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# Output regulation of a class of linear systems with switched exosystems

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In this article we investigate the output regulation (servomechanism) problem of linear systems with external inputs generated by switched linear systems. We use switching controllers and consider both the full information and the error feedback controllers under the assumption that there is a synchronization between exosystem and controller switching.

## 1. Introduction

The output regulation problem, or the so called servomechanism problem, has been studied extensively in the past few decades. In the 70s several authors considered and solved the problem for linear systems; see for example Francis and Wonham (1975, 1976), Davison and Goldenberg (1975), Davison (1976a,b) and Francis (1977). Davison (1976) considered the robust regulation problem for linear systems and developed a controller based on a parallel connection of a sevocompensator and a stabilizing compensator using error feedback. Davison and Goldenberg (1975) and Davison (1976b), on the other hand, considered decentralized servomechanism problem for linear systems and developed a similar controller. The solution they developed depends on the characteristics of the fixed modes of the system, a concept introduced earlier in Wang and Davison (1973). Francis (1977) developed necessary and sufficient conditions for the solvability of the output regulation problem. The solution achieved is in direct relation with the internal model principle, a concept introduced earlier (Francis and Wonhm 1975, 1976).

The problem of output regulation of non-linear systems was first pursued by Huang and Rugh (1990) for systems with constant exogenous signals and by Isidori and Byrnes (1990) for a more general class of exosystems (see also Isidori (1995)). In Huang and Rugh (1992a) the servomechanism problem for systems with slowly varying but not necessarily bounded exogenous signals was addressed, and a solution method based on the series expansion of the system functions and the solution of the regulator equations was presented. It was also shown that the solution of the problem depends also on the higher order harmonics of the system. Later in Huang and Rugh (1992b) the results were extended to present an approximate method for calculating the solution of the regulator equations and was shown that under the developed strategy a "guaranteed" bounded tracking is achieved where the bound on the tracking error depends on the quality of the approximation. Similarly, in Chu and Husang (1999) and Wang *et al.* (2000) neural networks were used to approximate the solutions of the regulator equations. Recent research in this area has been focusing on robust regional, semiglobal, and global regulation of non-linear systems (Khalil 2000, Serrani and Isidori 2000, Serrani *et al.* 2000) and systems with parameterized exosystems and adaptive internal models (Serrani *et al.* 2001).

Gazi and Passino (2006) extended the results by Davison (1976) for the regulation of non-linear systems in the framework of Isidori and Byrnes (1990). Later, in Ye and Huang (2003) Ye and Huang considered

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decentralized adaptive output regulation of large-scale systems. In particular, they considered systems composed of sub-systems of the form of those discussed in Serrani and Isidori (2000) and with adaptive internal model as in Serrani and Isidori (2001) and obtained global results.

One limitation of the results on output regulation is that the external signals (the reference inputs and the disturbances) are assumed to be generated by a neutrally stable exosystem. This limits the exogeneous signals to the class of constant and periodic signals. In this article we consider the problem of output regulation of a class of systems with switched external inputs (exosystems). Even if each of the exosystems, among which the switching occurs, is neutrally stable, its output is not necessarily periodic and can be more sophisticated. Moreover, in some systems we may have control over the design of the exosystem (at least over the part that generates the reference inputs). For example, consider the problem of a robot moving on a desired trajectory. The exosystem can serve as a virtual vehicle generating the desired trajectory for the actual robot. In such systems by appropriately switching among a set of neutrally stable exosystems (chosen again by the application designer) one can generate complex trajectories. In this article we will show that these trajectories can also be tracked by an appropriate switching controller, overcoming the above limitation.

Since the affect of the external signals are assumed to be switching, a natural approach to solve the problem is to use a switching controller and this is the approach we take in this article. We consider both the full information and the error feedback controllers under the assumption of simultaneous switching both in the exosystem and in the controller. To prove stability of the closed loop system and regulation of the output error to zero we use results from switched systems literature. To best of our knowledge no other author has considered the output regulation problem from the aspect we consider it here, i.e., under switched system signals.

## 2. The class of systems

Consider the problem of finding a controller for the output regulation of a class of systems described by

$$\left.\begin{aligned} \dot{x} &= Ax + E_{p^E}w + Bu, \\ e &= Cx + F_{p^F}w, \end{aligned}\right\} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input and $e \in \mathbb{R}^m$ is the output of the system. The signal $w \in \mathbb{R}^r$ represents the exogenous inputs, that includes the reference inputs that need to be tracked (due to the $F_{p_F}w$ term at the output), and the disturbances that need to be rejected (due to the $E_{p_E}w$ term in the system dynamics). It is assumed that all the exogenous signals are generated by a family of neutrally stable linear exosystems given by

$$\dot{w} = G_{p^G}w, \tag{2}$$

where $G_{p_G}$ has only simple eigenvalues on the imaginary axis. The matrices $E_{p_E}, F_{p_F}$, and $G_{p_G}$ are parameterized families which belong to the compact sets $\mathcal{E}_{p_E} = \{E_{p_E} : p_E \in \mathcal{P}_E\}$, $\mathcal{F}_{p_F} = \{F_{p_F} : p_F \in \mathcal{P}_F\}$, and $\mathcal{G}_{p_G} = \{G_{p_G} : p_G \in \mathcal{P}_G\}$. The sets $\mathcal{P}_E, \mathcal{P}_F$, and $\mathcal{P}_G$ are assumed to be compact (usually finite) index sets. Note once more that all $G_{p_G} \in \mathcal{G}_{p_G}$ have only simple eigenvalues on the imaginary axis.

