Linear Algebra

- Linear Transformations -

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Linear Transformations

Linear Transformations

- A mapping L from a vector space V into a vector space W is said to be a *linear transformation* if

Additivity

$$L(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2) = \underline{\alpha L(\boldsymbol{v}_1)} + \underline{\beta L(\boldsymbol{v}_2)}$$

Homogeneity

for all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and for all scalars α and β

- If L is a linear transformation mapping a vector space V into a vector space W,

$$L(\boldsymbol{v}_1 + \boldsymbol{v}_2) = L(\boldsymbol{v}_1) + L(\boldsymbol{v}_2)$$

$$L(\alpha \boldsymbol{v}) = \alpha L(\boldsymbol{v})$$

- Conversely, if L satisfies the above statements, then

$$L(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2) = \alpha L(\boldsymbol{v}_1) + \beta L(\boldsymbol{v}_2)$$

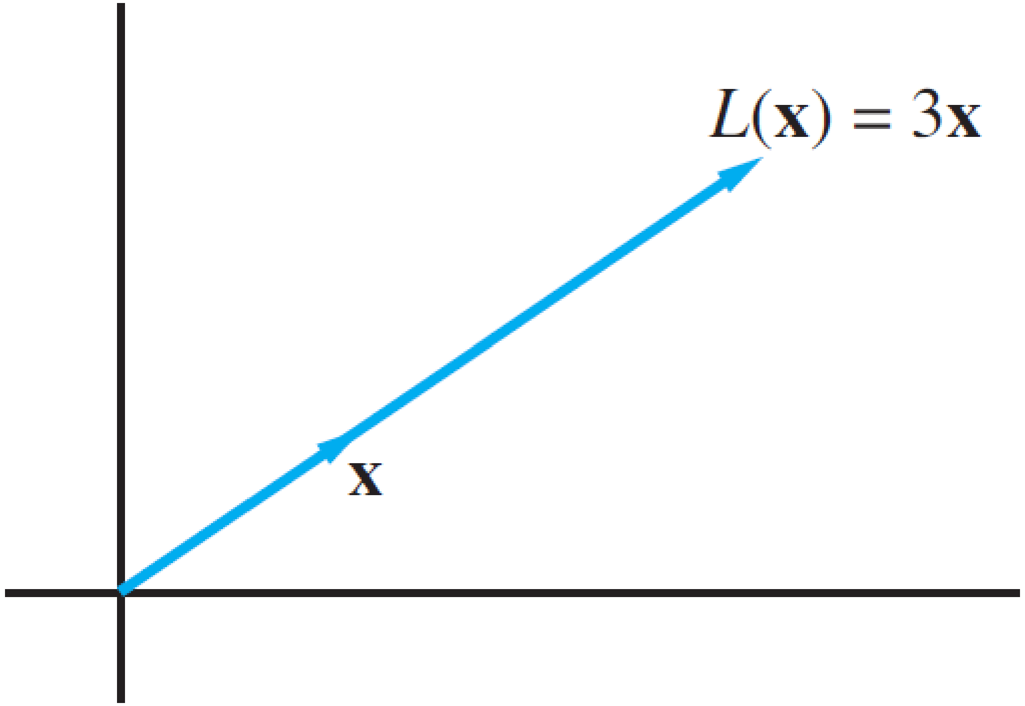
- Therefore, L is a linear transformation if and only if L satisfies *homogeneity* and *additivity*

Linear Operations on $\mathbb{R}^2$

- Scaling (Stretching or Shrinking)

$$L(\boldsymbol{x}) = \alpha \boldsymbol{x} \qquad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^2$$

- In general, if α is a positive scalar, the linear operator $F(\boldsymbol{x}) = \alpha \boldsymbol{x}$ can be thought of as a stretching($|\alpha| > 1$) or shrinking ($0 < |\alpha| < 1$) by a factor of α

