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Automatica 40 (2004) 917–927

automatica

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On reachability and minimum cost optimal control☆

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Received 27 March 2002; received in revised form 6 October 2003; accepted 15 January 2004

Abstract

Questions of reachability for continuous and hybrid systems can be formulated as optimal control or game theory problems, whose solution can be characterized using variants of the Hamilton–Jacobi–Bellman or Isaacs partial differential equations. The formal link between the solution to the partial differential equation and the reachability problem is usually established in the framework of viscosity solutions. This paper establishes such a link between reachability, viability and invariance problems and viscosity solutions of a special form of the Hamilton–Jacobi equation. This equation is developed to address optimal control problems where the cost function is the minimum of a function of the state over a specified horizon. The main advantage of the proposed approach is that the properties of the value function (uniform continuity) and the form of the partial differential equation (standard Hamilton–Jacobi form, continuity of the Hamiltonian and simple boundary conditions) make the numerical solution of the problem much simpler than other approaches proposed in the literature. This fact is demonstrated by applying our approach to a reachability problem that arises in flight control and using numerical tools to compute the solution.

Keywords: Nonlinear optimal control; L-infinity control; Hamilton–Jacobi equation; Viscosity solutions; Reachability; Viability

1. Introduction

Because of their importance in applications ranging from engineering to biology and economics, questions of reachability, viability and invariance have been studied extensively in the dynamics and control literature. Most recently, the study of these concepts has received renewed attention through the study of safety problems in hybrid systems. Reachability computations have been used in this context to address problems in the safety of ground transportation systems (Livadas & Lynch, 1998; Lygeros, Godbole, & Sastry, 1998), air traffic management systems (Livadas, Lygeros, & Lynch, 2000; Tomlin, Lygeros, & Sastry, 2000; Tomlin, Mitchell, & Ghosh, 2001), flight control (Lygeros, Tomlin, & Sastry, 1999; Oishi, Tomlin, Gopal, & Godbole, 2001), etc.

Direct characterization of reachability concepts is one of the topics addressed by viability theory (Aubin, 1991). The development of computational tools to support the numerous viability theory methods is an ongoing effort (see, for example, (Cardaliaguet, Quincampoix, & Saint-Pierre, 1999)). Methods for directly addressing reachability questions have also been proposed in the hybrid systems literature. For example, for certain classes of continuous dynamics, exact computation of the set of reachable states is possible (see Alur, Henzinger, Lafferriere, & Pappas (2000) for an overview). Motivated by this observation, computational tools to exactly compute the set of reachable states whenever possible have been developed (Alur & Kurshan, 1996; Bengtsson, Larsen, Larsson, Pettersson, & Yi, 1996; Daws, Olivero, Trypakis, & Yovine, 1996; Henzinger, Ho, & Toi, 1995). For more general classes of systems, tools have been developed to compute numerical approximations of these sets (Asarin, Bournez, Dang, Maler, & Pnueli, 2000; Botchkarev & Tripakis, 2000; Chutinam & Krogh, 1999; Greenstreet & Mitchell, 1998; Kurzhanski & Varaiya, 2000; Mitchell, Bayen, & Tomlin, 2001; Mitchell & Tomlin, 2000).

An alternative, indirect approach to reachability questions is using optimal control methods. In this case, the reachable, viable or invariant sets are characterized as level sets of the value function of an appropriate optimal control

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Kenko Uchida under the direction of Editor Tamer Başar.

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0005-1098/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2004.01.012

problem. Using dynamic programming, the value function can in turn be characterized as the solution to a partial differential equation. In this paper we adopt such an optimal control approach to the reachability problem. We discuss how reachability questions can be encoded as optimal control problems where the cost is the minimum of a function of the state over a given horizon. The objective of the controller is either to maximize this minimum (SUPMIN problem), or to minimize the minimum (INFMIN problem). We show how the value function of the former problem can be used to answer viability questions, whereas the value function of the latter problem can be used to answer invariance questions. We then proceed to characterize the two value functions as viscosity solutions to first order partial differential equations, which are variants of the standard Hamilton–Jacobi equation.

