Linear Algebra

- Determinants -

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The Determinant of a Matrix

Determinant of a Matrix

- For an $n \times n$ matrix A, a real number $\det(A)$ tells us whether the matrix is singular or not

- The *determinant*(행렬식) of a matrix A is represented by enclosing the array between vertical lines

$$\det(A) = |A| = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

- **Case 1. 1×1 Matrices**
  - If $A = (a)$ is a 1×1 matrix, then A will have a multiplicative inverse if and only if $a \neq 0$

$$\det(A) = a$$

  A will be nonsingular if and only if $\det(A) \neq 0$

Determinant of a Matrix

- **Case 2. 2×2 Matrices**

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

  - A will be nonsingular if and only if A is row equivalent to I (Assume $a_{11} \neq 0$)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \to \begin{bmatrix} a_{11} & a_{12} \\ a_{11}a_{21} & a_{11}a_{22} \end{bmatrix} \to \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

    The resulting matrix will be row equivalent to I if and only if $a_{11}a_{22} - a_{21}a_{12} \neq 0$

  - The determinant of a 2×2 matrix is defined as follows:

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

    A will be nonsingular if and only if $\det(A) \neq 0$

  - What about $a_{11} = 0$ case?

Determinant of a Matrix

- **Case 3. 3×3 Matrices**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}} & \frac{a_{11}a_{23} - a_{13}a_{21}}{a_{11}} \\ 0 & \frac{a_{11}a_{32} - a_{12}a_{31}}{a_{11}} & \frac{a_{11}a_{33} - a_{13}a_{31}}{a_{11}} \end{bmatrix}$$

A'

- The matrix on the right will be row equivalent to I if and only if

$$a_{11} \neq 0 \ \wedge \ \det(A') \neq 0 \ \Leftrightarrow \ a_{11} \begin{vmatrix} \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}} & \frac{a_{11}a_{23} - a_{13}a_{21}}{a_{11}} \\ \frac{a_{11}a_{32} - a_{12}a_{31}}{a_{11}} & \frac{a_{11}a_{33} - a_{13}a_{31}}{a_{11}} \end{vmatrix} \neq 0$$

- The determinant of a 3×3 matrix is defined as follows:

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$$

A will be nonsingular if and only if $\det(A) \neq 0$

Determinant of a Matrix

- **Case 3. 3×3 Matrices** - What if $a_{11} = 0$?

  (i) $a_{11} = 0, a_{21} \neq 0$

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ 0 & a_{12} & a_{13} \\ a_{21}a_{31} & a_{21}a_{32} & a_{21}a_{33} \end{bmatrix}$$

  (ii) $a_{11} = a_{21} = 0, a_{31} \neq 0$

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \end{bmatrix}$$

  (iii) $a_{11} = a_{21} = a_{31} = 0$

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \rightarrow \quad ?$$

Minor and Cofactor

- Let $A = \left(a_{ij}\right)$ be an $n \times n$ matrix and let M_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the row and column containing a_{ij} (소행렬)

- The determinant of M_{ij} is called the *minor*(소행렬식) of a_{ij}

- The *cofactor*(여인수) A_{ij} of a_{ij} is defined as follows:

$$A_{ij} = (-1)^{i+j} \det\left(M_{ij}\right)$$

- Example: 2×2 Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \det(A) = a_{11}a_{22} + a_{12}(-a_{21}) = a_{11}A_{11} + a_{12}A_{12}$$

  - This is called the *cofactor expansion*(여인수 전개) of $\det(A)$ along the first row

  - Cofactor expansion along one of the columns is also possible

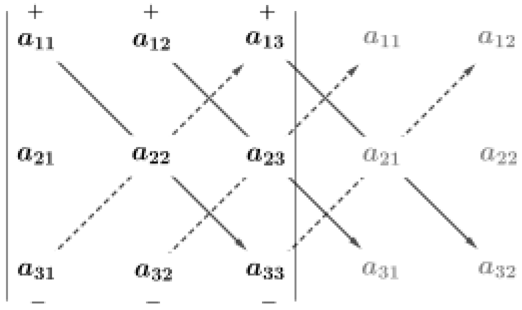
Minor and Cofactor

- Example: 3×3 Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$$

- Rule of Sarrus



https://en.wikipedia.org/wiki/Rule_of_Sarrus

Determinant

- The *determinant* of an $n \times n$ matrix A, denoted $\det(A)$, is a scalar associated with the matrix A that is defined inductively as follows:

$$\det(A) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{i1}A_{i1} + \cdots + a_{in}A_{in} & \text{if } n > 1 \end{cases}$$

  where

$$A_{ij} = (-1)^{i+j} \det\left(M_{ij}\right) \qquad j = 1, \cdots, n$$

  are the cofactors associated with the entries in the i-th row of A

- If A is an $n \times n$ matrix with $n \geq 2$, then $\det(A)$ can be expressed as a cofactor expansion using any row or column of A

