Row Echelon Form

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$$\begin{array}{cl}\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right) & \begin{array}{l} \leftarrow \text{pivotal row} \\ \\ \\ \\ \\ \end{array}\end{array}$$

$$\begin{array}{cl}\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{0} \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right) & \begin{array}{l} \\ \leftarrow \text{pivotal row} \\ \\ \\ \\ \end{array}\end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{3} \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right)$$

Staircase (or Echelon) Form

- The variables corresponding to the first nonzero elements in each row of the reduced matrix are *lead variables* (*i.e.*, x_1, x_3, x_5)
- The remaining variables corresponding to the columns skipped in the reduction process are *free variables* (*i.e.*, x_2, x_4)
- A matrix is said to be in *row echelon form* if:
  1) The first nonzero entry in each nonzero row is 1
  2) If row k does not consists entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k
  3) If there are rows whose entries are all zero, they are below the rows having nonzero entries

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Row Echelon Form

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right) \leftarrow \text{pivotal row}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{0} \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right) \leftarrow \text{pivotal row}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{3} \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right)$$

Staircase (or Echelon) Form

- The process of using row operations to transform a linear system into one whose augmented matrix is in row echelon form is called *Gaussian elimination*
- If the row echelon form of the augmented matrix contains a row of the form $(0 \quad \cdots \quad 0|1)$, the system is inconsistent. Otherwise, the system will be consistent
- If the system is consistent and the nonzero rows of the row echelon form of the matrix form a strictly triangular system, the system will have a unique solution

Overdetermined System

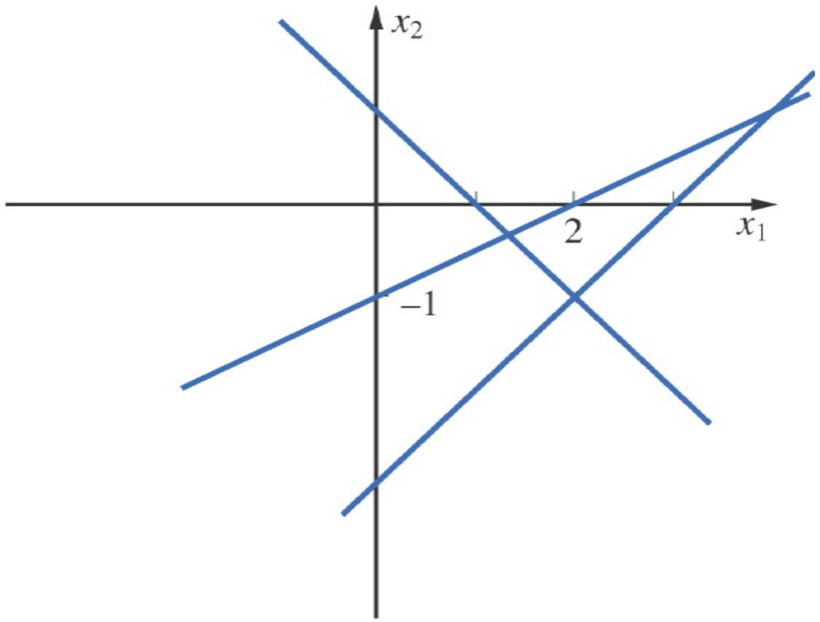
- A linear system is said to be *overdetermined* if there are more equations than unknowns
  - Overdetermined systems are usually (but not always) inconsistent

- Example

$$
\begin{aligned}
x_1 + x_2 &= 1 \\
x_1 - x_2 &= 3 \\
-x_1 + 2x_2 &= -2
\end{aligned}
$$

$$
\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{array}\right)
$$

$$
\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right)
$$



No Solution: Inconsistent System

Underdetermined System

- A system of m linear equations in n unknowns is said to be *underdetermined* if there are fewer equations than unknowns $(m < n)$
  - Usually consistent with infinitely many solutions
  - It is not possible to have a unique solution because arbitrary values can be assigned to the free variables