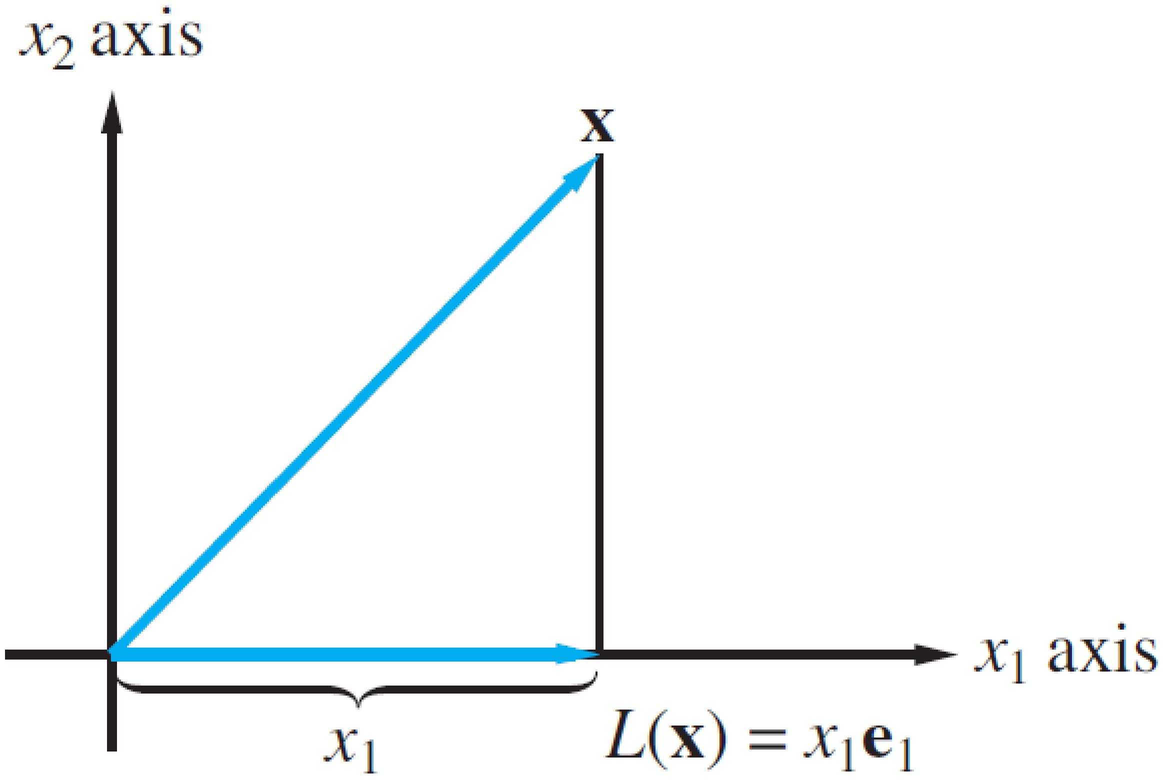


Linear Operations on $\mathbb{R}^2$

- Projection

$$L(\boldsymbol{x}) = x_1\boldsymbol{e}_1 \qquad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^2$$

- L is a projection onto the x_1-axis

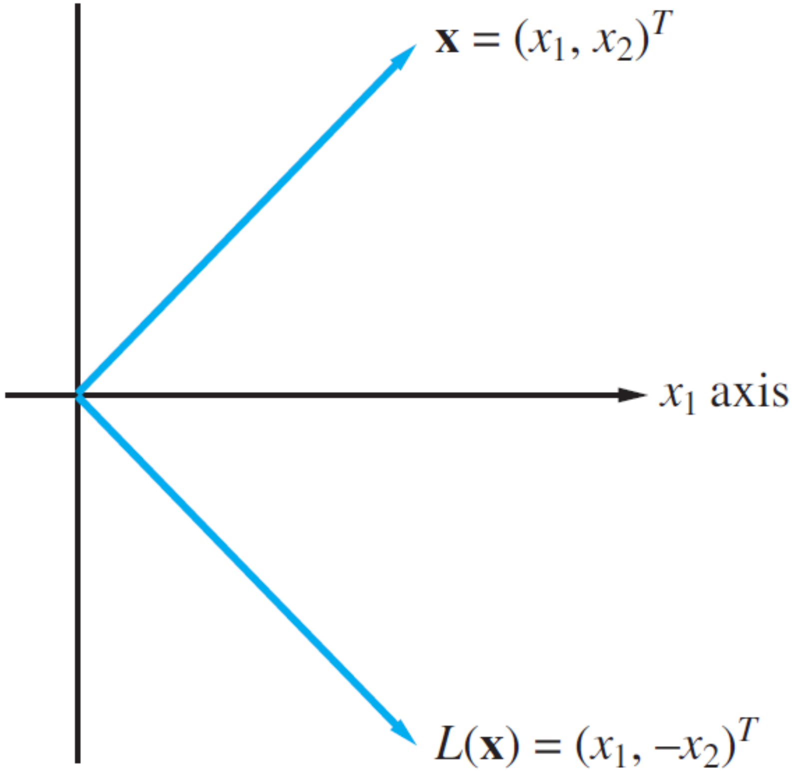


Linear Operations on $\mathbb{R}^2$

- Reflection

$$L(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} \qquad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^2$$

