NAME (PRINT):

Last/Surname First /Given Name

STUDENT #: **SIGNATURE:**

UNIVERSITY OF TORONTO MISSISSAUGA
DECEMBER 2018 FINAL EXAMINATION
STA314H5F
Introduction to Statistical Learning
Stanislav Volgushev
Duration - 2 hours
Aids: None

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*Please note, once this exam has begun, you **CANNOT** re-write it.*

Marks possible: 45 **Marks achieved**:

Problem	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Possible	2	1	1.5	1	1	1	1	1.5	3	4	2.5	1.5	1	3	3	4	1	1	1	1
Achieved																				

Problem	21	22	23	24	25
Possible	1	1	1	4	2
Achieved					

1. (0.5 marks each) Assume that `D` is a data frame that contains 4 columns with names `Y`, `X`, `V`, `W`. For each of the following specifications, describe the regression function that corresponds to the `lm` call in **R**. (you may use terms such as step function, spline, piecewise polynomial, local polynomial regression, smoothing spline etc.)

   (a) `lm( Y ~ . -V - W, data = D)`
   $f(x, v, w) = b_1 + b_2 x$, a linear function in x

   (b) `lm( Y ~ V + X + V:W, data = D)`
   $f(x, v, w) = b_1 + b_2 v + b_3 x + b_4 vw$

   (c) `lm( Y ~ X + bs(W,knots=c(1,3),degree = 2), data = D)`
   An additive model with linear effect of X and effect of W described by a polynomial spline of degree 2 with knots in $1, 3$.

   (d) `lm( Y ~ ns(X,knots = c(2,5)) + I(V^3), data = D)`
   An additive model with terms V^3 and effect of W described by a natural cubic spline with knots in $2, 5$.

2. (0.5 marks each) Assume that `D` is a data frame that contains 4 columns with names `Y`, `X`, `V`, `Z`. For each of the following regression functions, decide if they can be formulated as a linear regression in **R** (here, $b_1, ..., b_3$ are unknown). If yes, write the **R** call you would use.

   (a) $f(x, v, z) = b_1 + b_2 z + b_3 vz$

   `lm( Y ~ Z + V:Z, data = D)`

   (b) $f(x, v, z) = b_1 + b_2\sqrt{z + b_3 v}$
   Not possible

3. *Running a linear regression in* ***R*** *and applying the summary function you get the following output*

```
Call:
lm(formula = y ~ x1 + x2 + x1:x2)

Residuals:
     Min       1Q   Median       3Q      Max
-0.29881 -0.07232 -0.01262  0.07602  0.28263

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -1.100000   0.011326  -1.048    0.297
x1           0.900000   0.011910  83.384   <2e-16 ***
x2          -0.050000   0.011095  -0.762    0.448
x1:x2        0.100000   0.011635  0.359     0.672
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.111 on 96 degrees of freedom
Multiple R-squared:  0.7929,    Adjusted R-squared:  0.7927
F-statistic:  4487 on 3 and 96 DF,  p-value: < 2.2e-16
```

(a) (1 mark) Given the output above, what is your prediction for a new observation with predictor values `x1 = 1, x2 = 2`? You don't need to simplify your answer, it is enough if you write down the correct formula.

Solution $-1.1 + 0.9 * 1 - 0.05 * 2 + 0.1 * 1 * 2$

(b) (0.5 marks) What is the value for R^2 in the model above?
0.7929

4. (1 mark) Which output will running the following code in **R** give?

```
x = array(0,3)
for(k in 1:length(x)){
 x[k] = k-3
}
which.min(x)
```

5. (1 mark) Which output will running the following code in **R** give (you don't need to simplify the answer, writing down the correct formula is enough)?

```
x = array(0,3)
for(k in 1:3){
 x[k] = k^2
}
x[-1]
```

6. (1 mark) Assume that you observe data (x_i, y_i) with values $(1,2), (2,3), (4,5), (0,2), (3,7)$. Compute the 3-nn estimator for $x = 10$.

   $(3 + 5 + 7)/3$

7. (1 mark) You run 12-fold cross-validation on a data set and obtain the following values for the cross-validated error and standard error for different numbers d of predictors in linear models selected by best subset selection.

d	0	1	2	3	4	5	6	7	8	9	10
cv(d)	13	12	7	6	2	3	4	1	6	15	30
$12^{-1/2}\widehat{se}(d)$	8	5	4	3	3	5	6	7	5	6	5

   Which d would you select based on the one standard error rule?