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots a_{in}A_{in} = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots a_{nj}A_{nj} \qquad \text{for} \quad i, j \in [1, n]$$

Determinant

- Example
  - Expanding along the row or column containing zeros will save work

$$\begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{vmatrix}$$

Determinant

- If A is an $n \times n$ matrix, then $\det(A^T) = \det(A)$

The proof is by induction on n. Clearly, the result holds if $n = 1$, since a 1×1 matrix is necessarily symmetric. Assume that the result holds for all $k \times k$ matrices and that A is a $(k+1) \times (k+1)$ matrix. Expanding $\det(A)$ along the first row of A, we get

$$\det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + - \cdots \pm a_{1,k+1}\det(M_{1,k+1})$$

Since the M_{ij}'s are all $k \times k$ matrices, it follows from the induction hypothesis that

$$\det(A) = a_{11}\det(M_{11}^T) - a_{12}\det(M_{12}^T) + - \cdots \pm a_{1,k+1}\det(M_{1,k+1}^T)$$

(9)

The right-hand side of (9) is just the expansion by minors of $\det(A^T)$ using the first column of A^T. Therefore,

$$\det(A^T) = \det(A)$$

Determinant

- If A is an $n \times n$ triangular matrix, then the determinant of A equals the product of the diagonal elements of A

$$\begin{vmatrix} a_{11} & & & & & & \\ \vdots & \ddots & & & & O & \\ a_{i1} & \cdots & a_{ip} & & & & \\ \vdots & & & \ddots & & & \\ a_{j1} & \cdots & a_{jp} & \cdots & a_{jq} & & \\ \vdots & & & & & \ddots & \\ a_{n1} & \cdots & a_{np} & \cdots & a_{nq} & \cdots & a_{nn} \end{vmatrix}$$

Determinant

- Let A be an $n \times n$ matrix

  (i) If A has a row or column consisting entirely of zeros, then $\det(A) = 0$

  (ii) If A has two identical rows or columns, then $\det(A) = 0$

Exercises

6. Find all values of λ for which the following determinant will equal to 0

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix}$$

Properties of Determinants

Cofactor Expansion

- Let A be an $n \times n$ matrix. If A_{jk} denotes the cofactor of a_{jk} for $k = 1, \cdots, n$, then

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + A_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

If $i = j$, (1) is just the cofactor expansion of det(A) along the ith row of A. To prove (1) in the case $i \neq j$, let A^* be the matrix obtained by replacing the jth row of A by the ith row of A:

$$A^* = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \vdots & & & \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \\ j\text{th row} \\ \\ \\ \end{array}$$

Since two rows of A^* are the same, its determinant must be zero. It follows from the cofactor expansion of det(A^*) along the jth row that

$$\begin{aligned} 0 &= \det(A^*) = a_{i1}A^*_{j1} + a_{i2}A^*_{j2} + \cdots + a_{in}A^*_{jn} \\ &= a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} \end{aligned}$$

Effects of Elementary Operations

- **Type I**: Two rows of A are interchanged

If A is a 2×2 matrix and

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then

$$\det(EA) = \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{21}a_{12} - a_{22}a_{11} = -\det(A)$$

For $n > 2$, let E_{ij} be the elementary matrix that switches rows i and j of A. An induction proof can show that $\det(E_{ij}A) = -\det(A)$. We illustrate the idea behind the proof for the case $n = 3$. Suppose that the first and third rows of a 3×3 matrix A have been interchanged. Expanding $\det(E_{13}A)$ along the second row and making use of the result for 2×2 matrices, we see that

$$\begin{aligned} \det(E_{13}A) &= \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} \\ &= -a_{21}\begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix} + a_{22}\begin{vmatrix} a_{31} & a_{33} \\ a_{11} & a_{13} \end{vmatrix} - a_{23}\begin{vmatrix} a_{31} & a_{32} \\ a_{11} & a_{12} \end{vmatrix} \\ &= a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ &= -\det(A) \end{aligned}$$