The main advantage of the approach developed here is that the resulting partial differential equations have very good properties in terms of their numerical solution. The value functions of the optimal control problems we use can be shown to be uniformly continuous. Moreover, they are characterized as solutions to partial differential equations in the standard Hamilton–Jacobi form, with continuous Hamiltonians and simple boundary conditions. Therefore, very efficient algorithms developed for this class of equations (Mitchell et al., 2001; Mitchell & Tomlin, 2000; Osher & Sethian, 1988; Sethian, 1996), whose properties have been extensively tested in theory and in applications, can be directly applied to our problem. Even though the study of the properties of the numerical solutions is beyond the scope of this paper, we exploit this fact in Section 4, where the algorithms of Osher and Sethian (1988) and Sethian (1996) coded by Mitchell et al. (2001) and Mitchell and Tomlin (2000) are used to compute the solutions to a reachability problem that arises in flight control.

In Section 2 we pose two optimal control problems, which we refer to as the SUPMIN and INFMIN problems, and establish their relation to invariance and viability questions for continuous dynamical systems. To motivate the new solution we develop for these problems, we summarize some of the existing approaches that can be used to address viability and invariance questions in the framework of optimal control, and argue that the proposed solution of the SUPMIN and INFMIN problems has certain advantages. The characterization of the value functions as viscosity solutions of appropriate partial differential equations are given in Section 3 (for the SUPMIN problem) and Section 5 (for the INFMIN problem). To illustrate the possible applications of our approach, in Section 4 we encode a viability problem that arises in flight control as a SUPMIN problem and use the numerical tools of Mitchell et al. (2001) and Mitchell and Tomlin (2000) to provide a solution. Directions for future work are discussed in Section 6. Most of the proofs have been omitted in the interest of space. Complete proofs of all the facts (as well as simple motivating examples) can be found in Lygeros (2002).

2. SupMin and InfMin optimal control problems

2.1. Statement of the problems

Consider a continuous time control system,

$$\dot{x} = f(x,u) \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^m$, $f(\cdot,\cdot) : \mathbb{R}^n \times U \to \mathbb{R}^n$, a function,

$$l(\cdot) : \mathbb{R}^n \to \mathbb{R}, \tag{2}$$

and an arbitrary time horizon, $T \geqslant 0$. Let $\mathscr{U}_{[t,t']}$ denote the set of Lebesgue measurable functions from the interval $[t,t']$ to U. To eliminate technical difficulties we impose the following standing assumption.

Assumption 1. $U \subseteq \mathbb{R}^m$ is compact. f and l are bounded and Lipschitz continuous.

Assumption 1 is used in Sections 3 and 5 to ensure that the value functions of the optimal control problems considered there have certain continuity properties. In addition, it also ensures that for every $x \in \mathbb{R}^n$, $t \in [0,T]$ and $u(\cdot) \in \mathscr{U}_{[t,T]}$, system (1) admits a unique solution, denoted by $\phi(\cdot,t,x,u(\cdot)) : [t,T] \to \mathbb{R}^n$, with $\phi(t,t,x,u(\cdot)) = x$. Assumption 1 is sufficient for the results presented below. Not all of it, however, is necessary. Some parts can be relaxed, or replaced by alternative assumptions. We will not attempt such fine tuning of the results here. The reader is referred to Flemming and Soner (1993) for various improvements that can be pursued.

We introduce two optimal control problems with value functions $V_1 : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ and $V_2 : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ given by

$$V_1(x,t) = \sup_{u(\cdot)\in\mathscr{U}_{[t,T]}} \min_{\tau\in[t,T]} l(\phi(\tau,t,x,u(\cdot))), \tag{3}$$

$$V_2(x,t) = \inf_{u(\cdot)\in\mathscr{U}_{[t,T]}} \min_{\tau\in[t,T]} l(\phi(\tau,t,x,u(\cdot))). \tag{4}$$

The minimum with respect to time is well defined by continuity. In the first problem the objective of the input u is to maximize the minimum value attained by the function l along the state trajectory over the horizon $[t,T]$. In the second problem, on the other hand, the objective of u is to minimize this minimum. For obvious reasons we will subsequently refer to the first optimal control problem as the SUPMIN problem and to the second problem as the INFMIN problem.