- L has the effect of reflecting vectors about the x_1-axis

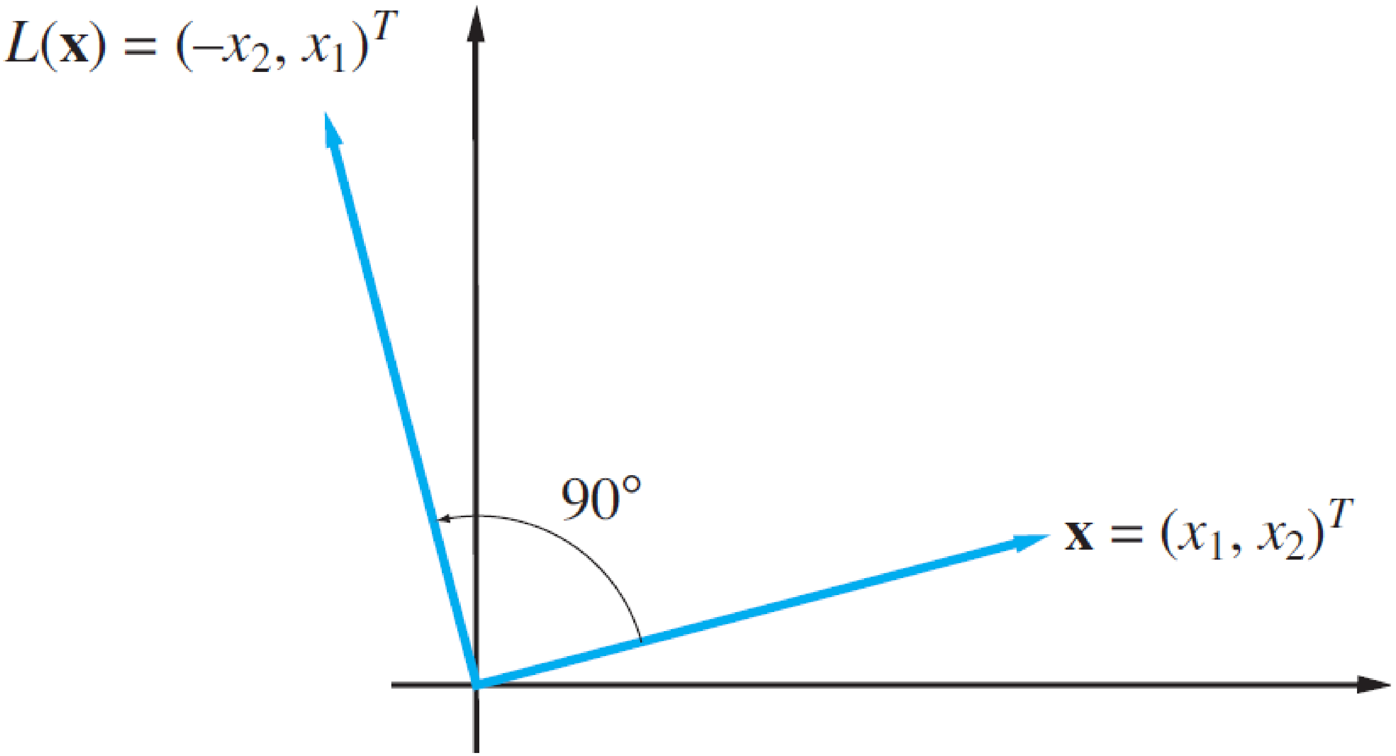


Linear Operations on $\mathbb{R}^2$

- Rotation

$$L(\boldsymbol{x}) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \qquad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^2$$

- L has the effect of rotating each vector in $\mathbb{R}^2$ by 90° in the counterclockwise direction



Linear Operations from $\mathbb{R}^n$ to $\mathbb{R}^m$

- The mapping $L: \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$L(\boldsymbol{x}) = x_1 + x_2$$

is a linear transformation, since

$$L(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) = (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) = \alpha(x_1 + x_2) + \beta(y_1 + y_2) = \alpha L(\boldsymbol{x}) + \beta L(\boldsymbol{y})$$

- The mapping $M(\boldsymbol{x}) = (x_1^2 + x_2^2)^{\frac{1}{2}}$ is not a linear operator, since

$$M(\alpha\boldsymbol{x}) = (\alpha^2 x_1^2 + \alpha^2 x_2^2)^{\frac{1}{2}} = |\alpha|(x_1^2 + x_2^2)^{\frac{1}{2}} = |\alpha| M(\boldsymbol{x}) \neq \alpha M(\boldsymbol{x})$$

Linear Operations from $\mathbb{R}^n$ to $\mathbb{R}^m$

- In general, if A is any $m \times n$ matrix, we can define a linear transformation L_A from $\mathbb{R}^n$ to $\mathbb{R}^m$ by

$$L_A(\boldsymbol{x}) = A\boldsymbol{x} \qquad \text{for} \quad x \in \mathbb{R}$$

- The transformation L_A is linear since,

$$L_A(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) = A(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) = \alpha A\boldsymbol{x} + \beta A\boldsymbol{y} = \alpha L_A(\boldsymbol{x}) + \beta L_A(\boldsymbol{y})$$

- Therefore, we can think of each $m \times n$ matrix A as defining a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

Linear Operations from V to W

- If L is a linear transformation mapping a vector space V to a vector space W, then

  i) $L(\mathbf{0}_V) = \mathbf{0}_W$ (where $\mathbf{0}_V$ and $\mathbf{0}_W$ are the zero vectors in V and W, respectively)

  ii) if $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ are elements of V and $\alpha_1, \cdots, \alpha_n$ are scalars, then

$$L(\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n) = \alpha_1 L(\boldsymbol{v}_1) + \cdots + \alpha_n L(\boldsymbol{v}_n)$$

  iii) $L(-\boldsymbol{v}) = -L(\boldsymbol{v})$ for $v \in V$

- Proof

$$\mathbf{0}_W = L(\mathbf{0}_V) = L\big(\boldsymbol{v} + (-\boldsymbol{v})\big) = L(\boldsymbol{v}) + L(-\boldsymbol{v}) \qquad \therefore L(-\boldsymbol{v}) = -L(\boldsymbol{v})$$

Theorem 3.1.1 If V is a vector space and $\mathbf{x}$ is any element of V, then

(i) $0\mathbf{x} = \mathbf{0}$.
(ii) $\mathbf{x} + \mathbf{y} = \mathbf{0}$ implies that $\mathbf{y} = -\mathbf{x}$ (i.e., the additive inverse of $\mathbf{x}$ is unique).
(iii) $(-1)\mathbf{x} = -\mathbf{x}$.

Image and Kernel

- Let $L: V \to W$ be a linear transformation. The *kernel* of L, denoted $\ker(L)$, is defined by

$$\ker(L) = \{\boldsymbol{v} \in V \mid L(\boldsymbol{v}) = \mathbf{0}_W\}$$

- Let $L: V \to W$ be a linear transformation and let S be a subspace of V. The *image* of S, denoted $L(S)$ is defined by

$$L(S) = \{\boldsymbol{w} \in W \mid \boldsymbol{w} = L(\boldsymbol{v}) \quad \text{for some} \quad \boldsymbol{v} \in S\}$$

- The image of the entire vector space, $L(V)$, is called the *range* of L

Image and Kernel

- If $L: V \to W$ is a linear transformation and S is a subspace of V, then

  i) $\ker(L)$ is a subspace of V
  ii) $L(S)$ is a subspace of W

Definition

If S is a nonempty subset of a vector space V, and S satisfies the conditions

(i) $\alpha\mathbf{x} \in S$ whenever $\mathbf{x} \in S$ for any scalar α
(ii) $\mathbf{x} + \mathbf{y} \in S$ whenever $\mathbf{x} \in S$ and $\mathbf{y} \in S$

then S is said to be a **subspace** of V.

