Evasion as a Team against a Faster Pursuer

Shih-Yuan Liu Zhengyuan Zhou Claire Tomlin Karl Hedrick

***Abstract*— In this paper, we present an open-loop formulation of a single-pursuer-multiple-evader pursuit-evasion game. In this game, the pursuer attempts to minimize the total capture time of all the evaders while the evaders, as a team, cooperate to maximize this time. The information pattern considered here is conservative towards the evaders. One important advantage of this open-loop approach over the geometrical approach in the literature is that it provides guaranteed survival time of the evader team for all initial conditions, without the limitation that the pursuer must capture the evaders in a specific sequence. Another advantage of this approach is that under the open-loop framework, we can quickly generate controls for multiple players that the classical Hamilton-Jacobi-Isaacs (HJI) approach, due to its computational infeasibility, cannot handle. We also relax the conservatism inherent in this open-loop formulation by presenting an iterative open-loop scheme of the evaders' evasion strategy. Simulations for the open-loop, the iterative open-loop as well as the HJI approaches are presented, with the results on performance analyzed and discussed.**

I. Introduction

Autonomous agent teams have attracted a lot of attention in the past decades due to their wide-ranging applications. One application of particular interest is to deploy a team of autonomous agents, such as unmanned aerial vehicles (UAVs), to gather information from a region of interest as in [1]. One of the challenges faced by these autonomous agent teams is that the region of interest might be an adversarial environment where hostile agents reside and can take down the vehicles in the team. In a scenario where a hostile agent has been detected and is attacking the team, the autonomous agents face a single-pursuer-multiple-evader problem. There are various ways to formulate this problem depending on the capabilities of the agents and the cost functions. In [2] the pursuer has a bounded detection domain and the team of 2 evaders cooperates to maximize the minimum distance to the pursuer; [3] also follows a similar formulation and [4] solves the problem in a 3-dimensional space. In [5], a special cost function is designed to encourage the evaders to be on the opposite side of the pursuer. [6] gives a good review on various different cost functions that are commonly utilized in the single-pursuer-multiple-evader problem.

In this paper we consider the game in a 2-dimensional space with the cost function being the survival time of the last captured evader. We impose no constraints on the sensing capability of the agents so that each agent knows the exact locations of other agents at all time. In [7], one of the seminal works on this topic by Breakwell *et al*., it is shown that for some initial conditions the optimal strategy for the evaders are simply straight lines with maximum speed. The resulting trajectory of the optimal pursuer is made up of line segments connecting the capture points of the evaders. We call this solution the geometrical solution. An important implication is that if both the pursuer and the evaders act optimally, then throughout the pursuit-evasion process neither the optimal capture sequence nor the optimal headings for the evaders change. However, for other initial conditions the geometrical solution is not optimal. The optimal strategy has to be computed by backwards integration of the Hamilton-Jacobi-Isaacs (HJI) equation and is hardly generalizable to cases with more than two evaders in terms of computational feasibility. In the work of Chikrii *et al.* [8], it is shown that if the pursuer is required to capture the evaders in a specific sequence, then the geometrical solution is always optimal. Based on this result, the fixed-sequence problem can be formulated as an optimization problem with the evaders' headings being the decision variables. This problem is solved numerically in [9] for cases with more than two evaders. Very recently in [10], Belousov *et al.* further exploited the optimality condition on the directional optimization problem to shrink the problem from a n-dimensional unconstrained optimization problem to a 2-dimensional root searching problem, which can be solved much more efficiently for a large number of evaders. [11] solved the fixed-sequence problem of a pursuer with limited turning rate.

The fixed-sequence formulation provides a conservative strategy for the pursuer in that the pursuer must pick a capture sequence first, and then the team of evaders can cooperatively optimize against the pursuer using the given capture sequence. In [8] it is shown that the optimal sequence depends not only on the initial condition but also on the speed ratios. In [12], Berdyshev utilized a space partitioning technique from the Traveling Salesman Problem (TSP) literature to solve for the optimal capture sequence under the fixed-sequence formulation more efficiently. It is worth noting that this problem can be viewed as a special case of the Generalized Traveling Salesman Problem (GTSP) where each city can pick its location from a given set after the salesman has declared its traveling route as pointed out in [13].

While the optimal evader actions can be efficiently computed in a fixed-sequence formulation, in a more realistic

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Shih-Yuan Liu and Karl Hedrick are with the Department of Mechanical Engineering of University of California, Berkeley. Email: sylíu@berkeley.edu and khedrick@me.berkeley.edu.

