A MMArkov Chain

Markov chain

A fight between fighters *blue* and *red* can be represented as a Markov chain. Although a real-world fight is very complex, we assume for this particular example that the following actions and states are possible:

- standing
- strike attempt blue
- strike landed blue
- strike attempt red
- strike landed red
- knockout blue
- knockout red

The transition matrix between the states and actions is constructed as follows:

$$T = \begin{bmatrix} 0.6 & 0.75 & 0.95 & 0.6 & 0.85 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.05 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0.15 & 0 & 1 \end{bmatrix}$$

The first column represents the probabilities of the fight moving from "standing" to one of the other states. So, the probability of the fight staying in the "standing"-state is equal to 60%. The last two columns represents the "knockout blue"- and "knockout red"-states. These states are absorbing states; once the fight is in this state, it is impossible to move to another state.

The vector $x_k = T^k x_0$ tell us the probability of being in any of the states after k steps.

$$x_1 = \begin{bmatrix} 0.6 \\ 0.3 \\ 0 \\ 0.1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0.645 \\ 0.18 \\ 0.075 \\ 0.06 \\ 0.04 \\ 0 \\ 0 \end{bmatrix}, x_{10} = \begin{bmatrix} 0.6232 \\ 0.1882 \\ 0.0474 \\ 0.0627 \\ 0.0253 \\ 0.0205 \\ 0.0328 \end{bmatrix}, x_{1000} = \begin{bmatrix} 0.001 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.384 \\ 0.614 \end{bmatrix}$$

After 1000 steps, there is an almost zero probability that the fight is still going on.

A transition matrix can be written in the following form

$$T = \begin{bmatrix} Q & O_{r \times s} \\ R & I_s \end{bmatrix}$$

where r represents the number of transient states and s the number of absorbing states. The elements are described as follows

- Q is an $r \times r$ matrix that holds the probabilities of moving from a transient state to another transient state
- R is an $s \times r$ matrix that holds the probabilities of moving from a transient state to an absorbing state
- O is an $r \times s$ matrix that holds the probabilities of moving from an absorbing state to a transient state (which is impossible)
- I is an $s \times s$ matrix that holds the probabilities of moving between absorbing states (which is also impossible)

In the simplified fight, the elements are constructed as follows

$$Q = \begin{bmatrix} 0.6 & 0.75 & 0.95 & 0.6 & 0.85 \\ 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix}$$

$$O_{r\times s} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & 0 & 0.05 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.15 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The variable of main interest is $\lim_{k\to\infty} T^K x_0$.

$$T^2 = \begin{bmatrix} Q & O_{r\times s} \\ R & I_s \end{bmatrix} \begin{bmatrix} Q & O_{r\times s} \\ R & I_s \end{bmatrix} = \begin{bmatrix} Q^2 & O_{r\times s} \\ RQ + R & I_s \end{bmatrix}$$

$$T^3 = \begin{bmatrix} Q & O_{r\times s} \\ R & I_s \end{bmatrix} \begin{bmatrix} Q^2 & O_{r\times s} \\ RQ + R & I_s \end{bmatrix} = \begin{bmatrix} Q^3 & O_{r\times s} \\ RQ^2 + RQ + R & I_s \end{bmatrix}$$

In general, the pattern can be written as

$$\begin{bmatrix} Q^k & O_{r\times s} \\ R + RQ + ... + RQ^{k-1} & I_s \end{bmatrix} = \begin{bmatrix} Q^k & O_{r\times s} \\ R\sum_{i=0}^{k-1} Q^i & I_s \end{bmatrix}$$

By calculating the lower left element of the matrix, we encouter the series

$$\sum_{i=0}^{\infty} Q^i = I + Q + Q^2 + Q^3 + ...$$

This series converges to $(I - Q)^{-1}$ when some power Q^k has columns sums that are less than 1.

The above holds for the example fight and, therefore,

$$(I - Q)^{-1} = \begin{bmatrix} 102.56 & 101.28 & 97.43 & 96.41 & 87.18 \\ 30.77 & 31.38 & 29.23 & 28.92 & 26.15 \\ 7.69 & 7.85 & 8.31 & 7.23 & 6.54 \\ 10.26 & 10.13 & 9.74 & 10.64 & 8.72 \\ 4.10 & 4.05 & 3.90 & 4.26 & 4.49 \end{bmatrix}$$

and

$$R(I-Q)^{-1} = \begin{bmatrix} 0.38 & 0.39 & 0.42 & 0.36 & 0.33 \\ 0.62 & 0.61 & 0.58 & 0.64 & 0.67 \end{bmatrix}$$

Consequently,

$$\lim_{k\to\infty} T^K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.38 & 0.39 & 0.42 & 0.36 & 0.33 & 1 & 0 \\ 0.62 & 0.61 & 0.58 & 0.64 & 0.67 & 0 & 1 \end{bmatrix}$$

A fight always starts with both fighters standing across each other. To find the probabilities of each state after "a very long time", we compute

$$\lim_{k\to\infty} T^K x_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.38 & 0.39 & 0.42 & 0.36 & 0.33 & 1 & 0 \\ 0.62 & 0.61 & 0.58 & 0.64 & 0.67 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.38 \\ 0.62 \end{bmatrix}$$

For an absorbing Markov chain, the (i, j)-entry of $R(I_r - Q)^{-1}$ is the probability of ending in absorbing state a_i given that we started in transient state t_j. In addition, the (i, j)-entry of $(I_r - Q)^{-1}$ contains the expected number of visits to transient state t_i given that we started in transient state t_j. With the help of these properties it is possible to determine the probabilities of different outcomes and forecast the number of actions in a fight.

Variable of interest	**blue**	**red**
Probability of knockout	38%	62%
Number of strikes attempted	30.77	10.26
Number of strikes landed	7.69	4.10