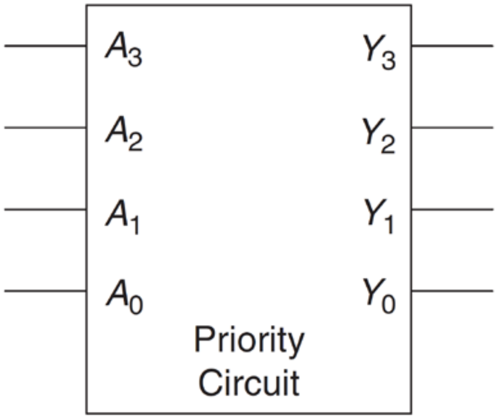




Multiple-Output Circuits

- **Example: Priority Circuit**

  Output asserted corresponding to most significant TRUE input

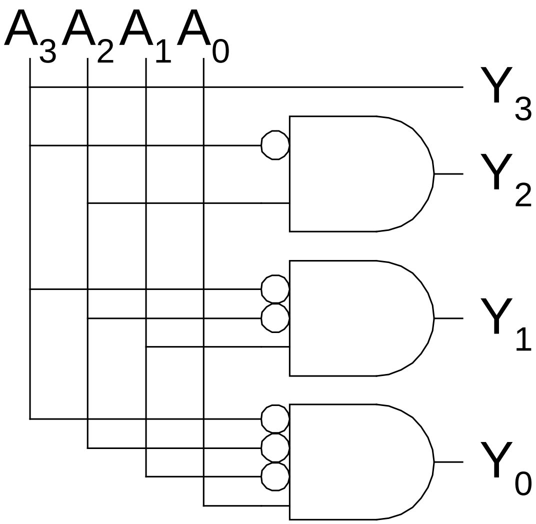


A_3	A_2	A_1	A_0	Y_3	Y_2	Y_1	Y_0
0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1
0	0	1	0	0	0	1	0
0	0	1	1	0	0	1	0
0	1	0	0	0	1	0	0
0	1	0	1	0	1	0	0
0	1	1	0	0	1	0	0
0	1	1	1	0	1	0	0
1	0	0	0	1	0	0	0
1	0	0	1	1	0	0	0
1	0	1	0	1	0	0	0
1	0	1	1	1	0	0	0
1	1	0	0	1	0	0	0
1	1	0	1	1	0	0	0
1	1	1	0	1	0	0	0
1	1	1	1	1	0	0	0

Priority Circuit Hardware

A_3	A_2	A_1	A_0	Y_3	Y_2	Y_1	Y_0
0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1
0	0	1	0	0	0	1	0
0	0	1	1	0	0	1	0
0	1	0	0	0	1	0	0
0	1	0	1	0	1	0	0
0	1	1	0	0	1	0	0
0	1	1	1	0	1	0	0
1	0	0	0	1	0	0	0
1	0	0	1	1	0	0	0
1	0	1	0	1	0	0	0
1	0	1	1	1	0	0	0
1	1	0	0	1	0	0	0
1	1	0	1	1	0	0	0
1	1	1	0	1	0	0	0
1	1	1	1	1	0	0	0

$$Y_3 = A_3$$
$$Y_2 = \overline{A_3}\,A_2$$
$$Y_1 = \overline{A_3}\,\overline{A_2}\,A_1$$
$$Y_0 = \overline{A_3}\,\overline{A_2}\,\overline{A_1}\,A_0$$



Don't Cares

A_3	A_2	A_1	A_0	Y_3	Y_2	Y_1	Y_0
0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1
0	0	1	0	0	0	1	0
0	0	1	1	0	0	1	0
0	1	0	0	0	1	0	0
0	1	0	1	0	1	0	0
0	1	1	0	0	1	0	0
0	1	1	1	0	1	0	0
1	0	0	0	1	0	0	0
1	0	0	1	1	0	0	0
1	0	1	0	1	0	0	0
1	0	1	1	1	0	0	0
1	1	0	0	1	0	0	0
1	1	0	1	1	0	0	0
1	1	1	0	1	0	0	0
1	1	1	1	1	0	0	0

$$Y_3 = A_3$$
$$Y_2 = \overline{A_3}\, A_2$$
$$Y_1 = \overline{A_3}\, \overline{A_2}\, A_1$$
$$Y_0 = \overline{A_3}\, \overline{A_2}\, \overline{A_1}\, A_0$$

A_3	A_2	A_1	A_0	Y_3	Y_2	Y_1	Y_0
0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1
0	0	1	X	0	0	1	0
0	1	X	X	0	1	0	0
1	X	X	X	1	0	0	0

Two-Level Logic Forms

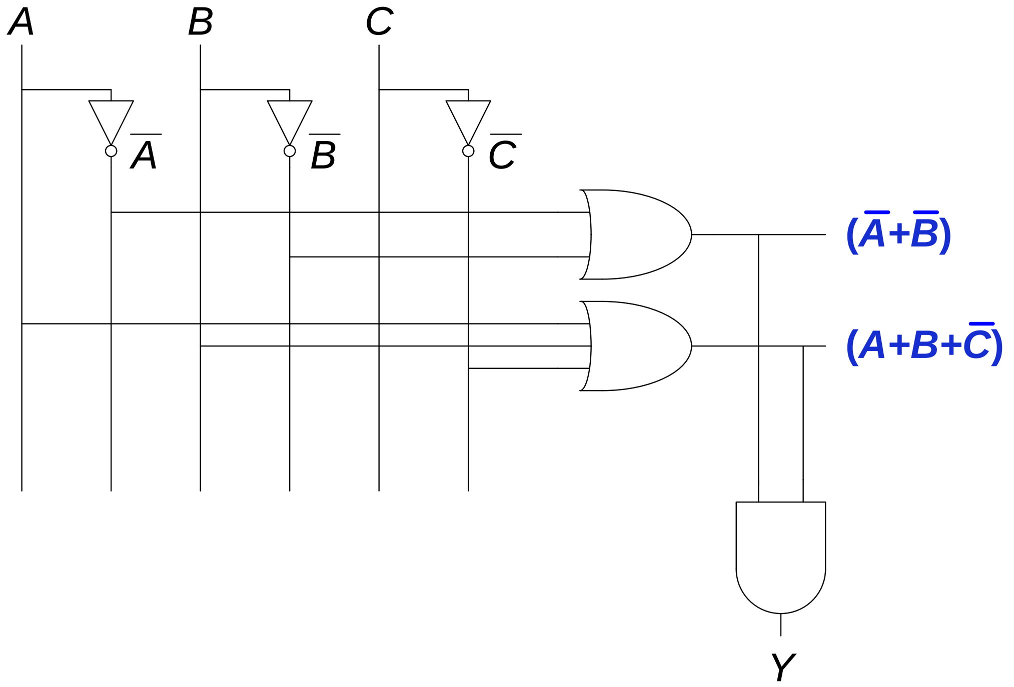
Two-Level Logic Variations

- **ANDs** followed by **ORs**: **SOP** form
- **ORs** followed by **ANDs**: **POS** form
- Only **NAND** gates: **SOP** form
- Only **NOR** gates: **POS** form