- Proof

It is obvious that $\ker(L)$ is nonempty since $\mathbf{0}_V$, the zero vector of V, is in $\ker(L)$. To prove (i), we must show that $\ker(L)$ is closed under scalar multiplication and addition of vectors. For closure under scalar multiplication let $\mathbf{v} \in \ker(L)$ and let α be a scalar. Then

$$L(\alpha\mathbf{v}) = \alpha L(\mathbf{v}) = \alpha\mathbf{0}_W = \mathbf{0}_W$$

Therefore, $\alpha\mathbf{v} \in \ker(L)$.

For closure under addition, let $\mathbf{v}_1, \mathbf{v}_2 \in \ker(L)$. Then

$$L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2) = \mathbf{0}_W + \mathbf{0}_W = \mathbf{0}_W$$

Therefore, $\mathbf{v}_1 + \mathbf{v}_2 \in \ker(L)$ and hence $\ker(L)$ is a subspace of V.

The proof of (ii) is similar. $L(S)$ is nonempty, since $\mathbf{0}_W = L(\mathbf{0}_V) \in L(S)$. If $\mathbf{w} \in L(S)$, then $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in S$. For any scalar α,

$$\alpha\mathbf{w} = \alpha L(\mathbf{v}) = L(\alpha\mathbf{v})$$

Since $\alpha\mathbf{v} \in S$, it follows that $\alpha\mathbf{w} \in L(S)$, and hence $L(S)$ is closed under scalar multiplication. If $\mathbf{w}_1, \mathbf{w}_2 \in L(S)$, then there exist $\mathbf{v}_1, \mathbf{v}_2 \in S$ such that $L(\mathbf{v}_1) = \mathbf{w}_1$ and $L(\mathbf{v}_2) = \mathbf{w}_2$. Thus,

$$\mathbf{w}_1 + \mathbf{w}_2 = L(\mathbf{v}_1) + L(\mathbf{v}_2) = L(\mathbf{v}_1 + \mathbf{v}_2)$$

and hence $L(S)$ is closed under addition. It follows that $L(S)$ is a subspace of W. ■

Exercises

3. Let $\boldsymbol{a}$ be a fixed nonzero vector in $\mathbb{R}^2$. A mapping of the form

$$L(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{a}$$

is called a *translation*. Show that a translation is not a linear operator

Matrix Representations of Linear Transformations

Matrix Representations of Linear Transformations

- If L is a linear transformation mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, there is an $m \times n$ matrix A such that

$$L(\boldsymbol{x}) = A\boldsymbol{x}$$

  for each $\boldsymbol{x} \in \mathbb{R}^n$.

- Proof

For $j = 1, \ldots, n$, define

$$\mathbf{a}_j = L(\mathbf{e}_j)$$

and let

$$A = (a_{ij}) = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$$

If

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

is an arbitrary element of $\mathbb{R}^n$, then

$$\begin{aligned} L(\mathbf{x}) &= x_1 L(\mathbf{e}_1) + x_2 L(\mathbf{e}_2) + \cdots + x_n L(\mathbf{e}_n) \\ &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n \\ &= (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= A\mathbf{x} \end{aligned}$$

Matrix Representations of Linear Transformations

- If L is a linear transformation mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, there is an $m \times n$ matrix A such that

$$L(\boldsymbol{x}) = A\boldsymbol{x}$$

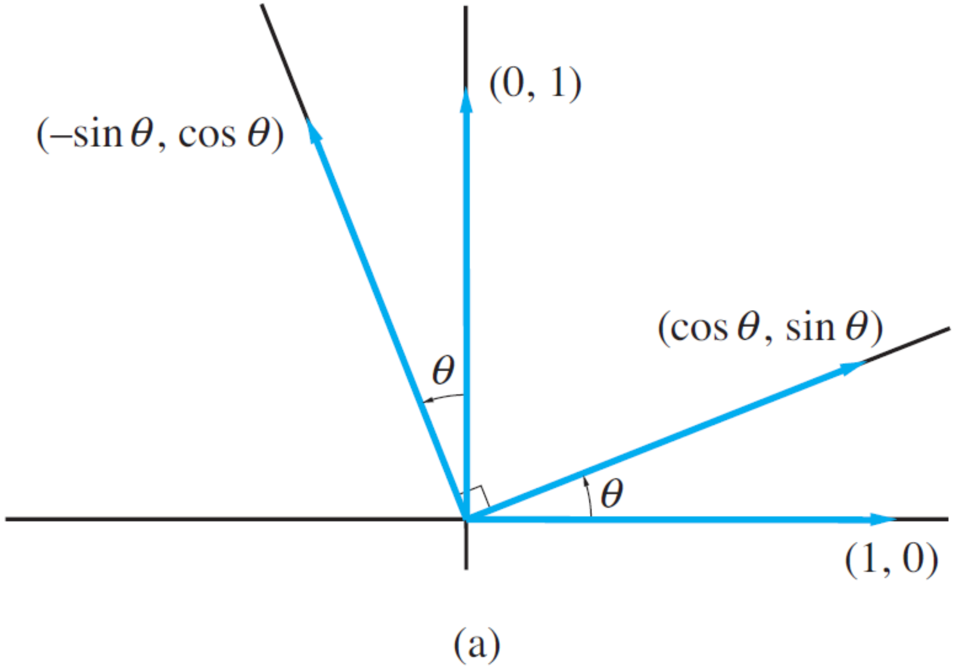
for each $\boldsymbol{x} \in \mathbb{R}^n$.

