Matrix Algebra

Algebraic Rules

- Each of the following statements is valid for any scalars α and β and for any matrices A, B, and C for which the indicated operations are defined

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C) = A + B + C$
3. $(AB)C = A(BC) = ABC$
4. $A(B + C) = AB + AC$
5. $(A + B)C = AC + BC$
6. $(\alpha\beta)A = \alpha(\beta A) = \alpha\beta A$
7. $\alpha(AB) = (\alpha A)B = A(\alpha B) = \alpha AB$
8. $(\alpha + \beta)A = \alpha A + \beta A$
9. $\alpha(A + B) = \alpha A + \alpha B$

- In general, $AB \neq BA$. Matrix multiplication is not commutative

- For an $n \times n$ matrix A, if k is a positive integer,

$$A^k = \underbrace{AA \cdots A}_{k\ \text{times}}$$

Matrix Inversion

- The $n \times n$ *identity matrix* is the matrix $I = (\delta_{ij})$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- I acts as an identity for matrix multiplication for any $n \times n$ matrix A

$$IA = AI = A$$

- The column vectors of the $n \times n$ identity matrix I are the standard vectors used to define a coordinate system in Euclidean n-space

- The standard notation for the j-th column vector of I is $\boldsymbol{e}_j$

$$I = (\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n)$$

Matrix Inversion

- An $n \times n$ matrix A is said to be *nonsingular* or *invertible* if there exists a matrix B such that $AB = BA = I$

- The matrix B is said to be a *multiplicative inverse* of A

- If B and C are both multiplicative inverses of A,

$$B = BI = B(AC) = (BA)C = IC = C$$

Therefore, a matrix can have at most one multiplicative inverse

- The inverse of A is denoted by A^{-1}

- An $n \times n$ matrix is said to be *singular* if it does not have a multiplicative inverse
  - Only square matrices have multiplicative inverses

Matrix Inversion

- If A and B are nonsingular $n \times n$ matrices, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$

$$(B^{-1}A^{-1})(AB) =$$
$$(AB)(B^{-1}A^{-1}) =$$

- If $A_1, \cdots, A_k$ are all nonsingular $n \times n$ matrices, then the product $A_1A_2 \cdots A_k$ is nonsingular and

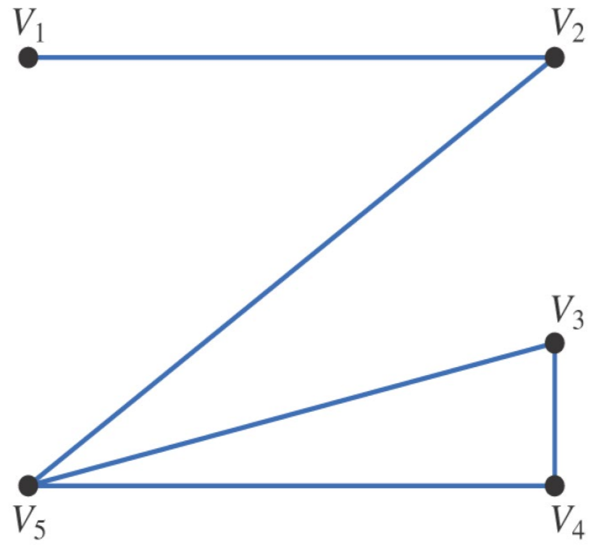
$$(A_1A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1}A_1^{-1}$$

- **Algebraic Rules for Transposes**

1. $(A^T)^T = A$
2. $(\alpha A)^T = \alpha A^T$
3. $(A + B)^T = A^T + B^T$
4. $(AB)^T = B^T A^T$

Application: Networks and Graphs

- A *graph* is defined to be a set of points called *vertices*, together with a set of unordered pairs of vertices, which are referred to as *edges*
  - Vertices: V_1, V_2, V_3, V_4, V_5
  - Edges: $\{V_1, V_2\}, \{V_2, V_5\}, \{V_3, V_4\}, \{V_3, V_5\}, \{V_4, V_5\}$



- An $n \times n$ matrix A is defined for the graph containing n vertices as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } \{V_i, V_j\} \text{ is an edge of the graph} \\ 0 & \text{if there is no edge joining } V_i \text{ and } V_j \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

