Homework Chapter 4.1~4.3

1.

Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. If

$$L((2,3)^T) = (3,-2)^T$$

and

$$L((-1,1)^T) = (1,4)^T$$

find the value of $L((8,7)^T)$.

2.

For each of the following linear transformations L mapping $\mathbb{R}^3$ into $\mathbb{R}^2$, find a matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^3$:

(a) $L((x_1, x_2, x_3)^T) = (x_3, x_3)^T$

(b) $L((x_1, x_2, x_3)^T) = (x_1 + x_2 + x_3, 0)^T$

(c) $L((x_1, x_2, x_3)^T) = (3x_1, -2x_2)^T$

3.

Let L be the linear operator on $\mathbb{R}^3$ defined by

$$L(\mathbf{x}) = \begin{bmatrix} x_1 + 2x_2 - 3x_3 \\ x_2 + 2x_1 - x_3 \\ x_3 + x_2 - 2x_1 \end{bmatrix}$$

Determine the standard matrix representation A of L, and use A to find $L(\mathbf{x})$ for each of the following vectors $\mathbf{x}$:

(a) $\mathbf{x} = (1,1,1)^T$ **(b)** $\mathbf{x} = (3,-2,1)^T$

(c) $\mathbf{x} = (4,5,1)^T$

4.

Find the standard matrix representation for each of the following linear operators:

(a) L is the linear operator that rotates each $\mathbf{x}$ in $\mathbb{R}^2$ by 45° in the clockwise direction.

(b) L is the linear operator that reflects each vector $\mathbf{x}$ in $\mathbb{R}^2$ about the x_1-axis and then rotates it 90° in the counterclockwise direction.

(c) L doubles the length of $\mathbf{x}$ and then rotates it 30° in the counterclockwise direction.

(d) L reflects each vector $\mathbf{x}$ about the line $x_2 = x_1$ and then projects it onto the x_1-axis.

5.

Let L be the linear transformation on $\mathbb{R}^3$ defined by

$$L(\mathbf{x}) = \begin{Bmatrix} 2x_1 - x_2 - x_3 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_1 - x_2 \end{Bmatrix}$$

and let A be the standard matrix representation of L (see Exercise 4 of Section 4.2). If $\mathbf{u}_1 = (1, 1, 0)^T$, $\mathbf{u}_2 = (1, 0, 1)^T$, and $\mathbf{u}_3 = (0, 1, 1)^T$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an ordered basis for $\mathbb{R}^3$ and $U = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is the transition matrix corresponding to a change of basis from $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Determine the matrix B representing L with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ by calculating $U^{-1}AU$.

6.

The *trace* of an $n \times n$ matrix A, denoted $\operatorname{tr}(A)$, is the sum of its diagonal entries; that is,

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

Show that

(a) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

(b) if A is similar to B, then $\operatorname{tr}(A) = \operatorname{tr}(B)$.