Most common form of two-level logic

Two-Level Logic Variation

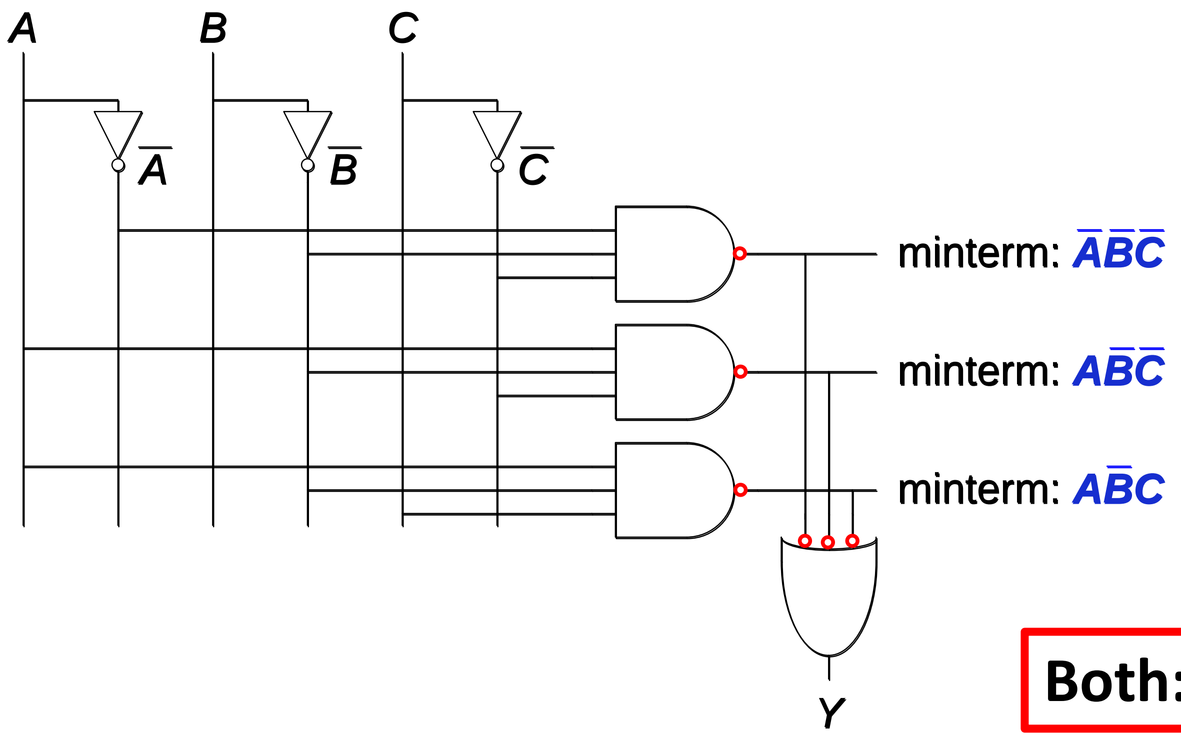
- Two-level logic variation: **ORs** followed by **ANDs**
- Example: $Y = (\overline{A}+\overline{B})(A+B+\overline{C})$



Implements functions in **POS form**

Two-Level Logic

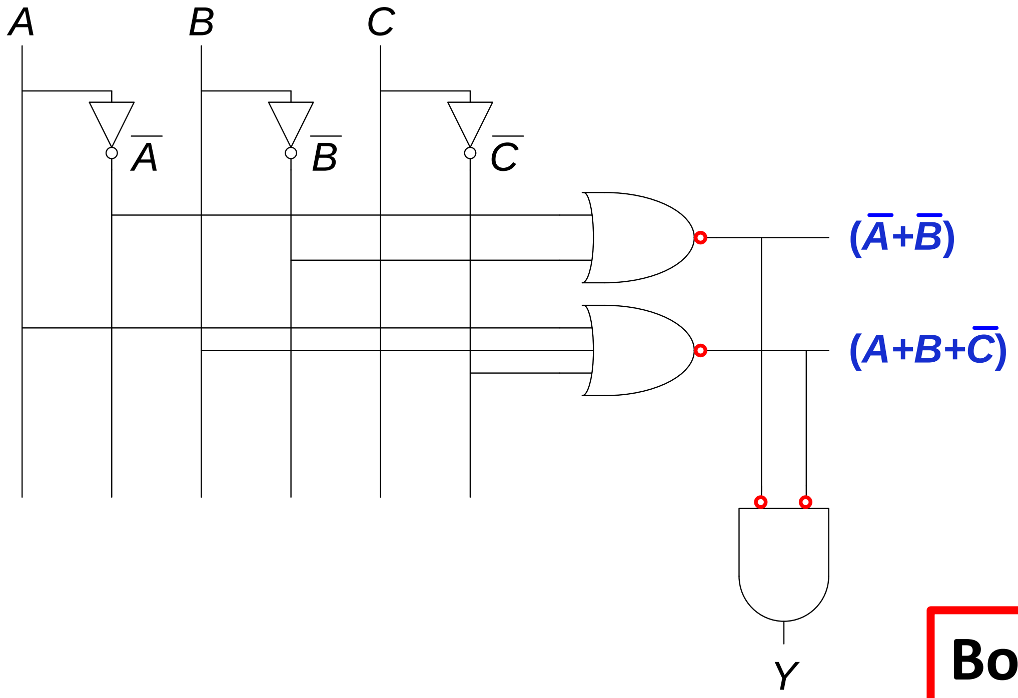
- Two-level logic: **ANDs** followed by **ORs → NANDs**
- Example: $Y = \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C} + A\overline{B}C$



Both: SOP form

Two-Level Logic Variation

- Two-level logic: **ORs** followed by **ANDs → NORs**
- Example: $Y = (\bar{A}+\bar{B})(A+B+\bar{C})$



Both: POS form

Put **bubbles** on internal nodes.

Bubble Pushing

De Morgan's Theorem

#	Theorem	Dual	Name
T12	$\overline{B \bullet C \bullet D\ldots} = \overline{B} + \overline{C} + \overline{D}\ldots$	$\overline{B+C+D\ldots} = \overline{B} \bullet \overline{C} \bullet \overline{D}\ldots$	De Morgan's Theorem

De Morgan's Theorem

Example D1:

$$Y = \overline{A + \overline{BC}}$$

$$= \overline{A} \bullet \overline{\overline{BC}}$$

$$= \overline{A} \bullet BC$$

$$= \overline{A}BC$$

- Work from the **outside in** (i.e., top bar, then down)
- Use **involution** when possible

De Morgan’s Theorem

Example D2:

$$Y = \overline{A + \overline{BC} + \overline{A}\,\overline{B}}$$

$$= \overline{A} \bullet \overline{\overline{BC}} \bullet \overline{\overline{A}\,\overline{B}}$$

$$= \overline{A} \bullet BC \bullet (\overline{\overline{A}} + \overline{\overline{B}})$$

$$= \overline{A}BC \bullet (A + B)$$

$$= \overline{A}BCA + \overline{A}BCB$$

$$= \overline{A}BC$$

- De Morgan **applies to**:
  - **Products** under a bar
  - **Sums** under a bar
- **Do not** try to apply DeMorgan’s to a **mix of operations**

De Morgan's Theorem

Example D2:

$$Y = \overline{A+\overline{BC}+\bar{A}\bar{B}}$$

$$= \bar{A} \bullet \overline{\overline{BC}} \bullet \overline{\bar{A}\bar{B}}$$

$$= \bar{A} \bullet BC \bullet (\bar{\bar{A}} + \bar{\bar{B}})$$

$$= \bar{A}BC \bullet (A + B)$$

$$= \bar{A}BCA + \bar{A}BCB$$

$$= \bar{A}BC$$

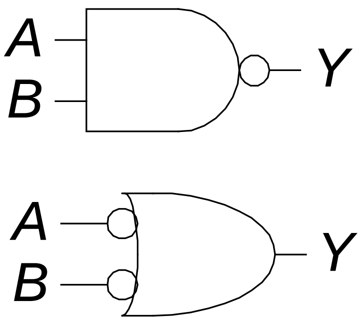
Don't forget these parentheses!