- In general, if A is an $n \times n$ matrix and E_{ij} is the $n \times n$ elementary matrix formed by interchanging the i-th and j-th rows of I, then

$$\det(E_{ij}A) = -\det(A)$$

$$\det(E_{ij}) = \det(E_{ij}I) = -\det(I) = -1$$

$$\det(EA) = -\det(A) = \det(E)\det(A)$$

Effects of Elementary Operations

- **Type II**: A row of A is multiplied by a nonzero scalar

Let E denote the elementary matrix of type II formed from I by multiplying the ith row by the nonzero scalar α. If $\det(EA)$ is expanded by cofactors along the ith row, then

$$\begin{aligned}\det(EA) &= \alpha a_{i1}A_{i1} + \alpha a_{i2}A_{i2} + \cdots + \alpha a_{in}A_{in} \\ &= \alpha(a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}) \\ &= \alpha \det(A)\end{aligned}$$

- In particular,

$$\det(E) = \det(EI) = \alpha \det(I) = \alpha$$

$$\det(EA) = \alpha \det(A) = \det(E)\det(A)$$

Effects of Elementary Operations

- **Type III**: A multiple of one row is added to another row

- E is triangular and its diagonal elements are all 1

$$E = \begin{array}{c} \begin{bmatrix} 1 & & & & & & \\ \vdots & \ddots & & & & O & \\ 0 & \cdots & 1 & & & & \\ \vdots & & & \ddots & & & \\ 0 & \cdots & c & \cdots & 1 & & \\ \vdots & & & & & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \mathrm{i-th\ row} \\ \\ \mathrm{j-th\ row} \\ \\ \\ \end{array}$$

If $\det(EA)$ is expanded by cofactors along the jth row, it follows from Lemma 2.2.1 that

$$\begin{aligned} \det(EA) &= (a_{j1} + ca_{i1})A_{j1} + (a_{j2} + ca_{i2})A_{j2} + \cdots + (a_{jn} + ca_{in})A_{jn} \\ &= (a_{j1}A_{j1} + \cdots + a_{jn}A_{jn}) + c(a_{i1}A_{j1} + \cdots + a_{in}A_{jn}) \\ &= \det(A) \end{aligned}$$

Thus,

$$\det(EA) = \det(A) = \det(E)\det(A)$$

Lemma 2.2.1: $a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + A_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Effects of Elementary Operations

- In summary, if E is an elementary matrix, then

$$\det(EA) = \det(E)\det(A)$$

where

$$\det(E) = \begin{cases} -1 & \text{for Type I} \\ \alpha \neq 0 & \text{for Type II} \\ 1 & \text{for Type III} \end{cases}$$

  - Interchanging two rows or columns of a matrix changes the sign of the determinant
  - Multiplying a single row or column of a matrix by a scalar has the effect of multiplying the value of the determinant by that scalar
  - Adding a multiple of one row or column to another does not change the value of the determinant

- Similar results hold for column operations

$$\det(AE) = \det\left((AE)^T\right) = \det(E^T A^T) = \det(E^T)\det(A^T) = \det(E)\det(A)$$

Determinant and Singularity

- An $n \times n$ matrix A is singular if and only if $\det(A) = 0$

The matrix A can be reduced to row echelon form with a finite number of row operations. Thus,

$$U = E_k E_{k-1} \cdots E_1 A$$

where U is in row echelon form and the E_i's are all elementary matrices. It follows that

$$\begin{aligned} \det(U) &= \det(E_k E_{k-1} \cdots E_1 A) \\ &= \det(E_k)\det(E_{k-1}) \cdots \det(E_1)\det(A) \end{aligned}$$

Since the determinants of the E_i's are all nonzero, it follows that $\det(A) = 0$ if and only if $\det(U) = 0$. If A is singular, then U has a row consisting entirely of zeros, and hence $\det(U) = 0$. If A is nonsingular, then U is triangular with 1's along the diagonal and hence $\det(U) = 1$.