Zhengyuan Zhou and Claire Tomlin are with the Department of Electrical Engineering & Computer Science of University of California, Berkeley. Email: zhengyuan@berkeley.edu and tomlin@eecs.berkeley.edu.

pursuit-evasion scenario the pursuer is not likely to be enforced to capture the evaders according to a specific sequence. In the fixed-sequence formulation literature it has been shown that for some initial conditions the fixed-sequence solution of the optimal sequence is the optimal solution for the sequence-free formulation. However there is not a way to identify these initial conditions other than actually simulating the whole chase for cases with more than 2 evaders. Moreover, it is not clear how the evaders should act if the initial condition does not satisfy this property. The evaders' controls in this scenario are somewhat incomplete in the literature. For this reason we focus on a conservative strategy for the evaders that is valid for all initial conditions and that provides guaranteed survival time. Our methodology is inspired by the recent work on the open-loop approach in a general setting for reach-avoid games [14], motion planning problems [15], and pursuit-evasion games [16].

II. Problem Formulation

Consider a pursuit-evasion problem taking place in $\mathbb{R}^2$, where N evaders attempt to evade from a single pursuer. The pursuer is faster than all evaders and its goal is to capture all the evaders as soon as possible. The evaders, as a team, attempt to delay the capture time of the last evader for as long as possible. An evader is considered captured if at some point, the pursuer coincides with the evader. We use β_i to denote the speed ratio of evader i to the pursuer and we have $0 \leq \beta_i < 1$, $i = 1, \ldots, N$. The point capture can always be achieved in finite time since the pursuer is strictly faster than all the evaders. Without loss of generality, we assume the pursuer's maximum speed to be 1. The control for agent j is $u_j(\cdot) : \mathbb{R}^+ \to \mathbb{R}^2$, where the subscript $j = p$ denotes the pursuer and the subscript $j = i$ ($1 \leq i \leq N$) denotes evader i. We often use u_j for notational simplicity. The trajectory of agent j is $x_j(\cdot) : \mathbb{R}^+ \to \mathbb{R}^2$. We also define the joint trajectories: $\mathbf{x}_e(\cdot) = [x_1(\cdot), \ldots, x_N(\cdot)]$ and $\mathbf{x}(\cdot) = [x_p(\cdot), \mathbf{x}_e(\cdot)]$. $\mathbf{u}_e(\cdot) = [u_1(\cdot), \ldots, u_N(\cdot)]$ denotes the joint control of all the evaders, which is taken from the set of admissible joint controls of all the evaders. We assume the following simple motion dynamics:

$$\dot{x}_p(t) = u_p(t) \tag{1}$$

$$\dot{x}_i(t) = \beta_i u_i(t), \qquad i = 1, \ldots, N \tag{2}$$

$$\mathbf{x}(0) = \mathbf{x}^0 \tag{3}$$

Note that each agent has full control over its velocity subject only to a constraint on the maximum speed. It is understood that each u_j is an element of the set $\mathbf{U}$ of admissible controls, where $\mathbf{U} = \{u_j(\cdot) | \|u_j(t)\| \leq 1, t \in [0, \infty)\}$. In other words, u_j represents the direction controlled by agent j as well as the ratio of agent j's speed over its maximum speed. The control set $\mathbf{U}$ is independent of which j we choose. $\mathbf{U}_N = \{\mathbf{u}_e(\cdot) | u_i(\cdot) \in \mathbf{U} \text{ for } i = 1, \ldots, N\}$.

We also characterize the notion of the fixed heading maximum speed control set $\mathbf{U}^{\theta_i} = \{u_i(\cdot) | u_i(t) = [\cos\theta_i, \sin\theta_i], t \in [0, \infty)\}$, which is the set of admissible controls with absolute heading θ_i and maximum speed. Similarly, $\mathbf{U}_N^{\theta} = \{\mathbf{u}_e(\cdot) | u_i(\cdot) \in \mathbf{U}^{\theta_i}, i = 1, \ldots, N\}$ denotes the set of admissible joint controls with absolute joint heading θ. Finally, since the game terminates when all evaders get captured, one interesting aspect to be explored is the sequence according to which the evaders are captured successively. We define $\mathbf{S}_N = \{[s_1, \ldots, s_N] | s_i \in \mathbb{N}_{>0}, s_i \neq s_j \text{ for } i \neq j, \sup_i s_i = N\}$ to be the set of all permutations of the sequence 1 through N, representing all possible sequences that the evaders can get captured.