Define $p = p_E \times p_F \times p_G$ and $\mathcal{P} = \mathcal{P}_E \times \mathcal{P}_F \times \mathcal{P}_G$ and consider the switched system

$$\left.\begin{aligned} \dot{x} &= Ax + E_\sigma w + Bu, \\ \dot{w} &= G_\sigma w, \\ e &= Cx + F_\sigma w, \end{aligned}\right\} \tag{3}$$

where $\sigma : [0, \infty) \to \mathcal{P}$ is a piecewise constant switching signal taking values in $\mathcal{P}$. Denote with $\{t_0, t_1, t_2, \ldots\}$ the sequence of switching instants of $\sigma(t)$ and with $\{p_0, p_1, p_2, \ldots\}$ the corresponding index sequence, where $p_i \in \mathcal{P}, i = 0, 1, 2, \ldots$. Then, we have

$$\sigma(t) = p_i \quad \text{for all } t \in [t_i, t_{i+1}).$$

In this article we are concerned with regulating the output $e(t)$ to zero of switched systems of the form of (3), despite the switching in the system. In other words, we are concerned with the following problem.

**Output regulation under switching:** Given the system in (3) determine a class of switching signals $\sigma(t)$ and an appropriate switching controller $u_\sigma(t)$ such that

- **(S)** The closed loop system is asymptotically stable and
- **(R)** The output satisfies $\lim_{t\to\infty} e(t) = 0$.

In the analysis below we will say that the problem of output regulation under switching is solvable if it is possible to find $u_\sigma(t)$ and $\sigma(t)$ such that conditions **(S)** and **(R)** are satisfied simultaneously. Since the system is switching, a natural approach to solve the problem is to use a switching controller and this is the approach we take here. Note that other approaches to the solution of the problem might also be possible. However, currently we are not aware of any work or formulation of the problem as we do it in this article.

Based on the available information to the controller two different approaches are possible for solving the above problem: the full information controller and the error feedback controller – both to be discussed below. Below we show that stability and output regulation is achieved provided that the switching signal $\sigma(t)$ is such that the zero error manifold is continuous at the switching points for the full information case and in addition it possesses an average dwell-time property for the error feedback case. We start with the full information case.

## 3. Full information controller

The full information controller can be designed if both $x(t)$ and $w(t)$ are available for feedback – hence the name full information. In this section we investigate this case. We consider the class of parameterized families of systems for which the problem is solvable using full information, which is expressed by the following assumption.

**Assumption 1:** *For all $p \in \mathcal{P}$ the output regulation problem is solvable using full information.*

We express this assumption because we do not want switching to systems for which the problem is not solvable, since this will render also the switched regulation problem unsolvable. To see this assume that there is a $\bar{p} \in \mathcal{P}$ for which the problem is not solvable. Then, if the switching signal is $\sigma(t) = \bar{p}$ for all $t$, it would not be possible to achieve output regulation. Therefore, we limit the problem to those parameterized systems satisfying Assumption 1. However, note that Assumption 1 is not a sufficient condition, since switching between stable linear systems may lead to an unstable composite system (Utkin 1977). Therefore, there are a few more conditions needed for the solvability of the output regulation problem, as will be discussed below.

Given $p \in \mathcal{P}$ Assumption 1 guarantees that there exist mappings $x = \Pi_p w$ and $u = K_p w$ such that the matrices $\Pi_p$ and $K_p$ satisfy the Francis equations (Francis 1977)

$$\left.\begin{aligned} \Pi_p G_p &= A\Pi_p + E_p + BK_p, \\ 0 &= C\Pi_p + F_p. \end{aligned}\right\} \tag{4}$$

Then, the controller

$$u_p = K_p w + L(x - \Pi_p w), \tag{5}$$

where $L$ is a matrix of appropriate dimensions such that $(A + BL)$ is Hurwitz, solves the problem, i.e., it leads to both **(S)** and **(R)** being satisfied. The mapping $x = \Pi_p w$ forms a manifold on which the output error is zero and the mapping $u = K_p w$ is the corresponding controller which renders this manifold invariant. The second term in the controller equation (5) drives the state of the system on the $x = \Pi_p w$ manifold (i.e., renders the manifold exponentially attractive), whereas the first term keeps the state on the manifold once it is reached (i.e., renders the manifold invariant). Note that uniqueness of the solutions of (4) is not an issue since if for a given $p$ there are multiple solutions $(\Pi_p, K_p)$ of (4), then it is possible to achieve tracking on more than one manifolds by choosing the corresponding controller in (5). Therefore, in the case of multiple solutions of (4) we have the freedom to choose any solution we want, or we can choose the one which satisfies the conditions needed for regulation under switching (to be discussed below).

Note also that the conditions in (4) are standard necessary and sufficient conditions for the solvability of the linear servomechanism (output regulation) problem given a particular parameter $p \in \mathcal{P}$. They are linear system of equalities which were derived by Francis in the 1970s (Francis 1977) and can be solved using standard methods. Their non-linear counterparts were also derived in the late 1980s by Isidori and Byrnes in the form of partial differential equations (Isidori and Byrnes 1990).

Before proceeding with the main result of this section we will discuss few more issues. Given index $p$, let $\tilde{x}_p = x - \Pi_p w$ be the difference between the system state and the desired zero error manifold. Then, applying the control input in (5) and using the fact in (4) one can show that the system can be represented in the form

$$\left.\begin{aligned} \dot{\tilde{x}}_p &= (A + BL)\tilde{x}_p \\ e &= C\tilde{x}_p. \end{aligned}\right\} \tag{6}$$

The fact that $(A + BL)$ is Hurwitz implies that there exist positive definite matrices $P$ and $Q$ such that

$$(A + BL)^\top P + P(A + BL) \leq -Q. \tag{7}$$

Moreover, if $\sigma(t) = p$ for all $t$, then $\tilde{x}_p \to 0$ as $t \to \infty$ exponentially fast implying the same for the output error $e(t)$.

