Linear Algebra

- 우리는 3차원 공간에 살고 있으며, 주위에서 일어나는 일을 잘 다루기 위해서는 3차원 공간을 잘 기술 할 수 있는 용어가 필요

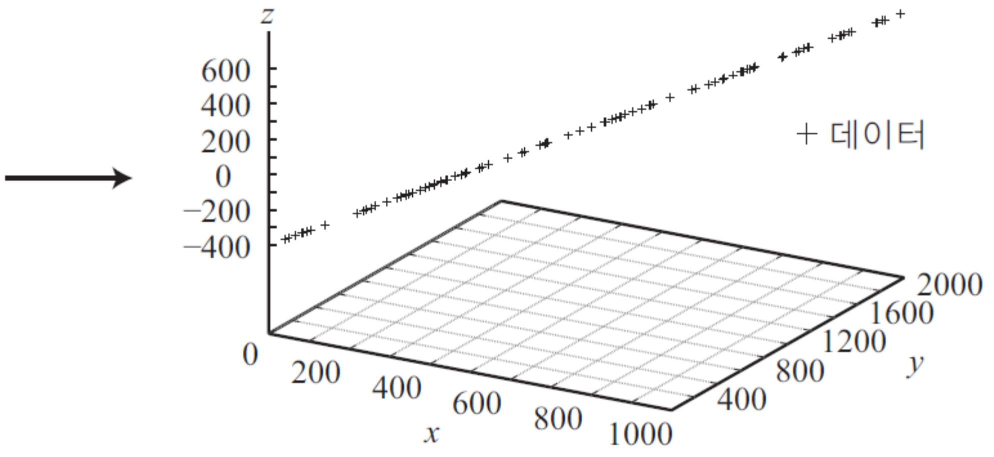
- 선형대수의 무대가 되는 Vector Space는 현실 공간의 성질을 특정 수준에서 추상화 한 것

- 공학에서는 많은 경우 다양한 숫자를 조합한 데이터를 다루는 경우에 마주치게 됨

- 공간과는 직접적인 관련이 없으나, 이러한 데이터를 고차원 공간 상의 점 으로 해석하여 공간에 관한 직관을 활용 할 수 있음

- 3차원 공간으로 부터 유추하여 직관적으로 일반적인 n차원에서 성립하는 현상을 이해 할 수 있음

x	y	z
731	1662	331
208	616	−192
540	1280	140
834	1864	434
217	634	−183
332	864	−68
31	262	−369
54	308	−346
717	1634	317
403	1006	3
⋮	⋮	⋮