   $d = 2$

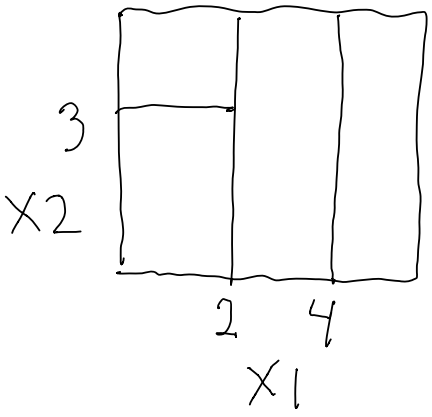
8. (1.5 marks) You run 3-fold cross-validation on a data set and obtain the following values for the cross-validated error and standard error for different values of m (number of principal components) in PCR.

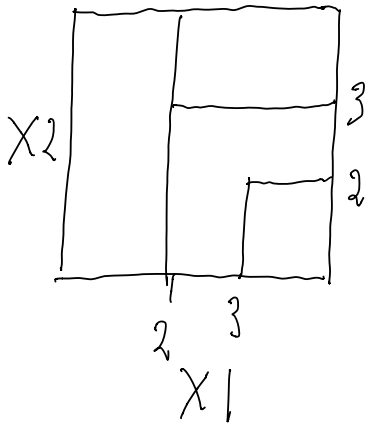
m	1	2	3	4	5	6	7	8	9	10
cv(m)	12	7	6	X	3	4	1	6	15	30
$3^{-1/2}\widehat{se}(m)$	15	14	13	13	8	6	7	5	16	15

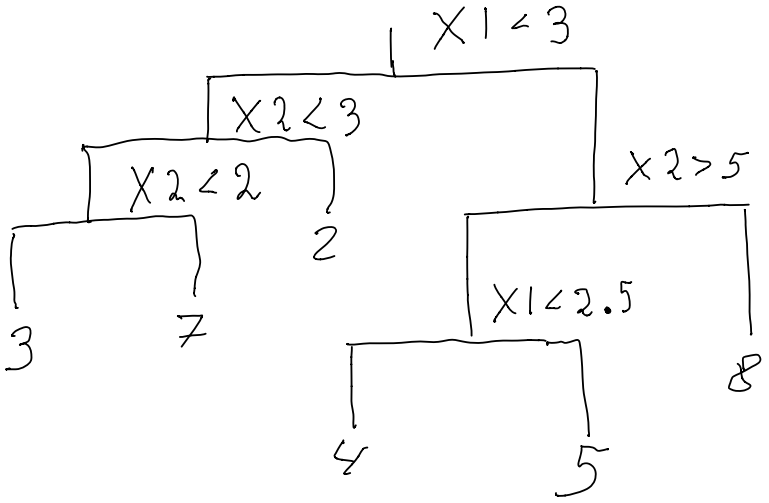
Is it possible to find a value for X such that $m = 1$ is selected by the one standard error rule in cross-validation? Justify your answer!

Answer: not possible

9. (1.5 marks each) For each of the two partitions of the predictor space given below, is it possible to represent them as a regression tree? If yes, draw the corresponding tree.







10. *For the regression tree shown above*

    (a) (1 mark) What is the predicted value for a new point with predictor X = (2, 1)?

    (b) (1.5 marks) Draw the corresponding partition of the predictor space next to the tree.

    (c) (1.5 marks) Assume that X = (1,$*$). Is it possible to find a value for $*$ so that the predicted value for a new point is 5? Justify your answer!

11. Consider a linear model with 3 predictors, A, B, C. The following table gives the RSS for each combination of predictors in a linear model.

none	A	B	C	A,B	A,C	B,C	A,B,C
7	3	4	5	2	2.5	1.5	1

(a) (0.5 marks) Which model with 2 predictors is selected by forward stepwise selection?

A,B

(b) (1 mark) Assume that additionally $\hat{\sigma}^2 = 0.6$ and $\log n = 2$. Which model will be selected by best subset selection with C_p?

B,C

(c) (1 mark) Assume that additionally $\hat{\sigma}^2 = 0.1$ and $\log n = 5$. Which model will be selected by backward stepwise selection with BIC?