An important issue to note here is that the controller gain $L$ and the corresponding positive definite matrices $P$ and $Q$ are independent of the index $p$. In fact, the existence of $L$ is a system property and depends only on the stabilizability of the pair $(A, B)$, which is implied by Assumption 1. Its value can be chosen as any value rendering $(A + BL)$ Hurwitz. Alternatively, an optimal value for $L$, which renders some predefined cost function minimal, could be chosen using the procedure discussed in Krener (1992).

Having established the necessary background, below we state our first result.

**Theorem 1:** *Consider the output regulation problem of the system in* (3). *If Assumption* 1 *holds, then given a piecewise constant switching signal* $\sigma : [0, \infty) \to \mathcal{P}$ *the output regulation problem is solved by the switching controller*

$$u_\sigma(t) = K_\sigma w(t) + L(x(t) - \Pi_\sigma w(t)). \tag{8}$$

*if there exists a finite time* $\bar{t}$ *such that at all switching instants* $t_i \geq \bar{t}$ *we have*

$$\Pi_{p_{i-1}} w(t_i) = \Pi_{p_i} w(t_i). \tag{9}$$

**Proof:** Define $\tilde{x}_\sigma = x - \Pi_\sigma w$ and

$$V(\tilde{x}_\sigma) = \tilde{x}_\sigma^\top P \tilde{x}_\sigma,$$

where $P > 0$ is the solution in (7). Note also that (6) holds with $p$ replaced with $\sigma$. Since at the switching instants $t_i$ we have $\Pi_{p_{i-1}} w(t_i) = \Pi_{p_i} w(t_i)$, $\tilde{x}_\sigma$ is continuous, implying that $V(\tilde{x}_\sigma)$ is also continuous. Since $V(\tilde{x}_\sigma)$ is also independent of the index $p$, it is a common quadratic Lyapunov function for the system with time derivative satisfying

$$\dot{V}(\tilde{x}_\sigma) \leq -\tilde{x}_\sigma^\top Q \tilde{x}_\sigma,$$

where $Q > 0$ is the matrix in (7). Therefore, the system is globally exponentially stable and we have $\tilde{x}_\sigma \to 0$ as $t \to \infty$ for any switching signal $\sigma(t)$ satisfying (9). This implies also that $e(t) \to 0$ as $t \to \infty$ for any such $\sigma(t)$ and the proof is complete. $\square$

The controller in (8) is a switching controller, which switches at the same time instants $t_i$ as the reference inputs. Therefore, for its implementation implicitly it is assumed that the switching signal $\sigma(t)$ is known. In other words, it is assumed that the sequence of switching instants $\{t_i\}$ and the sequence of parameter indices $\{p_i\}$ are known. Moreover, given a switching signal $\sigma(t)$ with a switching sequence $\{p_i\}$ it is required that the corresponding mappings $x = \Pi_{p_i} w$ and $u = K_{p_i} w$ are known a priory for all $p_i$ in $\sigma(t)$.

The statement that (9) is required to be satisfied only for $t_i \geq \bar{t}$ means that it is allowed not be satisfied for a finite number of times initially. This does not prevent achieving the objective $e(t) \to 0$ as $t \to \infty$. However, at the instants it is not satisfied there might be jumps in the output error. Note also that the assumption that

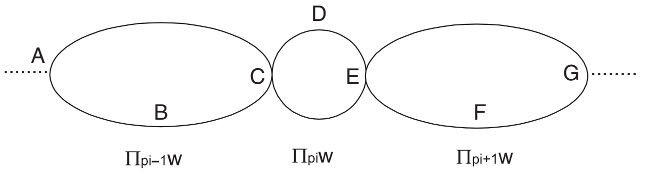


Figure 1. Zero error manifolds.

$\Pi_{p_{i-1}} w(t_i) = \Pi_{p_i} w(t_i)$ means that for the given switching signal $\sigma(t)$ at the switching instants the zero-error manifolds of two subsequent systems "touch" each other resulting in a continuous manifold $x = \Pi_\sigma w(t)$ for the switched system. We illustrate this with a simple drawing shown in figure 1. The first ellipse in the figure is the zero-error manifold for the system under the parameters for index $p_{i-1}$ and if there were no switchings the systems response would remain on that manifold moving with some periodicity based on the dynamics of $w$. With the switching of the system dynamics at $t_i$'s the system switches also the manifold. However, the requirement that $\Pi_{p_{i-1}} w(t_i) = \Pi_{p_i} w(t_i)$ guarantees that this switch does not create discontinuities and results in a "continuous reference trajectory" for the system (e.g., the "...A-B-C-D-E-F-G..." path in the figure). This is important since, as one can recall the Francis–Isidori–Byrnes framework is limited to systems with exogeneous inputs generated by neutrally stable exosystems. Therefore, within their framework without switching only constant or periodic reference trajectories can be tracked or such disturbances rejected. The above result suggests that this limitation can be overcome with appropriate switching. In other words, one might appropriately design the switching exosystem generating the reference inputs and determine an appropriate switching sequence so as to achieve tracking of some desired non-periodic reference trajectories. Usually we would not expect the disturbances to switch from one exosystem to another. However, if they do and the sequences of switching instants and indexes are known, the procedure will still allow tracking.

The procedure here can handle both signals which are switching finitely many times and those switching constantly and each exosystem is active for only a certain time. For systems in which there are only finite number of switchings a controller designed based on the last exosystem will also achieve the regulation objective. However, if such a controller is employed there might be large errors initially (before the last switch). If the controller discussed here is applied, on the other hand, and condition (9) is always satisfied regulation will be achieved even before the last switching. The beauty here is that asymptotic regulation is guaranteed also for systems with infinitely many switchings.

Note also that the two ellipses in figure 1 could have been generated by the same system $G_p$ due to different initial conditions. In other words, in figure 1 it might have been the case that $p_{i-1} = p_{i+1}$. Therefore, it is not required to have an infinite set of parameterized exosystems and with only a limited finite set of systems $\mathcal{P}$ and an appropriate switching sequence it is possible to generate many different reference trajectories. Then, using the procedure discussed here we can also track these trajectories.