Optimal control problems of this type have been studied in the literature for the last 15 years, starting with the pioneering work of Barron and co-workers (see Barron (1999) for an overview). The SUPMIN problem is a special case of the optimal control problem treated by Barron and Ishii (1989); Fialho and Georgiou (1999), where l is also allowed to depend on t and u. We will present a solution different from that of Barron and Ishii (1989); Fialho and Georgiou (1999) for this problem, which we believe has advantages in terms of numerical implementation. The INFMIN problem was also treated in Fialho and Georgiou (1999) and in Quincampoix and Serea (2002) from a viability point of view. As we will see in Section 2.2, the implications of the INFMIN problem from the point of reachability can also be deduced indirectly using standard, terminal cost, optimal control arguments. We will also present a direct solution to the INFMIN problem (Section 5) and establish a relation between the two. We believe that the direct solution has computational advantages over both the indirect approach and the solution discussed in Fialho and Georgiou (1999).

The main contribution of this paper is a characterization of the value functions V_1 and V_2 as viscosity solutions to appropriate partial differential equations. More specifically, we show that V_1 is a viscosity solutions of the terminal value problem

$$\frac{\partial V_1}{\partial t}(x,t) + \min\left\{0, \sup_{u\in U} \frac{\partial V_1}{\partial x}(x,t)\, f(x,u)\right\} = 0 \tag{5}$$

with $V_1(x,T) = l(x)$ over $(x,t) \in \mathbb{R}^n \times [0,T]$. Likewise, V_2 is a viscosity solution to the terminal value problem

$$\frac{\partial V_2}{\partial t}(x,t) + \min\left\{0, \inf_{u\in U} \frac{\partial V_2}{\partial x}(x,t)\, f(x,u)\right\} = 0 \tag{6}$$

with $V_2(x,T) = l(x)$ over $(x,t) \in \mathbb{R}^n \times [0,T]$.

Partial differential equations of the form (5) and (6) were also proposed in relation to reachability problems in Tomlin (1998) and Lygeros et al. (1999). The treatment in these references assumed the existence of differentiable (classical) solutions to the partial differential equations. It is well known that the value functions of optimal control problems are often not differentiable (or even continuous). We will study Eqs. (5) and (6) in the standard viscosity framework (Crandall & Lions, 1983). Recall that a viscosity solution is not necessarily a differentiable function. However, it can be shown that wherever the viscosity solution is differentiable it satisfies the partial differential equation in the classical sense (Crandall & Lions, 1983).

2.2. Connection to reachability

Given the control system of Eq. (1), the horizon $T \geqslant 0$ and a set of states $K \subseteq \mathbb{R}^n$, a number of questions can be naturally formulated regarding the relation between the set K and the state trajectories of (1) over the horizon T. Problems of interest include the following:

Viability. Does there exist a $u(\cdot) \in \mathscr{U}_{[0,T]}$ for which the trajectory $x(\cdot)$ satisfies $x(t) \in K$ for all $t \in [0,T]$?

Invariance. Do the trajectories $x(\cdot)$ for all $u(\cdot) \in \mathscr{U}_{[0,T]}$ satisfy $x(t) \in K$ for all $t \in [0,T]$?

Reachability. Does there exist a $u(\cdot) \in \mathscr{U}_{[0,T]}$ and a $t \in [0,T]$ such that the trajectory satisfies $x(t) \in K$?