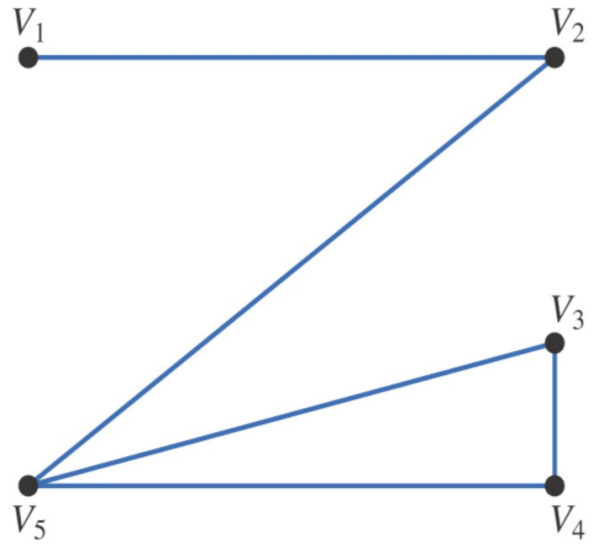
The matrix A is called the *adjacency matrix* of the graph

- Any adjacency matrix must be symmetric

Application: Networks and Graphs

- A *walk* in a graph is defined as a sequence of edges linking one vertex to another

- If A is an $n \times n$ adjacency matrix of a graph and $a_{ij}^{(k)}$ represents the (i, j) entry of A^k, then $a_{ij}^{(k)}$ is equal to the number of walks of length k from V_i to V_j



$$A^3 = \begin{bmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 & 4 \\ 1 & 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 2 & 4 \\ 0 & 4 & 4 & 4 & 2 \end{bmatrix}$$

- Proof

The proof is by mathematical induction. In the case $k = 1$, it follows from the definition of the adjacency matrix that a_{ij} represents the number of walks of length 1 from V_i to V_j. Assume for some m that each entry of A^m is equal to the number of walks of length m between the corresponding vertices. Thus, $a_{il}^{(m)}$ is the number of walks of length m from V_i to V_l. Now on the one hand, if there is an edge $\{V_l, V_j\}$, then $a_{il}^{(m)} a_{lj} = a_{il}^{(m)}$ is the number of walks of length $m+1$ from V_i to V_j of the form

$$V_i \to \cdots \to V_l \to V_j$$

On the other hand, if $\{V_l, V_j\}$ is not an edge, then there are no walks of length $m+1$ of this form from V_i to V_j and

$$a_{il}^{(m)} a_{lj} = a_{il}^{(m)} \cdot 0 = 0$$

It follows that the total number of walks of length $m+1$ from V_i to V_j is given by

$$a_{i1}^{(m)} a_{1j} + a_{i2}^{(m)} a_{2j} + \cdots + a_{in}^{(m)} a_{nj}$$

But this is just the (i, j) entry of A^{m+1}.

Exercises

19. Let A be an $n \times n$ matrix. Show that if $A^2 = O$, then $I - A$ is nonsingular and
$$(I - A)^{-1} = I + A$$

20. Let A be an $n \times n$ matrix. Show that if $A^{k+1} = O$, then $I - A$ is nonsingular and
$$(I - A)^{-1} = I + A + A^2 + \cdots + A^k$$

Elementary Matrices

Equivalent System

- Given an $m \times n$ linear system $A\boldsymbol{x} = \boldsymbol{b}$, an equivalent system can be obtained using a nonsingular $m \times m$ matrix M as follows:

$$A\boldsymbol{x} = \boldsymbol{b} \qquad (1)$$
$$MA\boldsymbol{x} = M\boldsymbol{b} \qquad (2)$$

  Any solution of (1) will also be a solution of (2). If $\hat{\boldsymbol{x}}$ is a solution of (2), then

$$M^{-1}(MA\hat{\boldsymbol{x}}) = M^{-1}(M\boldsymbol{b})$$
$$A\hat{\boldsymbol{x}} = \boldsymbol{b}$$

  Therefore, the two systems are equivalent

- Performing exactly one elementary row operation on the identity matrix I will give an *elementary matrix*