- Examples

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 + 2x_3 &= 3 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 2 \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 &= 3 \\ x_1 + x_2 + x_3 + 2x_4 + 3x_5 &= 2 \end{aligned}$$

Reduced Row Echelon Form

- A matrix is said to be in *reduced echelon form* if:
  - The matrix is in row echelon form
  - The first nonzero entry in each row is the only nonzero entry in its *column*

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\quad \begin{pmatrix}1 & 0 & 0 & 3\\0 & 1 & 0 & 2\\0 & 0 & 1 & 1\end{pmatrix},\quad \begin{pmatrix}0 & 1 & 2 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{pmatrix},\quad \begin{pmatrix}1 & 2 & 0 & 1\\0 & 0 & 1 & 3\\0 & 0 & 0 & 0\end{pmatrix}$$

- The process of using elementary operations to transform a matrix into reduced row echelon form is called *Gauss-Jordan reduction*

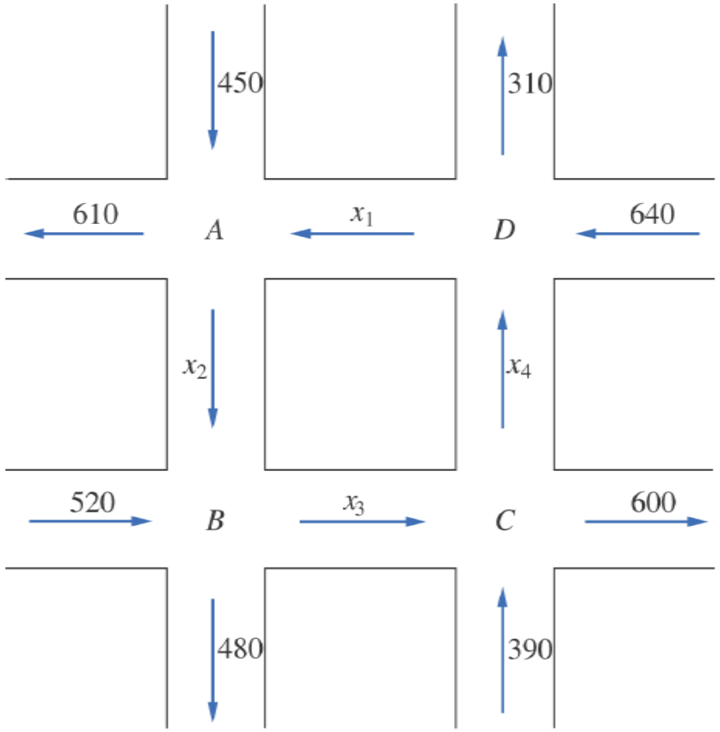
$$\begin{aligned}-x_1 + x_2 - x_3 + 3x_4 &= 0\\ 3x_1 + x_2 - x_3 - x_4 &= 0\\ 2x_1 - x_2 - 2x_3 - x_4 &= 0\end{aligned}$$

$$\left[\begin{array}{rrrr|r}-1 & 1 & -1 & 3 & 0\\3 & 1 & -1 & -1 & 0\\2 & -1 & -2 & -1 & 0\end{array}\right] \rightarrow \left[\begin{array}{rrrr|r}-1 & 1 & -1 & 3 & 0\\0 & 4 & -4 & 8 & 0\\0 & 1 & -4 & 5 & 0\end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrrr|r}-1 & 1 & -1 & 3 & 0\\0 & 4 & -4 & 8 & 0\\0 & 0 & -3 & 3 & 0\end{array}\right] \rightarrow \left[\begin{array}{rrrr|r}1 & -1 & 1 & -3 & 0\\0 & 1 & -1 & 2 & 0\\0 & 0 & 1 & -1 & 0\end{array}\right] \quad \text{row echelon form}$$

$$\rightarrow \left[\begin{array}{rrrr|r}1 & -1 & 0 & -2 & 0\\0 & 1 & 0 & 1 & 0\\0 & 0 & 1 & -1 & 0\end{array}\right] \rightarrow \left[\begin{array}{rrrr|r}1 & 0 & 0 & -1 & 0\\0 & 1 & 0 & 1 & 0\\0 & 0 & 1 & -1 & 0\end{array}\right] \quad \text{reduced row echelon form}$$