Note that in Theorem 1 the requirement that Assumption 1 holds could be relaxed to the statement: "Given a switching signal $\sigma(t)$ with the corresponding switching sequence $\{p_i\}$, the output regulation problem is solvable using full information for all $p_i$." This relaxation will not lead to any change in the statement, the proof or the consequences of the theorem.

Finally, we would like to mention that the result in Theorem 1 (with Assumption 1 satisfied) will still hold for arbitrary switching among $p \in \mathcal{P}$ satisfying Assumption 1. However, in that case (9) will essentially mean that $\Pi_p = \Pi_q$ for all $p, q \in \mathcal{P}$ implying that all parameterized systems possess the same zero error manifold. This may seem a limitation since it will imply that for arbitrary switching the main objective of generating and tracking more complex trajectories may not be achieved although the procedure will still allow for switching controllers due to the $K_p$ term. (The issue of whether one can find a non-switching controller and under which conditions it is possible to do so is not clear and may need to be investigated further. However, it is outside the scope of this article.) We would like to also emphasize that our objective here is not to investigate the system under arbitrary switching and develop new results on switching systems. Here we use switching as a design flexibility or as a mean to overcome a limitation of the classical output regulation framework.

As stated before, the signal $w$ contains both the reference and disturbances and is generated by (2). In the classical output regulation framework they are constants or periodic signals, while here they are possibly non-periodic continuous signals consisting of connected pieces of constant or periodic signals. In the full information case these signals are assumed to be known. Moreover, since the switching sequence is also known, computing the condition in (9) is fairly easy. In fact, from (9) one can see that it is satisfied if

$$w(t_i) \in \mathcal{N}\left(\Pi_{p_{i-1}} - \Pi_{p_i}\right)$$

or basically $w(t_i)$ is in the null space of the matrix $\left(\Pi_{p_{i-1}} - \Pi_{p_i}\right)$. Then given the indices $p_{i-1}$ and $p_i$ and the corresponding matrices $\Pi_{p_{i-1}}$ and $\Pi_{p_i}$ one can compute the values of $w(t)$ at which switching can occur and check at the switching instances $t_i$ whether it is satisfied. This can be used also as a design guideline and force the system to switch the reference trajectory and the corresponding controller at the instances at which (9) is satisfied (as will be done later in the numerical example).

## 4. Error feedback controller

In this section we assume that the exosystem state $w(t)$ is not available and only the output error $e(t)$ is measurable. Similarly to the full information controller case we assume that switching occurs among a family of systems for which the error feedback output regulation problem is solvable.

**Assumption 2:** *For all $p \in \mathcal{P}$ the output regulation problem is solvable using error feedback.*

As in the case of full information given a $p \in \mathcal{P}$ Assumption 2 still guarantees the existence of mappings $x = \Pi_p w$ and $u = K_p w$ with the matrices $\Pi_p$ and $K_p$ satisfying (4). Moreover, it guarantees that for each $p \in \mathcal{P}$, through a transformation $\xi = T_p w$, the system with outputs

$$\begin{aligned} \dot{w} &= G_p w, \\ u &= K_p w, \end{aligned}$$

is immersed into the system

$$\begin{aligned} \dot{\xi} &= \Phi_p \xi, \\ u &= \Gamma_p \xi, \end{aligned}$$

where the pair $(\Gamma_p, \Phi_p)$ is in observable canonical form see (Isidori (1995)). (Note that, since the system is linear, this would be satisfied even without Assumption 2.) Furthermore, $\Phi_p$ and $\Gamma_p$ are such that the pair

$$\begin{bmatrix} A & 0 \\ N_p C & \Phi_p \end{bmatrix}, \quad \begin{bmatrix} B \\ 0 \end{bmatrix}, \tag{10}$$

is stabilizable for some choice of $N_p$ and the pair

$$\begin{bmatrix} C & 0 \end{bmatrix}, \quad \begin{bmatrix} A & B\Gamma_p \\ 0 & \Phi_p \end{bmatrix}, \tag{11}$$

is detectable. With all these in mind, given $p \in \mathcal{P}$, the controller which solves the problem is given by

$$\left.\begin{aligned} \dot{\xi} &= \Phi_p \xi + N_p e, \\ \dot{\chi} &= \Psi_p \chi + L_p e, \\ u_p &= \Gamma_p \xi + M_p \chi, \end{aligned}\right\} \tag{12}$$

where the matrices $\Psi_p$, $L_p$, and $M_p$ are chosen such that the closed loop system is asymptotically stable, i.e., the matrix

$$\bar{A}_p = \begin{bmatrix} \begin{bmatrix} A & B\Gamma_p \\ N_p C & \Phi_p \end{bmatrix} & \begin{bmatrix} B \\ 0 \end{bmatrix} M_p \\ L_p \begin{bmatrix} C & 0 \end{bmatrix} & \Psi_p \end{bmatrix}$$

is Hurwitz or basically has its eigenvalues in the open left half plane.

An issue to be noticed here is that the existence of $\Pi_p$ and $K_p$, which solve (4), is still necessary for the solvability of the error feedback controller. (Recall, that their existence is guaranteed by Assumption 2.) From the immersion we know that

$$K_p w = \Gamma_p \xi = \Gamma_p T_p w$$

implying that

$$K_p = \Gamma_p T_p.$$

Then, at the manifold the following equations are satisfied

$$\left.\begin{aligned} \Pi_p G_p &= A\Pi_p + E_p + B\Gamma_p T_p, \\ T_p G_p &= \Phi_p T_p, \\ 0 &= C\Pi_p + F_p. \end{aligned}\right\} \tag{13}$$

Note that the triple $(\Pi_p w, T_p w, 0)$ serves as a zero-error manifold for the closed-loop system to which the triple $(x, \xi, \chi)$ must converge.