- If A is nonsingular, it is simpler to reduce A to triangular form using only row operations I and III

$$T = E_m E_{m-1} \cdots E_1 A = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{nn} \end{bmatrix}, \qquad \det(A) = \pm\det(T) = \pm t_{11} t_{12} \cdots t_{nn}$$

- If A is singular, the computed value of $\det(A)$ using exact arithmetic must be 0
- Since computers use a finite number system, roundoff errors are usually unavoidable
- In general, the value of $\det(A)$ is not a good indicator of nearness to singularity

Property of Determinant

- If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$

If B is singular, it follows from Theorem 1.5.2 that AB is also singular (see Exercise 14 of Section 1.5), and therefore,

$$\det(AB) = 0 = \det(A)\det(B)$$

If B is nonsingular, B can be written as a product of elementary matrices. We have already seen that the result holds for elementary matrices. Thus,

$$\begin{aligned} \det(AB) &= \det(AE_kE_{k-1}\cdots E_1) \\ &= \det(A)\det(E_k)\det(E_{k-1})\cdots\det(E_1) \\ &= \det(A)\det(E_kE_{k-1}\cdots E_1) \\ &= \det(A)\det(B) \end{aligned}$$

Operation Counts

n	Cofactors		Elimination	
	Additions	Multiplications	Additions	Multiplications and Divisions
2	1	2	1	3
3	5	9	5	10
4	23	40	14	23
5	119	205	30	44
10	3,628,799	6,235,300	285	339

Exercises

6. Let A be a nonsingular matrix. Show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Additional Topics and Applications

Adjoint of a Matrix

- Let A be an $n \times n$ matrix. The *adjoint*(수반행렬) of A is defined as follows:

$$adj(A) = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

- Each term is replaced by its cofactor and the resulting matrix is transposed

$$A\big(adj(A)\big) = \det(A)\, I$$

- If A is nonsingular, then

$$A\left(\frac{1}{\det(A)} adj(A)\right) = I, \qquad A^{-1} = \frac{1}{\det(A)} adj(A) \quad \text{when} \quad \det(A) \neq 0$$

Cramer's Rule

- Let A be a nonsingular $n \times n$ matrix, and let $\boldsymbol{b} \in \mathbb{R}^n$. Let A_i be the matrix obtained by replacing the i-th column of A by $\boldsymbol{b}$. If x is the unique solution of $A\boldsymbol{x} = \boldsymbol{b}$, then

$$x_i = \frac{\det(A_i)}{\det(A)} \quad \text{for } i = 1,2,\cdots,n, \qquad A_i = \begin{bmatrix} a_{11} & \cdots & b_1(=a_{1i}) & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{k1} & \cdots & b_k(=a_{ki}) & \cdots & a_{kn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & b_n(=a_{ni}) & \cdots & a_{nn} \end{bmatrix}$$

- Proof

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \frac{1}{\det(A)} adj(A)\boldsymbol{b}$$

$$x_i = \frac{b_1 A_{1i} + b_2 A_{2i} + \cdots b_n A_{ni}}{\det(A)} = \frac{\det(A_i)}{\det(A)}$$

Lemma 2.2.1: $a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + A_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Example: Cramer's Rule

$$x_1 + 2x_2 + x_3 = 5$$
$$2x_1 + 2x_2 + x_3 = 6$$
$$x_1 + 2x_2 + 3x_3 = 9$$

- Solution

$$\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -4 \qquad \det(A_1) = \begin{vmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{vmatrix} = -4$$

$$\det(A_2) = \begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{vmatrix} = -4 \qquad \det(A_3) = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{vmatrix} = -8$$

Therefore,

$$x_1 = \frac{-4}{-4} = 1, \quad x_2 = \frac{-4}{-4} = 1, \quad x_3 = \frac{-8}{-4} = 2$$

- $n + 1$ determinants of order n must be calculated

- Involves more computation than solving the system by Gaussian elimination

Cross Product

- Given two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^3$, the cross product denoted by $\boldsymbol{x} \times \boldsymbol{y}$ is defined as follows:

$$\boldsymbol{x} \times \boldsymbol{y} = \begin{bmatrix} x_2 y_3 - y_2 x_3 \\ y_1 x_3 - x_1 y_3 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