Application: Traffic Flow

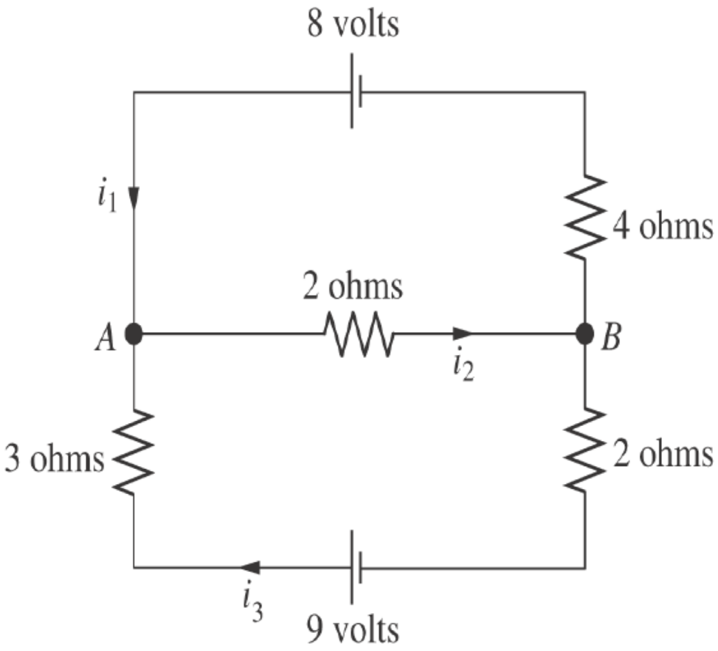


$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ -1 & 0 & 0 & 1 & -330 \end{array}\right)$$

$$\begin{aligned} x_1 + 450 &= x_2 + 610 \quad &\text{(intersection } A) \\ x_2 + 520 &= x_3 + 480 \quad &\text{(intersection } B) \\ x_3 + 390 &= x_4 + 600 \quad &\text{(intersection } C) \\ x_4 + 640 &= x_1 + 310 \quad &\text{(intersection } D) \end{aligned}$$

Application: Kirchhoff's Laws

- *Kirchhoff's Laws*
  1) At every node, the sum of the incoming currents equals the sum of the outgoing currents
  2) Around every closed loop, the algebraic sum of the voltage gains must equal the algebraic sum of the voltage drops



$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 4 & 2 & 0 & 8 \\ 0 & 2 & 5 & 9 \end{array}\right)$$

$$\begin{aligned} i_1 - i_2 + i_3 &= 0 && \text{(node } A\text{)} \\ -i_1 + i_2 - i_3 &= 0 && \text{(node } B\text{)} \end{aligned}$$

$$\begin{aligned} 4i_1 + 2i_2 &= 8 && \text{(top loop)} \\ 2i_2 + 5i_3 &= 9 && \text{(bottom loop)} \end{aligned}$$

Homogeneous System

- A system of linear equations is said to be *homogeneous* if the constants on the right-hand side are all zero

- Homogeneous systems are always consistent
  - The trivial solution $(0, 0, \cdots, 0)$

- An $m \times n$ homogeneous system of linear equations has a nontrivial solution if $n > m$
  - At most m lead variables and some free variables

Application: Chemical Equations

- *Photosynthesis*

$$x_1 CO_2 + x_2 H_2O \rightarrow x_3 O_2 + x_4 C_6H_{12}O_6$$

Carbon dioxide **Water** **Oxygen** **Glucose**

$$x_1 = 6x_4$$

$$2x_1 + x_2 = 2x_3 + 6x_4$$

$$2x_2 = 12x_4$$

Matrix Arithmetic

Matrix and Vector Notations

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (a_{ij}) \in \mathbb{R}^{m \times n}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