Define

$$\bar{x}_p = \begin{bmatrix} x - \Pi_p w \\ \xi - T_p w \\ \chi \end{bmatrix} = \begin{bmatrix} \tilde{x}_p \\ \tilde{\xi}_p \\ \chi \end{bmatrix}.$$

Then, with a simple manipulation the system can be converted into the form

$$\left.\begin{aligned} \dot{\bar{x}}_p &= \bar{A}_p \bar{x}_p, \\ e &= \bar{C}\bar{x}_p = C\tilde{x}_p, \end{aligned}\right\} \tag{14}$$

where $\bar{C} = [C^\top, 0, 0]^\top$. Since $\bar{A}_p$ is Hurwitz, provided that $\sigma(t) = p$ for all time, we have $\bar{x}_p \to 0$ as $t \to \infty$ exponentially fast implying the same for the output error $e(t)$. In other words, given an index $p \in \mathcal{P}$ and the controller in (12), the zero-error manifold $(\Pi_p w, T_p w, 0)$ of the closed-loop system is exponentially attractive.

As in the case of full information, since $\bar{A}_p$ is Hurwitz, there exist positive definite matrices $P_p$ and $Q_p$ such that

$$\bar{A}_p^\top P_p + P_p \bar{A}_p \leq -Q_p, \tag{15}$$

and

$$V_p(\bar{x}_p) = \bar{x}_p^\top P_p \bar{x}_p, \tag{16}$$

is a Lyapunov function for the closed-loop system given the particular index $p \in \mathcal{P}$. The difference from the full information case is that here $\bar{A}_p$ and therefore $P_p$ and $Q_p$ depend on $p$, which was not the case before. Other issues to note from (15) and (16) are that we have

$$a_p \|\bar{x}_p\|^2 \leq V_p(\bar{x}_p) \leq b_p \|\bar{x}_p\|^2, \tag{17}$$

and

$$\dot{V}_p(\bar{x}_p) \leq -c_p \|\bar{x}_p\|^2, \tag{18}$$

where $a_p = \lambda_{\min}(P_p)$, $b_p = \lambda_{\max}(P_p)$, and $c_p = \lambda_{\min}(Q_p)$. Moreover, for any $x$ and any $p \in \mathcal{P}$ and $q \in \mathcal{P}$ we have

$$V_p(x) \leq \mu V_q(x), \tag{19}$$

where

$$\mu = \sup_{p,q \in \mathcal{P}} \left\{ \frac{b_p}{a_q} \right\}.$$

Here $\mu$ is finite since $\mathcal{P}$ is a compact set.

Define $\bar{x}_\sigma = \bar{x}_p$ and $V_\sigma(\bar{x}_\sigma) = V_p(\bar{x}_p)$ for $t \in [t_i, t_{i+1})$. Since $P_p$ in $V_p(\bar{x}_p)$ depends on $p$ the function $V_\sigma(\bar{x}_\sigma)$ can incur jumps at the switching instants and therefore cannot directly serve as a common Lyapunov function for the switched system. For this reason, stability is not directly implied. There are articles in the switching systems literature which deal with determining conditions for existence of a common Lyapunov function for systems in the form of (14). See for example the survey in Liberzon and Morse (1999) and the references therein. Such conditions put further restrictions on the parameterized family $\bar{A}_p$. Instead, since here we are not concerned with arbitrary switching, the fact that the jumps are bounded and a simple restriction on the switching signal $\sigma(t)$ in terms of the frequency of switching are sufficient for achieving stability and regulation.

Before stating the main result of this section, we need to introduce certain classes of switching signals. These definitions are taken mainly from Hespanha and Morse (1999a,b). A switching signal is said to possess the property of dwell-time $\tau_D$ if the interval between any two consecutive switchings satisfy $\tau_i \triangleq t_{i+1} - t_i \geq \tau_D$. We will denote with $S[\tau_D]$ the set of all switching signals with dwell-time $\tau_D$. For each switching signal $\sigma$ and each $t \geq \tau \geq 0$, let $N_\sigma(t,\tau)$ denote the number of discontinuities of $\sigma$ in the open interval $(\tau, t)$. For given positive constants $\tau_D$ and $N_0$ the switching signals satisfying

$$N_\sigma(t,\tau) \leq N_0 + \frac{t-\tau}{\tau_D},$$

are said to possess the property of average dwell-time $\tau_D$ with chatter bound $N_0$. We will denote with $S_{ave}[\tau_D, N_0]$ the set of all signals having this property. Note that we have $S[\tau_D] \subset S_{ave}[\tau_D, 1]$. Now, we have the following result.

**Theorem 2:** *Consider the output regulation problem of the system in* (3) *using error feedback. If Assumption* 2 *holds, then given a piecewise constant switching signal* $\sigma : [0,\infty) \to \mathcal{P}$ *the output regulation problem is solved by the switching controller*

$$\left.\begin{aligned} \dot{\xi} &= \Phi_\sigma \xi + N_\sigma e, \\ \dot{\chi} &= \Psi_\sigma \chi + L_\sigma e, \\ u_\sigma &= \Gamma_\sigma \xi + M_\sigma \chi, \end{aligned}\right\} \tag{20}$$

*if there exists a finite time* $\bar{t}$ *such that at all switching instants* $t_i \geq \bar{t}$ *we have*

$$\left.\begin{aligned} \Pi_{p_{i-1}} w(t_i) &= \Pi_{p_i} w(t_i), \\ T_{p_{i-1}} w(t_i) &= T_{p_i} w(t_i), \end{aligned}\right\} \tag{21}$$

*at the switching instants* $t_i$ *and* $\sigma(t) \in S_{ave}[\tau_D, N_0]$ *for some finite positive constants* $N_0$ *and* $\tau_D$.

