Linear Algebra

- Vector Spaces -

Jinsun Park

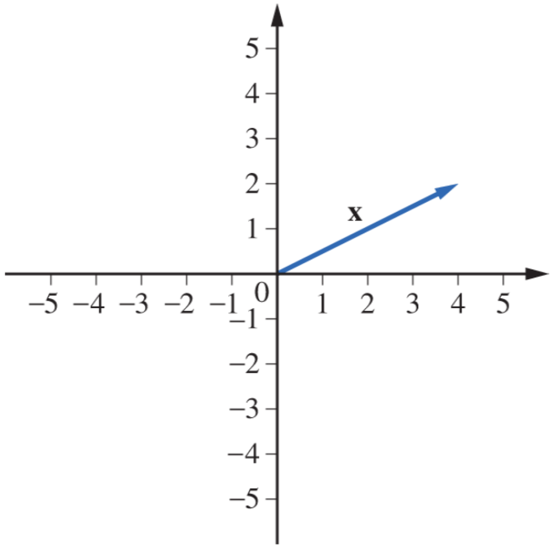
Visual Intelligence and Perception Lab., CSE, PNU

Vector Spaces

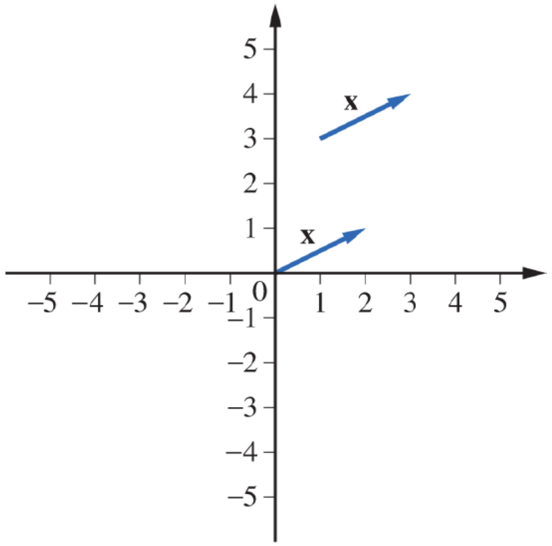
Euclidean Vector Space $\mathbb{R}^n$

- A *vector space* is a set whose elements (*i.e.*, vectors) may be added together (addition) and multiplied by scalars (scalar multiplication)
  - ex) Euclidean Vector Spaces $\mathbb{R}^n$

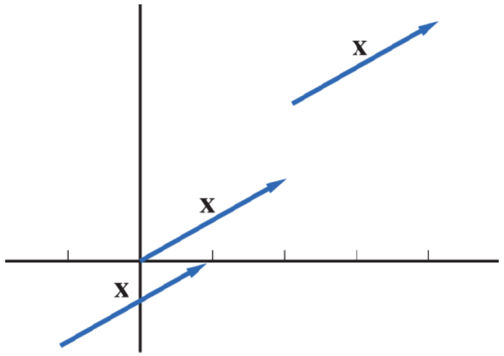
- Example: Euclidean Vector Space $\mathbb{R}^2$
  - A nonzero vector $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ can be associated with the directed line segment in the xy-plane



$x_1 = 4 \quad x_2 = 2$



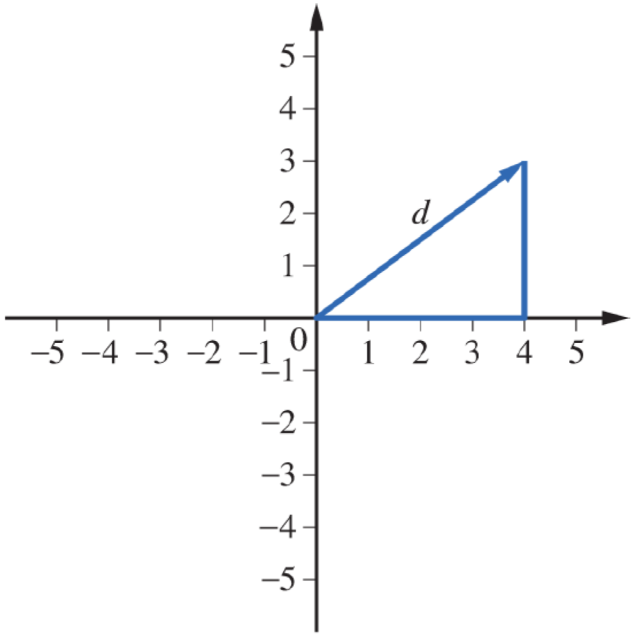
$x_1 = 2 \qquad x_2 = 1$
$a = 1 \qquad b = 3$
$a + x_1 = 3 \qquad b + x_2 = 4$



$\boldsymbol{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Euclidean Vector Space $\mathbb{R}^n$

