NAME (PRINT): ______________________________
Last/Surname First /Given Name

STUDENT #: ______________ **SIGNATURE:** ______________

UNIVERSITY OF TORONTO MISSISSAUGA
DECEMBER 2017 FINAL EXAMINATION
STA312H5F
Topics in Statistics: Applied Statistical Modelling
Stanislav Volgushev
Duration - 2 hours
Aids: None

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*Please note, once this exam has begun, you **CANNOT** re-write it.*

Marks possible: 45 **Marks achieved**:

Problem	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Possible	1.5	1.5	3	1	1	1	2	3	1	4	5	3	2	3	2.5	3	1	1	1	1	3.5
Achieved																					

1. (**0.5 marks each**) Assume that `D` is a data frame that contains 4 columns with names `Y`, `X`, `V`, `Z`. For each of the following specifications, write down the regression function that corresponds to the `lm` call in **R**. Example:

```
lm( Y ~ V, data = D)
```

corresponds to $f(x, v, z) = b_1 + b_2 v$.

   (a) `lm( Y ~ . - X, data = D)`
   $f(x, v, z) = b_1 + b_2 v + b_3 z$

   (b) `lm( Y ~ V +  V:Z, data = D)`
   $f(x, v, z) = b_1 + b_2 v + b_3 vz$

   (c) `lm( Y ~ I(X^2) + I(V^2) + Z, data = D)`
   $f(x, v, z) = b_1 + b_2 x^2 + b_3 v^2 + b_4 z$

2. (**0.5 marks each**) Assume that `D` is a data frame that contains 4 columns with names `Y`, `X`, `V`, `Z`. For each of the following regression functions, decide if they can be formulated as a linear regression in **R** (here, $b_1, ..., b_3$ are unknown). If yes, write the **R** call you would use.

   (a) $f(x, v, z) = b_1 + b_2 \sin(x + b_3)$
   Not possible

   (b) $f(x, v, z) = b_1 + b_2 z + b_3 x^2$

   ```
   lm( Y ~ Z + I(X^2), data = D)
   ```

   (c) $f(x, v, z) = b_1 + b_2 xv + b_3 x^7$

   ```
   lm( Y ~ X:V + I(X^7), data = D)
   ```

3. Running a linear regression in **R** and applying the summary function you get the following output

```
Call:
lm(formula = y ~ x1 + x2 + I(x2^2))

Residuals:
     Min       1Q   Median       3Q      Max
-0.29881 -0.07232 -0.01262  0.07602  0.28263

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.010000   0.011326  -1.048    0.297
x1          -0.010000   0.011095  -0.762    0.448
x2           0.900000   0.011910  83.384   <2e-16 ***
I(x2^2)      1.200000   0.011635  86.359   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.111 on 96 degrees of freedom
Multiple R-squared:  0.9929,    Adjusted R-squared:  0.9927
F-statistic:  4487 on 3 and 96 DF,  p-value: < 2.2e-16
```

(a) (**1 mark**) Given the output above, what is your prediction for a new observation with predictor values `x1 = 1, x2 = 2`? You don't need to simplify your answer, it is enough if you write down the correct formula.
$-0.01 - 0.01 * 1 + 0.9 * 2 + 1.2 * 2^2$

(b) (**0.5 marks**) What is the value for R^2 in the model above?
0.9929

(c) (**1.5 marks**) Given the output above, would it make sense to consider a linear model without the predictor `x1`? Why/why not?
Yes. The p-value of the test for the null 'predictor $x1$ is not helpful' is 0.448, so we can not reject the hypothesis that $x1$ is not helpful.

4. (**1 mark**) Assume that you observe data (x_i, y_i) with values $(1,2),(2,1),(3,3),(4,7)$. Compute the 2-nn estimator for $x = 5/4$.