**Proof:** The assumptions that $\Pi_{p_{i-1}} w(t_i) = \Pi_{p_i} w(t_i)$ and $T_{p_{i-1}} w(t_i) = T_{p_i} w(t_i)$ guarantee that $\bar{x}_\sigma$ is continuous. From Assumption 2 we can conclude that for each $p \in \mathcal{P}$ an error feedback controller in (12) exists and that $\bar{A}_p$ is Hurwitz. Then, for each $p \in \mathcal{P}$ $V_p(\bar{x}_p)$ in (16) is a Lyapunov function satisfying (17), (18), and (19) for all $p, q \in \mathcal{P}$. Applying Theorem 2 in Hespanha and Morse (1999a,b) we obtain the result. □

The reasoning behind the proof of the mentioned result from Hespanha and Morse (1999a,b) is basically as follows. The Lyapunov function $V_\sigma(\bar{x}_\sigma)$ may possess discontinuous jumps at the switching instants. However, (19) bounds the relative amount of the possible jumps and (17) and (18) guarantee that the Lyapunov function will decrease exponentially fast between consecutive jumps. Then, if the switching frequency is bounded disallowing infinitely fast switching and allowing for the system to *dwell* for some time for each parameter $p$ during which exponential decay occurs, then exponential stability of the switched system (i.e., the system in (14) where $p$ is replaced with $\sigma$) is still guaranteed which implies that $e(t) \to 0$ as $t \to \infty$ despite the switching. Since the switching signals with dwell-time are a subset of the signals with average dwell-time, i.e., since $S[\tau_D] \subset S_{ave}[\tau_D, 1]$, the above result automatically holds if $\sigma(t)$ possesses a dwell-time property $\tau_i \geq \tau_D$ for some $\tau_D$. In fact, note from (17) and (18) that in each interval $[t_i, t_{i+1})$ the Lyapunov function $V_\sigma(\bar{x}_\sigma)$ is exponentially decaying and at the switching instants satisfies

$$V_\sigma(\bar{x}_\sigma(t_{i+1})) \leq \frac{b_{p_{i+1}}}{a_{p_i}} e^{-(c_{p_i}/b_{p_i})\tau_i} V_\sigma(\bar{x}_\sigma(t_i)).$$

Therefore, if we have

$$\tau_i > \frac{b_{p_i}}{c_{p_i}} \ln\left(\frac{b_{p_{i+1}}}{a_{p_i}}\right),$$

then we will have

$$V_\sigma(\bar{x}_\sigma(t_{i+1})) < V_\sigma(\bar{x}_\sigma(t_i)).$$

Define

$$\tau^* = \sup_{p,q \in \mathcal{P}} \left\{ \frac{b_p}{c_p} \ln\left(\frac{b_q}{a_p}\right) \right\}.$$

Then, if $\tau_i \geq \tau^*$ for all $i$, the Lyapunov function $V_\sigma(\bar{x}_\sigma)$ will be decaying independent of the switching index sequence $\{p_i\}$ and the system will be asymptotically stable.

Note that it is possible to decrease the value of $\tau^*$ by selecting appropriate controller parameters. In particular, for each $p$ one can choose the triple of matrices $(M_p, \Psi_p, L_p)$ in the controller (12) to assign the eigenvalues of $\bar{A}_p$. This will result in different $Q_p$ and $P_p$, and therefore in a different $\tau^*$.

As in the full information case from (21) we see that in order for the condition to be satisfied we need

$$w(t_i) \in \mathcal{N}\left(\Pi_{p_{i-1}} - \Pi_{p_i}\right) \cap \mathcal{N}\left(T_{p_{i-1}} - T_{p_i}\right)$$

or basically $w(t_i)$ has to be in the intersection of the null spaces of the matrices $\left(\Pi_{p_{i-1}} - \Pi_{p_i}\right)$ and $\left(T_{p_{i-1}} - T_{p_i}\right)$. This information could be used to check condition (21) or to determine the allowed switching instances $t_i$.

As in the full information case in Theorem 2 the requirement that Assumption 2 holds could be relaxed to the statement: "Given a switching signal $\sigma(t)$ with the corresponding switching sequence $\{p_i\}$, the output regulation problem is solvable using error feedback for all $p_i$." This relaxation will not lead to any change in the statement, the proof or the consequences of the theorem.

Finally, note that we assumed that in the system in (1) only $E_p$ and $F_p$ depend on $p$. The procedure discussed will still hold even if the other system matrices depend on $p$. In particular, if $A$ and/or $B$ depend on $p$, then in order to achieve output regulation one will need $\sigma(t)$ to satisfy a dwell-time property for both the full information and the error feedback controllers.

We would like also to emphasize that the results developed in this article can easily be extended to problems with exosystem with piecewise constant inputs discussed in Krener (1992). Then, appropriate combination of a set of piecewise constant inputs and switching will lead to zero-error manifolds similar to the one in figure 1 and can be used to generate and track many reference trajectories.

## 5. Simulation examples

In this section we will provide numerical simulations. In particular, first we consider the system

$$\left.\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ e &= \begin{bmatrix} 3 & 2 \end{bmatrix} x - \begin{bmatrix} 1 & 0 \end{bmatrix} w \end{aligned}\right\} \tag{22}$$

which is taken from Nikiforov (1997). Note that the eigenvalues of the system matrix have positive real parts and the system is unstable. The reference trajectories are assumed to be generated by (2) where $p \in \{1, 2, 3\}$ and

$$G_1 = \begin{bmatrix} 0 & \alpha_1 \\ -\alpha_1 & 0 \end{bmatrix}, G_2 = \begin{bmatrix} 0 & -\alpha_2 \\ \alpha_2 & 0 \end{bmatrix}, \text{ and } G_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Basically the reference trajectory is generated by a switched combination of a constant and two sinosoids with two different frequencies. Consider the full information case. Solving the Francis equations for each $p$ one can easily calculate