Definition 1: For a given initial condition $\mathbf{x}(0)$, a fixed pursuer's control $u_p(\cdot)$ and a fixed joint control $\mathbf{u}_e(\cdot)$ of all the evaders, we define the i-th evader capture time τ_i to be:

$$\tau_i(\mathbf{x}(0), u_p(\cdot), \mathbf{u}_e(\cdot)) = \inf\{t | x_p(t) = x_i(t), \text{given } \mathbf{x}(0), u_p(\cdot), \mathbf{u}_e(\cdot)\} \tag{4}$$

In other words, τ_i is the time it takes for the pursuer to capture evader i with the initial condition $\mathbf{x}(0)$ if each agent adopts the respective prescribed control. If the prescribed control does not lead to the capture of evader i, then the value of τ_i is defined to be infinity. When the $\mathbf{x}(0)$, $u_p(\cdot)$, and $\mathbf{u}_e(\cdot)$ lead to the capture of all evaders, the following must hold:

$$x_i(\tau_i) = x_p(\tau_i) \;\; i = 1, \ldots, N \tag{5}$$

The capture time of the last evader can be represented as $\sup_{i \in \{1, \ldots, N\}} \tau_i$.

Ideally, we would like to solve this differential pursuit-evasion game using the closed-loop formulation, where at each time instant, each agent chooses its control input based on the past and current states of all the other agents as well as of itself. The optimal controls for all agents can be obtained by solving the Hamilton-Jacobi-Isaacs (HJI) differential equations, using backwards integration. However, this approach scales exponentially with the number of agents and become computationally intractable for cases above even 2 evaders. To address this difficulty, we propose in this paper an open-loop formulation of the game, where the controls are selected in a conservative way towards the evaders. To this end, we characterize the following two notions that quantify the open-loop capture time.

Definition 2: For a given initial condition $\mathbf{x}^0$ and a given joint control $\mathbf{u}_e(\cdot)$ for all the evaders,

$$T^\star(\mathbf{x}^0, \mathbf{u}_e(\cdot)) = \inf_{u_p(\cdot) \in \mathbf{U}} \sup_{i \in \{1, \ldots, N\}} \tau_i(\mathbf{x}^0, u_p(\cdot), \mathbf{u}_e(\cdot)) \tag{6}$$

That is, $T^\star$ is the minimum time it takes for the pursuer to capture all the evaders if the pursuer knows the prescribed control functions of all the evaders.

Definition 3: For a given initial condition, we define the open-loop capture time of the game to be

$$\tau^{ol\star}(\mathbf{x}^0) = \sup_{\mathbf{u}_e(\cdot) \in \mathbf{U}_N} T^\star(\mathbf{x}^0, \mathbf{u}_e(\cdot)) \tag{7}$$

We note that $\tau^{ol\star}$ essentially encodes the conservatism towards the evaders. The information pattern embodied in the definition is that the evaders first jointly choose the optimal control functions over the entire time horizon assuming the worst case. Thereafter, all the evaders must commit to the control functions chosen initially. Then, with the chosen joint control of all the evaders revealed to the pursuer, the pursuer then selects its best control to minimize the total capture time.

III. Solution, Computation, and Simulation

A. Solution

To compute $T^\star(\mathbf{x}^0, \mathbf{u}_e(\cdot))$, we shall first look at a sub-problem where the pursuer has to capture the evaders in a specific capture sequence $s = [s_1, s_2, \ldots, s_N] \in \mathbf{S}_N$. Here we have that the capture order is dictated by the sequence s, with evader s_N being the last to be captured. This problem can be formulated as the following variational optimization problem.

$$T_s^\star(\mathbf{x}^0, \mathbf{u}_e(\cdot)) = \inf_{u_p(\cdot)\in\mathbf{U}} \quad \tau_{s_N}(\mathbf{x}^0, u_p(\cdot), \mathbf{u}_e(\cdot)) \tag{8a}$$

$$\text{subject to} \quad \dot{x}_p(t) = u_p(t) \tag{8b}$$

$$\dot{x}_i(t) = \beta_i \hat{u}_i(t) \quad i = 1, \ldots, N \tag{8c}$$

$$x_i(\tau_i) = x_p(\tau_i) \quad i = 1, \ldots, N \tag{8d}$$

$$\tau_{s_i} \geq \tau_{s_{i-1}} \quad i = 2, \ldots, N \tag{8e}$$

$$\mathbf{x}(0) = \mathbf{x}^0 \tag{8f}$$

We can solve this seemingly complicated problem efficiently by decomposing this problem into a series of single-pursuer-single-evader minimum time capture problems with given evader trajectories, without compromising the optimality based on the following lemmas.