Remember:

$$\overline{AB} = (\bar{A} + \bar{B})$$

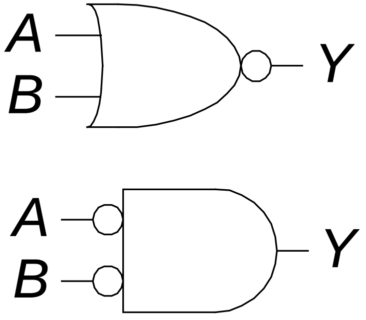
De Morgan's Theorem: Gates

- $Y = \overline{AB} = \overline{A} + \overline{B}$



NAND gate
two forms

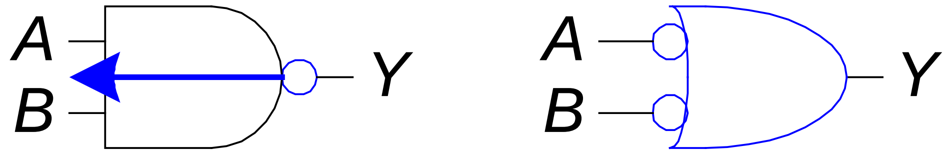
- $Y = \overline{A + B} = \overline{A} \cdot \overline{B}$



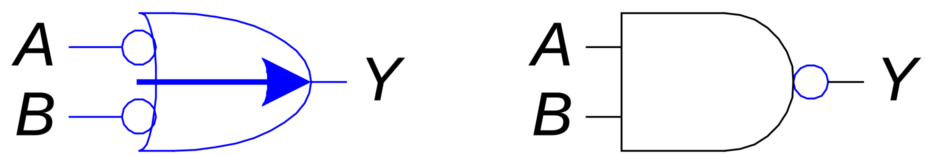
NOR gate
two forms

Bubble Pushing

- **Backward:**
  - Body changes
  - Adds bubbles to inputs

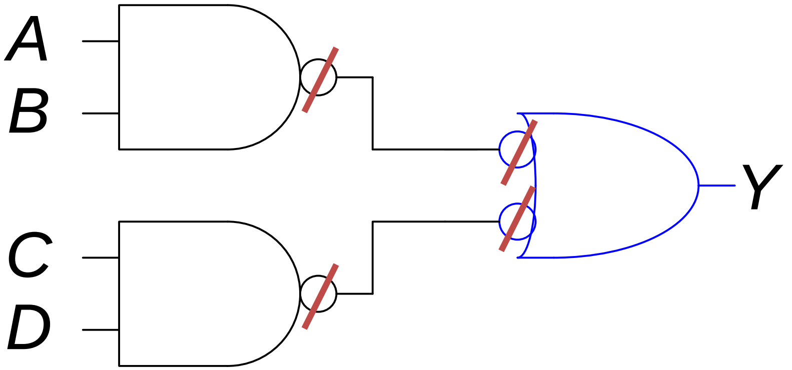


- **Forward:**
  - Body changes
  - Adds bubble to output



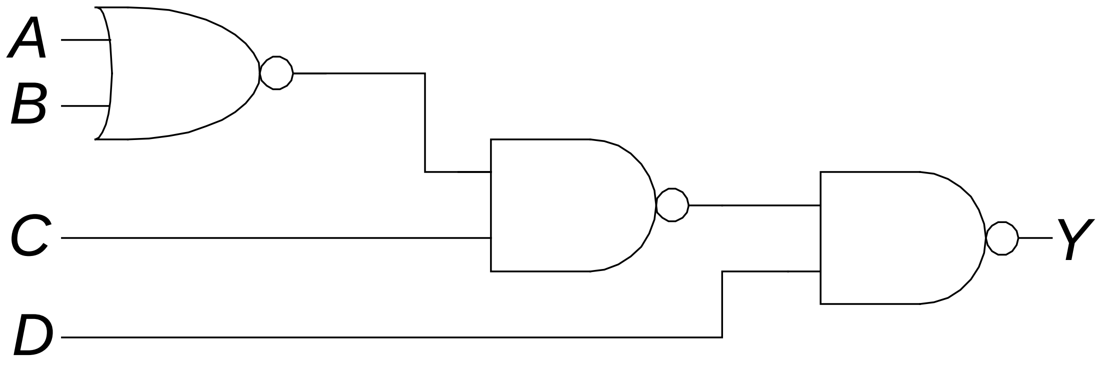
Bubble Pushing

- What is the Boolean expression for this circuit?

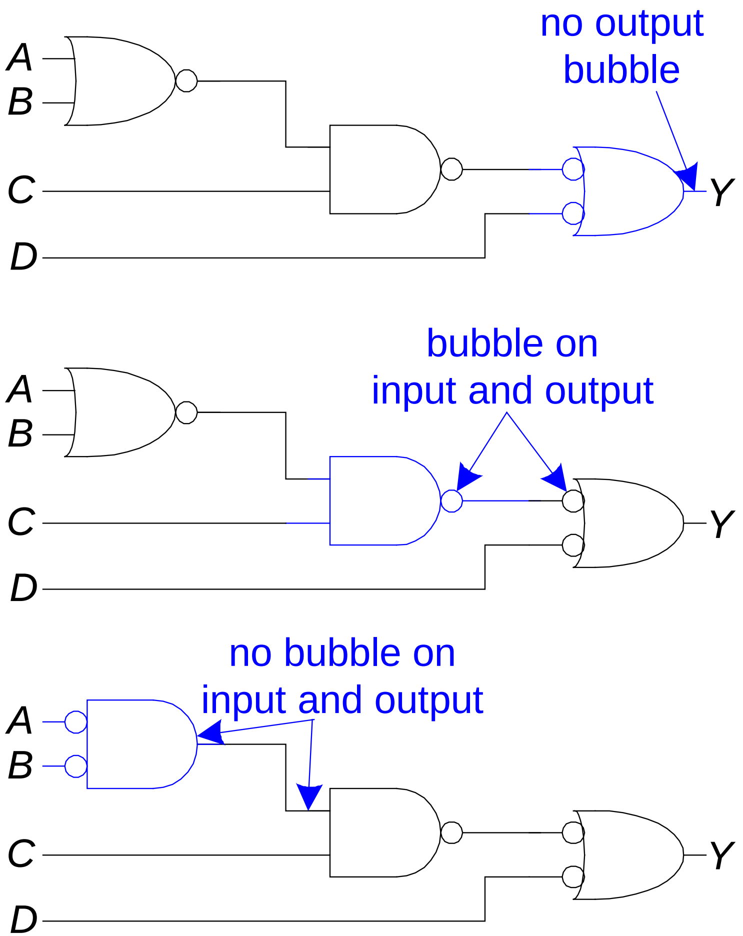


Bubble Pushing Rules

- Begin at output, then work toward inputs
- Push bubbles on final output back
- Draw gates in a form so bubbles cancel