One would typically like to characterize the set of initial states for which the answer to the viability/invariance/reachability questions is "yes". Or, more generally, one would like to characterize the sets

$$\mathrm{Viab}(t,K) = \{x \in \mathbb{R}^n \mid \exists u(\cdot) \in \mathscr{U}_{[t,T]},\ \forall \tau \in [t,T]$$
$$\phi(\tau,t,x,u(\cdot)) \in K\},$$

$$\mathrm{Inv}(t,K) = \{x \in \mathbb{R}^n \mid \forall u(\cdot) \in \mathscr{U}_{[t,T]},\ \forall \tau \in [t,T]$$
$$\phi(\tau,t,x,u(\cdot)) \in K\},$$

$$\mathrm{Reach}(t,K) = \{x \in \mathbb{R}^n \mid \exists u(\cdot) \in \mathscr{U}_{[t,T]},\ \exists \tau \in [t,T]$$
$$\phi(\tau,t,x,u(\cdot)) \in K\}.$$

The notation used here differs from the standard notation used in viability theory (see, for example, Aubin, 1991). $\mathrm{Viab}(t,K)$ is not the set of states that can remain in K for time t, it is the set of states that can remain in K for time $T-t$. This difference in notation is motivated by the connection we seek to establish between these sets and certain Hamilton–Jacobi equations, which are solved backwards in time from a given terminal condition. Clearly, $\mathrm{Reach}(t,K) = (\mathrm{Inv}(t,K^c))^c$, therefore, the invariance and reachability problems are duals of one another and need not be treated separately.[1]

We first establish a connection between viability and the SUPMIN optimal control problem. Assume that the set K is open[2] and is related to the zero level set of a continuous function $l : \mathbb{R}^n \to \mathbb{R}$ by $K = \{x \in \mathbb{R}^n \mid l(x) > 0\}$. A natural choice for the function l is the signed distance to the set K. To ensure that l satisfies Assumption 1 one can impose a saturation to the distance function at some value. The following fact is easy to establish.

Proposition 1. $\mathrm{Viab}(t,K) = \{x \in \mathbb{R}^n \mid V_1(x,t) > 0\}$.

Using the Hamilton–Jacobi characterization in the next section, it is easy to show that the level sets of $V_1(x,t)$ are independent of the choice of the function l.

To establish the connection between invariance and the INFMIN optimal control problem, consider a closed set[3] L, that can be written as the level set of a continuous function $l : \mathbb{R}^n \to \mathbb{R}$, i.e. $L = \{x \in \mathbb{R}^n \mid l(x) \geqslant 0\}$.

Proposition 2. $\mathrm{Inv}(t,L) = \{x \in \mathbb{R}^n \mid V_2(x,t) \geqslant 0\}$.

2.3. Alternative characterizations

We first point out that the set $\mathrm{Inv}(t,K)$ can be computed using the standard Hamilton–Jacobi–Bellman equation (see Tomlin (1998) and Lygeros et al. (1999), for

[1] As usual, K^c stands for the complement of the set K in $\mathbb{R}^n$.

[2] The argument when K is closed is somewhat more complicated. Since it is tangential to the main theme of the paper it is omitted.

[3] The argument when L is open is again somewhat more complicated and is omitted.

more on this observation). Consider again that the closed set $L = \{x \in \mathbb{R}^n \mid l(x) \geqslant 0\}$ and let

$$V_3(x,t) = \inf_{u(\cdot)\in\mathscr{U}_{[t,T]}} l(\phi(T,t,x,u(\cdot))).$$

A standard optimal control argument (see for example Evans (1998); Flemming & Soner (1993)) shows that V_3 is a viscosity solution for the terminal value problem

$$\frac{\partial V_3}{\partial t}(x,t) + \inf_{u\in U} \frac{\partial V_3}{\partial x}(x,t) f(x,u) = 0 \qquad (7)$$

over $(x,t) \in \mathbb{R}^n \times [0,T]$ with $V_3(x,T) = l(x)$.

Proposition 3. *For all* $(x,t) \in \mathbb{R}^n \times [0,T]$,

$$V_2(x,t) = \min_{\tau\in[t,T]} V_3(x,\tau).$$

Moreover,

$$\mathrm{Inv}(t,L) = \bigcap_{\tau\in[t,T]} \{x \in \mathbb{R}^n \mid V_3(x,\tau) \geqslant 0\}.$$