Lemma 1 (Single Evader with A Given Trajectory): In the minimum time capture problem of one slower evader with a given trajectory, the trajectory resulting from the optimal control of the pursuer must be a straight line traversed with maximum speed.

Proof: Given an initial condition $\mathbf{x}^0 = [x_p^0, x_1^0]$, a speed ratio $\beta_1 < 1$, and the evader control $u_1(\cdot) \in \mathbf{U}$, the evader trajectory in time is defined by $x_1(t) = x_1^0 + \beta_1 \int_{\tau=0}^{t} u_1(\tau) d\tau$. Assume on the contrary that the optimal trajectory of the pursuer is not a straight line with maximum speed and the evader is captured at $x_1(t_1^\star)$ at time $t_1^\star$. There exists a straight-line-maximum-speed trajectory such that the pursuer can reach $x_1(t_1^\star)$ at time $t_1' < t_1^\star$. The pursuer can then backtrack on the evader trajectory $x_1(\cdot)$ starting from $x_1(t_1^\star)$ and capture the evader before $t_1^\star$. This contradicts that $t_1^\star$ is the minimum capture time and hence completes the proof. ■

Lemma 2 (Multiple Evaders with a Given Sequence): Given a joint initial condition $\mathbf{x}^0$, the speed ratios $\{\beta_i\}$, the joint control of the evaders $\mathbf{u}_e \in \mathbf{U}_N$, and a capture sequence s, the pursuer, in order to achieve the minimum capture time of N evaders, must capture all the evaders by successively doing the following until the last evader is captured: travel on a line with maximum speed to capture the first evader in s that has not been captured yet.

Proof: We will prove this by induction. The base case is validated by the previous lemma. Assume that it is true for $N-1$ evaders. That is, the optimal pursuer control must follow the procedures stated in the lemma for $N-1$ evaders. Now consider the case of capturing N evaders. First denote the joint trajectory of the evaders resulting from the prescribed controls by $\mathbf{x}_e(\cdot) = \{x_1(\cdot), \ldots, x_N(\cdot)\}$. Let the minimum time capture point of the first evader be $x_1(t_1^\star)$ which happens at time $t_1^\star$. Assume for contradiction that to achieve minimum capture time of the whole team, the pursuer should capture the first evader not at the minimum time capture position but instead at $x_1(t_1')$ with $t_1' > t_1^\star$, and then capture the rest $N-1$ evaders who are at $\{x_2(t_1'), \ldots, x_N(t_1')\}$ when the first evader is captured. Note that the pursuer can achieve exactly the same minimum capture time of the whole evader team by using the following alternative control. The pursuer can capture the first evader at $x_1(t_1^\star)$ in minimum time $t_1^\star$, then reach $x_1(t_1')$ at time $t_1'' < t_1'$ and wait for $t_1' - t_1''$ second. At time t_1', the pursuer can then start to pursuit the rest of the $N-1$ evaders who are again at $\{x_2(t_1'), \ldots, x_N(t_1')\}$. By the principle of optimality, the pursuer trajectory starting from time t_1'' qualifies as an optimal trajectory to capture the $N-1$ evaders. However this trajectory includes an non-maximum-speed section which is waiting at $x_1(t')$ from t_1'' to t_1'. This contradicts the induction assumption that the optimal trajectory to capture $N-1$ evaders is only composed of straight lines traversed with maximum speed and hence completes the proof. ■

With the above lemmas, we can compute the minimum capture time of a specific joint control of evaders with a specific capture sequence efficiently. We are now ready to present the following theorem that will enable us to compute the optimal open-loop control for evaders.