- Example: Euclidean Vector Space $\mathbb{R}^2$

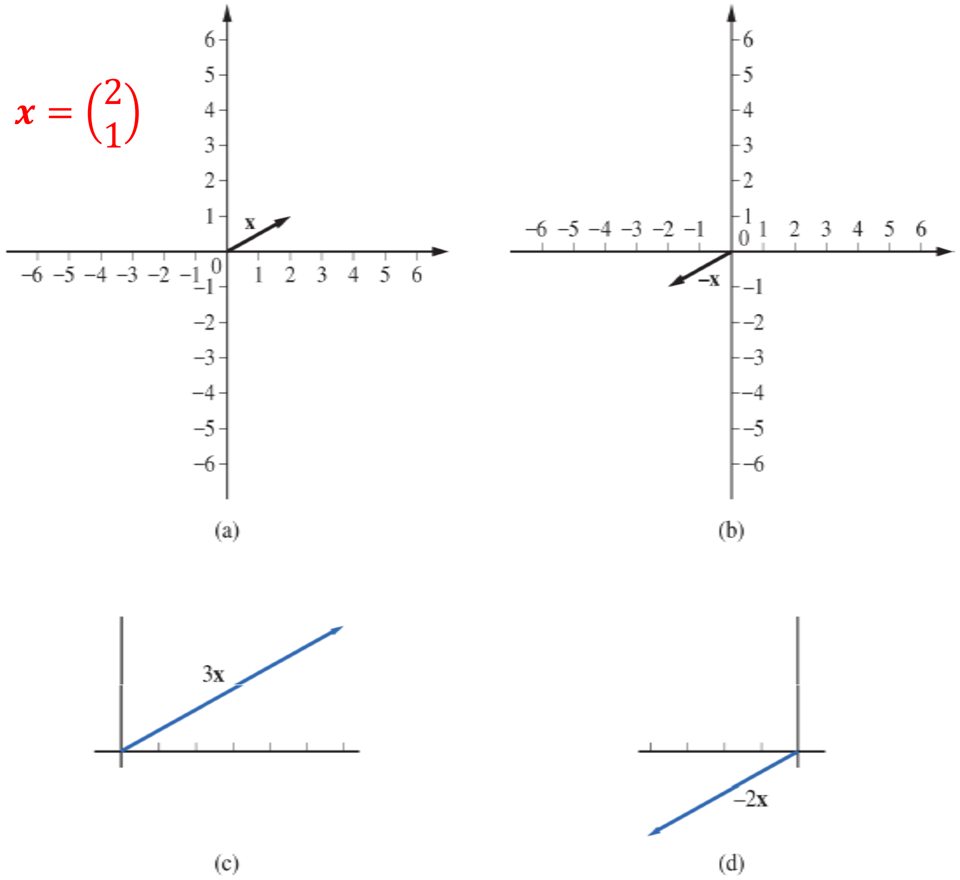


$x_1 = 4 \quad x_2 = 3 \quad \text{length } d = 5$

$$d = \sqrt{x_1^2 + x_2^2}$$

Length of a vector

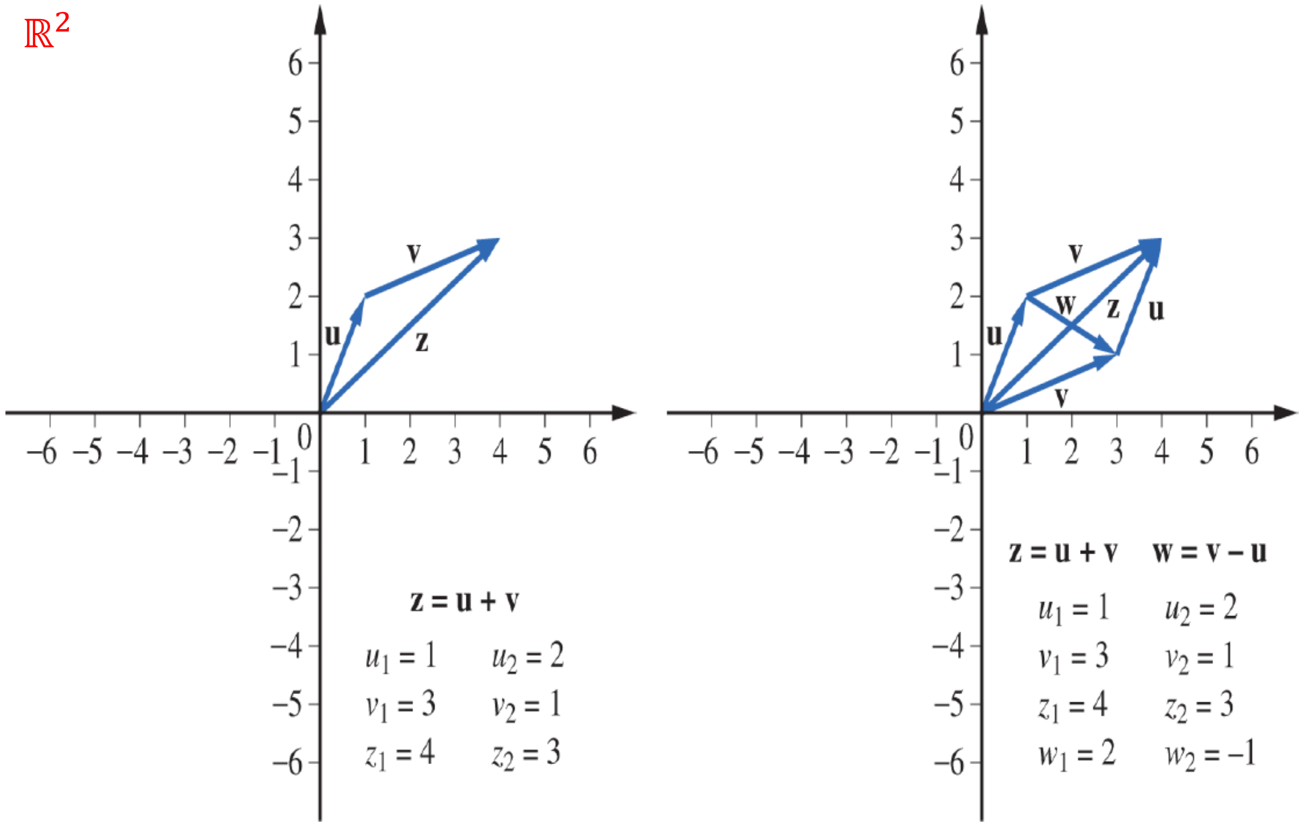
$\boldsymbol{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



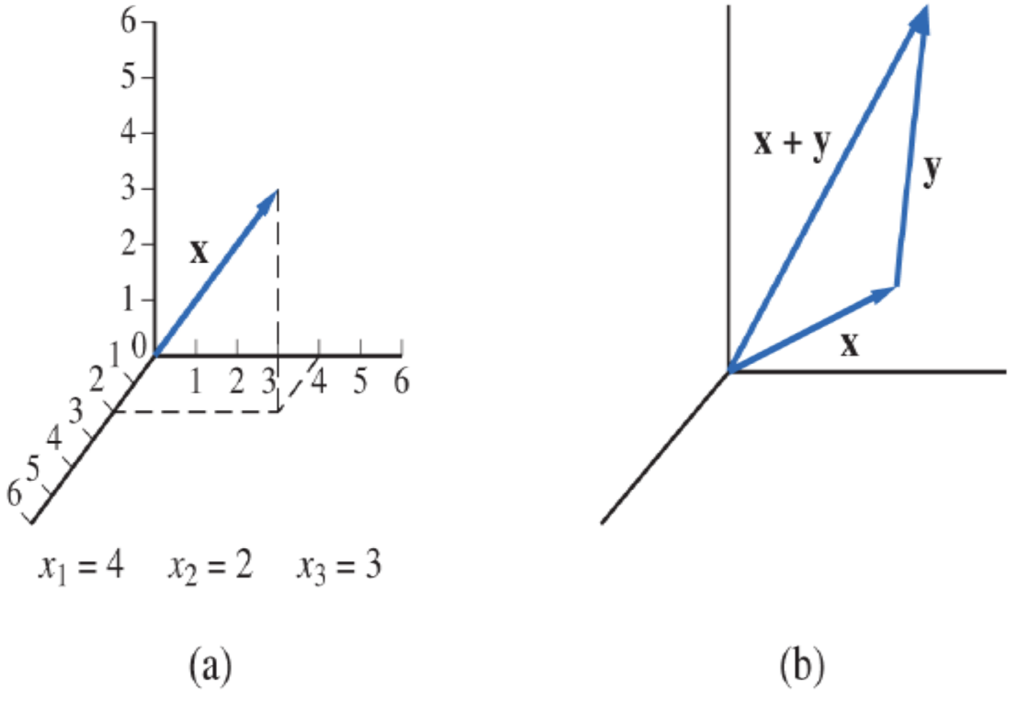
Euclidean Vector Space $\mathbb{R}^n$

- Example: Euclidean Vector Spaces $\mathbb{R}^2$ and $\mathbb{R}^3$

$\mathbb{R}^2$



$\mathbb{R}^3$



Vector Space $\mathbb{R}^{\mathbf{m}\times\boldsymbol{n}}$

- Let $\mathbb{R}^{m\times n}$ denote the set of all $m \times n$ matrices with real entries