Euclidean n-space

- The set of all $n \times 1$ matrices of real numbers

- The entries of a matrix are called *scalars*
  - Usually either real or complex numbers
- a_{ij} will denote the entry of the matrix A that is in the i-th row and j-th column (*i.e.*, (i, j) entry of A)
- Vector is an n-tuple of real numbers
  - $1 \times n$ matrix is a row vector ($\vec{\boldsymbol{x}}$)
  - $n \times 1$ matrix is a column vector ($\boldsymbol{x}$)
- Vectors are used to represent solutions of linear systems

Matrix and Vector Notations

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [\boldsymbol{a}_1, \quad \boldsymbol{a}_2, \quad \cdots, \quad \boldsymbol{a}_n] = \begin{bmatrix} \vec{\boldsymbol{a}}_1 \\ \vec{\boldsymbol{a}}_2 \\ \vdots \\ \vec{\boldsymbol{a}}_m \end{bmatrix}$$

- A matrix can be represented in terms of either its column vectors or row vectors

- Example

$$A = \begin{bmatrix} 3 & 2 & 5 \\ -1 & 8 & 4 \end{bmatrix}$$

Some Definitions

- **Equality**
  - Two $m \times n$ matrices A and B are said to be equal if $a_{ij} = b_{ij}$ for each i and j
- **Scalar Multiplication**
  - If A is an $m \times n$ matrix and α is a scalar, then αA is the $m \times n$ matrix whose (i, j) entry is αa_{ij}
- **Matrix Addition**
  - If $A = (a_{ij})$ and $B = (b_{ij})$ are both $m \times n$ matrices, then the sum $A + B$ is the $m \times n$ matrix whose (i, j) entry is $a_{ij} + b_{ij}$ for each ordered pair (i, j)
  - $A - B$ is defined to be $A + (-1)B$
- **Zero Matrix (Additive Identity)**
  - A matrix O whose entries are all zero
  - $A + O = O + A = A$
- **Additive Inverse**
  - $A + (-1)A = O = (-1)A + A$
  - $-A = (-1)A$

Matrix Multiplication and Linear Systems

- We can represent an $m \times n$ linear system by a single matrix equation of the form

$$A\boldsymbol{x} = \boldsymbol{b}$$

where A is an $m \times n$ matrix, $\boldsymbol{x}$ is an unknown vector in $\mathbb{R}^n$, and $\boldsymbol{b}$ is in $\mathbb{R}^m$

- ***M* equations in *N* unknowns**

$$A\boldsymbol{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} \vec{\boldsymbol{a}}_1\boldsymbol{x} \\ \vec{\boldsymbol{a}}_2\boldsymbol{x} \\ \vdots \\ \vec{\boldsymbol{a}}_m\boldsymbol{x} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \boldsymbol{b}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n$$

Linear Combination

- If $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ are vectors in $\mathbb{R}^m$ and $c_1, c_2, \cdots, c_n$ are scalars, then a sum of the form

$$c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + \cdots + c_n\boldsymbol{a}_n$$

  is said to be a *linear combination* of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$

- If A is an $m \times n$ matrix and $\boldsymbol{x}$ is a vector in $\mathbb{R}^n$, then

$$A\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n$$

- **Consistency Theorem for Linear Systems**
  - A linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent *if and only if* $\boldsymbol{b}$ can be written as a linear combination of the column vectors of A

  - Example

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 4x_2 &= 1 \end{aligned}$$

Matrix Multiplication

- If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times r$ matrix, then the product $AB = C = (c_{ij})$ is the $m \times r$ matrix whose entries are defined by

$$AB = (A\boldsymbol{b}_1, A\boldsymbol{b}_2, \cdots, A\boldsymbol{b}_r) = C$$

$$c_{ij} = \vec{\boldsymbol{a}}_i \boldsymbol{b}_j = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- Multiplication of matrices is not commutative

Matrix Transpose

- The *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix B defined by

$$b_{ji} = a_{ij}$$

for $j = 1, \cdots, n$ and $i = 1, \cdots, m$

- The transpose of A is denoted by A^T

- An $n \times n$ matrix A is said to be *symmetric* if $A^T = A$