Theorem 1:

$$\sup_{\mathbf{u}_e(\cdot)\in\mathbf{U}_N} T^\star(\mathbf{x}^0, \mathbf{u}_e(\cdot)) = \sup_{\mathbf{u}_e(\cdot)\in\mathbf{U}_N^\theta} T^\star(\mathbf{x}^0, \mathbf{u}_e(\cdot)) \tag{9}$$

Proof: Due to the space limitation, we briefly sketch the outline of the proof. For any $u_i(\cdot)$ that is not constant heading and/or not full speed, there exists a $\bar{u}_i(\cdot)$ that is composed of waiting at the beginning of the chase for a finite amount of time and then moving in a fixed direction with maximum speed that can achieve the exact same capture time and capture position for that evader. Since waiting at the beginning of the chase is sub-optimal against an optimal pursuer, the optimal control functions for evaders for Eq. (7) have to be those control functions with fixed headings and maximum speed. That is $\mathbf{u}_e^\star(\cdot) \in \mathbf{U}_N^\theta$ ■

Theorem 1 implies that we can change the admissible control set in Eq. (7) from $\mathbf{U}_N$ to $\mathbf{U}_N^\theta$ containing only straight-line-maximum-speed controls, without compromising the optimality of the solution. By parameterizing an evader control $u_i(\cdot) \in \mathbf{U}^{\theta_i}$ by the direction angle θ_i of the trajectory and a slight abuse of notation on T_s, we can rewrite Eq. (7) to its final form

$$\tau^{ol\star}(\mathbf{x}) = \sup_{\theta} \inf_{s\in\mathbf{S}_N} \quad T_s^\star(\mathbf{x}, \theta) \tag{10}$$

where the joint heading of evaders $\theta = [\theta_1, \theta_2, \ldots, \theta_N]$ is a vector of length N denoting the heading directions of the evaders. This is a finite dimensional unconstrained max-min optimization problem. The optimizer, which is the optimal open-loop joint heading, is also defined as a function of the

TABLE I

AVERAGE COMPUTATION TIME

Num. of Evaders	1	2	3	4	5
Avg. Time (sec)	0.069	0.071	0.089	0.188	0.957

layout $\mathbf{x}$ (with a given speed ratio set $\{\beta_i\}$).

$$\theta^{ol\star}(\mathbf{x}) = \underset{\theta}{\operatorname{argsup}} \inf_{s\in\mathbf{S}_N} T_s^\star(\mathbf{x},\theta) \tag{11}$$

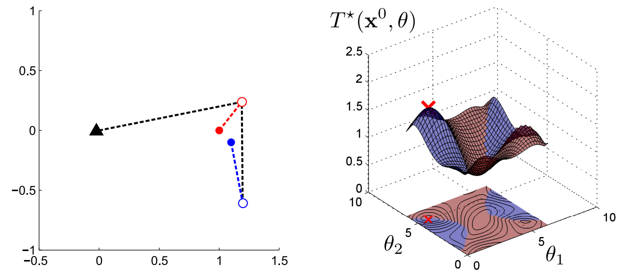
B. Computation

In this section, we describe how the optimization problem in Eq. (10) is solved numerically.

To compute the value of $T^\star(\mathbf{x}^0,\theta)$, we first note that for an evader with a constant heading and maximum speed, the pursuer can achieve minimum capture time by applying the parallel pursuit strategy. In this case the minimum capture time can be computed in a closed-form formula as shown in [10]. This greatly simplifies the computation of $T_s^\star$. We simply compute the parallel pursuit capture time $t_{s_1}^\star$ and the capture point $x_{s_1}(t_{s_1}^\star)$ of the first evader in the sequence. We then compute the minimum capture time of the second evader starting from $t_{s_1}^\star$ with the pursuer starting at $x_{s_1}(t_{s_1}^\star)$ and the second evader starting at $x_{s_2}(t_{s_1}^\star)$. By repeating this process in the order of the capture sequence s, we can then compute the value of $T_s^\star(\mathbf{x}^0,\theta)$. Notice that for a given capture sequence s, speed ratios $\{\beta_i\}$, and initial condition $\mathbf{x}^0$, $T_s^\star$ is purely a function of θ and the complexity of the computation scales linearly with the number of evaders. So to compute the value of $T^\star(\mathbf{x}^0,\theta)$, we compute the value of $T_s^\star(\mathbf{x}^0,\theta)$ for all possible capture sequences $s\in\mathbf{S}_N$ and then take the minimum. Fig. 1(b) shows the value of $T^\star(\mathbf{x}^0,\theta)$ as a function of θ with an initial layout of 2 evaders shown in Fig. 1(a).