   $(2+1)/2$

5. (**1 mark**) Which output will running the following code in **R** give?

```
x = array(0,3)
for(i in 1:length(x)){
 x[i] = i -2
}
x[c(1,3)]
```

6. (**1 mark**) Which output will running the following code in **R** give?

```
x = 0
for(i in 1:3){
 x = i-1
}
x
```

7. (**2 marks**) Consider the data set with observations (x_i, y_i) given by $(1,2),(3,4),(1.5,5)$, $(3.5,2),(4,0)$ (here y_i are outcomes and $x_i \in R$ are predictors). If you fit a model using a step function with intervals $(0,2],(2,5]$ as regression function, what is your prediction for a new observation with predictor $x = 3$? Show your computations to get full marks.

   Answer: $(4+2+0)/3$

8. The table below provides values of estimated coefficients in a linear model of the form

$$f(x) = b_1 + b_2 x$$

for either lasso or ridge regression for different values of λ (you will need to figure out which one). All estimators were computed on the same data set.

λ	0	1	2	3	10	20
$\widehat{b}_1$	1	1	1	1	1	1
$\widehat{b}_2$	2	1.5	1	0	0	0

(a) (**1 mark**) Based on this table, what is the usual least squares estimator for b_1, b_2 for this data set?
$\hat{b}_1 = 1, \hat{b}_2 = 2$

(b) (**2 marks**) Is this lasso or ridge regression (it is one of the two)? Justify your answer.
Answer: lasso.

9. (**1 mark**) The table below gives estimated MSE values (vie 5-fold cross-validation) for k-nn regression with different values of k. Which of the k in the table would you select based on cross-validation?

k	1	2	3	4	5	6	7
CV error	10	8	1.5	3	1	2.5	5

$k = 5$

10. Assume you want to model the influence of a predictor `x` on a response `y` by the regression function f which is a cubic spline (i.e. polynomial spline of degree 3) with knots in the points 0 and 1.

   (a) (**1 mark**) How many degrees of freedom (**counting the way we counted in lectures**) does this model have?

   6

   (b) (**1 mark**) Assume that `D` is data frame with columns `y` (response) and `x` (predictor). Which `R` input would you use to fit this model?

   ```
   lm( y ~ bs(x,knots=c(0,1),degree=3), data = D)
   ```

   (c) (**2 marks**) Write down functions $g_1, g_2, ..., g_d$ such that f is a a cubic spline with knots in the points 0 and 1 if and only if

   $$f(x) = b_1 + b_2 g_1(x) + ... + b_{d+1} g_d(x)$$

   for some $b_1, ..., b_{d+1} \in R$.

   $g_1(x) = x, g_2(x) = x^2, g_3(x) = x^3, g_4(x) = (x-0)^3_+, g_5(x) = (x-1)^3_+$

11. Consider a linear model with 3 predictors, A, B, C. The following table gives the RSS for each combination of predictors in a linear model.

none	A	B	C	A,B	A,C	B,C	A,B,C
6	4	4.5	3	1	2	2.5	0.5

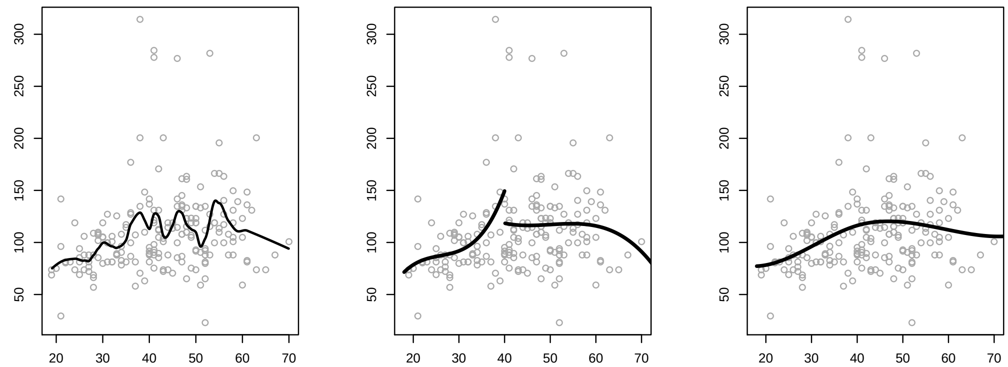
(a) (**1 mark**) Which model with 2 predictors is selected by best subset selection?
A,B