- If $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $B = (b_{ij}) \in \mathbb{R}^{m\times n}$,
  - The sum $A + B$ is defined to be the $m \times n$ matrix $C = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$
  - Given a scalar α, αA is defined to be the $m \times n$ matrix whose (i, j) entries is αa_{ij}

- With $m \times n$ matrices and two operations (*i.e.*, addition and scalar multiplication) on the set $\mathbb{R}^{m\times n}$, we can define a mathematical system

Vector Space Axioms

- Let V be a set on which the operations of addition and scalar multiplication are defined
  - Each pair of elements $\boldsymbol{x}, \boldsymbol{y} \in V$ is associated with a unique element $\boldsymbol{x} + \boldsymbol{y} \in V$
  - Each element $\boldsymbol{x} \in V$ and each scalar α are associated with a unique element $\alpha\boldsymbol{x} \in V$
- The set V together with the operations of addition and scalar multiplication is said to form a *vector space* if the following axioms are satisfied:

A1. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$ for any $\boldsymbol{x}, \boldsymbol{y} \in V$
A2. $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$ for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$
A3. There exists an element $\boldsymbol{0} \in V$ such that $\boldsymbol{x} + \boldsymbol{0} = \boldsymbol{x}$ for each $\boldsymbol{x} \in V$
A4. For each $\boldsymbol{x} \in V$, there exists an element $-\boldsymbol{x} \in V$ such that $\boldsymbol{x} + (-\boldsymbol{x}) = \boldsymbol{0}$
A5. $\alpha(\boldsymbol{x} + \boldsymbol{y}) = \alpha\boldsymbol{x} + \alpha\boldsymbol{y}$ for each scalar α and any $\boldsymbol{x}, \boldsymbol{y} \in V$
A6. $(\alpha + \beta)\boldsymbol{x} = \alpha\boldsymbol{x} + \beta\boldsymbol{x}$ for any scalars α, β and any $\boldsymbol{x} \in V$
A7. $(\alpha\beta)\boldsymbol{x} = \alpha(\beta\boldsymbol{x})$ for any scalars α, β and any $\boldsymbol{x} \in V$
A8. $1\boldsymbol{x} = \boldsymbol{x}$ for all $\boldsymbol{x} \in V$

Vector Space Axioms

- A vector space over a *field* $F(=\mathbb{R})$ is a non-empty set V together with two binary operations that satisfy the following axioms (Field: 체)
  - The elements of V are called *vectors*, and the elements of F are called *scalars*

A1. $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x}$ for any $\boldsymbol{x},\boldsymbol{y}\in V$ (Commutativity of vector addition)

A2. $(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z}=\boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})$ for any $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in V$ (Associativity of vector addition)

A3. There exists an element $\mathbf{0}\in V$ such that $\boldsymbol{x}+\mathbf{0}=\boldsymbol{x}$ for each $\boldsymbol{x}\in V$

(Identity element of vector addition)

A4. For each $\boldsymbol{x}\in V$, there exists an element $-\boldsymbol{x}\in V$ such that $\boldsymbol{x}+(-\boldsymbol{x})=\mathbf{0}$

(Inverse elements of vector addition)

A5. $\alpha(\boldsymbol{x}+\boldsymbol{y})=\alpha\boldsymbol{x}+\alpha\boldsymbol{y}$ for each scalar α and any $\boldsymbol{x},\boldsymbol{y}\in V$

(Distributivity of scalar multiplication with respect to vector addition)

A6. $(\alpha+\beta)\boldsymbol{x}=\alpha\boldsymbol{x}+\beta\boldsymbol{x}$ for any scalars α,β and any $\boldsymbol{x}\in V$

Field Addition Vector Addition

(Distributivity of scalar multiplication with respect to field addition)

A7. $(\alpha\beta)\boldsymbol{x}=\alpha(\beta\boldsymbol{x})$ for any scalars α,β and any $\boldsymbol{x}\in V$

Field Multiplication Scalar Multiplication

(Compatibility of scalar multiplication with field multiplication)

A8. $1\boldsymbol{x}=\boldsymbol{x}$ for all $\boldsymbol{x}\in V$

(Identity element of scalar multiplication)

Vector Space Axioms

- Closure properties of the two operations

  C1. If $\boldsymbol{x} \in V$ and α is a scalar, then $\alpha\boldsymbol{x} \in V$

  C2. If $\boldsymbol{x}, \boldsymbol{y} \in V$, then $\boldsymbol{x} + \boldsymbol{y} \in V$

- Example

$$W = \{(a, 1) \mid a \text{ is real}\}$$

$$(3,1) + (5,1) =$$

Vector Space $C[a, b]$

- Let $C[a, b]$ denote the set of all real-valued functions that are defined and continuous on the closed interval $[a, b]$
  - The universal set is a set of functions, therefore, vectors are the functions in $C[a, b]$

- The sum $f + g$ of two functions in $C[a, b]$ is defined as follows:

$$(f + g)(x) = f(x) + g(x) \quad \text{for all } x \in [a, b]$$

- If f is a function in $C[a, b]$ and α is a real number, αf is defined as follows:

$$(\alpha f)(x) = \alpha f(x) \quad \text{for all } x \in [a, b]$$

- Proof
  - A1. $(f + g)(x) = (g + f)(x) \quad \text{for every } x \in [a, b]$
  - A3. $z(x) = 0 \quad \text{for all } x \in [a, b]$

Vector Space P_n

- Let P_n denote the set of all polynomials of degree less than n

$$(p+q)(x) = p(x) + q(x)$$

$$(\alpha p)(x) = \alpha p(x)$$

- Proof
  - A3. $z(x) = 0x^{n-1} + \cdots 0x + 0$

Additional Properties of Vector Spaces

- If V is a vector space and $\boldsymbol{x}$ is any element of V:

  i) $0\boldsymbol{x} = \mathbf{0}$
  ii) $\boldsymbol{x} + \boldsymbol{y} = \mathbf{0}$ implies that $\boldsymbol{y} = -\boldsymbol{x}$ (*i.e.*, the additive inverse of $\boldsymbol{x}$ is unique)
  iii) $(-1)\boldsymbol{x} = -\boldsymbol{x}$

- Proof

It follows from axioms A6 and A8 that

$$\mathbf{x} = 1\mathbf{x} = (1+0)\mathbf{x} = 1\mathbf{x} + 0\mathbf{x} = \mathbf{x} + 0\mathbf{x}$$

Thus,

$$-\mathbf{x} + \mathbf{x} = -\mathbf{x} + (\mathbf{x} + 0\mathbf{x}) = (-\mathbf{x} + \mathbf{x}) + 0\mathbf{x}$$
$$\mathbf{0} = \mathbf{0} + 0\mathbf{x} = 0\mathbf{x}$$

To prove (ii), suppose that $\mathbf{x} + \mathbf{y} = \mathbf{0}$. Then

$$-\mathbf{x} = -\mathbf{x} + \mathbf{0} = -\mathbf{x} + (\mathbf{x} + \mathbf{y})$$

Therefore,

$$-\mathbf{x} = (-\mathbf{x} + \mathbf{x}) + \mathbf{y} = \mathbf{0} + \mathbf{y} = \mathbf{y}$$