Bubble Pushing Example

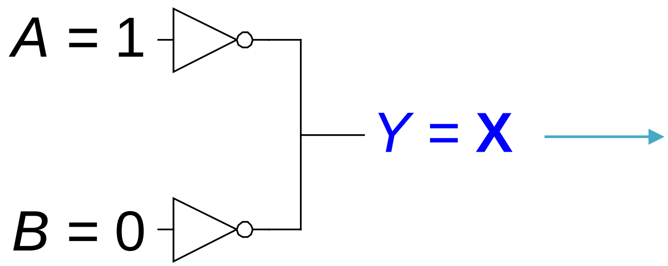


$$Y = \overline{A}\,\overline{B}C + \overline{D}$$

X's and Z's, Oh My

Contention: X

- **Contention:** circuit tries to drive output to 1 **and** 0 at the same time
  - Actual value somewhere in between 0 and 1
  - Could be 0, 1, or in forbidden zone
  - Might change with voltage, temperature, time, noise
  - Often causes excessive power dissipation



X indicates that the circuit node has an *unknown* or *illegal* value

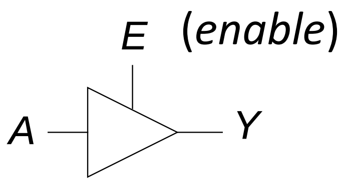
- X is also used for:
  - Uninitialized values
  - “Don’t care” values
- **Warnings:**
  - Contention or uninitialized outputs usually indicate a **bug**.
  - Look at the context to tell meaning

Floating: Z

- *Floating, high impedance, open, high Z*
- Floating output might be 0, 1, or somewhere in between
  - A voltmeter **won't** indicate whether a node is floating
  - But if you touch the node or your instructor walks over for a checkoff, it may change randomly

Tristate Buffer

(High, Low, floating)

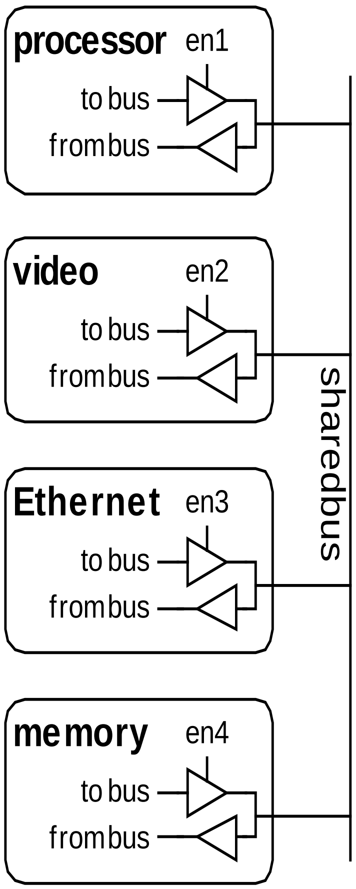


E	A	Y
0	0	Z
0	1	Z
1	0	0
1	1	1

Tristate Buffers on Busses

Floating nodes are used in tristate busses

- Many different drivers
  - E.g., a microprocessor, a video controller, and an Ethernet controller all communicate with the memory system in a PC.
- Exactly one is active at once



Karnaugh Maps

Karnaugh Maps (K-Maps)

- Boolean expressions can be minimized by combining terms
- K-maps minimize equations graphically
  - $PA + P\overline{A} = P$

A	B	C	Y
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

Y — C \ AB	00	01	11	10
0				
1				

Y — C \ AB	00	01	11	10
0	$\overline{A}\overline{B}\overline{C}$	$\overline{A}B\overline{C}$	$AB\overline{C}$	$A\overline{B}\overline{C}$
1	$\overline{A}\overline{B}C$	$\overline{A}BC$	ABC	$A\overline{B}C$

K-Map

- Circle 1's in adjacent squares
- In Boolean expression: include only literals whose true **and** complement forms are ***not*** in the circle

A	B	C	Y
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

Y (C \ AB)	00	01	11	10
0	1	0	0	0
1	1	0	0	0

Y (C \ AB)	00	01	11	10
0	$\bar{A}\bar{B}\bar{C}$	$\bar{A}B\bar{C}$	$AB\bar{C}$	$A\bar{B}\bar{C}$
1	$\bar{A}\bar{B}C$	$\bar{A}BC$	ABC	$A\bar{B}C$

$$Y = \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C = \bar{A}\bar{B}$$

3-Input K-Map

- Circle 1's in adjacent squares
- In Boolean expression: include only literals whose true **and** complement forms are ***not*** in the circle

Truth Table

A	*B*	*C*	*Y*
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

K-Map

Y: *C* \ *AB*	00	01	11	10
0				
1				

Y =

Some Definitions

- **Complement**: variable with a bar over it

  $\bar{A}, \bar{B}, \bar{C}$

- **Literal**: variable or its complement

  $A, \bar{A}, B, \bar{B}, C, \bar{C}$

- **Implicant**: product of literals

  $AB\bar{C}, \bar{A}C, BC$

- **Prime implicant:** implicant corresponding to the **largest circle** in a K-map

K-Map Rules

- **Every 1 must be circled** at least once
- Each circle must span a **power of 2** (i.e. 1, 2, 4) squares in each direction
- Each circle must be as **large** as possible
- A circle may **wrap around the edges**

4-Input K-Map

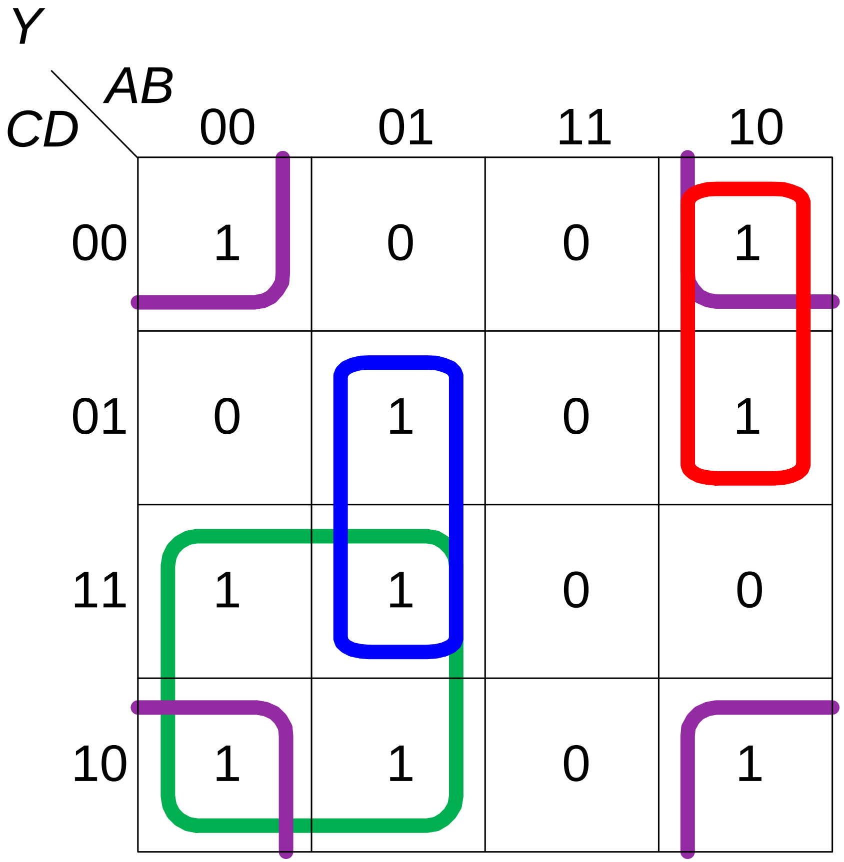
A	*B*	*C*	*D*	*Y*
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

