Similarity

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- If L is a linear operator on an n-dimensional vector space V, the matrix representation of L will depend on the ordered basis chosen for V

- By using different bases, it is possible to represent L by different $n \times n$ matrices

- Let L be the linear transformation mapping $\mathbb{R}^2$ into itself defined by

$$L(\boldsymbol{x}) = (2x_1, x_1 + x_2)^T$$
$$L(\boldsymbol{e}_1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad L(\boldsymbol{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- The matrix representing L with respect to $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

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- If we use a different basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ for $\mathbb{R}^2$, where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

then we must determine $L(\boldsymbol{u}_1)$ and $L(\boldsymbol{u}_2)$ and express these vectors as linear combinations of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ (with respect to $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$)

$$L(\boldsymbol{u}_1) = A\boldsymbol{u}_1 = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$L(\boldsymbol{u}_2) = A\boldsymbol{u}_2 = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

- To express these vectors in terms of $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, we use a transition matrix to change from the ordered basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ to $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$

$$U = (\boldsymbol{u}_1, \boldsymbol{u}_2) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

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- The transition matrix from $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ to $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ will then be

$$U^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

- To determine the coordinates of $L(\boldsymbol{u}_1)$ and $L(\boldsymbol{u}_2)$ with respect to $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, we multiply the vectors by U^{-1}

$$U^{-1}L(\boldsymbol{u}_1) = U^{-1}A\boldsymbol{u}_1 = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad U^{-1}L(\boldsymbol{u}_2) = U^{-1}A\boldsymbol{u}_2 = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$L(\boldsymbol{u}_1) = 2\boldsymbol{u}_1 + 0\boldsymbol{u}_2 \qquad L(\boldsymbol{u}_2) = -1\boldsymbol{u}_1 + 1\boldsymbol{u}_2$$

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- The matrix representing L with respect to $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is

$$B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = [U^{-1}A\boldsymbol{u}_1 \quad U^{-1}A\boldsymbol{u}_2] = U^{-1}A[\boldsymbol{u}_1 \quad \boldsymbol{u}_2] = U^{-1}AU$$

- If
  1) B is the matrix representing L with respect to $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$
  2) A is the matrix representing L with respect to $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$
  3) U is the transition matrix corresponding to the change of basis from $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ to $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$

  then, $B = U^{-1}AU$

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- Let $E = \{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$ and $F = \{\boldsymbol{w}_1, \cdots, \boldsymbol{w}_n\}$ be two ordered bases for a vector space V, and let L be a linear operator on V. Let S be the transition matrix representing the change from F to E. If A is the matrix representing L with respect to E, and B is the matrix representing L with respect to F, then $B = S^{-1}AS$

- Proof

Let $\mathbf{x}$ be any vector in $\mathbb{R}^n$ and let

$$\mathbf{v} = x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + \cdots + x_n\mathbf{w}_n$$

Let

$$\mathbf{y} = S\mathbf{x}, \quad \mathbf{t} = A\mathbf{y}, \quad \mathbf{z} = B\mathbf{x} \tag{2}$$

It follows from the definition of S that $\mathbf{y} = [\mathbf{v}]_E$ and hence

$$\mathbf{v} = y_1\mathbf{v}_1 + \cdots + y_n\mathbf{v}_n$$

Since A represents L with respect to E, and B represents L with respect to F, we have

$$\mathbf{t} = [L(\mathbf{v})]_E \qquad \text{and} \qquad \mathbf{z} = [L(\mathbf{v})]_F$$

The transition matrix from E to F is S^{-1}. Therefore,

$$S^{-1}\mathbf{t} = \mathbf{z} \tag{3}$$

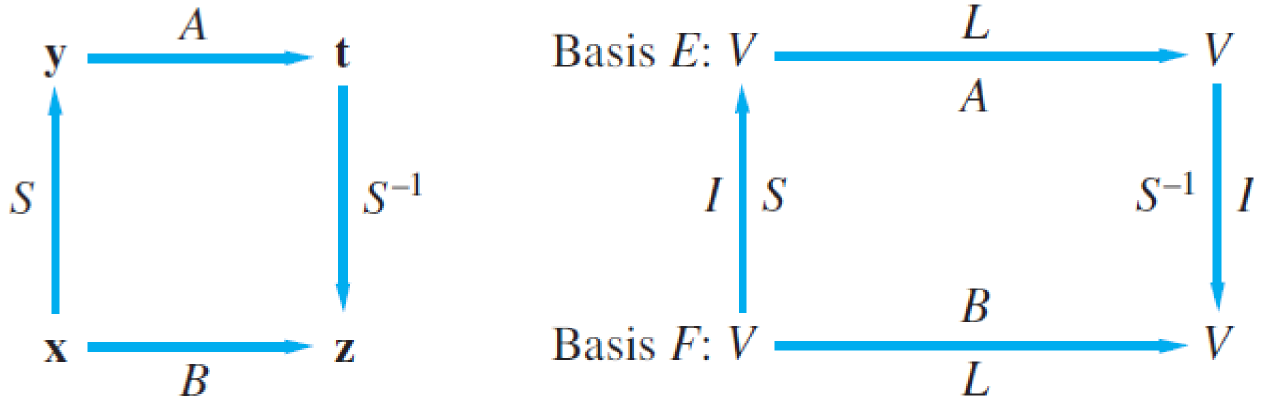
It follows from (2) and (3) that

$$S^{-1}AS\mathbf{x} = S^{-1}A\mathbf{y} = S^{-1}\mathbf{t} = \mathbf{z} = B\mathbf{x}$$

(see Figure 4.3.1). Thus,

$$S^{-1}AS\mathbf{x} = B\mathbf{x}$$

for every $\mathbf{x} \in \mathbb{R}^n$, and hence $S^{-1}AS = B$. ■



Similarity

- Let A and B be $n \times n$ matrices. B is said to be *similar* to A if there exists a nonsingular matrix S such that $B = S^{-1}AS$

- Note that if B is similar to A, then $A = (S^{-1})^{-1}BS^{-1}$ is similar to B. Thus, we may simply say that A and B are similar matrices

Similarity

Ex 2. Let L be the linear operator mapping $\mathbb{R}^3$ into $\mathbb{R}^3$ defined by $L(\boldsymbol{x}) = A\boldsymbol{x}$, where $A = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$.

Thus, the matrix A represents L with respect to $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. Find the matrix representing L with respect to $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$, where

$$\boldsymbol{y}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{y}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{y}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Solution

$$\begin{aligned} L(\mathbf{y}_1) &= A\mathbf{y}_1 = \mathbf{0} = 0\mathbf{y}_1 + 0\mathbf{y}_2 + 0\mathbf{y}_3 \\ L(\mathbf{y}_2) &= A\mathbf{y}_2 = \mathbf{y}_2 = 0\mathbf{y}_1 + 1\mathbf{y}_2 + 0\mathbf{y}_3 \\ L(\mathbf{y}_3) &= A\mathbf{y}_3 = 4\mathbf{y}_3 = 0\mathbf{y}_1 + 0\mathbf{y}_2 + 4\mathbf{y}_3 \end{aligned}$$

Thus, the matrix representing L with respect to $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

We could have found D by using the transition matrix $Y = (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)$ and computing

$$D = Y^{-1}AY$$

This was unnecessary due to the simplicity of the action of L on the basis $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$. **why?**

Exercises

11. Show that if A and B are similar matrices, then $\det(A) = \det(\mathrm{B})$.

Thank You