Finally, to prove (iii), note that

$$\mathbf{0} = 0\mathbf{x} = (1 + (-1))\mathbf{x} = 1\mathbf{x} + (-1)\mathbf{x}$$

Thus

$$\mathbf{x} + (-1)\mathbf{x} = \mathbf{0}$$

and it follows from part (ii) that

$$(-1)\mathbf{x} = -\mathbf{x}$$

Exercises

12. Let R^+ denote the set of positive real numbers. Define the operations of scalar multiplication $\circ$ by

$$\alpha \circ x = x^{\alpha} \quad \text{for each } x \in R^+ \text{ and for any real number } \alpha$$

Define the operation of addition $\oplus$ by

$$x \oplus y = x \cdot y \quad \text{for all } x, y \in R^+$$

Is R^+ a vector space with these operations?

ex) $-3 \circ \frac{1}{2} = \left(\frac{1}{2}\right)^{-3} = 8, \qquad 2 \oplus 5 = 2 \cdot 5 = 10$

Subspaces

Subspaces

- Given a vector space V, it is possible to form another vector space by taking a subset S of V and using the operations of V

- The set S must be closed under the operations of V

- Example
  - S is a subset of $\mathbb{R}^2$

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, x_2 = 2x_1 \right\}$$

i) Addition

ii) Scalar Multiplication

Subspaces

- If S is a nonempty subset of a vector space V, and S satisfies the conditions

  i) $\alpha \boldsymbol{x} \in S$ whenever $\boldsymbol{x} \in S$ for any scalar α (Closed under scalar multiplication)

  ii) $\boldsymbol{x} + \boldsymbol{y} \in S$ whenever $\boldsymbol{x} \in S$ and $\boldsymbol{y} \in S$ (Closed under addition)

  then S is said to be a *subspace* of V

- Every subspace of a vector space is a vector space

- Remarks
  - In a vector space V, $\{\mathbf{0}\}$ and V are subspaces of V. All other subspaces are referred to as *proper subspaces*. $\{\mathbf{0}\}$ is the *zero subspace*
  - Every subspace must contain the zero vector, therefore, we can verity that S is nonempty by showing that $\mathbf{0} \in S$

Subspaces

Ex 4. Let $S = \{A \in \mathbb{R}^{2\times 2} \mid a_{12} = -a_{21}\}$. The set S is nonempty, since O (the zero matrix) is in S.

(i) If $A \in S$, then A must be of the form

$$A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$$

and hence

$$\alpha A = \begin{bmatrix} \alpha a & \alpha b \\ -\alpha b & \alpha c \end{bmatrix}$$

Since the (2, 1) entry of αA is the negative of the (1, 2) entry, $\alpha A \in S$.

(ii) If $A, B \in S$, then they must be of the form

$$A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d & e \\ -e & f \end{bmatrix}$$

It follows that

$$A + B = \begin{bmatrix} a+d & b+e \\ -(b+e) & c+f \end{bmatrix}$$

Hence, $A + B \in S$.

Subspaces

Ex 5. Let S be the set of all polynomials of degree less than n with the property that $p(0) = 0$. The set is nonempty since it contains the zero polynomial. We claim that S is a subspace of P_n

(i) if $p(x) \in S$ and α is a scalar, then

$$\alpha p(0) = \alpha \cdot 0 = 0$$

and hence $\alpha p \in S$; and

(ii) if $p(x)$ and $q(x)$ are elements of S, then

$$(p+q)(0) = p(0) + q(0) = 0 + 0 = 0$$

and hence $p + q \in S$.

Ex 6. Let $C^n[a,b]$ be the set of all functions f that have a continuous n-th derivative on $[a,b]$. Verify that $C^n[a,b]$ is a subspace of $C[a,b]$

The Null Space of a Matrix

- Let A be an $m \times n$ matrix. Let $N(A)$ denote the set of all solutions to the homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$

$$N(A) = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \boldsymbol{0}\}$$

- $N(A)$ is a subspace of $\mathbb{R}^n$ and $\boldsymbol{0} \in N(A)$, thus, $N(A)$ is nonempty
  i) Scalar Multiplication

$$A(\alpha\boldsymbol{x}) = \alpha A\boldsymbol{x} = \alpha\boldsymbol{0} = \boldsymbol{0} \qquad \therefore \alpha\boldsymbol{x} \in N(A)$$

  ii) Addition

$$A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y} = \boldsymbol{0} + \boldsymbol{0} = \boldsymbol{0} \qquad \therefore \boldsymbol{x} + \boldsymbol{y} \in N(A)$$

- The set of all solutions of the homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ forms a subspace of $\mathbb{R}^n$

- The subspace $N(A)$ is called the *null space* of A

The Null Space of a Matrix

Ex 9. Determine $N(A)$ if

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

Using Gauss–Jordan reduction to solve $A\mathbf{x} = \mathbf{0}$, we obtain

$$\left(\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{rrrr|r} 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{rrrr|r} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right)$$

The reduced row echelon form involves two free variables, x_3 and x_4:

$$\begin{aligned} x_1 &= x_3 - x_4 \\ x_2 &= -2x_3 + x_4 \end{aligned}$$

Thus, if we set $x_3 = \alpha$ and $x_4 = \beta$, then

$$\mathbf{x} = \begin{pmatrix} \alpha - \beta \\ -2\alpha + \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

is a solution of $A\mathbf{x} = \mathbf{0}$. The vector space $N(A)$ consists of all vectors of the form

$$\alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

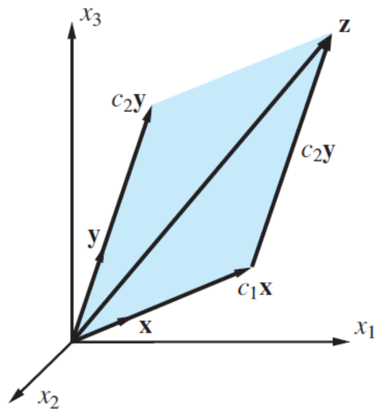
where α and β are scalars.