Y

CD \ *AB*	00	01	11	10
00				
01				
11				
10				

4-Input K-Map

A	*B*	*C*	*D*	*Y*
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0



$$Y = \bar{A}C + \bar{A}BD + A\bar{B}\bar{C} + \bar{B}\bar{D}$$

Karnaugh Maps with Don’t Cares

K-Map Rules

- **Every 1 must be circled** at least once
- Each circle must span a **power of 2** (i.e. 1, 2, 4) squares in each direction
- Each circle must be as **large** as possible
- A circle may **wrap around the edges**
- Circle a **“don't care”** (X) **only if it helps** minimize the equation

K-Maps with Don't Cares

A	B	C	D	Y
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	X
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	X
1	0	1	1	X
1	1	0	0	X
1	1	0	1	X
1	1	1	0	X
1	1	1	1	X

Y

CD \ AB	00	01	11	10
00				
01				
11				
10				

K-Maps with Don’t Cares

A	B	C	D	Y
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	X
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	X
1	0	1	1	X
1	1	0	0	X
1	1	0	1	X
1	1	1	0	X
1	1	1	1	X

Y — CD \ AB	00	01	11	10
00	1	0	X	1
01	0	X	X	1
11	1	1	X	X
10	1	1	X	X

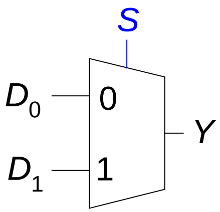
$$Y = A + \overline{B}\,\overline{D} + C$$

Combinational Building Blocks: Multiplexers

Multiplexer (*mux*)

- Selects between one of *N* inputs to connect to output
- **Select** input is **log$_2$*N* bits** – control input
- **Example:**

2:1 *mux*



S	D_1	D_0	Y
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

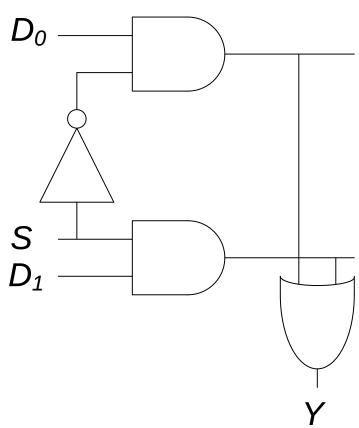
S	Y
0	D_0
1	D_1

2:1 Multiplexer Implementations

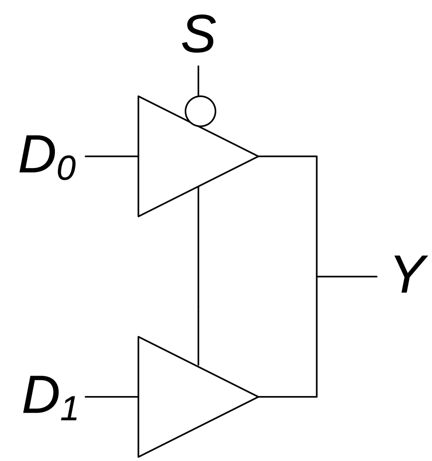
- **Logic gates**
  - Sum-of-products form

Y: S \ D_0D_1	00	01	11	10
0	0	0	1	1
1	0	1	1	0

$Y = D_0\overline{S} + D_1S$

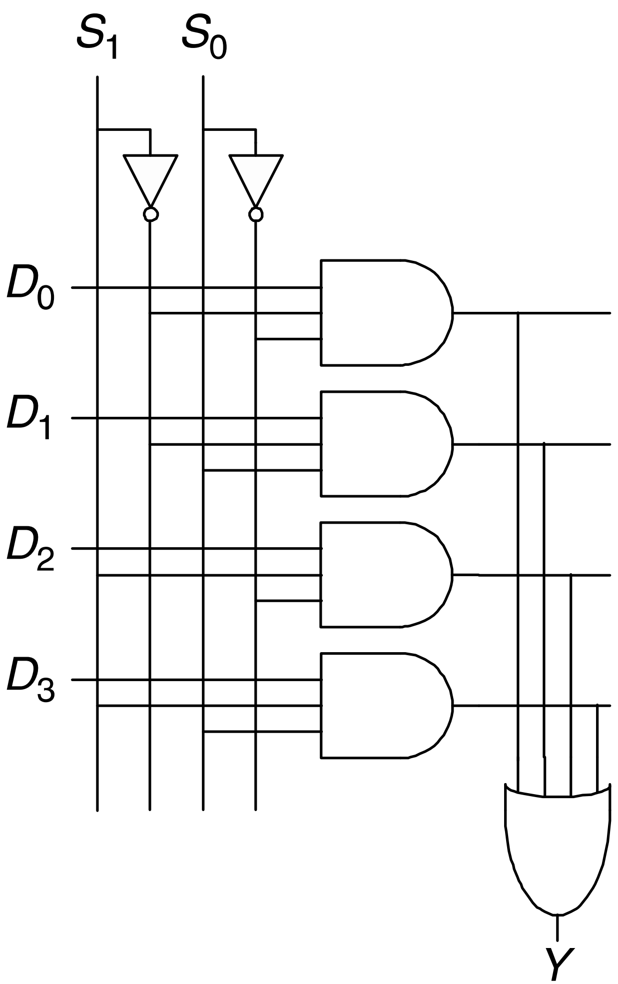


- **Tristates**
  - Two tristate buffers
  - Turn on exactly one to select the appropriate input