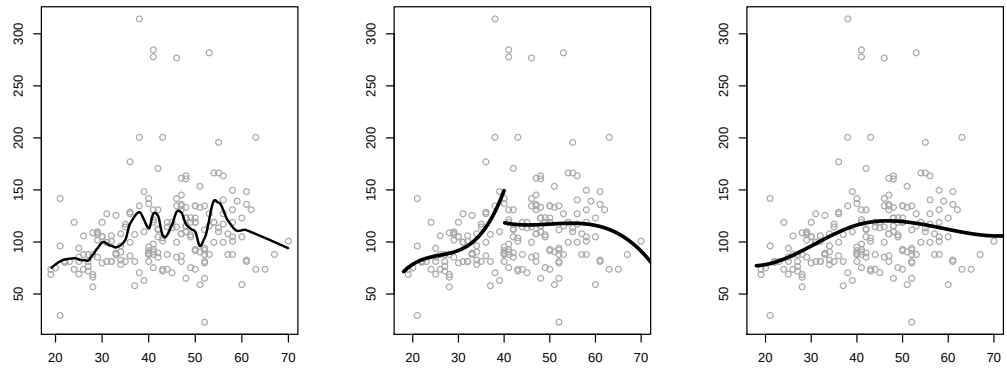
A,B,C or B,C (either answer would have been correct)

12. (1.5 marks) Assume you have 7 predictors. Will best subset selection with AIC and backward stepwise selection with AIC always result in the same model with 6 predictors? Justify your answer.

Answer: yes.

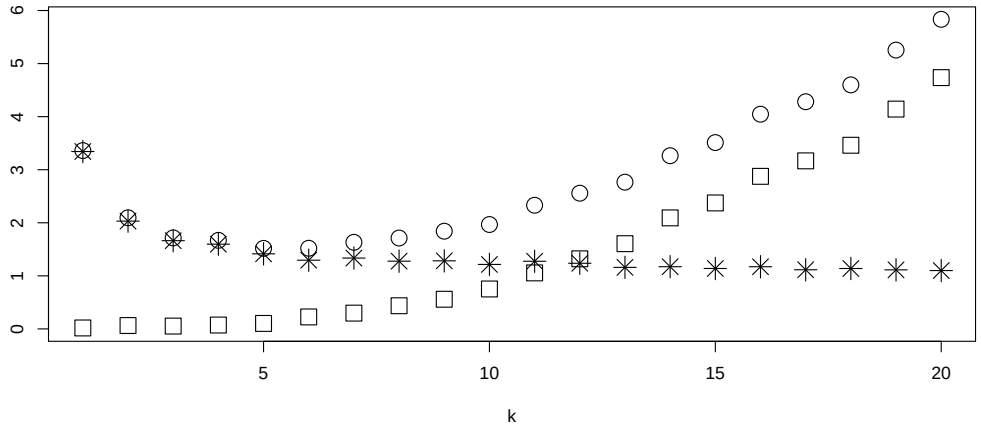
13. (1 mark) In the notation given in lectures, assume that $\tilde{X}^T\tilde{X}$ has eigenvectors v_1, v_2, v_3 corresponding to eigenvalues $2, 1.5, 3$. What are the first and second principal directions (principal components) in PCR?

First: v_3. Second: v_1.



14. (3 marks) The plots at the top of this page show the graphs of 3 different regression functions: a spline of degree 3 with 1 knot, a spline of degree 3 with 20 knots, and a piecewise polynomial of degree 3 with one knot. All functions were learned from the displayed data set by least squares. Which plot is which? Justify your answer.

    Answer: left plot: spline of degree 3 with 30 knots, middle plot: piecewise polynomial, right plot: spline with one knot.



15. (3 marks) The plot on top of this page shows squared bias, variance, and MSE of k-nn on a test set as a function of k. Which one is which? Justify your answer!

    Answer: circles are MSE, squares are squared bias, stars are variance.

16. Assume you want to model the influence of a predictor `x` on a response `y` by the regression function f which is a cubic spline (i.e. polynomial spline of degree 3) with a knot in the point 3.

