Partitioned Matrices

Partitioned Matrices

- A matrix is composed of a number of submatrices called *blocks*

$$C = \begin{bmatrix} 1 & -2 & 4 & 1 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & -1 & 2 \\ 4 & 6 & 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \left[\begin{array}{ccc|cc} 1 & -2 & 4 & 1 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ \hline 3 & 3 & 2 & -1 & 2 \\ 4 & 6 & 2 & 2 & 4 \end{array}\right]$$

$$B = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 4 & 1 \end{bmatrix} = \left[\begin{array}{c|c|c} -1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 4 & 1 \end{array}\right] = [\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \boldsymbol{b}_3]$$

- In general, if A is an $m \times n$ matrix and B is an $n \times r$ times matrix that has been partitioned into columns $[\boldsymbol{b}_1 \quad \cdots \quad \boldsymbol{b}_r]$, then the block multiplication of AB is given by

$$AB = [A\boldsymbol{b}_1 \quad A\boldsymbol{b}_2 \quad \cdots \quad A\boldsymbol{b}_r] = \begin{bmatrix} \vec{\boldsymbol{a}}_1 B \\ \vec{\boldsymbol{a}}_2 B \\ \vdots \\ \vec{\boldsymbol{a}}_m B \end{bmatrix}$$

Block Multiplication

- Let A be an $m \times n$ matrix and B an $n \times r$ matrix

- **Case I**: $B = [B_1 \quad B_2], B_1 \in \mathbb{R}^{n\times t}, B_2 \in \mathbb{R}^{n\times(r-t)}$

$$\begin{aligned} AB &= A[\boldsymbol{b}_1 \quad \cdots \quad \boldsymbol{b}_t \quad \boldsymbol{b}_{t+1} \quad \cdots \quad \boldsymbol{b}_r] = [A\boldsymbol{b}_1 \quad \cdots \quad A\boldsymbol{b}_t \quad A\boldsymbol{b}_{t+1} \quad \cdots \quad A\boldsymbol{b}_r] \\ &= [A(\boldsymbol{b}_1 \quad \cdots \quad \boldsymbol{b}_t) \quad A(\boldsymbol{b}_{t+1} \quad \cdots \quad \boldsymbol{b}_r)] = [AB_1 \quad AB_2] = A[B_1 \quad B_2] \end{aligned}$$

- **Case II**: $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, A_1 \in \mathbb{R}^{k\times n}, A_2 \in \mathbb{R}^{(m-k)\times n}$

$$AB = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} B = \begin{bmatrix} \vec{\boldsymbol{a}}_1 \\ \vdots \\ \vec{\boldsymbol{a}}_k \\ \vec{\boldsymbol{a}}_{k+1} \\ \vdots \\ \vec{\boldsymbol{a}}_m \end{bmatrix} B = \begin{bmatrix} \vec{\boldsymbol{a}}_1 B \\ \vdots \\ \vec{\boldsymbol{a}}_k B \\ \vec{\boldsymbol{a}}_{k+1} B \\ \vdots \\ \vec{\boldsymbol{a}}_m B \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} \vec{\boldsymbol{a}}_1 \\ \vdots \\ \vec{\boldsymbol{a}}_k \end{pmatrix} B \\ \begin{pmatrix} \vec{\boldsymbol{a}}_{k+1} \\ \vdots \\ \vec{\boldsymbol{a}}_m \end{pmatrix} B \end{bmatrix} = \begin{bmatrix} A_1 B \\ A_2 B \end{bmatrix}$$

Block Multiplication

- Let A be an $m \times n$ matrix and B an $n \times r$ matrix

- **Case III**: $A = [A_1 \quad A_2], A_1 \in \mathbb{R}^{m \times s}, B_2 \in \mathbb{R}^{m \times (n-s)}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, B_1 \in \mathbb{R}^{s \times r}, B_2 \in \mathbb{R}^{(n-s) \times r}$

$$C = AB = (c_{ij}) = \left(\sum_{l=1}^{n} a_{il} b_{lj}\right) = \left(\sum_{l=1}^{s} a_{il} b_{lj} + \sum_{l=s+1}^{n} a_{il} b_{lj}\right) = A_1 B_1 + A_2 B_2 = [A_1 \quad A_2] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

- **Case IV**: $A = \begin{array}{cc} \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] & \begin{array}{c} k \\ m-k \end{array} \\ \begin{array}{cc} s & n-s \end{array} & \end{array}$ $\quad B = \begin{array}{cc} \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] & \begin{array}{c} s \\ n-s \end{array} \\ \begin{array}{cc} t & r-t \end{array} & \end{array}$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Block Multiplication