Proof. The second claim is easy to establish from the first. We show that for all $(x,t) \in \mathbb{R}^n \times [0,T]$ and for all $\varepsilon > 0$, $V_2(x,t) \geqslant \min_{\tau\in[t,T]} V_3(x,\tau) - \varepsilon$ and $V_2(x,t) \leqslant \min_{\tau\in[t,T]} V_3(x,\tau) + \varepsilon$. If these hold for all $\varepsilon > 0$, the first claim follows. Choose $u(\cdot) \in \mathscr{U}_{[t,T]}$ such that $V_2(x,t) \geqslant \min_{\tau\in[t,T]} l(\phi(\tau,t,x,u(\cdot))) - \varepsilon$ and notice that

$$\begin{aligned}\min_{\tau\in[t,T]} V_3(x,\tau) &\leqslant \min_{\tau\in[t,T]} l(\phi(\tau,t,x,u(\cdot)))\\ &\leqslant V_2(x,t) + \varepsilon.\end{aligned}$$

To show the second inequality, choose $s \in [t,T]$ and $u(\cdot) \in \mathscr{U}_{[s,T]}$ such that $l(\phi(T,s,x,u(\cdot))) \leqslant \min_{\tau\in[t,T]} V_3(x,\tau) + \varepsilon$. Define $\hat{u}(\cdot) \in \mathscr{U}_{[t,T]}$ by $\hat{u}(\tau) = u(\tau+s-t)$ if $\tau \in [t, t+T-s)$ and $\hat{u}(\tau) = u(T)$ if $\tau \in [t+T-s, T]$. By uniqueness, $\phi(\tau,t,x,\hat{u}(\cdot)) = \phi(\tau+s-t,s,x,u(\cdot))$ for all $\tau \in [t, t+T-s]$. Therefore,

$$\begin{aligned}V_2(x,t) &\leqslant \min_{\tau\in[t,T]} l(\phi(\tau,t,x,\hat{u}(\cdot)))\\ &\leqslant l(\phi(t+T-s,t,x,\hat{u}(\cdot)))\\ &= l(\phi(T,s,x,u(\cdot)))\\ &\leqslant \min_{\tau\in[t,T]} V_3(x,\tau) + \varepsilon. \qquad \square\end{aligned}$$

Proposition 3 shows that one can compute $\mathrm{Inv}(t,L)$ by solving a standard Hamilton–Jacobi–Bellman equation (7) and then taking the intersection of the level sets of the solution (or, equivalently, computing the minimum of the value function over time horizon $[t,T]$). In this case, the only advantage of solving the INFMIN optimal control problem to characterize the set $\mathrm{Inv}(t,L)$ is that one does not have to perform the extra step of taking the intersection of the level sets.

One might expect a similar approach to work for the set $\mathrm{Viab}(t,K)$. This is not the case in general, however. Roughly speaking, the argument in Proposition 3 breaks down because it is impossible to exchange the order of the universal quantification over $\tau \in [t,T]$ with the existential quantification over $u(\cdot) \in \mathscr{U}_{[t,T]}$ (or, equivalently, exchange the min over t with the sup over $u(\cdot)$); in Proposition 3 both quantifiers were universal and could therefore be exchanged.

Even if this direct assault does not work, there are other methods in the optimal control literature that can be adapted to characterize the set $\mathrm{Viab}(t,K)$. For example, one can treat the problem as maximizing the "exit time" from the open set K. It can be shown that this involves solving a standard Hamilton–Jacobi–Bellman equation over the set K (and possibly pieces of its boundary), with rather complicated boundary conditions (Bardi & Capuzzo-Dolcetta, 1997; Flemming & Soner, 1993). Moreover, the value function will not be continuous in general. These features suggest that numerical computations are likely to be more difficult with this approach. By contrast, the characterization proposed here is such that the value function is continuous (Lemma 2 below) and the terminal value problem (5) is solved over the entire $\mathbb{R}^n \times [0,T]$ with very simple boundary conditions. This allows one to deal with the problem using well established numerical methods like those of Osher and Sethian (1988) and Sethian (1996).