(a) (1 mark) How many degrees of freedom (**counting the way we counted in lectures**) does this model have?
5

(b) (1 mark) Assume that `D` is data frame with columns `y` (response) and `x` (predictor). Which `R` input would you use to fit this model?

```
lm( Y ~ X + bs(W,knots=c(1,3),degree = 2), data = D)
```

(c) (2 marks) Write down functions $g_1, g_2, ..., g_d$ such that f is a a cubic spline with knots in the points 0 and 1 if and only if

$$f(x) = b_1 + b_2 g_1(x) + ... + b_{d+1} g_d(x)$$

for some $b_1, ..., b_{d+1} \in R$.
$g_1(x) = x, g_2(x) = x^2, g_3(x) = x^3, g_4(x) = (x-0)_+^3, g_5(x) = (x-1)_+^3$

17. (**1 mark if and only if ticks are only next to correct answers. Any number of answers (0-3) can be correct**) Which of the following methods can still give reasonable predictions when the number of predictors $p = 567$ is larger than the number of observations $n = 100$?

- ☐ PCR with number of principal components $m > n$.
- ☐ An additive model with polynomials of degree 3 for each predictor.
- ☐ Ridge regression with $\lambda = 0$.

18. (**1 mark if and only if ticks are only next to correct answers. Any number of answers (0-3) can be correct**) Compared to growing a large single tree, using bagging

- ☐ Can be better because it helps to reduce bias.
- X Can be better because it helps to reduce the variance.
- ☐ Will only be better if the number of trees is small.

19. (**1 mark if and only if ticks are only next to correct answers. Any number of answers (0-3) can be correct**) Using 5-fold cross validation with a data set of size $n = 30$ to select a tuning parameter

- X Will usually be computationally less expensive than leave-one-out cross-validation on the same data set.
- X Might give different answers since some randomness is involved.
- ☐ Does not make sense since $n > 5$.

20. (**1 mark if and only if ticks are only next to correct answers. Any number of answers (0-3) can be correct**) Which of the following changes correspond to *increasing* the flexibility of a model (usually this corresponds to decreasing the training error)

- ☐ Removing predictors from a linear model.
- ☐ Increasing λ in lasso.
- X Decreasing the span in local regression.
- ☐ Pruning back large regression trees.

21. (**1 mark if and only if ticks are only next to correct answers. Any number of answers (0-3) can be correct**) Interaction effects between the predictors x_1, x_2 in linear regression

- X Can be incorporated by including a term of the form bx_1x_2.
- ☐ Can be incorporated by including a term of the form $b(x_1 + x_2)$.
- ☐ Can not be incorporated since the model would become nonlinear.

22. (**1 mark if and only if ticks are only next to correct answers. Any number of answers (0-3) can be correct**) A high value of R^2 (a value close to one) in a linear regression model

- ☐ Means that the regression model is incorrect.
- ☐ Means that there is no relationship between the predictors and the response.
- ☐ Means that using the predictors to construct a linear model leads to better predictions than ignoring the predictors. **This is false because R^2 can be large even if there are many irrelevant predictors**

23. (**1 mark if and only if ticks are only next to correct answers. Any number of answers (0-3) can be correct**) Increasing the parameter λ in smoothing spline regression

- X Can increase the training error.
- X Will never decrease the training error.
- ☐ Will never affect the training error.

24. (4 marks) Consider a linear regression with one-dimensional predictor x and data $(x_i, y_i)_{i=1,\dots,n}$. Assume that the predictor is standardized such that $\bar{x} := \frac{1}{n}\sum_{i=1}^n x_i = 0$.

(a) (1 mark) Write down the minimization problem that correspond to ridge regression with parameter λ (recall: the intercept is not penalized).
See lectures.

(b) (3 marks) Compute the explicit form of the corresponding estimators $\hat{b}_1^R, \hat{b}_2^R$ from scratch (i.e. without using any results from lectures). The following notation might be useful:

$$\overline{xy} = \frac{1}{n}\sum_{i=1}^n x_i y_i, \quad \overline{y} = \frac{1}{n}\sum_{i=1}^n y_i, \quad \overline{x^2} = \frac{1}{n}\sum_{i=1}^n x_i^2$$

See derivations in lectures.

25. (2 marks) Write down the definition of RSS and TSS in a linear model with a one-dimensional predictor x (using formulas). Prove that $TSS \geq RSS(\hat{b})$ where $\hat{b}$ is the least squares estimator (the estimator we discussed in class) in the linear model **without using properties of R^2 that were derived in lectures.**

See derivations in lectures.

End of Exam