- In general, if the blocks have proper dimensions, the block multiplication can be carried out in the same manner as ordinary matrix multiplication

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1t} \\ \vdots & \ddots & \vdots \\ A_{s1} & \cdots & A_{st} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & \cdots & B_{1r} \\ \vdots & \ddots & \vdots \\ B_{t1} & \cdots & B_{tr} \end{bmatrix}$$

$$AB = \begin{bmatrix} C_{11} & \cdots & C_{1r} \\ \vdots & \ddots & \vdots \\ C_{s1} & \cdots & B_{sr} \end{bmatrix} = (C_{ij}) = \left(\sum_{k=1}^{t} A_{ik} B_{kj} \right)$$

- The multiplication can be carried out in this manner only if the number of columns of A_{ik} equals the number of rows of B_{kj} for each k

Outer Product Expansions

- Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

- A scalar product (or an inner product) is defined as follows:

$$\boldsymbol{x}^T\boldsymbol{y} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

- An outer product is defined as follows:

$$\boldsymbol{x}\boldsymbol{y}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & \cdots & x_ny_n \end{bmatrix}$$

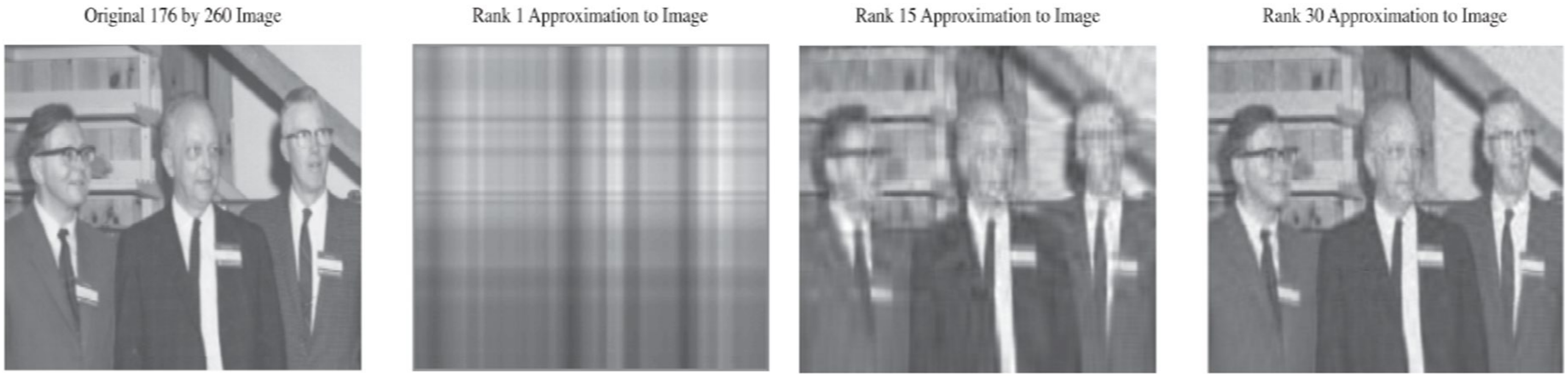
Outer Product Expansions

- Let $X \in \mathbb{R}^{m \times n}, Y \in \mathbb{R}^{k \times n}$

- A outer product expansion is defined as follows:

$$XY^T = [\boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \cdots \quad \boldsymbol{x}_n]\begin{bmatrix} \boldsymbol{y}_1^T \\ \boldsymbol{y}_2^T \\ \vdots \\ \boldsymbol{y}_n^T \end{bmatrix} = \boldsymbol{x}_1\boldsymbol{y}_1^T + \boldsymbol{x}_2\boldsymbol{y}_2^T + \cdots + \boldsymbol{x}_n\boldsymbol{y}_n^T$$

- Will be used in digital imaging and in information retrieval applications



Exercises

10. Let U be an $m \times m$ matrix, let V be an $n \times n$ matrix, and let

$$\Sigma = \begin{bmatrix} \Sigma_1 \\ O \end{bmatrix}$$

where Σ_1 is an $n \times n$ diagonal matrix with diagonal entries $\sigma_1, \sigma_2, \cdots, \sigma_n$ and O is the $(m-n) \times n$ zero matrix

(a) Show that if $U = (U_1, U_2)$, where U_1 has n columns, then $U\Sigma = U_1\Sigma_1$

(b) Show that if $A = U\Sigma V^T$, then A can be expressed as an outer product expansion of the form

$$A = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^T + \cdots + \sigma_n \boldsymbol{u}_n \boldsymbol{v}_n^T$$

Thank You