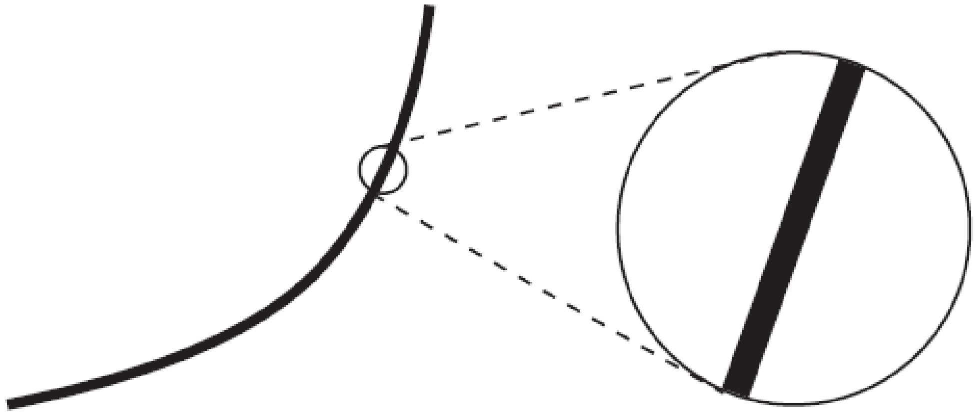


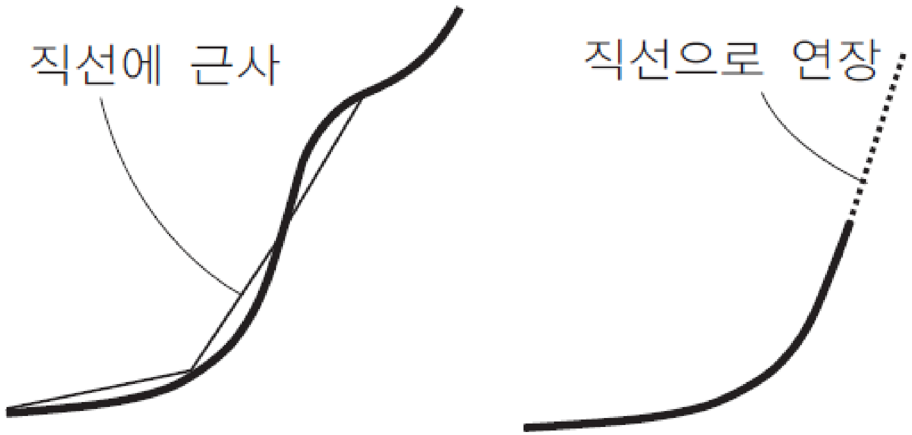
Linear Algebra

- 선형대수가 다루는 대상은 선형적(Linear) 이므로 다루기 쉽고, 예측이 편함 (e.g., 직선, 평면 등)

- 대부분의 실제 문제는 비선형적(Non-Linear)이나, 좁은 영역만을 고려한다면 선형으로 근사 할 수 있음

- 이 경우, 좁은 범위 내에서는 선형으로 근사해도 유효한 분석 및 결과를 얻을 수 있음
  - 요구 정확도 등의 다양한 조건에 따라 선형 근사 적용 판단





Vectors and Matrices

Vectors and Matrices

- Linear algebra is a branch of mathematics widely used throughout science and engineering
- The study of linear algebra involves several types of mathematical objects
  - Scalars: A single number x
  - Vectors: An array of numbers

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

  - Matrices: 2-D array of numbers

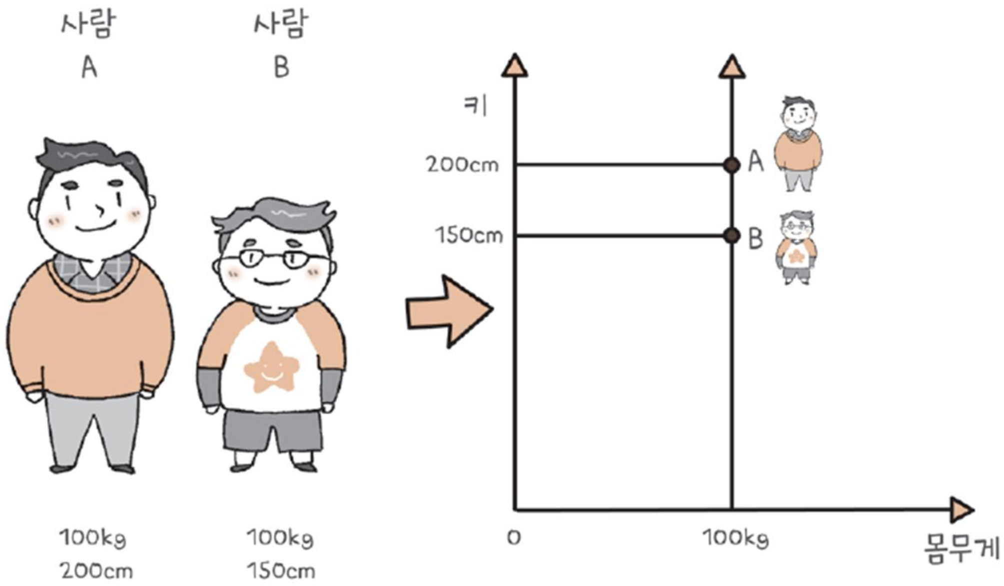
$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