(b) (**1 mark**) Which model with 2 predictors is selected by forward stepwise selection?
A,C

(c) (**1 mark**) Which model with one predictor is selected by backward stepwise selection?
A

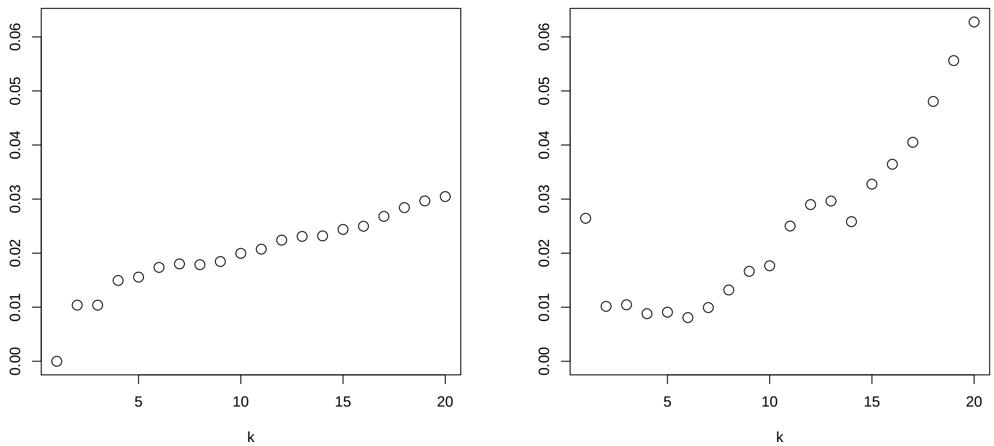
(d) (**2 marks**) Assume that additionally $\hat{\sigma}^2 = 1$ and $\log n = 10$. Which model will be selected by best subset selection with BIC penalty? Show your computations to get full marks.

Answer: model with no predictors



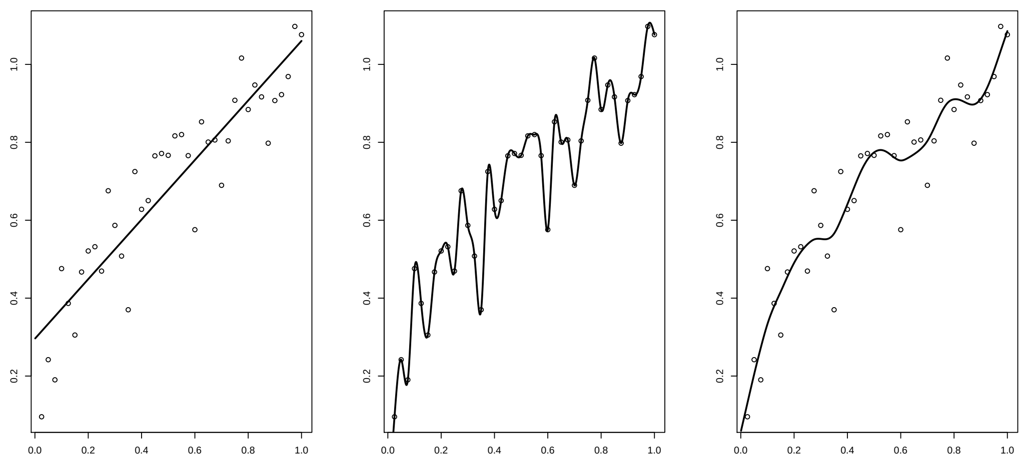
12. (**3 marks**) The plots at the top of this page show the graphs of 3 different regression functions: a spline of degree 3 with 1 knot, local regression, and a piecewise polynomial of degree 3 with one knot. All functions were fitted to the displayed data set by least squares. Which plot is which? Justify your answer.

    Answer: left plot: local regression, middle plot: piecewise polynomial of degree 3, right plot: spline of degree 3



13. (**2 marks**) The plots above show the test and training MSE for k-nn regression as a function of k. Which one is which? Justify your answer.