The optimization problem in Eq. (10) has the special structure of maximizing the minimum over a set of functions. The decision variable θ of the maximization problem is continuous and unconstrained, and the set of functions are the optimal capture time under the given θ with different capture sequences. This optimization problem can be reformulated as a constrained nonlinear optimization problem which can then be solved by the sequential quadratic programming (SQP) method. For the details of the reformulation and the SQP method, see [17, Chap. 2]. It is worth pointing out that since $T^\star(\mathbf{x}^0,\theta)$ is not concave in θ, it is possible that a local maximum will be found instead of a global maximum. This issue is dealt with by solving the problem with multiple randomly generated initial guesses and then choosing the maximum among the solutions. In our implementation we use the `fminimax` function in MATLAB.

Table I shows the average time to solve for the optimal open-loop headings of initial conditions with different amount of evaders. The data are averaged over 500 data points gathered from simulations with random initial conditions. It was run in MATLAB on a laptop with Intel Core i7-720QM CPU and 8GB of RAM. The computation time scales gracefully for teams with 3 evaders and below. For a team



(a) The layout $\mathbf{x}$ (b) The value of $T^\star(\mathbf{x},\theta)$

Fig. 1. (a) Shows the optimal open loop trajectories of the evaders and the pursuer. The triangle and the filled circles are the initial position of the pursuer and the evaders. The hollow circles are the resulting capture points. (b) Shows the value of $T^\star(\mathbf{x},\theta)$ given different headings of the evaders in both the surface and the contour form. The colors indicate the optimal capture sequence of a specific θ and the red cross marks the optimal open-loop headings of the evaders.

with 5 evaders the computation time is still at the sub-second level. However for more than 5 evaders the computation time grows factorially.

C. Simulation Approaches

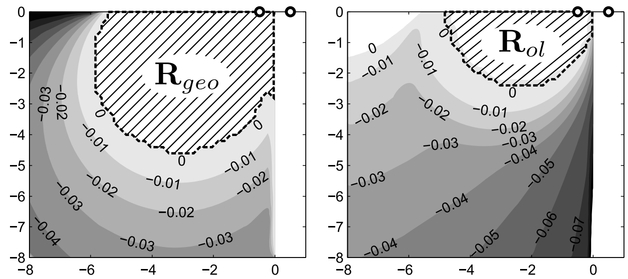
With the ability to compute the optimal open-loop headings for an arbitrary initial condition, we propose the following two different implementation approaches, with the second built on top of the first.

1) The Open-Loop Approach: The team of evaders solve for the optimal open-loop joint heading $\theta^{ol\star}$ of the initial condition $\mathbf{x}^0$ and travel according to the headings with maximum speed. Fig. 1(a) shows the resulting optimal headings and the optimal trajectories of the pursuer and the evaders. Using the open-loop approach, the team of evaders will achieve at least the guaranteed survival time $\tau^{ol\star}(\mathbf{x}^0)$ no matter what the pursuer does; any sub-optimal action on the pursuer side will result in a longer survival time for the evader team.

2) The Iterative Open-loop Approach: In the iterative open-loop approach, the evaders apply the joint control $\theta^{ol\star}(\mathbf{x}(t))$ derived by the state at time t instead of the initial condition $\mathbf{x}^0$. This approach effectively closes the loop in a way similar to that of the receding horizon control method. The benefit of the iterative approach is that on one hand, the conservatism towards the evaders is relaxed; and on the other hand, the sub-optimal move on the pursuer side will often be punished more severely than that in the open-loop approach. In our implementation the heading is updated for every Δt; between time t and $t+\Delta t$, the evaders keep the joint heading $\theta^{ol\star}(\mathbf{x}(t))$. The Δt is set to be 0.01 for all the simulations.

IV. RESULTS AND DISCUSSION

In the literature [7], [8], [9], and [10] the evaders determine their headings after the pursuer has picked and declared a capture sequence. We name this formulation the geometrical formulation. For a given speed ratio set $\{\beta_i\}$, the optimal



(a) $(\tau^{HJI\star} - \tau^{geo\star})/\tau^{geo\star}$ (b) $(\tau^{ol\star} - \tau^{HJI\star})/\tau^{HJI\star}$

Fig. 2. Survival time of the geometrical approach and the optimal open-loop approach compared to that of the HJI approach in a 2-evader layout with different pursuer positions. The 2 hollow circles are the positions of the evaders. The level sets indicate the relative survival time given the different positions of the pursuer.