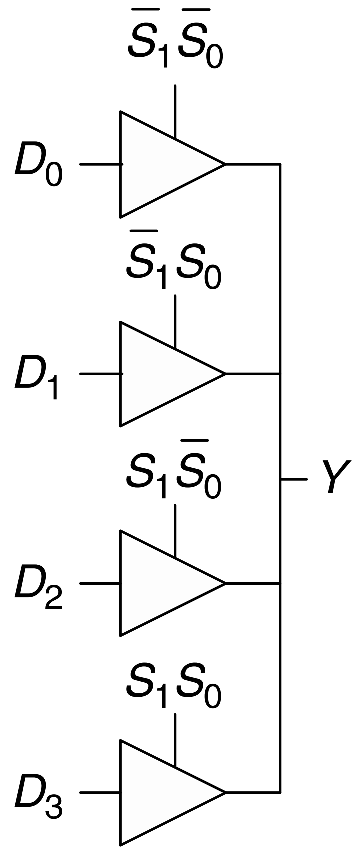


4:1 Multiplexer Implementations

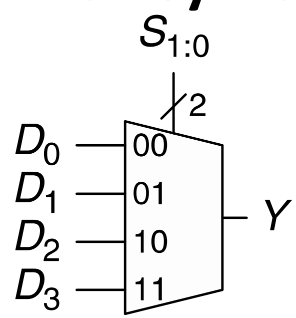
2-Level Logic



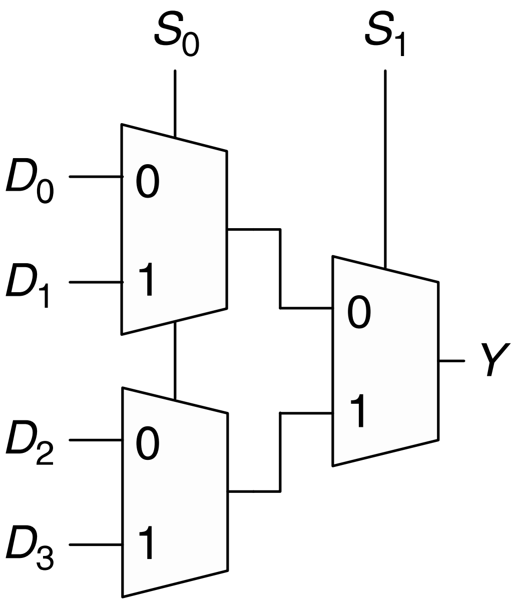
Tristates



4:1 *mux* symbol



Hierarchical

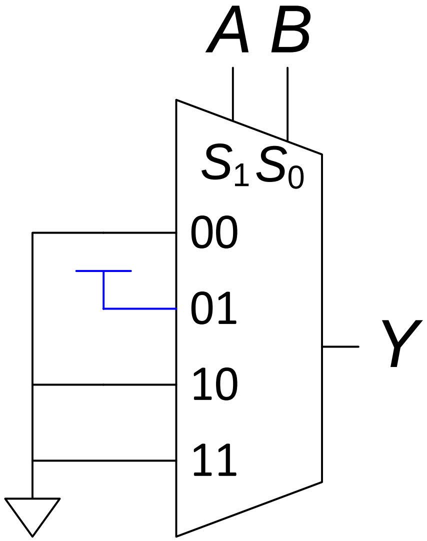


Logic using Multiplexers

Using *mux* as a **lookup table**

A	B	Y
0	0	0
0	1	1
1	0	0
1	1	0

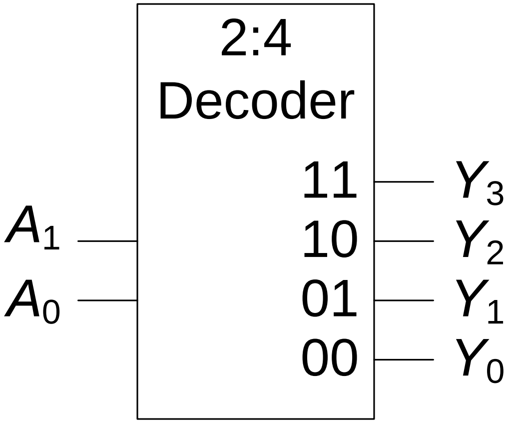
$$Y = \overline{A}B$$



Combinational Building Blocks: Decoders

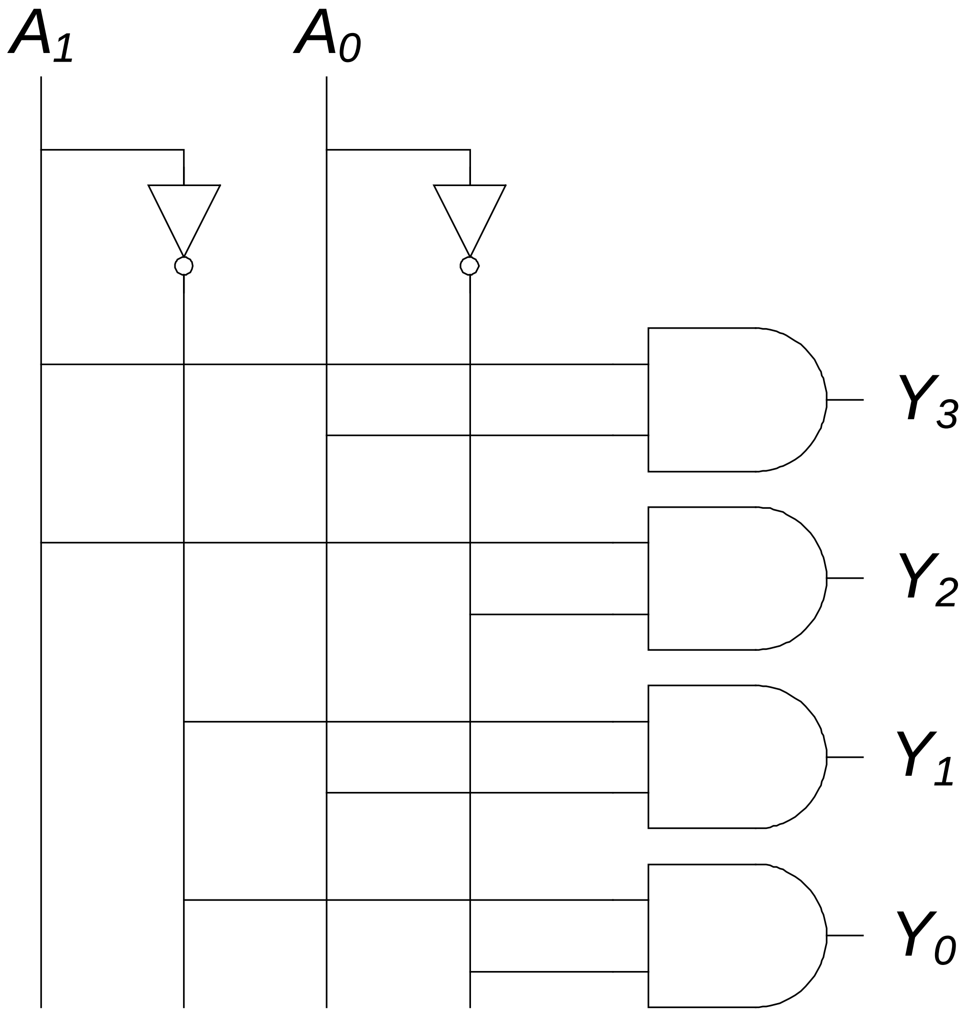
Decoders

- N inputs, 2^N outputs
- **One-hot output:** only one output **HIGH** at once



A_1	A_0	Y_3	Y_2	Y_1	Y_0
0	0	0	0	0	**1**
0	1	0	0	**1**	0
1	0	0	**1**	0	0
1	1	**1**	0	0	0

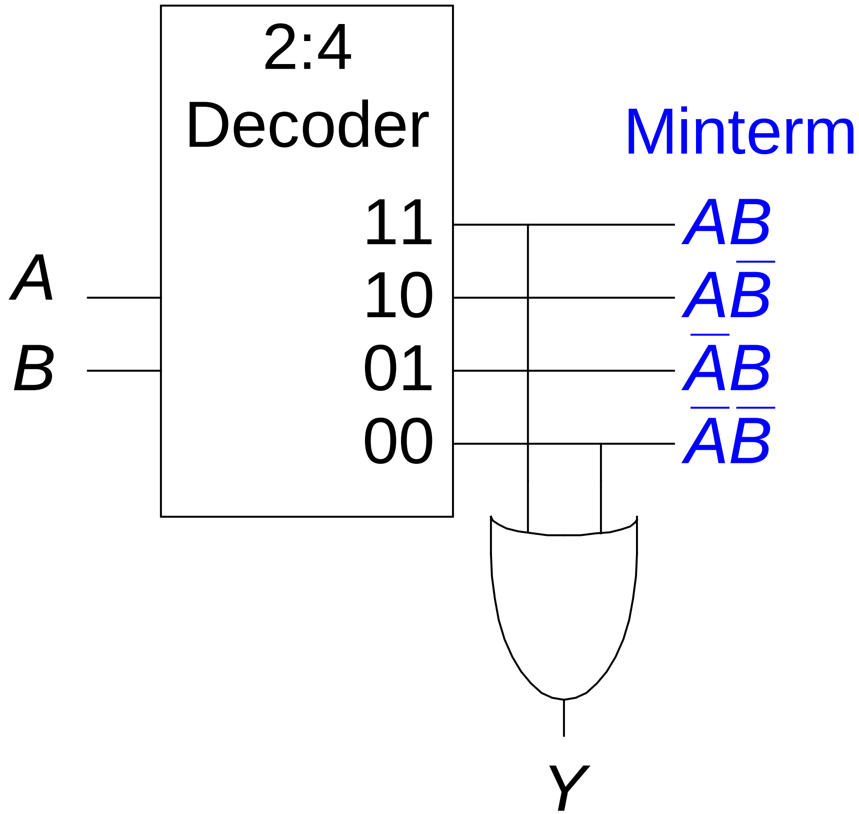
Decoder Implementation



Logic Using Decoders

- Two-input XNOR using decoder

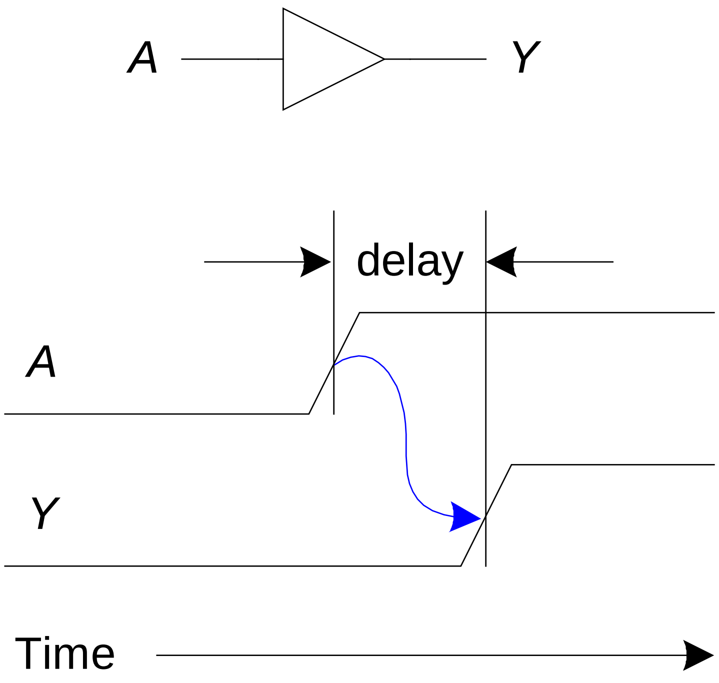
$$Y = AB + \overline{A}\,\overline{B} = \overline{A \oplus B}$$