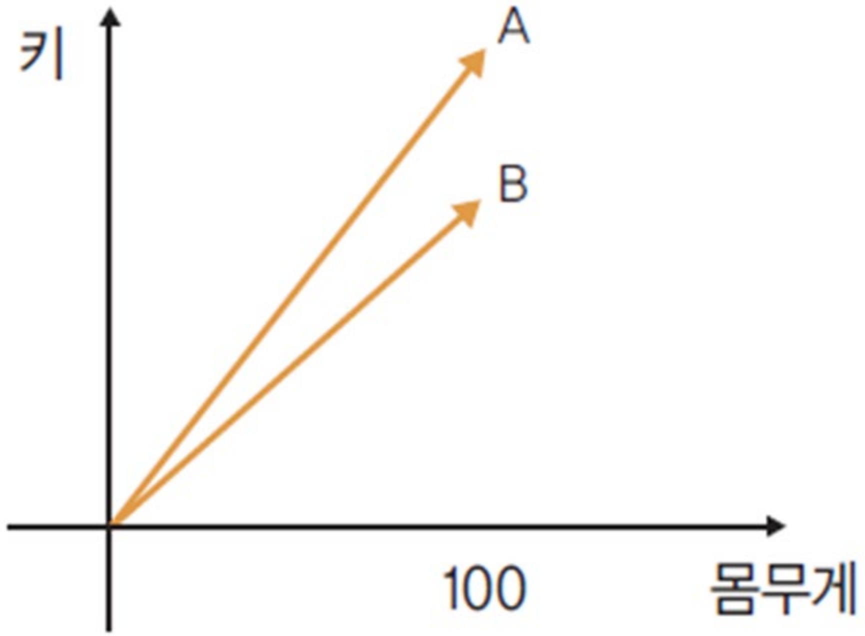
  - Tensors: N-D array of numbers

Vectors

- Vector Examples

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix} \text{ 또는 } \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix} \text{ 또는 } \begin{pmatrix} 2.9 \\ -0.3 \\ 1/7 \\ \sqrt{\pi} \\ 42 \end{pmatrix}$$





Vectors

- Column Vector (열벡터/종벡터) and Row Vector (행벡터/횡벡터)

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n\times 1}, \qquad \boldsymbol{y} = [y_1 \quad y_2 \quad \cdots \quad y_m] \in \mathbb{R}^{1\times m}$$

- Vector Transpose / Scalar Multiplication

$$[1\ 2\ 3\ 5]^T = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}, \qquad a\boldsymbol{x} = \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix}$$

- Zero Vector

$$\boldsymbol{o} = [0 \quad 0 \quad \cdots \quad 0]^T$$

Vector Arithmetic

- Addition / Subtraction

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \rightarrow \quad \boldsymbol{a} \pm \boldsymbol{b} = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \end{bmatrix}$$

- Element-wise Multiplication (Hadamard Product)

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \rightarrow \quad \boldsymbol{a} \circ \boldsymbol{b} = \begin{bmatrix} a_1 b_1 \\ a_2 b_2 \end{bmatrix}$$

- Inner Product (Dot Product)

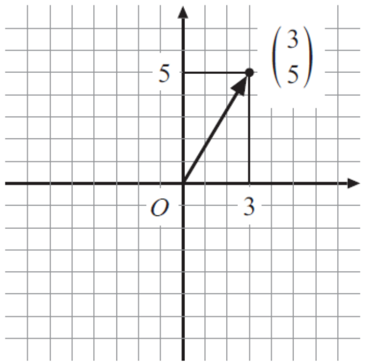
$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \rightarrow \quad \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^T \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

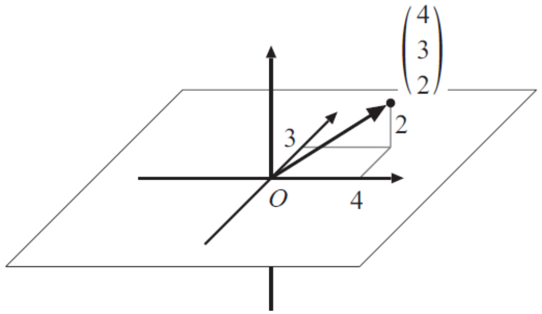
Vector Arithmetic

- $(cc')\boldsymbol{x} = c(c'\boldsymbol{x})$ 예) $(2\cdot 3)\begin{pmatrix}1\\5\end{pmatrix}=\begin{pmatrix}6\\30\end{pmatrix}=2\left(3\begin{pmatrix}1\\5\end{pmatrix}\right)$
- $1\boldsymbol{x} = \boldsymbol{x}$ 예) $1\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}$
- $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x}$ 예) $\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}1\\5\end{pmatrix}=\begin{pmatrix}3\\8\end{pmatrix}=\begin{pmatrix}1\\5\end{pmatrix}+\begin{pmatrix}2\\3\end{pmatrix}$
- $(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z}=\boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})$ 예) $\left(\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}1\\5\end{pmatrix}\right)+\begin{pmatrix}10\\20\end{pmatrix}=\begin{pmatrix}13\\28\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}+\left(\begin{pmatrix}1\\5\end{pmatrix}+\begin{pmatrix}10\\20\end{pmatrix}\right)$
- $\boldsymbol{x}+\boldsymbol{o}=\boldsymbol{x}$ 예) $\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}$
- $\boldsymbol{x}+(-\boldsymbol{x})=\boldsymbol{o}$ 예) $\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}-2\\-3\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$
- $c(\boldsymbol{x}+\boldsymbol{y})=c\boldsymbol{x}+c\boldsymbol{y}$ 예) $10\left(\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}6\\4\end{pmatrix}\right)=\begin{pmatrix}80\\70\end{pmatrix}=10\begin{pmatrix}2\\3\end{pmatrix}+10\begin{pmatrix}6\\4\end{pmatrix}$
- $(c+c')\boldsymbol{x}=c\boldsymbol{x}+c'\boldsymbol{x}$ 예) $(4+5)\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}18\\27\end{pmatrix}=4\begin{pmatrix}2\\3\end{pmatrix}+5\begin{pmatrix}2\\3\end{pmatrix}$