The Span of a Set of Vectors

- Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ be vectors in a vector space V. A sum of the form $\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 \cdots + \alpha_n \boldsymbol{v}_n$, where $\alpha_1, \cdots, \alpha_n$ are scalars, is called a *linear combination* of $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$

- The set of all linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ is called the span of $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$

- The span of $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ will be denoted by $Span(\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n)$



* In general, if two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ can be used to determine a plane in 3-space, that plane is the geometrical representation of $Span(\boldsymbol{x}, \boldsymbol{y})$

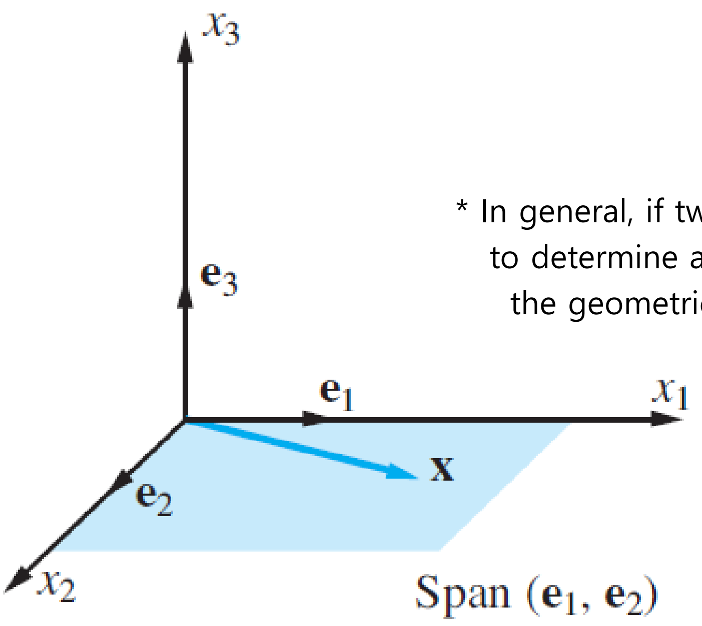
- Ex 10.

In $\mathbb{R}^3$, the span of $\mathbf{e}_1$ and $\mathbf{e}_2$ is the set of all vectors of the form

$$\alpha \mathbf{e}_1 + \beta \mathbf{e}_2 = \begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}$$

The reader may verify that Span$(\mathbf{e}_1, \mathbf{e}_2)$ is a subspace of $\mathbb{R}^3$. The subspace can be interpreted geometrically as the set of all vectors in 3-space that lie in the x_1x_2-plane (see Figure 3.2.1). The span of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is the set of all vectors of the form

$$\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$



The Span of a Set of Vectors

- If $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ are elements of a vector space V, then $Span(\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n)$ is a subspace of V

- Proof
  - The $Span(\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n)$ must be closed under the addition and scalar multiplication

Let β be a scalar and let $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$ be an arbitrary element of $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$. Since

$$\beta\mathbf{v} = (\beta\alpha_1)\mathbf{v}_1 + (\beta\alpha_2)\mathbf{v}_2 + \cdots + (\beta\alpha_n)\mathbf{v}_n$$

it follows that $\beta\mathbf{v} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Next, we must show that any sum of elements of $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is in $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Let $\mathbf{v} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n$ and $\mathbf{w} = \beta_1\mathbf{v}_1 + \cdots + \beta_n\mathbf{v}_n$. Then

$$\mathbf{v} + \mathbf{w} = (\alpha_1 + \beta_1)\mathbf{v}_1 + \cdots + (\alpha_n + \beta_n)\mathbf{v}_n \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$$

Therefore, $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a subspace of V. ■

Spanning Set for a Vector Space

- Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ be vectors in a vector space V. We say that the subspace $Span(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n)$ is *spanned* by $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$

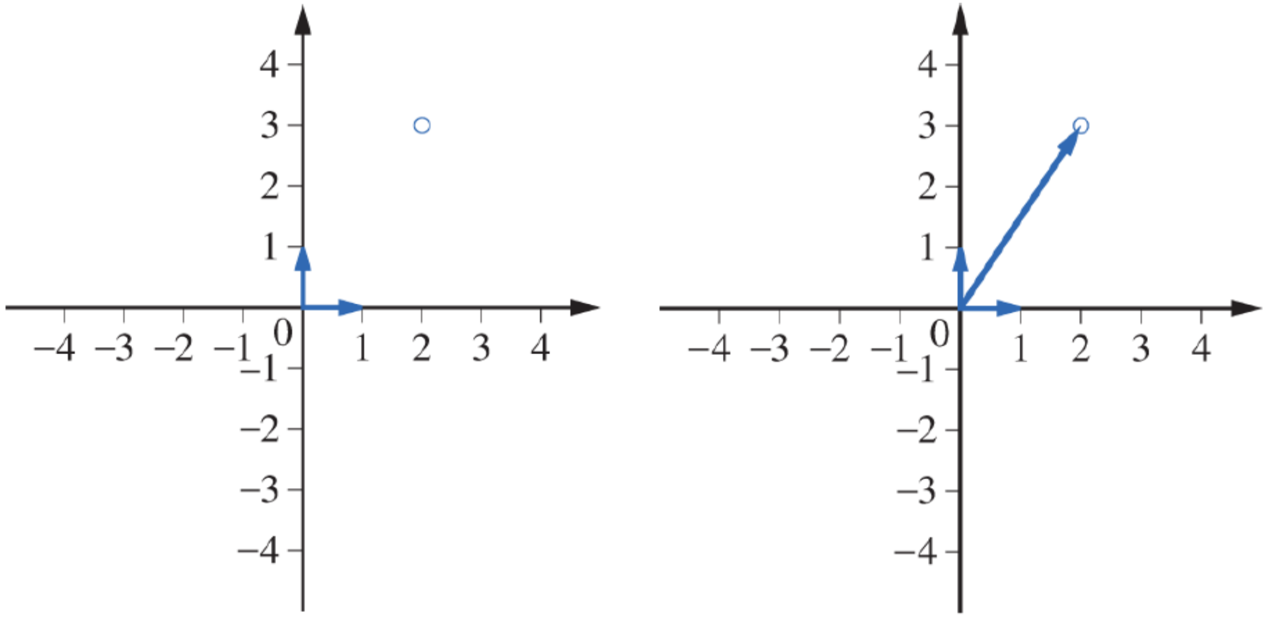
- The set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ is a *spanning set* for V if and only if every vector in V can be written as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$

- Ex) Euclidean space $\mathbb{R}^2$

$$\boldsymbol{x} = a\boldsymbol{e}_1 + b\boldsymbol{e}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$$

  - Any vector $\boldsymbol{x} \in \mathbb{R}^2$ can be represented as a linear combination of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$

  - $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is a spanning set for $\mathbb{R}^2$



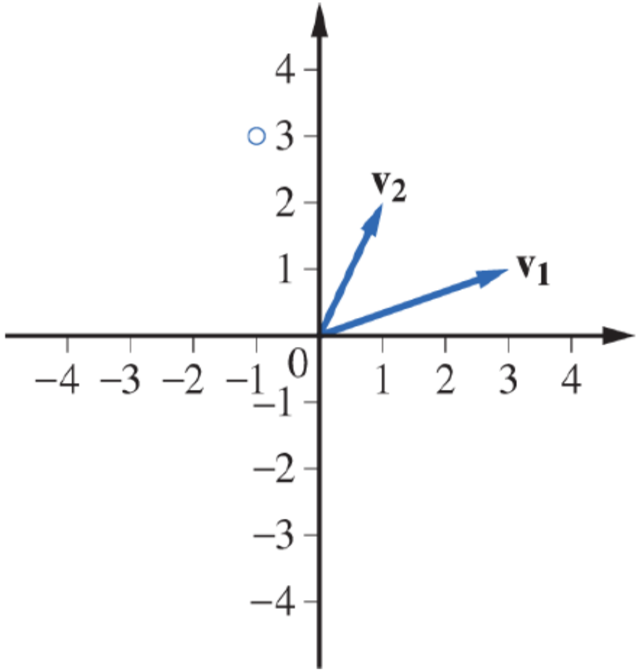
Terminal point of first vector (1, 0)
Terminal point of second vector (0, 1)
Target point (2, 3)

