Linear Algebra

- Matrices and Systems of Equations -

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Matrices and Systems of Linear Equations

- One of the most important problems in mathematics is that of solving a system of linear equations
- Over 75% of all mathematical problems encountered in scientific or industrial applications involve solving a linear system at some stage
- It is often possible to take a sophisticated problem and reduce it to a single system of linear equations
- Linear systems arise in applications to such areas as business, economics, sociology, ecology, demography, genetics, electronics, engineering, and physics

Systems of Linear Equations

Linear System

- A *linear equation* in n unknowns is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \dots, a_n$ and b are real numbers and $x_1, x_2, \dots, x_n$ are variables

- A *linear system* of m equations is a system of the form

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{gathered}$$

where the a_{ij}'s and the b_i's are all real numbers

- A system of this form is called an $m \times n$ linear system

Linear System

- Examples of linear systems

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 8 \end{aligned}$$

(a) 2×2 system

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 4 \end{aligned}$$

(b) 2×3 system

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 \quad\;\; &= 4 \end{aligned}$$

(c) 3×2 system

- A *solution* of an $m \times n$ system is an ordered n-tuple of numbers $(x_1, x_2, \cdots, x_n)$ that satisfies all the equations of the system

$$(1, 2)$$

$$\begin{gathered} (2, 0, 0) \\ (2, 1, 1) \\ (2, 3, 3) \\ \vdots \end{gathered}$$

???

- A linear system is:
  - *inconsistent* if the system has no solution
  - *consistent* if the system has at least one solution

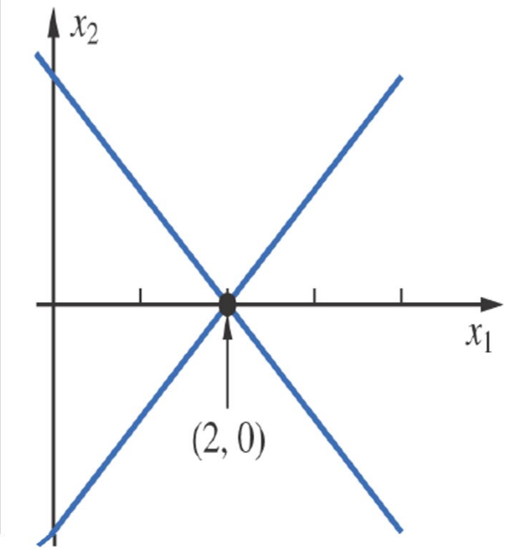
- The set of all solutions of a linear system is called the *solution set* of the system

Linear System

- Geometric interpretation of 2×2 systems

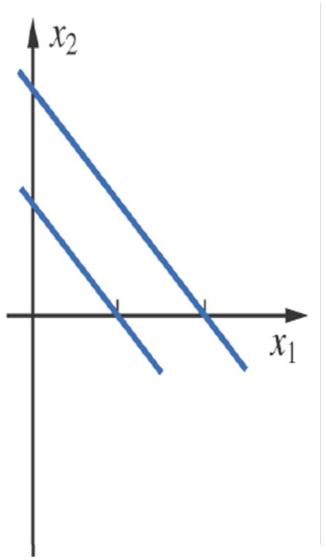
$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 &= b_1 \\
a_{21}x_1 + a_{22}x_2 &= b_2
\end{aligned}
$$

$$
\begin{aligned}
x_1 + x_2 &= 2 \\
x_1 - x_2 &= 2
\end{aligned}
$$



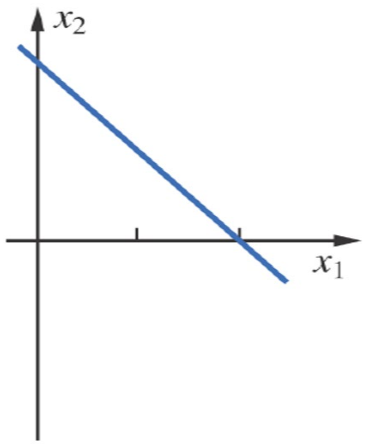
(i) Unique Solution: Intersecting Lines
Intersecting Point (2, 0)

$$
\begin{aligned}
x_1 + x_2 &= 2 \\
x_1 + x_2 &= 1
\end{aligned}
$$



(ii) No Solution: Parallel Lines

$$
\begin{aligned}
x_1 + x_2 &= 2 \\
-x_1 - x_2 &= -2
\end{aligned}
$$



(iii) Infinite Solutions: Same Line

Equivalent System

- Two systems of equations involving the same variables are said to be *equivalent* if they have the same solution set

