Homework Chapter 6

1.

Find the eigenvalues and the corresponding eigenspaces for each of the following matrices:

(a) $\begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 5 \\ -4 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$ (h) $\begin{bmatrix} 1 & -2 & 2 \\ 2 & 0 & 2 \\ 3 & -2 & 4 \end{bmatrix}$

(i) $\begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ (j) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(k) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ (l) $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2.

Let $A = (a_{ij})$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that

$$\lambda_j = a_{jj} + \sum_{i \neq j} (a_{ii} - \lambda_i) \quad \text{for} \quad j = 1, \ldots, n$$

3.

Solve each of the following initial value problems:

(a) $y_1' = -y_1 + 2y_2$
$y_2' = 2y_1 - y_2$
$y_1(0) = 3, y_2(0) = 1$

(b) $y_1' = y_1 - 2y_2$
$y_2' = 2y_1 + y_2$
$y_1(0) = 1, y_2(0) = -2$

(c) $y_1' = 2y_1 - 6y_3$
$y_2' = y_1 - 3y_3$
$y_3' = y_2 - 2y_3$
$y_1(0) = y_2(0) = y_3(0) = 2$

(d) $y_1' = y_1 + 2y_3$
$y_2' = y_2 - y_3$
$y_3' = y_1 + y_2 + y_3$
$y_1(0) = y_2(0) = 1, y_3(0) = 4$

4.

For each of the following, find a matrix B such that $B^2 = A$:

(a) $A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 9 & -5 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

5.

Let

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{3} & \frac{2}{5} \\ \frac{1}{4} & \frac{1}{3} & \frac{2}{5} \end{bmatrix}$$

be a transition matrix for a Markov process.

(a) Compute $\det(A)$ and $\operatorname{trace}(A)$ and make use of those values to determine the eigenvalues of A.

(b) Explain why the Markov process must converge to a steady-state vector.

(c) Show that $\mathbf{y} = (16, 15, 15)^T$ is an eigenvector of A. How is the steady-state vector related to $\mathbf{y}$?

6.

Given that

$$A = \begin{bmatrix} 1 & 0 & -i \\ 0 & 4 & 0 \\ i & 0 & 1 \end{bmatrix}$$

Find a matrix B such that $B^H B = A$.

7.

Use the method of Example 1 to find the singular value decomposition of each of the following matrices:

(a) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 5 & -3 \\ 0 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

8.

The matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

has singular value decomposition

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \sqrt{18} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(a) Use the singular value decomposition to find orthonormal bases for $R(A^T)$ and $N(A)$.

(b) Use the singular value decomposition to find orthonormal bases for $R(A)$ and $N(A^T)$.