$$\Pi_1 = \begin{bmatrix} \dfrac{3}{4\alpha_1^2+9} & \dfrac{-2\alpha_1}{4\alpha_1^2+9} \\ \dfrac{2\alpha_1^2}{4\alpha_1^2+9} & \dfrac{3\alpha_1}{4\alpha_1^2+9} \end{bmatrix} \text{ and}$$
$$K_1 = \begin{bmatrix} \dfrac{-5\alpha_1+12}{4\alpha_1^2+9} & \dfrac{2\alpha_1^3-11\alpha_1}{4\alpha_1^2+9} \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} \dfrac{3}{4\alpha_2^2+9} & \dfrac{2\alpha_2}{4\alpha_2^2+9} \\ \dfrac{2\alpha_2^2}{4\alpha_2^2+9} & \dfrac{-3\alpha_2}{4\alpha_2^2+9} \end{bmatrix} \text{ and}$$
$$K_2 = \begin{bmatrix} \dfrac{-5\alpha_2+12}{4\alpha_2^2+9} & \dfrac{-2\alpha_2^3+11\alpha_2}{4\alpha_2^2+9} \end{bmatrix},$$

$$\Pi_3 = \begin{bmatrix} \dfrac{1}{3} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } K_3 = \begin{bmatrix} \dfrac{4}{3} & 0 \end{bmatrix}.$$

For switching between two different indexes we need (9) to be satisfied from which one easily obtains the conditions (Here $(p=i) \Leftrightarrow (p=j)$ means switching from index i to j and vice versa)

$$(p=1) \Leftrightarrow (p=3) \Rightarrow w_2 = -\left(\frac{2\alpha_1}{3}\right) w_1,$$

$$(p=2) \Leftrightarrow (p=3) \Rightarrow w_2 = \left(\frac{2\alpha_2}{3}\right) w_1,$$

$$(p=1) \Leftrightarrow (p=2) \text{ and } \alpha_1 = \alpha_2 \Rightarrow w_2 = 0, w_1 = \text{ any value},$$

$$p=1 \Leftrightarrow p=2 \text{ and } \alpha_1 \neq \alpha_2 \Rightarrow w_2 = \left(\frac{6(\alpha_2-\alpha_1)}{4\alpha_1\alpha_2+9}\right) w_1.$$

In other words, if we switch between index $p=1$ and $p=3$ and vice versa when $w_2 = -(2\alpha_1/3)w_1$, then condition (9) will be satisfied and no discontinuity in the state will occur rendering the conditions of Theorem 1 satisfied. The same holds for the other cases as well.

Figure 2 shows the trajectories of a simulation for the index sequence $\{1, 3, 1, 2\}$. The frequencies used for the sinusoids are $\alpha_1 = 1$ and $\alpha_2 = 3$ and the simulation is

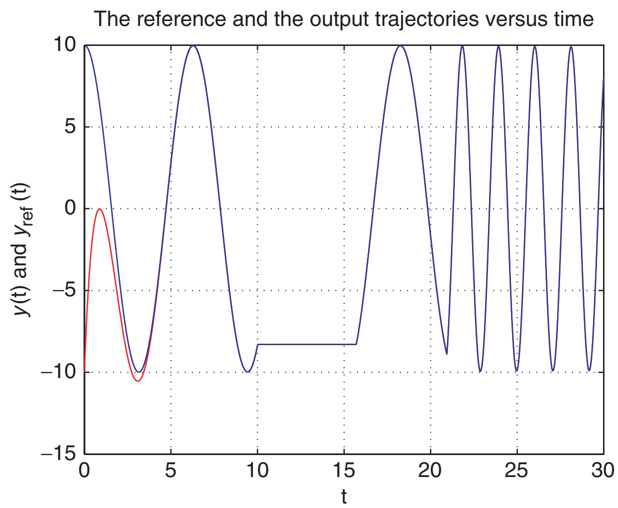


Figure 2. The reference and actual outputs.

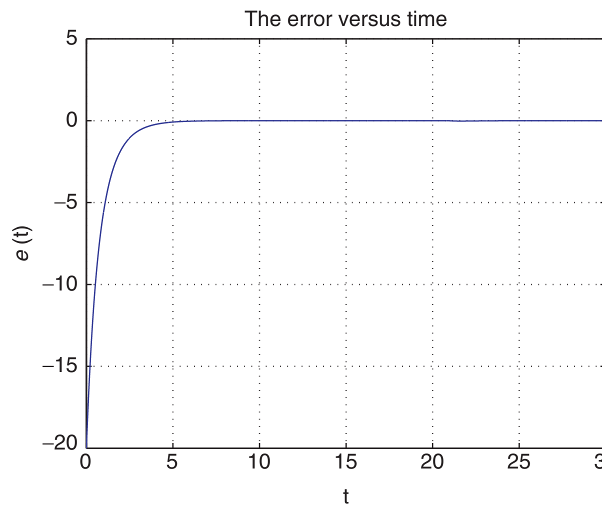


Figure 3. The ouput error for proper switching.

run for 30 sec. The switching in this simulation was performed at the instants where the above conditions were satisfied. As one can see the output ($y = Cx = \begin{bmatrix} 3 & 2 \end{bmatrix} x$) of the system can perfectly track the reference trajectory ($y_{ref} = Fw = \begin{bmatrix} 1 & 0 \end{bmatrix} w$). In fact, the output error $e(t)$ converges exponentially fast to zero as shown in figure 3. We would like to mention that if at the switching instants (9) is not satisfied, then a new transient occurs after the switching as shown in figure 4 where we intentionally switched at a different time instant (see the bump near $t = 20$ sec). This may not be a problem if the switching signal has a sufficiently large dwell time since the system is exponentially stable in between. However, it may lead to instability in fast switching systems.