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ x_2 &= 3 \\ 2x_3 &= 4 \end{aligned}$$

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ -3x_1 - x_2 + x_3 &= 5 \\ 3x_1 + 2x_2 + x_3 &= 2 \end{aligned}$$

$$\begin{array}{r} 3x_1 + 2x_2 - x_3 = -2 \\ 2x_3 = 4 \\ \hline 3x_1 + 2x_2 + x_3 = 2 \end{array}$$

(1)+(3)

$$\begin{array}{r} 3x_1 + 2x_2 - x_3 = -2 \\ -3x_1 - x_2 + x_3 = 5 \\ \hline x_2 = 3 \end{array}$$

(1)+(2)

$$\begin{array}{r} x_2 = 3 \\ 3x_1 + 2x_2 - x_3 = -2 \\ \hline -3x_1 - x_2 + x_3 = 5 \end{array}$$

(2)-(1)

$$\begin{array}{r} 3x_1 + 2x_2 + x_3 = 2 \\ 3x_1 + 2x_2 - x_3 = -2 \\ \hline 2x_3 = 4 \end{array}$$

(3)-(1)

Solution set = $\{(-2, 3, 2)\}$

Equivalent System

- Three operations that can be used on a system to obtain an equivalent system

I. The order in which any two equations are written may be interchanged
II. Both sides of an equation may be multiplied by the same nonzero real number
III. A multiple of one equation may be added to (or subtracted from) another

I.

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 - x_2 &= 2 \\ 4x_1 + x_2 &= 6 \end{aligned} \quad \text{and} \quad \begin{aligned} 4x_1 + x_2 &= 6 \\ 3x_1 - x_2 &= 2 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

II.

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ -2x_1 - x_2 + 4x_3 &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 6 \\ -2x_1 - x_2 + 4x_3 &= 1 \end{aligned}$$

III.

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\ a_{j1}x_1 + \cdots + a_{jn}x_n &= b_j \end{aligned}$$

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\ (a_{j1} + \alpha a_{i1})x_1 + \cdots + (a_{jn} + \alpha a_{in})x_n &= b_j + \alpha b_i \end{aligned}$$

Back Substitution

- A system is said to be in *strict triangular form* if, in the k-th equation:
  - the coefficients of the first $k-1$ variables are all zero
  - the coefficient of x_k is nonzero $(k = 1, \cdots, n)$

- Any $n \times n$ strictly triangular system can be solved as follows:
  1) The n-th equation is solved for the value of x_n
  2) This value is used in the $(n-1)$-th equation to solve for x_{n-1}, and so on

- This method is called *back substitution*

- Example

$$
\begin{aligned}
2x_1 - x_2 + 3x_3 - 2x_4 &= 1 \\
x_2 - 2x_3 + 3x_4 &= 2 \\
4x_3 + 3x_4 &= 3 \\
4x_4 &= 4
\end{aligned}
$$

Solution = $(1, -1, 0, 1)$

Back Substitution

- Coefficient Matrix

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{pmatrix}$$

- Augmented Matrix
  - A system can be solved by performing operations on the augmented matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right)$$

$$A_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

$$B_{m\times r} = \begin{pmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mr} \end{pmatrix}$$

$$\Rightarrow \quad (A|B) = \left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & b_{11} & \cdots & b_{1r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_{m1} & \cdots & b_{mr} \end{array}\right)$$

Back Substitution

- Pivotal Row and Pivot

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 3 \\
3x_1 - x_2 - 3x_3 &= -1 \\
2x_1 + 3x_2 + x_3 &= 4
\end{aligned}
\quad \Longrightarrow \quad
\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right)
$$

$$
\left.\begin{array}{r} (\text{pivot } a_{11} = 1) \\ \text{entries to be eliminated} \\ a_{21} = 3 \text{ and } a_{31} = 2 \end{array}\right\} \rightarrow
\left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{3} \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right]
\begin{array}{l} \leftarrow \text{pivotal row} \\ \\ \\ \end{array}
$$

$$
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ \mathbf{0} & \mathbf{-7} & \mathbf{-6} & \mathbf{-10} \\ 0 & -1 & -1 & -2 \end{array}\right]
\begin{array}{l} \\ \leftarrow \text{pivotal row} \\ \\ \end{array}
$$