geometrical capture time can be defined as $\tau^{geo\star}(\mathbf{x}) = \inf_{s\in\mathbf{S}_N} \sup_\theta T_s^\star(\mathbf{x},\theta)$. The optimal capture sequence under this formulation is defined as a function of the layout $\mathbf{x}$: $s^{geo\star}(\mathbf{x}) = \underset{s\in\mathbf{S}_N}{\operatorname{arginf}} \sup_\theta T_s^\star(\mathbf{x},\theta)$. The optimizer of a specific capture sequence s given the layout is defined as: $\theta^{geo}(\mathbf{x},s) = \underset{\theta}{\operatorname{argsup}}\, T_s^\star(\mathbf{x},\theta)$ and the optimal geometrical headings of a layout is defined as $\theta^{geo\star}(\mathbf{x}) = \theta^{geo}(\mathbf{x}, s^{geo\star}(\mathbf{x}))$.

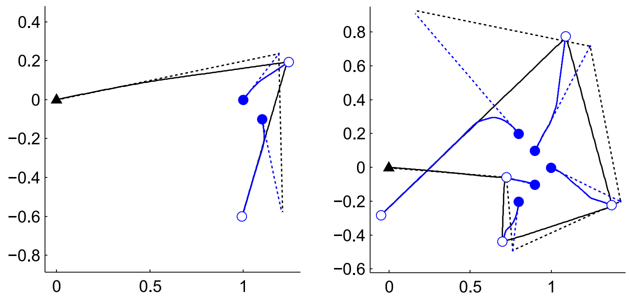
We compare the team survival time of the geometrical approach, the open-loop approach, and the iterative open-loop approach with that of the HJI approach in the 2-evader case. Fig. 2(a) shows the survival time of the geometrical approach compared to that of the HJI approach for various pursuer positions given a 2-evader layout. $\mathbf{R}_{geo}$ denotes the region where $\tau^{HJI\star} = \tau^{geo\star}$. In this region the geometrical solution is the HJI-optimal solution which implies that the optimal joint heading of the evaders never changes as pointed out in [7]. Outside the region $\mathbf{R}_{geo}$, the geometrical approach overestimates the survival time due to the fact that it does not consider the possibility of the pursuer changing the capture sequence after the chase starts. Fig. 2(b) shows the survival time of the optimal open-loop approach compared to that of the HJI approach. $\mathbf{R}_{ol}$ denotes the region where $\tau^{ol\star} = \tau^{HJI\star}$. Note that $\mathbf{R}_{ol}$ is a subset of $\mathbf{R}_{geo}$ so we have $\tau^{geo\star}(\mathbf{x}) = \tau^{ol\star}(\mathbf{x}) = \tau^{HJI\star}(\mathbf{x})$ for $\mathbf{x} \in \mathbf{R}_{ol}$.

Through simulations we discovered that when the layout $\mathbf{x}$ lies within $\mathbf{R}_{ol}$, the optimal open-loop joint heading stays constant along the trajectory generated by playing against an optimal pursuer. This implies that for layouts in $\mathbf{R}_{ol}$, the iterative open-loop approach achieves the same survival time as the open-loop approach. Denote the team survival time of the iterative open-loop approach by $\tau^{iter^\star}$, we have $\tau^{geo\star}(\mathbf{x}) = \tau^{ol\star}(\mathbf{x}) = \tau^{iter\star}(\mathbf{x}) = \tau^{HJI\star}(\mathbf{x})$ for $\mathbf{x} \in \mathbf{R}_{ol}$. Therefore, it suffices to just compare the performance of different approaches outside of $\mathbf{R}_{ol}$ in the 2-evader case.

The computations of $\tau^{ol\star}$ and $\tau^{HJI\star}$ have already been described in the previous section. To compute $\tau^{iter\star}$ outside of $\mathbf{R}_{ol}$, we need an optimal strategy for the pursuer against the iterative open-loop strategy of the evaders: a strategy that takes into account the changes in evaders' headings in response to the movement of the pursuer. Unfortunately, computing such a strategy for pursuer in a tractable way is still an open problem. Hence, we employ a near optimal strategy for the pursuer described as follows. At the beginning of each iteration step, the pursuer is given the headings of evaders; based on these headings the pursuer computes the minimum-time-capture sequence and the trajectory as if the evaders would travel in these headings with maximum speed until being captured. The pursuer will then follow this trajectory until the next iteration and then re-computes a new trajectory. Due to the nature of the optimal open-loop headings of the evaders, the pursuer will often face the choice between multiple capture sequences that will achieve the same minimum capture time. In these situations the pursuer favors capturing the closer evader first if the evaders are not equidistant to the pursuer. When they are equidistant to the pursuer, the capture sequence of the last iteration will be favored to prevent the pursuer from constantly switching back and forth between different capture sequences.