$c_1 = 2 \qquad c_2 = 3$

Spanning Set for a Vector Space

- Ex) Euclidean space $\mathbb{R}^2$

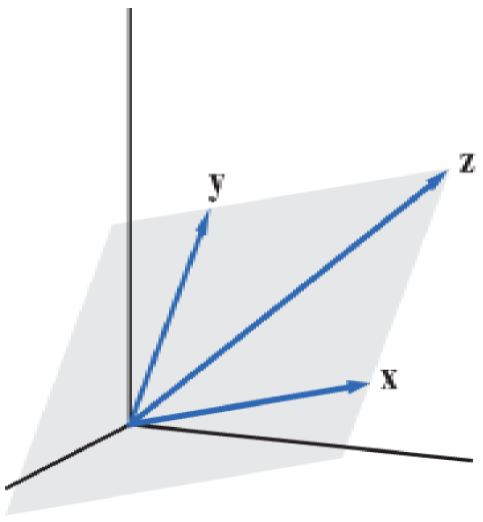
$$\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$



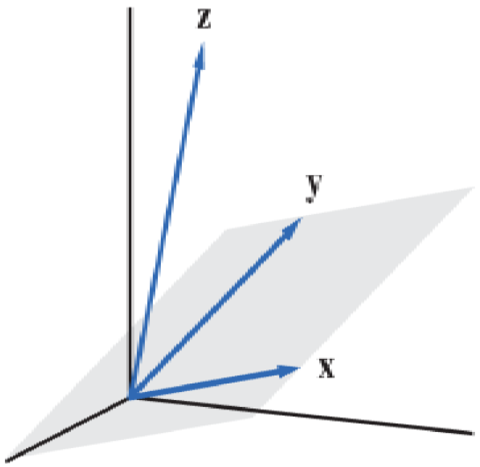
  - $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a spanning set for $\mathbb{R}^2$

Spanning Set for a Vector Space

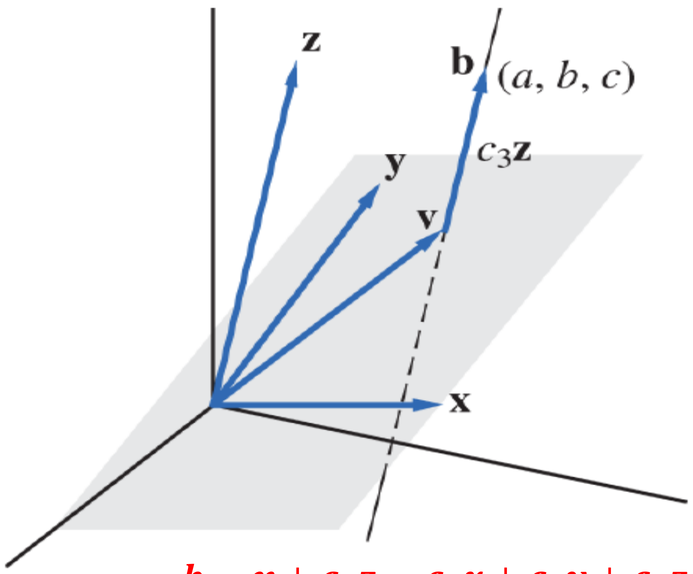
- Ex) Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$
  - A single nonzero vector $\boldsymbol{x}$ cannot span $\mathbb{R}^2$
  - Two nonzero vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ cannot span $\mathbb{R}^3$ (Note: $\boldsymbol{y} \neq \alpha\boldsymbol{x}$)



(a)



(b)



$\boldsymbol{b} = \boldsymbol{v} + \boldsymbol{c_3 z} = \boldsymbol{c_1 x} + \boldsymbol{c_2 y} + \boldsymbol{c_3 z}$

Spanning Set for a Vector Space

Ex 11. Which of the following are spanning sets for $\mathbb{R}^3$?

(b) $\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\}$

For part (b), we must determine whether it is possible to find constants α_1, α_2, and α_3 such that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This leads to the system of equations

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= a \\ \alpha_1 + \alpha_2 &= b \\ \alpha_1 &= c \end{aligned}$$

Since the coefficient matrix of the system is nonsingular, the system has a unique solution. In fact, we find that

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} c \\ b - c \\ a - b \end{bmatrix}$$

Thus,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (b - c) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (a - b) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

so the three vectors span $\mathbb{R}^3$.

Linear Systems Revisited

- Let S be the solution set to a consistent $m \times n$ linear system $A\boldsymbol{x} = \boldsymbol{b}$
  - If $\boldsymbol{b} = \mathbf{0}$, then $S = N(A)$, and consequently, the solution set forms a subspace of $\mathbb{R}^n$
  - If $\boldsymbol{b} \neq \mathbf{0}$, one can find a particular solution $\boldsymbol{x}_0$, then it is possible to represent any solution vector in terms of $\boldsymbol{x}_0$ and a vector $\boldsymbol{z} \in N(A)$

Let $A\mathbf{x} = \mathbf{b}$ be a consistent linear system and let $\mathbf{x}_0$ be a particular solution to the system. If there is another solution $\mathbf{x}_1$ to the system, then the difference vector $\mathbf{z} = \mathbf{x}_1 - \mathbf{x}_0$ must be in $N(A)$ since

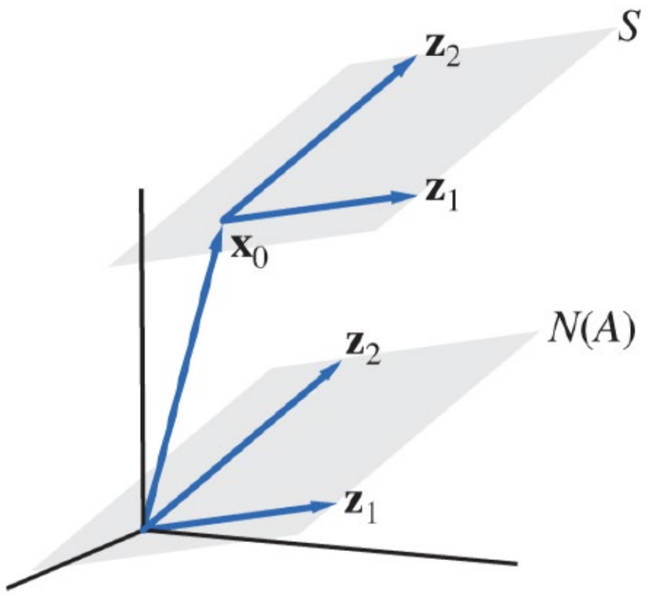
$$A\mathbf{z} = A\mathbf{x}_1 - A\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

Thus, if there is a second solution, it must be of the form $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{z}$, where $\mathbf{z} \in N(A)$.