Another approach is to solve the modified terminal value problem

$$-\frac{\partial V}{\partial t}(x,t) = \begin{cases} \sup_{u\in U} \frac{\partial V}{\partial x}(x,t) f(x,u) \\ \quad \text{if } x \in K, \\ \min\left\{0, \sup_{u\in U} \frac{\partial V}{\partial x}(x,t) f(x,u)\right\} \\ \quad \text{if } x \in K^c. \end{cases} \qquad (8)$$

This approach was proposed in Tomlin et al. (2000) in the context of differential games, where a relation to reachability problems was discussed for the case of classical solutions. Notice that with this approach the Hamiltonian will in general be discontinuous. Continuity of the Hamiltonian is desirable, because it simplifies both the theoretical analysis and the numerical solution of the partial differential equation; it is unclear how one would even define a viscosity solution for (8). Cardaliaguet, Quincampoix, and Saint-Pierre (2001) also consider a Hamiltonian similar to that of Eq. (8) for differential games. They do so in the context of viability theory, however, and do not establish a relation to a partial differential equation. Such a relation could be established indirectly, based on the characterization of the value functions of various exit time optimal control problems as capture basins (Cardaliaguet, Quincampoix, & Saint-Pierre, 2000).

The approach most closely related to the one proposed here is that of Barron and Ishii (1989) and Fialho and Georgiou (1999), where a generalized version of the SUPMIN optimal control problem is formulated and solved. Related work on differential games includes Barron (1990) (extending the results of Barron & Ishii, 1989) and Mitchell, Bayen, and Tomlin (preprint) (based on the classical results of

Evans & Souganidis, 1984). In Barron and Ishii (1989) the value function of the problem is shown to satisfy a set of discontinuous, quasi-variational inequalities. Though this approach is conceptually appealing, the discontinuity and the implicit dependence of the Hamiltonian on the value function severely limit its usefulness from the numerical computation point of view (as Barron & Ishii (1989) point out). Fialho and Georgiou (1999) simplify this characterization to the following continuous variational inequality

$$\sup_{u\in U} \min\{l(x) - V(x,t), \frac{\partial V}{\partial t}(x,t) + \frac{\partial V}{\partial x}(x,t)\, f(x,u)\} = 0. \tag{9}$$

The main advantage of Eq. (9) over the characterization of Barron and Ishii (1989) is that the Hamiltonian is continuous. In Fialho and Georgiou (1999) specialized numerical schemes were developed to exploit this fact and approximate the solutions to the variational inequality (9). The advantage of Eq. (5) over this approach is that the Hamiltonian is not only continuous, but the problem is also is in the standard Hamilton–Jacobi form. Eq. (5) can therefore be approached numerically using well established schemes for solving Hamilton–Jacobi equations. It should be noted that Barron and Ishii (1989) also establish a partial differential equation in standard Hamilton–Jacobi form whose solution is equivalent to that of the quasi-variational inequalities. Still, the Hamiltonian in this equation is discontinuous (may even take the value $+\infty$ in certain cases); therefore the Hamilton–Jacobi characterization suffers from the same drawbacks as the quasi-variational inequality characterization. In Fialho and Georgiou (1999) a continuous variational inequality was also proposed to address the INFMIN problem. A similar discussion extends to the relative merits of this approach over existing approaches and Eq. (6).

3. Solution to the SupMin problem

We start by showing that V satisfies an appropriate version of the optimality principle.

Lemma 1. *For all $(x,t)\in\mathbb{R}^n\times[0,T]$ and all $h\in[0,T-t]$, $V_1(x,t)\leqslant V_1(x,t+h)$ and $V_1(x,T)=l(x)$. Moreover*

$$V_1(x,t) = \sup_{u(\cdot)\in\mathscr{U}_{[t,t+h]}} \min\{\min_{\tau\in[t,t+h]} l(\phi(\tau,t,x,u(\cdot))), V_1(\phi(t+h,t,x,u(\cdot)),t+h)\}.$$