Elementary Matrices

- **Type I**: A matrix obtained by interchanging two rows of I

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Left multiplication (Row interchange)

$$E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Right multiplication (Column interchange)

$$AE_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}$$

Elementary Matrices

- **Type II**: A matrix obtained by multiplying a row of I by a nonzero constant

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- Left multiplication

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix}$$

- Right multiplication

$$AE_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} a_{12} & a_{11} & 3a_{13} \\ a_{22} & a_{21} & 3a_{23} \\ a_{32} & a_{31} & 3a_{33} \end{bmatrix}$$

Elementary Matrices

- **Type III**: A matrix obtained from I by adding a multiple of one row to another row

$$E_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Left multiplication

$$E_3A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} + 3a_{31} & a_{12} + 3a_{32} & a_{13} + 3a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Right multiplication

$$AE_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 3a_{11} + a_{13} \\ a_{21} & a_{22} & 3a_{21} + a_{23} \\ a_{31} & a_{32} & 3a_{31} + a_{33} \end{bmatrix}$$

Elementary Matrices

- If A is an $n \times r$ matrix, premultiplying A by E is equivalent to performing that same row operation on A

- If B is an $m \times n$ matrix, postmultiplying B by E is equivalent to performing that same column operation on B

- If E is an elementary matrix, then E is nonsingular and E^{-1} is an elementary matrix of the same type

- Proof

If E is the elementary matrix of type I formed from I by interchanging the ith and jth rows, then E can be transformed back into I by interchanging these same rows again. Therefore, $EE = I$ and hence E is its own inverse. If E is the elementary matrix of type II formed by multiplying the ith row of I by a nonzero scalar α, then E can be transformed into the identity matrix by multiplying either its ith row or its ith column by $1/\alpha$. Thus,

$$E^{-1} = \begin{bmatrix} 1 & & & & & O \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 1/\alpha & & \\ & & & & 1 & \\ & O & & & & \ddots \\ & & & & & 1 \end{bmatrix} \begin{array}{l} \\ \\ \\ i\text{th row} \\ \\ \\ \\ \end{array}$$

Finally, if E is the elementary matrix of type III formed from I by adding m times the ith row to the jth row, that is,

$$E = \begin{bmatrix} 1 & & & & & & \\ \vdots & \ddots & & & & & O \\ 0 & \cdots & 1 & & & & \\ \vdots & & & \ddots & & & \\ 0 & \cdots & m & \cdots & 1 & & \\ \vdots & & & & & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{array}{l} \\ \\ i\text{th row} \\ \\ j\text{th row} \\ \\ \\ \end{array}$$

then E can be transformed back into I either by subtracting m times the ith row from the jth row or by subtracting m times the jth column from the ith column.

Thus,

$$E^{-1} = \begin{bmatrix} 1 & & & & & & \\ \vdots & \ddots & & & & & O \\ 0 & \cdots & 1 & & & & \\ \vdots & & & \ddots & & & \\ 0 & \cdots & -m & \cdots & 1 & & \\ \vdots & & & & & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

Row Equivalent Matrix

- A matrix B is *row equivalent* to a matrix A if there exists a finite sequence $E_1, E_2, \cdots, E_k$ of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_1 A$$

- If two augmented matrices $(A|\boldsymbol{b})$ and $(B|\boldsymbol{c})$ are row equivalent, then $A\boldsymbol{x} = \boldsymbol{b}$ and $B\boldsymbol{x} = \boldsymbol{c}$ are equivalent systems

- If A is row equivalent to B, then B is row equivalent to A

- If A is row equivalent to B, and B is row equivalent to C, then A is row equivalent to C

Equivalent Conditions for Nonsingularity

- Let A be an $n \times n$ matrix. The following are equivalent:

  a. A is nonsingular
  b. $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$
  c. A is row equivalent to I

- Proof

We prove first that statement **(a)** implies statement **(b)**. If A is nonsingular and $\hat{\mathbf{x}}$ is a solution of $A\mathbf{x} = \mathbf{0}$, then

$$\hat{\mathbf{x}} = I\hat{\mathbf{x}} = (A^{-1}A)\hat{\mathbf{x}} = A^{-1}(A\hat{\mathbf{x}}) = A^{-1}\mathbf{0} = \mathbf{0}$$

Thus, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Next, we show that statement **(b)** implies statement **(c)**. If we use elementary row operations, the system can be transformed into the form $U\mathbf{x} = \mathbf{0}$, where U is in row echelon form. If one of the diagonal elements of U were 0, the last row of U would consist entirely of 0's. But then $A\mathbf{x} = \mathbf{0}$ would be equivalent to a system with more unknowns than equations and hence, by Theorem 1.2.1, would have a nontrivial solution. Thus, U must be a strictly triangular matrix with diagonal elements all equal to 1. It then follows that I is the reduced row echelon form of A and hence A is row equivalent to I.