    Answer: left: training, right: test



14. (**3 marks**) The plots above show the training set (dots, same data in all plots) and smoothing splines corresponding to $\lambda = 0$, $\lambda = 1$, $\lambda = 100$. Which plot corresponds to which smoothing spline? Justify your answer.

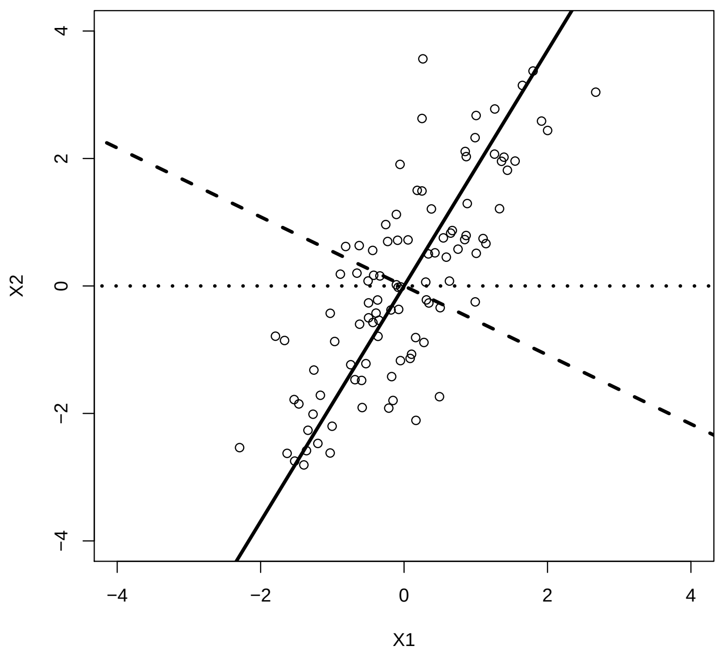
    Answer: left: $\lambda = 100$, middle: $\lambda = 0$, right: $\lambda = 1$

15. (**1.5 marks each**) For each of the two partitions of the predictor space given below, is it possible to represent them as a regression tree? If yes, draw the corresponding tree.

16. For the regression tree shown below

   (a) (**1 mark**) What is the predicted value for a new point with predictor $X = (1/4, 1)$?

   (b) (**1.5 marks**) Draw the corresponding partition of the predictor space.



17. (**1 mark**) Which of the lines in the plot above corresponds to the first principal component in PCR? (The circles correspond to predictors, predictors are two-dimensional with components `X1,X2`.)

    The solid line.

18. (**1 mark if and only if ticks are only next to correct answers. Any number of answers (0-3) can be correct**) Which of the following methods can still give reasonable predictions when the number of predictors p is larger than the number of observations n?

   X PCR with number of principal components smaller than n

   ☐ The usual least squares estimator with all predictors.

   X Ridge regression with suitably chosen value for λ.

19. (**1 mark if and only if ticks are only next to correct answers. Any number of answers (0-3) can be correct**) Compared to growing a large single tree, using bagging

   X Can be better because it helps to reduce variance.

   ☐ Will never be better.

   ☐ Will only be better if the number of trees is small.

20. (**1 mark if and only if ticks are only next to correct answers. Any number of answers (0-3) can be correct**) By pruning large trees we try to

   ☐ Reduce the bias.

   X Reduce the variance.

   ☐ Obtain a more flexible model.

21. (**Check the boxes next to correct statements. 0.5 marks for each correct box**) Which of the following changes correspond to *increasing* the flexibility of a model (usually this corresponds to decreasing the training error)

   ☐ Removing predictors from a linear model.

   ☐ Increasing λ in lasso.

   X Decreasing the span in local regression.

   ☐ Pruning back large regression trees.

   X Adding knots to a polynomial spline.

   X Increasing the degree of a polynomial spline while holding the knots fixed.

   X Choosing smaller k in k-nn regression.

END OF EXAM