In general, if $\mathbf{x}_0$ is a particular solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{z}$ is any vector in $N(A)$, then setting $\mathbf{y} = \mathbf{x}_0 + \mathbf{z}$, we have

$$A\mathbf{y} = A\mathbf{x}_0 + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

So $\mathbf{y} = \mathbf{x}_0 + \mathbf{z}$ must also be a solution to the system $A\mathbf{x} = \mathbf{b}$.



- If the linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent and $\boldsymbol{x}_0$ is a particular solution, then a vector $\boldsymbol{y}$ will also be a solution if and only if $\boldsymbol{y} = \boldsymbol{x}_0 + \boldsymbol{z}$, where $\boldsymbol{z} \in N(A)$

Exercises

13. Given

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$$

(a) Is $\boldsymbol{x} \in Span(\boldsymbol{x}_1, \boldsymbol{x}_2)$?

(b) Is $\boldsymbol{y} \in Span(\boldsymbol{x}_1, \boldsymbol{x}_2)$?

Linear Independence

Minimal Spanning Set

- It is desirable to find a *minimal spanning set* that is a spanning set with no unnecessary elements (*i.e.*, all the elements in the set are needed to span the vector space)

- Example

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{x}_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \qquad \boldsymbol{x}_3 = \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix}$$

$$\boldsymbol{x}_1 = -\frac{2}{3}\boldsymbol{x}_2 + \frac{1}{3}\boldsymbol{x}_3, \qquad \boldsymbol{x}_2 = -\frac{3}{2}\boldsymbol{x}_1 + \frac{1}{2}\boldsymbol{x}_3, \qquad \boldsymbol{x}_3 = 3\boldsymbol{x}_1 + 2\boldsymbol{x}_2$$

$$S = Span(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = Span(\boldsymbol{x}_1, \boldsymbol{x}_2) = Span(\boldsymbol{x}_2, \boldsymbol{x}_3) = Span(\boldsymbol{x}_1, \boldsymbol{x}_3)$$

Minimal Spanning Set

- If $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ span a vector space V and one of these vectors can be written as a linear combination of the other $n-1$ vectors, then those $n-1$ vectors span V

- Given n vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$, it is possible to write one of the vectors as a linear combination of the other $n-1$ vectors if and only if there exist scalars $c_1, \cdots, c_n$, not all zero, such that

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n = \mathbf{0}$$

Proof of (I) Suppose that $\mathbf{v}_n$ can be written as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$; that is,

$$\mathbf{v}_n = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_{n-1}\mathbf{v}_{n-1}$$

Let $\mathbf{v}$ be any element of V. Since

$$\begin{aligned}\mathbf{v} &= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_{n-1}\mathbf{v}_{n-1} + \alpha_n\mathbf{v}_n \\ &= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_{n-1}\mathbf{v}_{n-1} + \alpha_n(\beta_1\mathbf{v}_1 + \cdots + \beta_{n-1}\mathbf{v}_{n-1}) \\ &= (\alpha_1 + \alpha_n\beta_1)\mathbf{v}_1 + (\alpha_2 + \alpha_n\beta_2)\mathbf{v}_2 + \cdots + (\alpha_{n-1} + \alpha_n\beta_{n-1})\mathbf{v}_{n-1}\end{aligned}$$

Thus, any vector $\mathbf{v}$ in V can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$, and hence these vectors span V. ∎

Proof of (II) Suppose that one of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, say $\mathbf{v}_n$, can be written as a linear combination of the others; that is,

$$\mathbf{v}_n = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_{n-1}\mathbf{v}_{n-1}$$

Subtracting $\mathbf{v}_n$ from both sides of this equation, we get

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_{n-1}\mathbf{v}_{n-1} - \mathbf{v}_n = \mathbf{0}$$

If we set $c_i = \alpha_i$ for $i = 1, \ldots, n-1$, and set $c_n = -1$, then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

Conversely, if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

and at least one of the c_i's, say c_n, is nonzero, then

$$\mathbf{v}_n = \frac{-c_1}{c_n}\mathbf{v}_1 + \frac{-c_2}{c_n}\mathbf{v}_2 + \cdots + \frac{-c_{n-1}}{c_n}\mathbf{v}_{n-1}$$ ∎

Linear Independence

- The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ in a vector space V are said to be *linearly independent* if

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n = \mathbf{0}$$

  implies that all the scalars $c_1, \cdots, c_n$ must equal 0

- If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ is a minimal spanning set, then $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ are linearly independent

- A minimal spanning set is called a *basis*

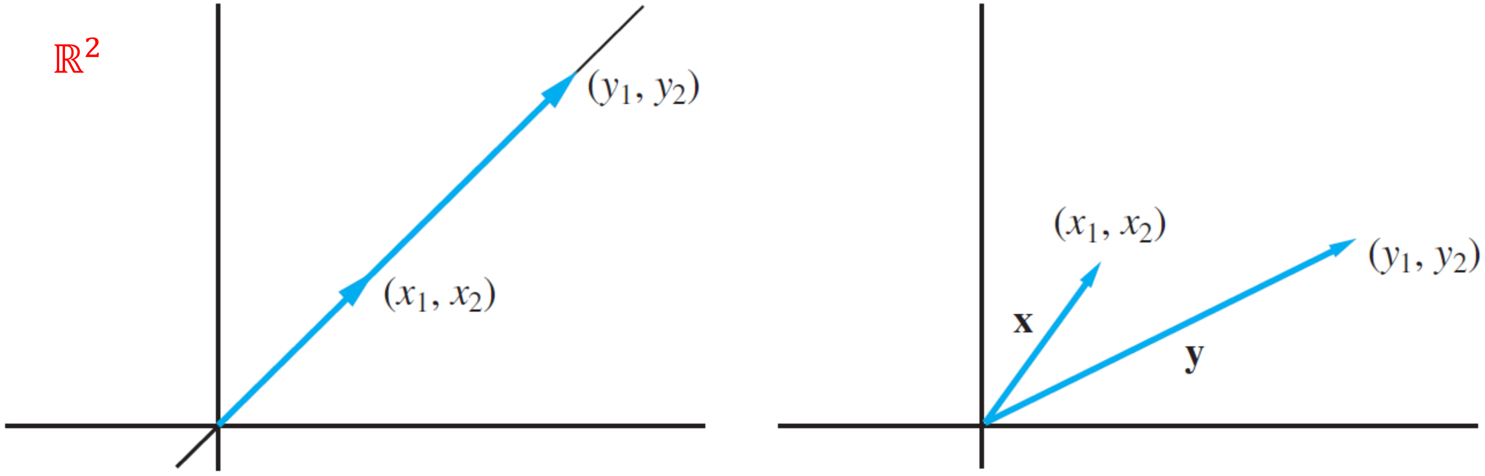
- The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ in a vector space V are said to be *linearly dependent* if there exists scalars $c_1, c_2, \cdots, c_n$, not all zero, such that