Finally, we will show that statement **(c)** implies statement **(a)**. If A is row equivalent to I, there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$A = E_k E_{k-1} \cdots E_1 I = E_k E_{k-1} \cdots E_1$$

But since E_i is invertible, $i = 1, \ldots, k$, the product $E_k E_{k-1} \ldots E_1$ is also invertible. Hence, A is nonsingular and

$$A^{-1} = (E_k E_{k-1} \cdots E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Uniqueness of Solution

- The system $A\boldsymbol{x} = \boldsymbol{b}$ of n linear equations in n unknowns has a unique solution if and only if A is nonsingular

- Proof

If A is nonsingular, and $\hat{\mathbf{x}}$ is any solution of $A\mathbf{x} = \mathbf{b}$, then

$$A\hat{\mathbf{x}} = \mathbf{b}$$

Multiplying both sides of this equation by A^{-1}, we see that $\hat{\mathbf{x}}$ must be equal to $A^{-1}\mathbf{b}$.

Conversely, if $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution $\hat{\mathbf{x}}$, then we claim that A cannot be singular. Indeed, if A were singular, then the equation $A\mathbf{x} = \mathbf{0}$ would have a solution $\mathbf{z} \neq \mathbf{0}$. But this would imply that $\mathbf{y} = \hat{\mathbf{x}} + \mathbf{z}$ is a second solution of $A\mathbf{x} = \mathbf{b}$, since

$$A\mathbf{y} = A(\hat{\mathbf{x}} + \mathbf{z}) = A\hat{\mathbf{x}} + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

Therefore, if $A\mathbf{x} = \mathbf{b}$ has a unique solution, then A must be nonsingular.

Inverse Matrix

- If A is nonsingular, then A is row equivalent to I and hence there exist elementary matrices $E_1, \cdots, E_k$ such that

  $$E_k E_{k-1} \cdots E_1 A = I$$

  Multiplying both sides of this equation on the right by A^{-1}:

  $$E_k E_{k-1} \cdots E_1 I = A^{-1}$$

  Thus, the same series of elementary row operations that transforms a nonsingular matrix A into I will transform I into A^{-1}

- The reduced row echelon form of the augmented matrix $(A|I)$ will be $(I|A^{-1})$

Diagonal and Triangular Matrices

- An $n \times n$ matrix A is said to be *upper triangular* if $a_{ij} = 0$ for $i > j$ and *lower triangular* if $a_{ij} = 0$ for $i < j$

- A is said to be *triangular* if it is either upper triangular or lower triangular

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 6 & 0 & 0 \\ 1 & 4 & 3 \end{bmatrix}$$

- An $n \times n$ matrix A is *diagonal* if $a_{ij} = 0$ whenever $i \neq j$
  - A diagonal matrix is both upper triangular and lower triangular

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Triangular Factorization

- If an $n \times n$ matrix A can be reduced to strict upper triangular form using only row operation III, then it is possible to represent the reduction process in terms of a matrix factorization

- Example

Let

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$$

and let us use only row operation **III** to carry out the reduction process. At the first step, we subtract $\frac{1}{2}$ times the first row from the second and then we subtract twice the first row from the third.

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix}$$

To keep track of the multiples of the first row that were subtracted, we set $l_{21} = \frac{1}{2}$ and $l_{31} = 2$. We complete the elimination process by eliminating the -9 in the $(3,2)$ position.

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

Let $l_{32} = -3$, the multiple of the second row subtracted from the third row. If we call the resulting matrix U and set

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

then it is easily verified that

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} = A$$

The matrix L in the previous example is lower triangular with 1's on the diagonal. We say that L is *unit lower triangular*. The factorization of the matrix A into a product of a unit lower triangular matrix L times a strictly upper triangular matrix U is often referred to as an *LU factorization*.

- In general, if an $n \times n$ matrix A can be reduced to strict upper triangular form using only row operation III, then A has an *LU factorization*