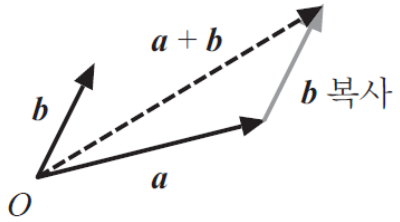
Vector and Space

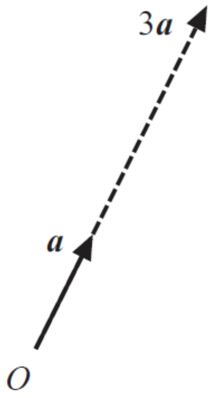
- Assume that we have a 2D coordinate system
  - $\boldsymbol{o} = [0 \quad 0]^T$ is at the origin

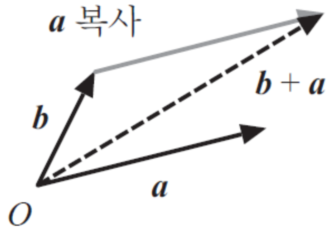




- Geometric interpretation of some vector arithmetic







- Roughly speaking, a vector space can be defined with addition and scalar multiplication

Matrices

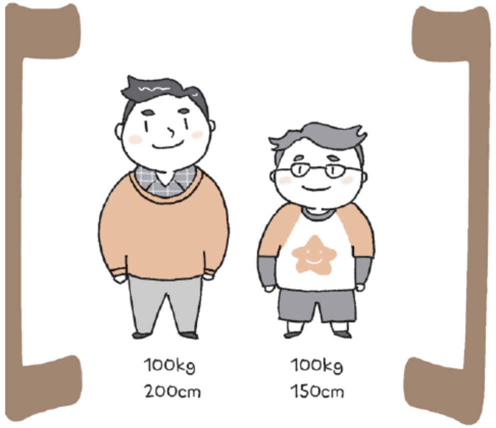
- A matrix is a 2-D array of numbers
  - a_{mn} is called an entry or an element

Row

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

Column

$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$이나 $\begin{pmatrix} 2.2 & -9 & 1/7 \\ \sqrt{7} & \pi & 42 \end{pmatrix}$이나 $\begin{pmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \\ 2 & 6 & 5 \\ 3 & 5 & 8 \\ 9 & 7 & 9 \end{pmatrix}$



	몸무게	키
사람A	100kg	200cm
사람B	100kg	150cm

$$M = \begin{bmatrix} 100 & 100 \\ 200 & 150 \end{bmatrix}$$

Matrix Arithmetic

- Addition / Subtraction

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\rightarrow \quad A \pm B = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} \end{bmatrix}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

예: $\begin{pmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 11 & 7 \\ 11 & 10 & 9 \end{pmatrix}$

- Scalar Multiplication

$$cA = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

$$c\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{pmatrix}$$

예: $3\begin{pmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 27 & 12 \\ 21 & 15 & 9 \end{pmatrix}$

Matrix Arithmetic

- Element-wise Multiplication (Hadamard Product)

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\rightarrow \quad A \circ B = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} \\ a_{21}b_{21} & a_{22}b_{22} \end{bmatrix}$$

- Inner Product (Dot Product)

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\rightarrow \quad A \cdot B = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Matrix Arithmetic

$$\underset{A}{\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}} \times \underset{B}{\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}} = \begin{bmatrix} (1\times2)+(0\times3) & (1\times1)+(0\times1) \\ (2\times2)+(3\times3) & (2\times1)+(3\times1) \end{bmatrix} = \begin{pmatrix} 2 & 1 \\ 13 & 5 \end{pmatrix}$$

$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$

Note:

$$\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}$$

Matrix Arithmetic Example

고기를 $x_{고}$그램, 콩을 $x_{콩}$그램, 쌀을 $x_{쌀}$그램 샀습니다. 합계는 얼마일까요? 또한, 총 몇 칼로리 일까요?