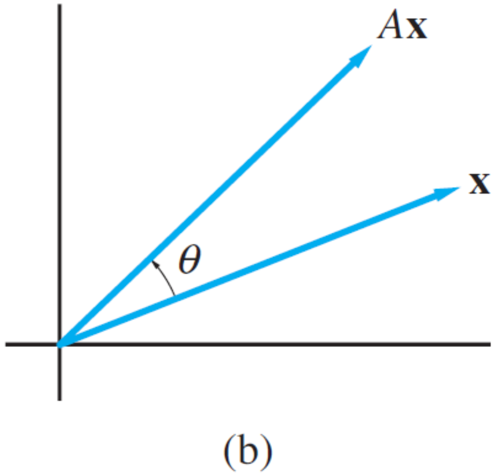
- To construct the matrix A corresponding to a particular linear transformation L:
  - See what L does to the first basis element $\boldsymbol{e}_1$ of $\mathbb{R}^n$, and set $\boldsymbol{a}_1 = L(\boldsymbol{e}_1)$
  - Repeat the process with $\boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$ and get $\boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$
  - Construct $A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n]$

Rotation Matrix

- Let L be the linear transformation operator $\mathbb{R}^2$ that rotates each vector by an angle θ in the counterclockwise direction

$$Ae_1 = ? \qquad Ae_2 = ?$$





$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Matrix Representation Theorem

- If $E = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ and $F = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_m\}$ are ordered bases for vector spaces V and W, respectively, then, corresponding to each linear transformation $L: V \to W$, there is an $m \times n$ matrix A such that

$$[L(\boldsymbol{v})]_F = A[\boldsymbol{v}]_E \quad \text{for each} \quad \boldsymbol{v} \in V$$

A is the matrix representing L relative to the ordered bases E and F. In fact,

$$\boldsymbol{a}_j = \left[L(\boldsymbol{v}_j)\right]_F \qquad j = 1,2,\cdots,n$$

$$\begin{tikzcd}
\mathbf{v} \in V \arrow[r, "L = L_A"] \arrow[d, leftrightarrow] & \mathbf{w} = L(\mathbf{v}) \in W \arrow[d, leftrightarrow] \\
\mathbf{x} = [\mathbf{v}]_E \in R^n \arrow[r, "A"] & A\mathbf{x} = [\mathbf{w}]_F \in R^m
\end{tikzcd}$$

Matrix Representation Theorem

- Proof

let

$E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for V and $F = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be an ordered basis for W. Let L be a linear transformation mapping V into W. If $\mathbf{v}$ is any vector in V, then we can express $\mathbf{v}$ in terms of the basis E:

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n$$

We will show that there exists an $m \times n$ matrix A representing the linear transformation L, in the sense that

$$A\mathbf{x} = \mathbf{y} \quad \text{if and only if} \quad L(\mathbf{v}) = y_1\mathbf{w}_1 + y_2\mathbf{w}_2 + \ldots + y_m\mathbf{w}_m$$

The matrix A characterizes the effect of the linear transformation L. If $\mathbf{x}$ is the coordinate vector of $\mathbf{v}$ with respect to E, then the coordinate vector of $L(\mathbf{v})$ with respect to F is given by

$$[L(\mathbf{v})]_F = A\mathbf{x}$$

The procedure for determining the matrix representation A is essentially the same as before. For $j = 1, \ldots, n$, let $\mathbf{a}_j = (a_{1j}, a_{2j}, \ldots, a_{mj})^T$ be the coordinate vector of $L(\mathbf{v}_j)$ with respect to $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$; that is,

$$L(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + a_{2j}\mathbf{w}_2 + \ldots + a_{mj}\mathbf{w}_m \quad 1 \le j \le n$$

Let $A = (a_{ij}) = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$. If

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n$$

then

$$\begin{aligned} L(\mathbf{v}) &= L\left(\sum_{j=1}^{n} x_j\mathbf{v}_j\right) \\ &= \sum_{j=1}^{n} x_j L(\mathbf{v}_j) \\ &= \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} a_{ij}\mathbf{w}_i\right) \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij}x_j\right)\mathbf{w}_i \end{aligned}$$

For $i = 1, \ldots, m$, let

$$y_i = \sum_{j=1}^{n} a_{ij}x_j$$

Thus,

$$\mathbf{y} = (y_1, y_2, \ldots, y_m)^T = A\mathbf{x}$$

is the coordinate vector of $L(\mathbf{v})$ with respect to $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$.