In the simulations here we used only finite sequence of indexes. However, the procedure will hold also for constantly switching signals. It is not possible to simulate signals switching infinitely many times. Moreover, even finitely switching sequences illustrate the effectiveness of the method since there are no discontinuities at the output error at the switching instants and it decays to zero exponentially fast.

Below we provide two more simulations examples for higher dimensional systems. For these simulations we omit the corresponding manifold and the controller equations. The first simulation is a system whose two independent dimensions are composed of systems given in (22). However, the exosystems in this case are different from the previous example. Figure 5 shows an example trajectory with proper switching applied. As one can see it is a non-periodic trajectory which cannot be tracked by the classical non-switching

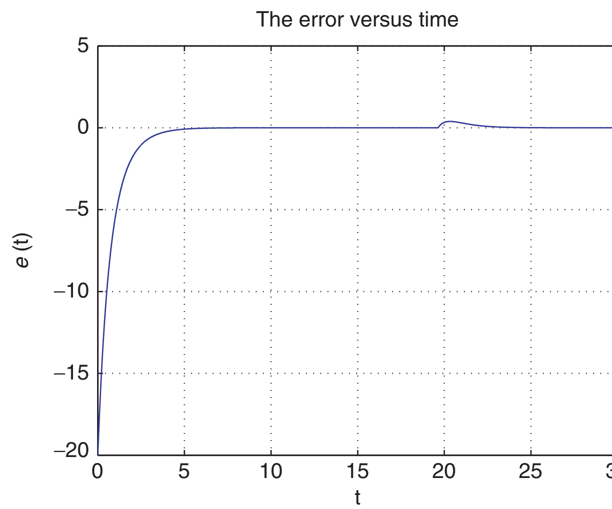


Figure 4. The output error for improper switching.

output regulation methods. The output error exponentially converges to zero.

The condition in (9) puts a restriction on switching and if zero error for all time is desired then only at instants or points at which it is satisfied one can switch the controller. However, if small errors are tolerable then it is possible to switch the controller at other time instants too. (This does not mean that we are switching between controllers for arbitrary parameters $p$ since this may lead to large errors during transient. We mean that, for example, for switching from $p_i$ to $p_{i+1}$ we may not strictly enforce (9) or basically do not switch at exactly $t_{i+1}$ but may be switch little before or little after it. The result for improper switching was illustrated in

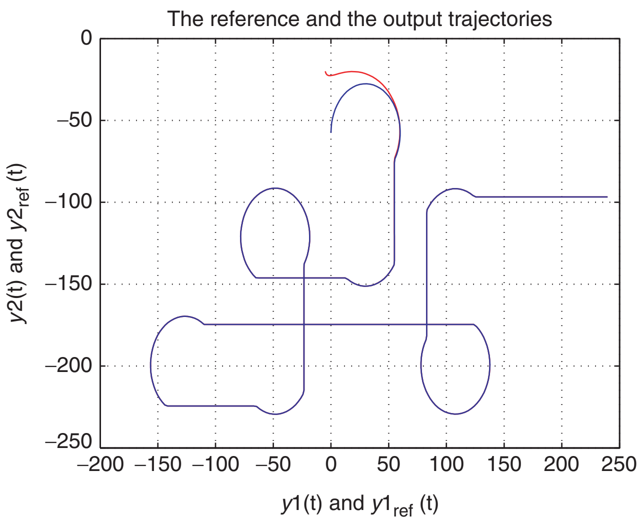


Figure 5. The reference and the actual outputs.

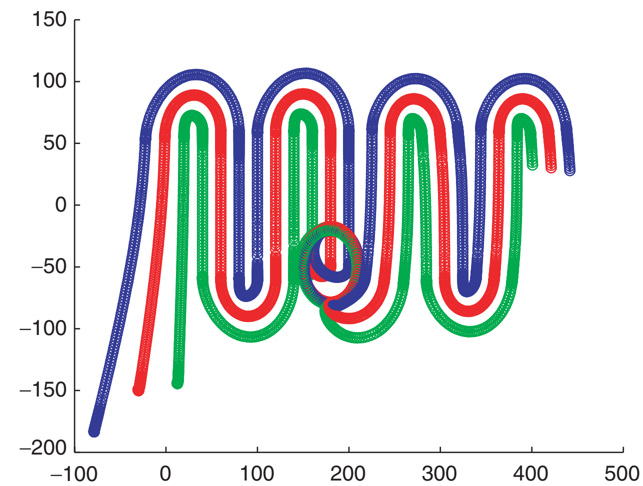

Figure 6. Three robots search an area in a formation.

figure 4.) Figure 6 shows a simulation for which we did not enforce (9). It is a simulation of three robots searching an area in a formation similar to the simulations in Gazi (2004). The trajectory is generated by a switching exosystem. The circle in the middle of the trajectory represents the behavior of encircling an object found to get a better look at it. When the procedure is applied to robot control each index $p$ may represent a different "behavior" to be performed by the robot. This is just to illustrate that the applicability of the method might be wider than initially thought. Moreover, note also that given starting and destination points and starting and destination orientations the corresponding minimum length path is a combination of arcs of circles and straight lines (Dubins 1957) and the procedure here might be applicable to follow/trace such paths. However, in order to be able to state this with certainty further considerations and research is needed and is out of the scope of this article.

## 6. Concluding remarks

In this article we considered output regulation of systems with switched external inputs under the assumption of simultaneous switching in the controller and the exosystem. The results obtained are relevant since they allow generating and tracking more complex trajectories, a limitation of the classical output regulation literature. In our analysis we applied existing results from the switching systems literature. The procedure discussed provides a more flexible design framework for linear output regulation controllers. The ideas from this article can be extended to nonlinear problems as well. This could be an avenue for future research direction. Another direction for future research could be to derive conditions for successful output regulation under arbitrary switching.

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