Timing

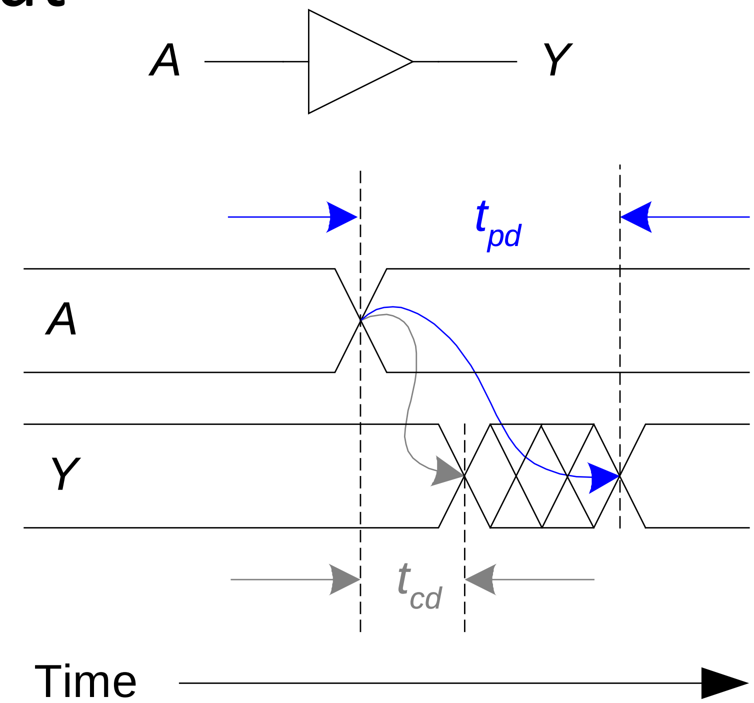
Timing

- **Delay:** time between input change and subsequent output change
- How to build fast circuits?



Propagation & Contamination Delay

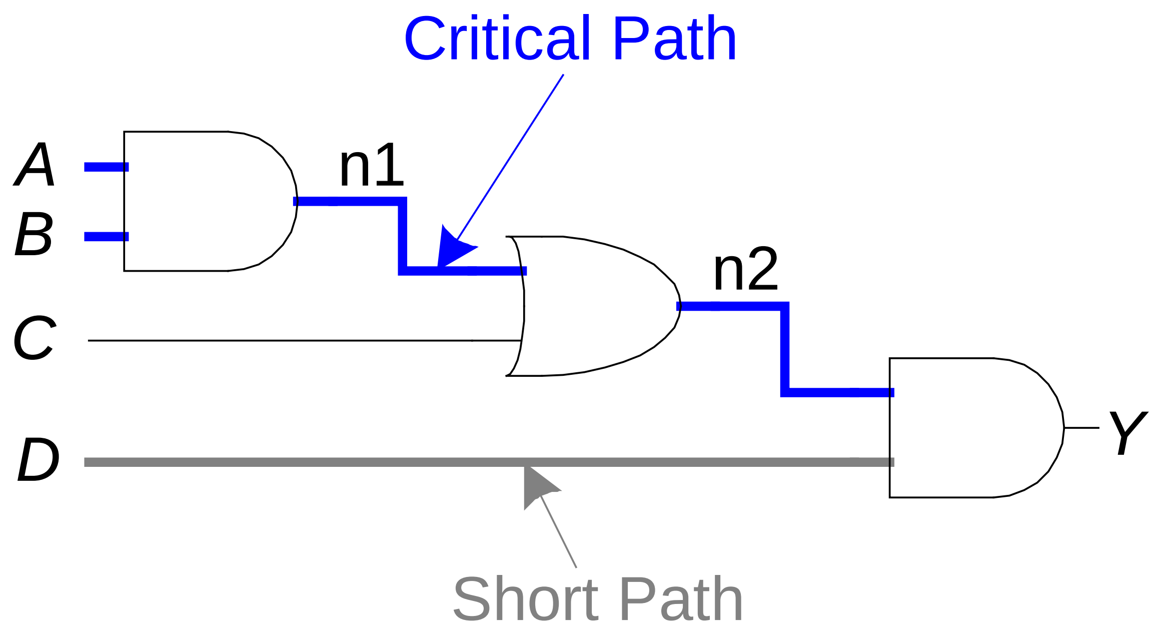
- **Propagation delay:** t_{pd} = **max** delay from input to output
- **Contamination delay:** t_{cd} = **min** delay from input to output



Propagation & Contamination Delay

- **Delay is caused by**
  - Capacitance charging and resistance in a circuit
  - Speed of light limitation
- **Reasons why t_{pd} and t_{cd} may be different:**
  - Different rising and falling delays
  - Multiple inputs and outputs, some of which are faster than others
  - Circuits slow down when hot and speed up when cold

Critical (Long) & Short Paths



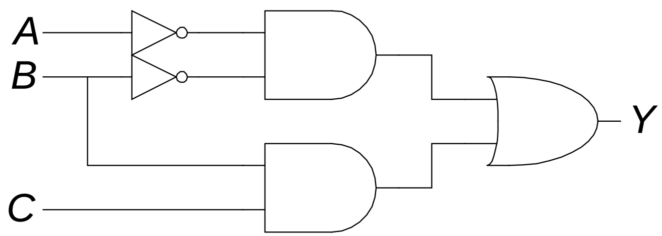
Critical (Long) Path: $t_{pd} = 2t_{pd\_AND} + t_{pd\_OR}$ **(max delay)**