Matrix Representation Theorem

Ex 3. Let L be the linear transformation mapping $\mathbb{R}^3$ into $\mathbb{R}^2$ defined by

$$L(\boldsymbol{x}) = x_1\boldsymbol{b}_1 + (x_2 + x_3)\boldsymbol{b}_2$$

for each $\boldsymbol{x} \in \mathbb{R}^3$, where

$$b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Find the matrix A representing L with respect to the ordered bases $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$

- Solution

$$L(\boldsymbol{e}_1) = 1\boldsymbol{b}_1 + 0\boldsymbol{b}_2, \qquad L(\boldsymbol{e}_2) = 0\boldsymbol{b}_1 + 1\boldsymbol{b}_2, \qquad L(\boldsymbol{e}_3) = 0\boldsymbol{b}_1 + 1\boldsymbol{b}_2$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Matrix Representation Theorem

- Let $E = \{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_n\}$ and $F = \{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_m\}$ be ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and A is the matrix representing L with respect to E and F, then

$$\boldsymbol{a}_j = B^{-1} L(\boldsymbol{u}_j) \qquad \text{for } j = 1, \cdots, n$$

where $B = (\boldsymbol{b}_1, \cdots, \boldsymbol{b}_m)$

- Proof

If A is representing L with respect to E and F, then, for $j = 1, \ldots, n$, $\quad [L(\boldsymbol{v})]_F = \boldsymbol{y} = A\boldsymbol{x} = A[\boldsymbol{v}]_E$

$$L(\boldsymbol{u}_j) = a_{1j}\boldsymbol{b}_1 + a_{2j}\boldsymbol{b}_2 + \cdots + a_{mj}\boldsymbol{b}_m = B\boldsymbol{a}_j$$

$\because$ $\boldsymbol{a}_j$ is the coordinate vector of $\boldsymbol{u}_j$ in F. Therefore, $L(\boldsymbol{u}_j)$ is represented by the linear combination of the basis F.

The matrix B is nonsingular since its column vectors form a basis for $\mathbb{R}^m$. Hence,

$$\mathbf{a}_j = B^{-1}L(\mathbf{u}_j) \quad j = 1, \ldots, n$$

Matrix Representation Theorem

- If A is the matrix representing the linear transformation $L: \mathbb{R}^n \to \mathbb{R}^m$ with respect to the bases

$$E = \{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_n\} \text{ and } F = \{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n\}$$

then the reduced row echelon form of $(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_m | L(\boldsymbol{u}_1), \cdots, L(\boldsymbol{u}_n))$ is $(I|A)$

- Proof

Let $B = (\mathbf{b}_1, \ldots, \mathbf{b}_m)$. The matrix $(B \mid L(\mathbf{u}_1), \ldots, L(\mathbf{u}_n))$ is row equivalent to

$$\begin{aligned} B^{-1}(B \mid L(\mathbf{u}_1), \ldots, L(\mathbf{u}_n)) &= (I \mid B^{-1}L(\mathbf{u}_1), \ldots, B^{-1}L(\mathbf{u}_n)) \\ &= (I \mid \mathbf{a}_1, \ldots, \mathbf{a}_n) \\ &= (I \mid A) \end{aligned}$$

Matrix Representation Theorem

Ex 6. Let $L: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$L(\boldsymbol{x}) = (x_2, x_1 + x_2, x_1 - x_2)^T$$

Find the matrix representations of L with respect to the ordered bases $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$, where

$$\boldsymbol{u}_1 = (1,2)^T \qquad \boldsymbol{u}_2 = (3,1)^T \qquad \boldsymbol{b}_1 = (1,0,0)^T \qquad \boldsymbol{b}_2 = (1,1,0)^T \qquad \boldsymbol{b}_3 = (1,1,1)^T$$

- Solution

We must compute $L(\mathbf{u}_1)$ and $L(\mathbf{u}_2)$ and then transform the matrix $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \mid L(\mathbf{u}_1), L(\mathbf{u}_2))$ to reduced row echelon form:

$$L(\mathbf{u}_1) = (2, 3, -1)^T \quad \text{and} \quad L(\mathbf{u}_2) = (1, 4, 2)^T$$

$$\left[\begin{array}{ccc|cc} 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right]$$

The matrix representing L with respect to the given ordered bases is

$$A = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}$$

The reader may verify that

$$L(\mathbf{u}_1) = -\mathbf{b}_1 + 4\mathbf{b}_2 - \mathbf{b}_3$$
$$L(\mathbf{u}_2) = -3\mathbf{b}_1 + 2\mathbf{b}_2 + 2\mathbf{b}_3$$

Application: Computer Graphics and Animation

- A picture in the plane can be stored in the computer as a set of vertices
- The vertices can then be plotted and connected by lines to produce the picture
- If there are n vertices, they are stored in a $2 \times n$ matrix