Proof. For the first part, the fact that $V_1(x,T)=l(x)$ is immediate from the definition of V_1. Moreover, $V_1(x,t) = \sup_{u(\cdot)\in\mathscr{U}_{[t,T]}} \min_{\tau\in[t,T]} l(\phi(\tau,t,x,u(\cdot)))$ and $V_1(x,t+h) = \sup_{u(\cdot)\in\mathscr{U}_{[t+h,T]}} \min_{\tau\in[t+h,T]} l(\phi(\tau,t+h,x,u(\cdot)))$. Assume, for the sake of contradiction, that $V_1(x,t)>V_1(x,t+h)$. Then there exists $u_1(\cdot)\in\mathscr{U}_{[t,T]}$ such that for all $u_2(\cdot)\in\mathscr{U}_{[t+h,T]}$, $\min_{\tau\in[t,T]} l(\phi(\tau,t,x,u_1(\cdot))) > \min_{\tau\in[t+h,T]} l(\phi(\tau,t+h,x,u_2(\cdot)))$. Choose $u_2(\cdot)\in\mathscr{U}_{[t+h,T]}$ according to $u_2(\tau)=u_1(\tau-h)$ for $\tau\in[t+h,T]$. By uniqueness, $\phi(\tau,t+h,x,u_2(\cdot))=\phi(\tau-h,t,x,u_1(\cdot))$ for all $\tau\in[t+h,T]$. Therefore, $\min_{\tau\in[t,T]} l(\phi(\tau,t,x,u_1(\cdot))) > \min_{\tau\in[t,T-h]} l(\phi(\tau,t,x,u_1(\cdot)))$, which is a contradiction. The last part is a special case of Proposition 3.1 of Barron and Ishii (1989). □

Lemma 1 makes two assertions. The first is that the "value" of a given state x can only decrease as the "time to go" increases. Starting from x the minimum value that l experiences over a certain time horizon is less than or equal to the minimum value that l would experience if we stopped the evolution at any time before the horizon expires. This is the reason why the extra min was introduced in the Hamilton–Jacobi–Bellman equation to produce the terminal value problem (5). The second part of the lemma is a variant of the standard principle of optimality: it relates the optimal cost to go from (x,t) to the optimal cost to go from $(x(t+h),t+h)$ and the minimum value experienced by l over the interval $[t,t+h]$.

One can show that under Assumption 1 the value function V_1 is bounded and uniformly continuous.

Lemma 2. *There exists a constant $C>0$ such that* $|V_1(x,t)|\leqslant C$ and $|V_1(x,t)-V_1(\hat{x},\hat{t})|\leqslant C(|x-\hat{x}|+|t-\hat{t}|)$, *for all $(x,t),(\hat{x},\hat{t})\in\mathbb{R}^n\times[0,T]$.*

Lemma 2 is a special case of Proposition 3.1 of Barron and Ishii (1989).

Next, introduce the Hamiltonian $H_1:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ defined by

$$H_1(p,x) = \min\left\{0, \sup_{u\in U} p^{\mathrm{T}} f(x,u)\right\}. \tag{10}$$

Lemma 3. *There exists a constant $C>0$ such that for all $p,q\in\mathbb{R}^n$ and all $x,y\in\mathbb{R}^n$, $|H_1(p,x)-H_1(q,x)|<C|p-q|$ and $|H_1(p,x)-H_1(p,y)|<C|p|\,|x-y|$.*

The proof is straightforward. Finally, the following fact (see, for example, Evans (1998), p. 546) saves us the trouble of checking the viscosity conditions at the initial time.

Lemma 4. *Assume that V_1 satisfies the viscosity conditions for Eq. (5) over $\mathbb{R}^n\times(0,T)$. Then for all $W:\mathbb{R}^n\times[0,T]\to\mathbb{R}$ such that V_1-W attains a local maximum (minimum) at $(x_0,t_0)\in\mathbb{R}^n\times[0,T)$, $(\partial W/\partial t)(x_0,t_0)+H_1((\partial W/\partial x)(x_0,t_0),x_0)\geqslant 0\ (\leqslant 0)$.*

We are now in a position to prove the main result of this section.

Theorem 1. *V_1 is the unique bounded and uniformly continuous viscosity solution of the terminal value problem*

$$\frac{\partial V}{\partial t}(x,t) + H_1\left(\frac{\partial V}{\partial x}(x,t),x\right) = 0$$

over $(x,t)\in\mathbb{R}^n\times[0,T]$ *with boundary condition* $V(x,T)=l(x)$.

