Let $X_0, X_1, ..., X_n$ be a Markov chain with state spaces S, initial probability distribution μ and transition probability matrix $P = \{p_{i,j}\}$. There are A absorbing states.

Passage and hitting times

The first passage time from state i to j, $T_{i,j}$ is the number of steps taken by the chain until it arrives for the first time in state j given that $X_0 = i$. This is a random variable and its probability distribution is given as

$$h_{i,j}^{(n)} = P(T_{i,j} = n) = P(X_n = j, X_{n-1} \neq j, ..., X_1 \neq j | X_0 \neq j)$$

The first passage times can be found recursively as follows: $h_{i,j}^{(1)} = p_{i,j}$ and for $n \geq 2$

$$h_{i,j}^{(n)} = \sum_{k \in S - \{j\}} p_{i,k} h_{k,j}^{(n-1)}$$

Expected number of visits to transient states

Let $I_{i,j}(n)$ to be 1 if $X_n = j$ given that $X_0 = i$ and 0 otherwise. The number of visits to state j, starting at state i by time n is defined as

$$N_{i,j}(n) = \sum_{k=1}^{n} I_{I,j}(k)$$

By linearity of expectation

$$E[N_{i,j}(n)] = \sum_{k=1}^{n} p_{i,j}^{(k)}$$

The initial passage time from i to j is distributed according to $h_{i,j}^{(n)}$ and all the subsequent return times to state j follow the distribution $h_{j,j}^{(n)}$. Thanks to the Markov property, once the chain visits state j, it either returns to this state with probability $h_{j,j}$ or leaves it with probability $1 - h_{j,j}$. If state j is transient, then $h_{j,j}(n) < 1$ and $N_{j,j}(\infty)$ is geometric distributed.

Reversible absorbing probability

The variable of interest is the joint probability that the chain was in transient state j at step $n-1$ and it entered absorbing state k at time n

$$P(X_{n-1} = j, X_n = k)$$

Let $\tau = \min\{t \geq 0 : X_t = k\}$ be the first hitting time of state k. Because k is absorbing

$$P(X_{n-1} = j, X_n = k) = P(X_{n-1} = j, X_n = k, \tau = n)$$

By the definition of τ, this is equivalent to

$$P(X_0 \neq k, X_1 \neq k, ..., X_{n-1} = j, X_n = k)$$

The probability of a conjunction is not affected by the order of the events. By smart ordering and using the product rule

$$P(X_{n-1} = j, X_n = k, \tau = n) = P(X_{n-1} = j, X_m \neq k \forall m < n) \cdot P(X_n = k | X_{n-1} = j)$$

Via the Markov property, the probability of going to k from j is determined solely by the transition matrix, T. Hence

$$P(X_{n-1} = j, X_n = k, \tau = n) = P(X_{n-1} = j, X_m \neq k \forall m < n) \cdot T_{jk}$$

Next, we need to determine $P(X_{n-1} = j, X_m \neq k \forall m < n)$. Let S' be the set of states in S excluding k. Furthermore, let $Q^{(k)}$ be the restricted transition matrix where transitions between non-k states are kept, and transitions to and from k are removed and μ be the initial distribution restricted to non-k states. Each multiplication by $Q^{(k)}$ simulates a one-step transition while avoing state k:

$$P(X_{n-1} = j, X_m \neq k \forall m < n) = \mu \cdot (Q^{(k)})^{n-1}$$

All of the above implies that

$$P(X_{n-1} = j, X_n = k, \tau = n) = \mu \cdot (Q^{(k)})^{n-1} \cdot P_{jk}$$

For any fixed time n, the events:

$$X_{n-1} = j,\ X_n = k,\ \tau = n, \quad \text{for } j \in S'$$

are mutually exclusive, because the chain can only be in one state at time $\tau - 1$, and it can only enter k once. Therefore, these events cannot occur simultaneously. Hence, the sum of their probabilities gives the total probability that the chain is absorbed in k at time n:

$$\sum_{j \in S'} P(X_{n-1} = j,\ X_n = k,\ \tau = n) = P(X_n = k, \tau = n)$$