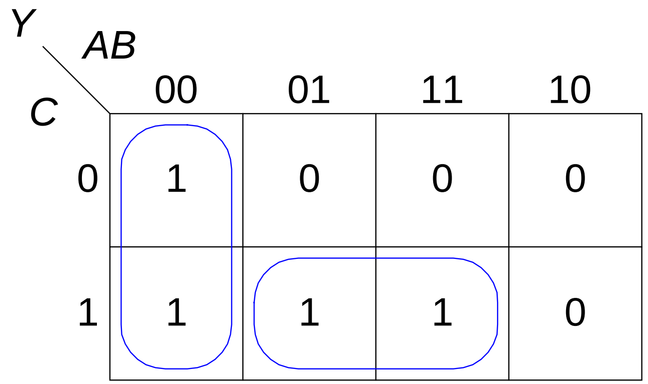
Short Path: $t_{cd} = t_{cd\_AND}$ **(min delay)**

Glitches

- Def.: a single input change causes an output to change multiple times

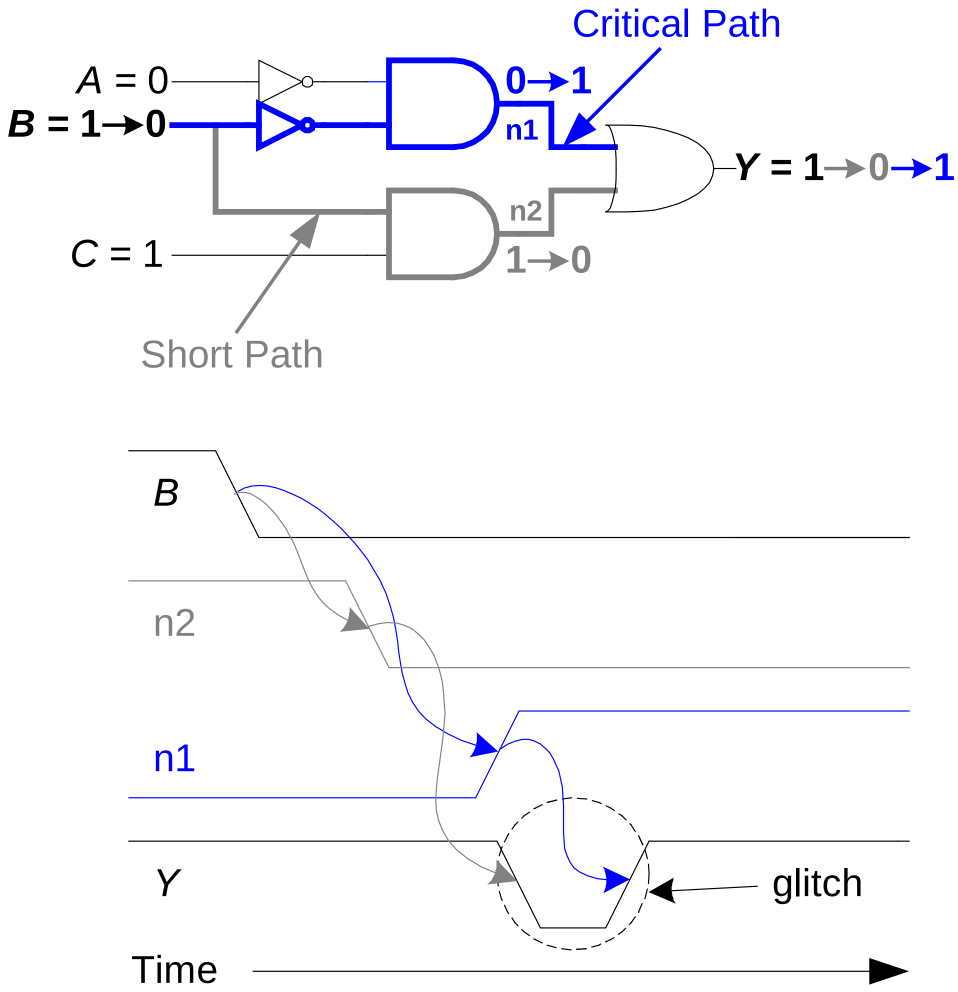
- Q: What happens when A = 0, C = 1, B falls?



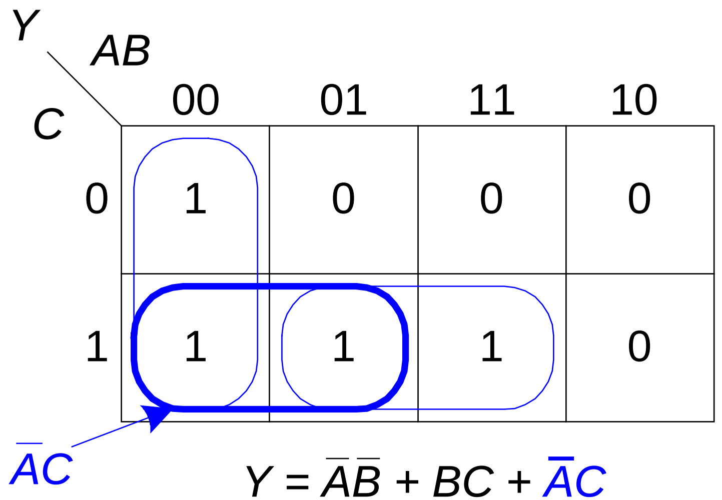


$Y = \overline{A}\,\overline{B} + BC$

Glitches (cont.)

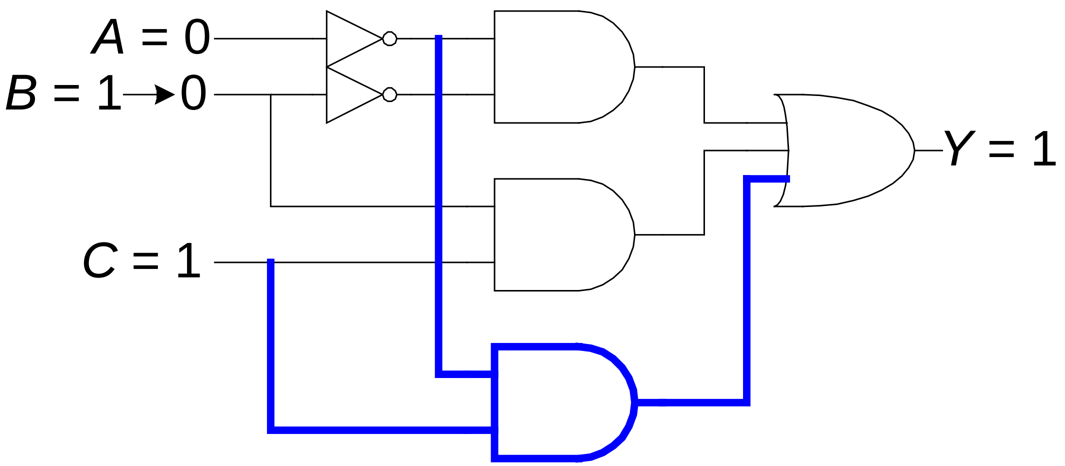


Fixing the Glitch



$Y = \overline{A}\overline{B} + BC + \overline{A}C$

.. covers the prime implicant boundary



Why Understand Glitches?

- Because of **synchronous design** conventions (see Chapter 3), glitches don't cause problems.
- We **can't get rid of all glitches** – simultaneous transitions on multiple inputs can also cause glitches.
- It's important to **recognize** a glitch: in simulations or on oscilloscope.

About these Notes