$$y_{돈} = a_{돈고} x_{고} + a_{돈콩} x_{콩} + a_{돈쌀} x_{쌀}$$
$$y_{칼} = a_{칼고} x_{고} + a_{칼콩} x_{콩} + a_{칼쌀} x_{쌀}$$

$$\begin{pmatrix} y_{돈} \\ y_{칼} \end{pmatrix} = \begin{pmatrix} a_{돈고} & a_{돈콩} & a_{돈쌀} \\ a_{칼고} & a_{칼콩} & a_{칼쌀} \end{pmatrix} \begin{pmatrix} x_{고} \\ x_{콩} \\ x_{쌀} \end{pmatrix}$$

Matrix Arithmetic Example

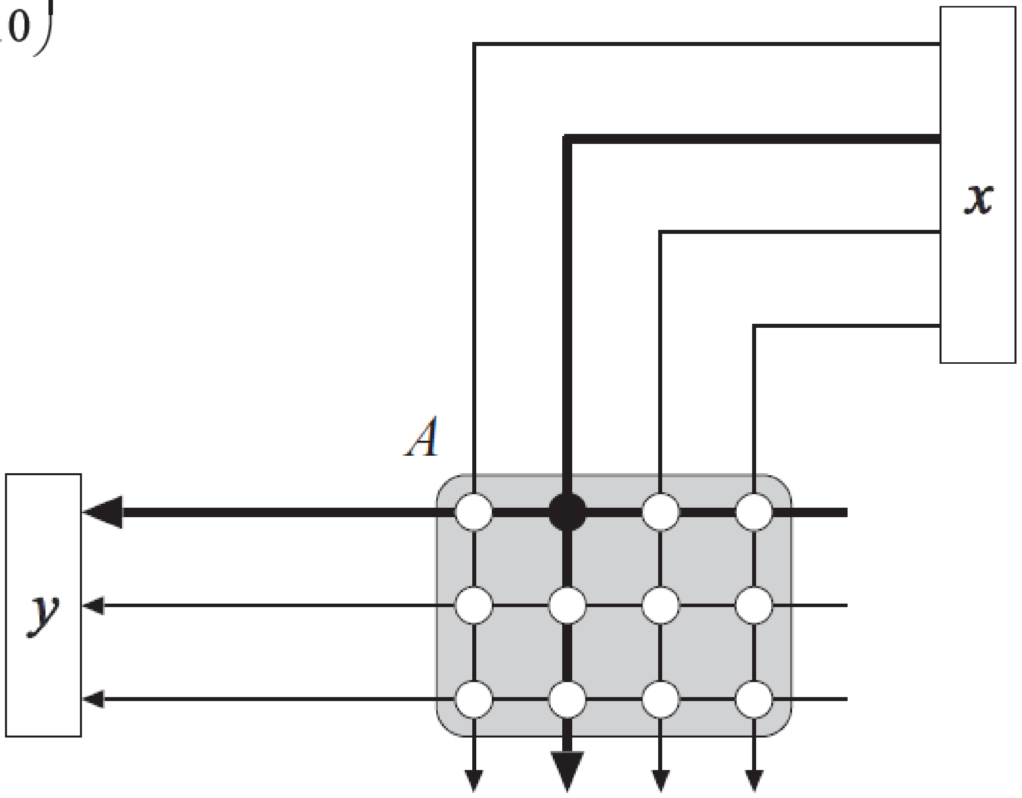
$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}$$

예: $\begin{pmatrix} 2 & 7 \\ 9 & 5 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 + 7\cdot 2 \\ 9\cdot 1 + 5\cdot 2 \\ 4\cdot 1 + 3\cdot 2 \end{pmatrix} = \begin{pmatrix} 16 \\ 19 \\ 10 \end{pmatrix}$

- 행렬과 벡터의 곱은 벡터

$$\boldsymbol{y} = A\boldsymbol{x}$$

- 행렬의 Column 수가 입력의 차원 수와 일치
- 행렬의 Row 수가 출력의 차원 수와 일치
- a_{mn}은 m번째 출력에 n번째 입력이 기여하는 정도를 의미함



Special Matrices

- Zero Matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Square Matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Upper/Lower Triangular Matrix

$$\begin{bmatrix} a & f & e \\ 0 & b & d \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{bmatrix}$$

- Diagonal Matrix

$$diag\left(\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{bmatrix}\right) = diag\left(a_1, a_2, a_3, a_4, a_5\right)$$