- If C is any matrix of the form

$$C = \begin{bmatrix} w_1 & w_2 & w_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}, \qquad \boldsymbol{x} \times \boldsymbol{y} = C_{11}\boldsymbol{e}_1 + C_{12}\boldsymbol{e}_2 + C_{13}\boldsymbol{e}_3 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}$$

$$\det(C) = w_1 C_{11} + w_2 C_{12} + w_3 C_{13} = \boldsymbol{w}^T(\boldsymbol{x} \times \boldsymbol{y})$$

- If $\boldsymbol{w} = \boldsymbol{x}$ or $\boldsymbol{w} = \boldsymbol{y}$, then the matrix C will have two identical rows

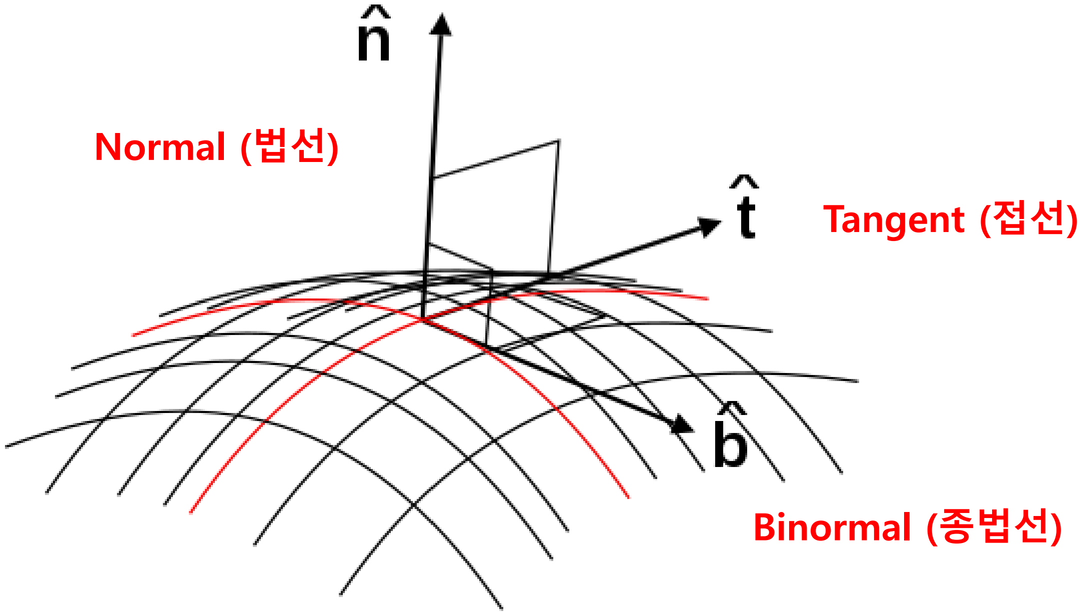
$$\boldsymbol{x}^T(\boldsymbol{x} \times \boldsymbol{y}) = \boldsymbol{y}^T(\boldsymbol{x} \times \boldsymbol{y}) = 0$$

Cross Product

- The cross product can be represented in terms of the determinant of a matrix whose first row's entries are $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

$$\boldsymbol{x} \times \boldsymbol{y} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

- The cross product can be used to define a binormal direction(종법선)



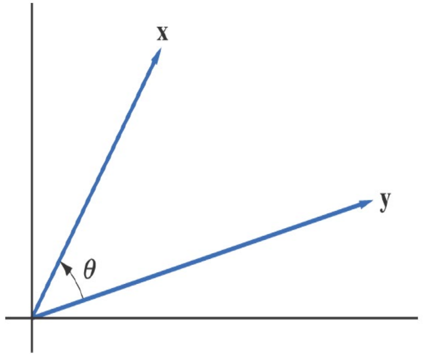
https://commons.wikimedia.org/wiki/File:Tangent_normal_binormal_unit_vectors.svg

Application: Newtonian Mechanics

- If $\boldsymbol{x}$ is a vector in either $\mathbb{R}^2$ or $\mathbb{R}^3$, the length of $\boldsymbol{x}$ denoted by $\|\boldsymbol{x}\|$ is defined as follows:

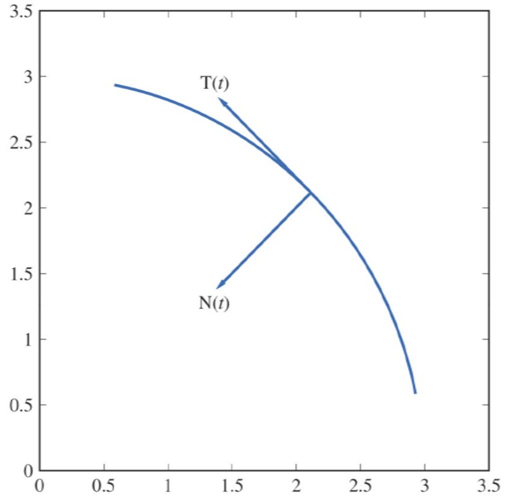
$$\|\boldsymbol{x}\| = (\boldsymbol{x}^T\boldsymbol{x})^{\frac{1}{2}}$$

- A vector $\boldsymbol{x}$ is called a *unit vector* if $\|\boldsymbol{x}\| = 1$



- The angle θ between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^2$ is the smallest angle of rotation necessary to rotate one of the two vectors clockwise so that it ends up in the same direction as the other vector

- Newton found it convenient to represent the position of vectors at time t as linear combination of the vectors $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$



Application: Newtonian Mechanics

- If $\boldsymbol{x}$ and $\boldsymbol{y}$ are nonzero vectors and θ is the angle between the vectors, then

$$\boldsymbol{x}^T\boldsymbol{y} = \|\boldsymbol{x}\|\|\boldsymbol{y}\|\cos\theta$$

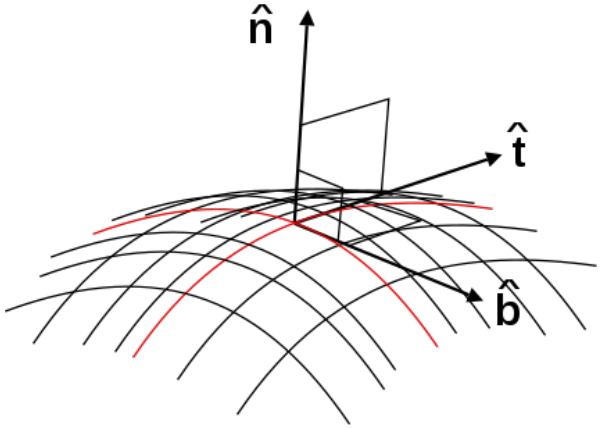
- The vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are called *orthogonal*(직교) if and only if $\boldsymbol{x}^T\boldsymbol{y} = 0$

$$\boldsymbol{T}(t)^T\boldsymbol{N}(t) = 0$$

- If $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors in $\mathbb{R}^3$ and θ is the angle between the vectors, then

$$\|\boldsymbol{x}\times\boldsymbol{y}\| = \|\boldsymbol{x}\|\|\boldsymbol{y}\|\sin\theta$$

- If $\boldsymbol{z}$ is any nonzero vector in the direction of the normal line to the plane containing $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$, then the angles between the vectors $\boldsymbol{z}$ and $\boldsymbol{T}(t)$ and $\boldsymbol{z}$ and $\boldsymbol{N}(t)$ are right angles



$$\boldsymbol{B}(t) = \boldsymbol{T}(t)\times\boldsymbol{N}(t)$$
$$\|\boldsymbol{B}(t)\| = \|\boldsymbol{T}(t)\times\boldsymbol{N}(t)\| = 1$$

https://commons.wikimedia.org/wiki/File:Tangent_normal_binormal_unit_vectors.svg

Exercises

16. Let x and y be vectors in $\mathbb{R}^3$ and define the skew-symmetric matrix A_x by

$$A_x = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

(a) Show that $\boldsymbol{x} \times \boldsymbol{y} = \begin{bmatrix} x_2y_3 - y_2x_3 \\ y_1x_3 - x_1y_3 \\ x_1y_2 - y_1x_2 \end{bmatrix} = A_x\boldsymbol{y}$

(b) Show that $\boldsymbol{y} \times \boldsymbol{x} = A_x^T\boldsymbol{y}$ (Note: $\boldsymbol{y} \times \boldsymbol{x} = -\boldsymbol{x} \times \boldsymbol{y}$)

Thank You