Fig. 3 shows the resulting trajectory of the iterative open-loop approach under this pursuer strategy for a 2-evader case and a 5-evader case. Observe from Fig. 3(a) that the trajectories of the agents consist of curved portions in the beginning of the evasion and straight-line portions towards the end. This is because the layout starts from the outside of $\mathbf{R}_{ol}$ and enters $\mathbf{R}_{ol}$ during the chase and stays within $\mathbf{R}_{ol}$ thereafter. Recall that for layouts within $\mathbf{R}_{ol}$, the optimal open-loop headings stay constant and the optimal pursuer trajectory is available. Hence the sub-optimal pursuer trajectory is only sub-optimal during the curved phase of the chase.

Table II is generated by sampling 400 random initial pursuer positions outside $\mathbf{R}_{ol}$ and inside the plotting region in Fig. 2. The $\tau^{HJI\star}$ values are used as references. For every sample point we compute the ratios of the survival time achieved by the iterative open-loop approach and the open-loop approach to that of the HJI approach. We can see that both approaches achieve similar performance to the HJI approach on average while the iterative open-loop does better than the open-loop. This is to be expected in that the iterative open-loop further exploits the sub-optimal moves of the pursuer. Also shown in the table are the percentages of the sample points where the iterative open-loop approach and the open-loop approach achieve longer survival time than the HJI approach. Theoretically the HJI should always achieve the best performance in that it is the optimal strategy for both the pursuer and evaders. However the open-loop approach does better than HJI about 4% of the time; this is due to numerical issues of the HJI backward integration process. The iterative open-loop approach does better than the HJI approach 24% of the time, which is mainly due to the sub-optimality of the pursuer's strategy against the iterative open-loop. Using the average performance as an indicator, the pursuer strategy described in the previous paragraph is very close to optimal in the 2-evader case.



(a) 2-evader case

(b) 5-evader case

Fig. 3. Resulting trajectory of the open-loop and iterative open-loop approaches. The triangle indicates the initial position of the pursuer. Filled circles are the initial positions of the evaders. Dashed lines and the solid lines are the resulting trajectories of the open-loop and iterative open-loop approach respectively. The hollow circles are the resulting capture points of the iterative open-loop approach.

TABLE II

PERFORMANCE OUTSIDE OF $\mathbf{R}_{ol}$ IN 2 EVADERS CASE

	Avg. Ratio to $\tau^{HJI\star}$	Better than $\tau^{HJI\star}$
$\tau^{ol\star}$	0.970	4%
$\tau^{iter\star}$	0.996	29%

V. CONCLUSIONS AND FUTURE WORK

The open-loop approach strikes a good balance between the computational efficiency of the geometrical approach and the optimality of the HJI approach. The geometrical approach, while can be computed very efficiently, is only applicable when the initial condition is within certain region and does not provide any guarantee on the survival time outside the region. The open-loop approach is computationally more expensive than the geometrical one, but provides guaranteed performance for every initial condition and performs equally well as the geometrical approach in $\mathbf{R}_{ol}$. The HJI approach achieves the optimal survival time for the 2-evader case, but is not computationally feasible for more than 2 evaders. The open-loop approach achieves almost optimal performance in the 2-evader case and can be computed within a second for a team of 5 evaders. The performance can be further enhanced by applying the open-loop iteratively.

For the initial conditions where the open-loop approach achieves the same survival time as the HJI approach, the optimal headings stay constant throughout the chase and the iterative open-loop approach performs exactly the same as the open-loop approach. We observed that for these initial conditions, the optimal capture sequence for the optimal open-loop joint heading is always unique. One of our directions is to investigate the relationship between the uniqueness of the optimal capture sequence and the closed-loop optimality of the open-loop optimal headings.

The bottleneck of computing the optimal open-loop headings for a team with more evaders lies in the evaluation of the minimum capture time over all possible capture sequences. The number of capture sequences grows factorially with the number of evaders. Since the problem itself can been viewed as a Generalized Traveling Salesman Problem (GTSP), we intend in future work to utilize the pruning heuristics from the GTSP literature to decrease the number of capture sequences that have to be evaluated. Another interesting problem we would like to investigate is the optimal pursuer strategy against a team of evaders employing the iterative open-loop approach.

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