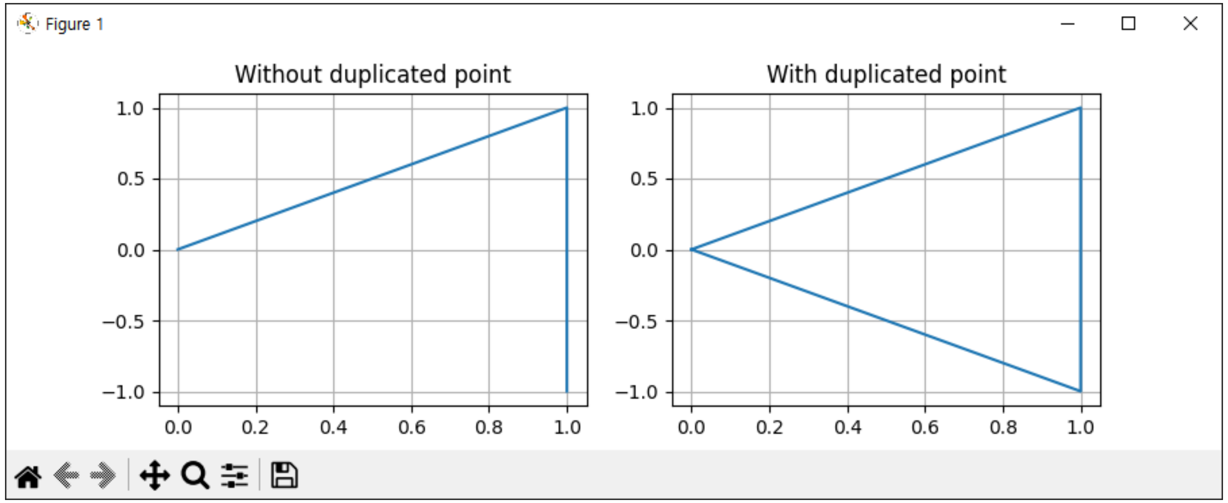
```python
import matplotlib.pyplot as plt
import numpy as np

if __name__ == '__main__':
    # Without duplicated first point
    T1 = np.array([[0, 1, 1], [0, 1, -1]])
    # With duplicated first point
    T2 = np.array([[0, 1, 1, 0], [0, 1, -1, 0]])

    plt.figure(figsize=(9, 3))
    plt.subplot(121)
    plt.plot(T1[0, :], T1[1, :])
    plt.title('Without duplicated point')
    plt.grid()
    plt.subplot(122)
    plt.plot(T2[0, :], T2[1, :])
    plt.title('With duplicated point')
    plt.grid()
    plt.show()
```

$$T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

Duplicated first point for plot



Application: Computer Graphics and Animation

- **Dilations and contractions**

$$L(\boldsymbol{x}) = c\boldsymbol{x} \qquad A = cI$$

is a dilation if $c > 1$ and a contradiction if $0 < c < 1$

- **Reflections about the x-axis and y-axis**

$$L_x(\boldsymbol{e}_1) = \boldsymbol{e}_1 \qquad L_x(\boldsymbol{e}_2) = -\boldsymbol{e}_2 \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$L_y(\boldsymbol{e}_1) = -\boldsymbol{e}_1 \qquad L_y(\boldsymbol{e}_2) = \boldsymbol{e}_2 \qquad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Rotations (Counterclockwise)**

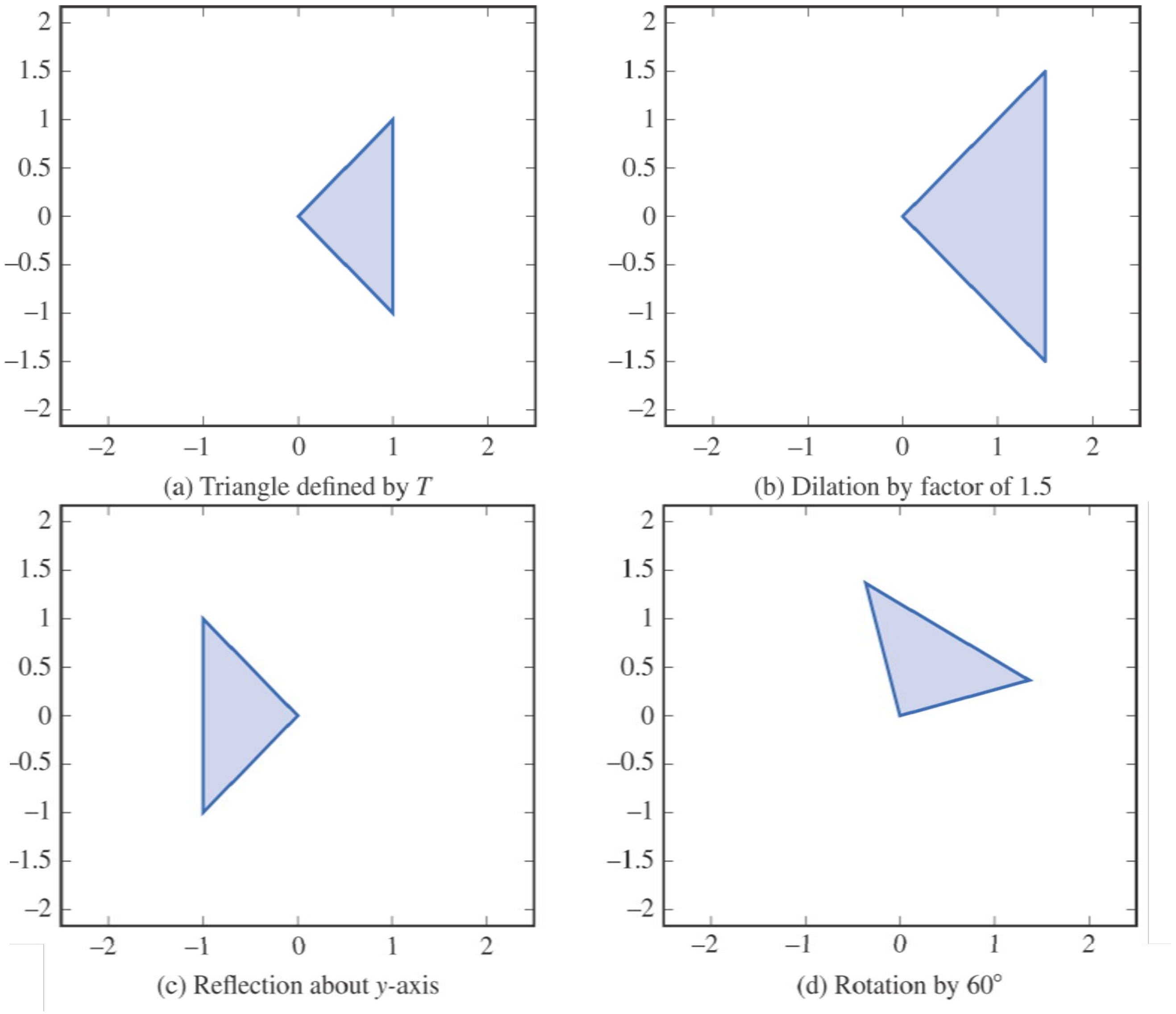
$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- **Translations**