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n = \mathbf{0}$$

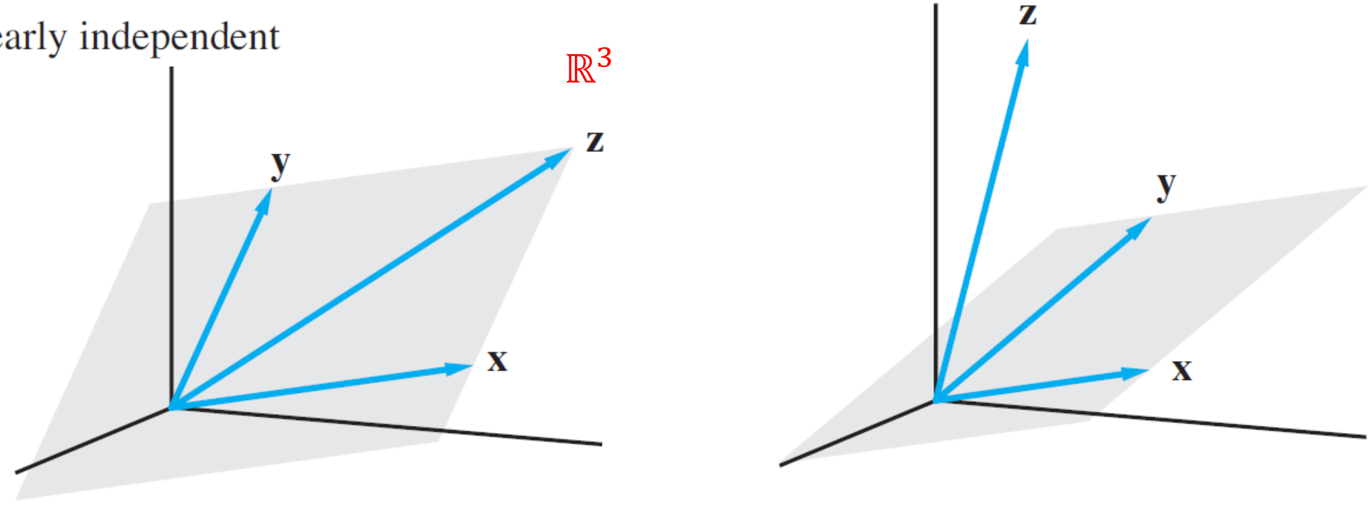
- If the *only* way the linear combination $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$ can equal to the zero vector is for all the scalars $c_1, c_2, \cdots, c_n$ to be 0, then $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ are linearly independent

Geometric Interpretation

- If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$ (or $\mathbb{R}^3$) are linearly dependent, then $c_1\boldsymbol{x} + c_2\boldsymbol{y} = \mathbf{0}$ where c_1 and c_2 are not both 0



(a) **x** and **y** linearly dependent (b) **x** and **y** linearly independent



(a) (b)

Theorems and Examples

- Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n$ be n vectors in $\mathbb{R}^n$ and let $X = (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$. The vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n$ will be linearly dependent if and only if X is singular

- Proof

The equation

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0}$$

can be rewritten as a matrix equation

$$X\mathbf{c} = \mathbf{0}$$

This equation will have a nontrivial solution if and only if X is singular. Thus, $\mathbf{x}_1, \ldots, \mathbf{x}_n$ will be linearly dependent if and only if X is singular. ■

- Ex 5. Determine whether the following vectors are linearly dependent or not

$$(4, 2, 3)^T, \qquad (2, 3, 1)^T, \qquad (2, -5, 3)^T$$

Theorems and Examples

- Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ be vectors in a vector space V. A vector $\boldsymbol{v} \in Span(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n)$ can be written uniquely as a linear combination of $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ if and only if $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ are linearly independent

- Proof

If $\mathbf{v} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, then $\mathbf{v}$ can be written as a linear combination

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n \tag{5}$$

Suppose that $\mathbf{v}$ can also be expressed as a linear combination

$$\mathbf{v} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_n\mathbf{v}_n \tag{6}$$

We will show that, if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, then $\beta_i = \alpha_i$, $i = 1, \ldots, n$, and if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent, then it is possible to choose the β_i's different from the α_i's.

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, then subtracting (6) from (5) yields

$$(\alpha_1 - \beta_1)\mathbf{v}_1 + (\alpha_2 - \beta_2)\mathbf{v}_2 + \cdots + (\alpha_n - \beta_n)\mathbf{v}_n = \mathbf{0} \tag{7}$$

By the linear independence of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, the coefficients of (7) must all be 0. Hence,

$$\alpha_1 = \beta_1,\ \alpha_2 = \beta_2, \ldots,\ \alpha_n = \beta_n$$

Thus, the representation (5) is unique when $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

On the other hand, if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent, then there exist $c_1, \ldots, c_n$, not all 0, such that

$$\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \tag{8}$$

Now if we set

$$\beta_1 = \alpha_1 + c_1,\ \beta_2 = \alpha_2 + c_2, \ldots,\ \beta_n = \alpha_n + c_n$$

then, adding (5) and (8), we get

$$\begin{aligned}\mathbf{v} &= (\alpha_1 + c_1)\mathbf{v}_1 + (\alpha_2 + c_2)\mathbf{v}_2 + \cdots + (\alpha_n + c_n)\mathbf{v}_n \\ &= \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_n\mathbf{v}_n\end{aligned}$$

Since the c_i's are not all 0, $\beta_i \neq \alpha_i$ for at least one value of i. Thus, if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent, the representation of a vector as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is not unique. ■

Vector Space of Functions

- The Vector Space P_n

To test whether the following polynomials $p_1, p_2, \ldots, p_k$ are linearly independent in P_n, we set

$$c_1 p_1 + c_2 p_2 + \cdots + c_k p_k = z \qquad (9)$$

where z represents the zero polynomial; that is,

$$z(x) = 0x^{n-1} + 0x^{n-2} + \cdots + 0x + 0$$

If the polynomial on the left-hand side of equation (9) is rewritten in the form $a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n$, then, since two polynomials are equal if and only if their coefficients are equal, it follows that the coefficients a_i must all be 0. But each of the a_i's is a linear combination of the c_j's. This leads to a homogeneous linear system with unknowns $c_1, c_2, \cdots, c_k$. If the system has only the trivial solution, the polynomials are linearly independent; otherwise, they are linearly dependent.

- Ex 7.

To test whether the vectors

$$p_1(x) = x^2 - 2x + 3, \quad p_2(x) = 2x^2 + x + 8, \quad p_3(x) = x^2 + 8x + 7$$

are linearly independent, set

$$c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) = 0x^2 + 0x + 0$$

Grouping terms by powers of x, we get

$$(c_1 + 2c_2 + c_3)x^2 + (-2c_1 + c_2 + 8c_3)x + (3c_1 + 8c_2 + 7c_3) = 0x^2 + 0x + 0$$

Equating coefficients leads to the system

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 0 \\ -2c_1 + c_2 + 8c_3 &= 0 \\ 3c_1 + 8c_2 + 7c_3 &= 0 \end{aligned}$$

The coefficient matrix for this system is singular and hence there are nontrivial solutions. Therefore, p_1, p_2, and p_3 are linearly dependent. ■

Exercises

16. Let A be an $m \times n$ matrix. Show that if A has linearly independent columns, then $N(A) = \{\mathbf{0}\}$
(Hint: For any $\boldsymbol{x} \in \mathbb{R}^n$, $A\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n$)