$$L(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{a}$$

If $\boldsymbol{a} \neq \boldsymbol{0}$, then L is not a linear transformation

Application: Computer Graphics and Animation



(a) Triangle defined by T

(b) Dilation by factor of 1.5

(c) Reflection about y-axis

(d) Rotation by 60°

Homogeneous Coordinates

- The homogeneous coordinate system is formed by equating each vector in $\mathbb{R}^2$ with a vector in $\mathbb{R}^3$ having the same first two coordinates and having 1 as its third coordinate

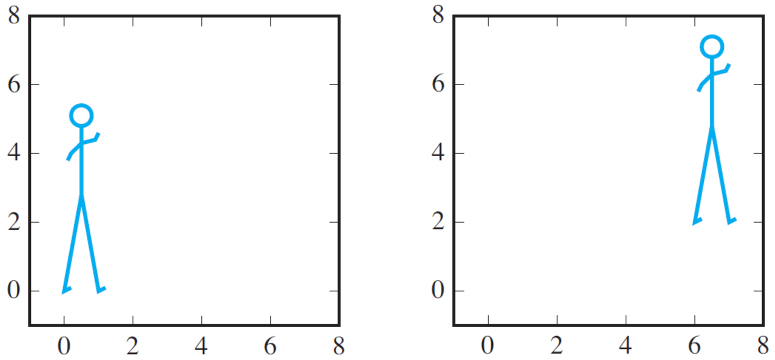
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \left(= \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)$$

- When we want to plot a point represented by the homogeneous coordinate vector, we scale the vector with $\frac{1}{x_3}$ and use $\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right)$ to plot the point

- Translation with homogeneous coordinates

$$a = (6,2)^T$$

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + 6 \\ x_2 + 2 \\ 1 \end{bmatrix}$$



(a) Graph of 3×81 matrix S

(b) Graph of translated figure AS

Exercises

9. Let

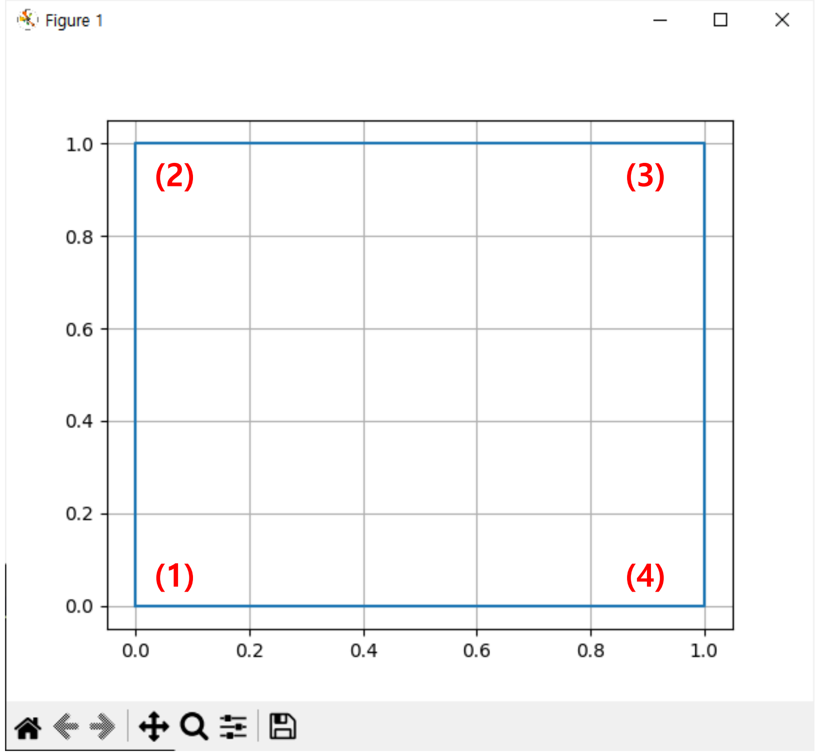
$$R = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The column vectors of R represent the homogeneous coordinates of points in the plane

(a) Draw the figure

```
R = np.array([[0, 0, 1, 1, 0], [0, 1, 1, 0, 0], [1, 1, 1, 1, 1]])

plt.figure(figsize=(9, 3))
plt.plot(R[0, :], R[1, :])
plt.grid()
plt.show()
```



Exercises

9. Let

$$R = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The column vectors of R represent the homogeneous coordinates of points in the plane

(b-ii) $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

```
v = 1/np.sqrt(2)
R = np.array([[0, 0, 1, 1, 0], [0, 1, 1, 0, 0], [1, 1, 1, 1, 1]])
A = np.array([[v, v, 0], [-v, v, 0], [0, 0, 1]])
R = np.matmul(A, R) # Equals to R = A@R

plt.figure(figsize=(9, 3))
plt.plot(R[0, :], R[1, :])
plt.grid()
plt.show()
```