Proof. By Lemma 1, $V_1(x,T)=l(x)$. Therefore, under Lemma 4, it suffices to show that

(1) For all $(x_0,t_0)\in\mathbb{R}^n\times(0,T)$ and for all smooth $W:\mathbb{R}^n\times(0,T)\to\mathbb{R}$, if V_1-W attains a local maximum at (x_0,t_0), then $(\partial W/\partial t)(x_0,t_0)+\min\{0,\sup_{u\in U}(\partial W/\partial x)(x_0,t_0)f(x_0,u)\}\geqslant 0$.
(2) For all $(x_0,t_0)\in\mathbb{R}^n\times(0,T)$ and for all smooth $W:\mathbb{R}^n\times(0,T)\to\mathbb{R}$, if V_1-W attains a local minimum at (x_0,t_0), then $(\partial W/\partial t)(x_0,t_0)+\min\{0,\sup_{u\in U}(\partial W/\partial x)(x_0,t_0)f(x_0,u)\}\leqslant 0$.

Uniqueness then follows by Lemmas 2 and 3 and a standard uniqueness result for viscosity solutions (Evans, 1998).

Part 1. Consider an arbitrary $(x_0,t_0)\in\mathbb{R}^n\times(0,T)$ and a smooth $W:\mathbb{R}^n\times(0,T)\to\mathbb{R}$ such that V_1-W attains a local maximum at (x_0,t_0). Then, there exists $\delta_1>0$ such that for all $(x,t)\in\mathbb{R}^n\times(0,T)$ with $|x-x_0|^2+(t-t_0)^2<\delta_1$, we have that $(V_1-W)(x_0,t_0)\geqslant(V_1-W)(x,t)$. We would like to show that $(\partial W/\partial t)(x_0,t_0)+\min\{0,\sup_{u\in U}(\partial W/\partial x)(x_0,t_0)f(x_0,u)\}\geqslant 0$. Assume, for the sake of contradiction, that this is not the case. Then, for some $\theta>0$,

$$\frac{\partial W}{\partial t}(x_0,t_0)+\min\left\{0,\sup_{u\in U}\frac{\partial W}{\partial x}(x_0,t_0)f(x_0,u)\right\}<-2\theta. \tag{11}$$

We distinguish two cases.

Case 1.1: $\sup_{u\in U}(\partial W/\partial x)(x_0,t_0)f(x_0,u)<0$. Then $(\partial W/\partial t)(x_0,t_0)+\sup_{u\in U}(\partial W/\partial x)(x_0,t_0)f(x_0,u)<-2\theta$. Moreover, there exists an $\varepsilon>0$ such that for all $u\in U$, $(\partial W/\partial x)(x_0,t_0)f(x_0,u)<-\varepsilon$. Therefore, since W is smooth, there exists $\delta_2\in(0,\delta_1)$ such that for all $(x,t)\in\mathbb{R}^n\times(0,T)$ with $|x-x_0|^2+(t-t_0)^2<\delta_2$ and all $u\in U$,

$$\begin{aligned}&\frac{\partial W}{\partial t}(x,t)+\min\left\{0,\frac{\partial W}{\partial x}(x,t)f(x,u)\right\}\\&\quad=\frac{\partial W}{\partial t}(x,t)+\frac{\partial W}{\partial x}(x,t)f(x,u)\\&\quad<-\theta<0.\end{aligned}$$

Consider an arbitrary $u(\cdot)\in\mathscr{U}_{[t_0,T]}$. By continuity of the solution with respect to time, there exists $\delta_3>0$ such that $|\phi(t,t_0,x_0,u(\cdot))-x_0|^2+(t-t_0)^2<\delta_2$ for all $t\in[t_0,t_0+\delta_3]$. Therefore,

$$\begin{aligned}&V_1(\phi(t_0+\delta_3,t_0,x_0,u(\cdot)),t_0+\delta_3)-V_1(x_0,t_0)\\&\quad\leqslant W(\phi(t_0+\delta_3,t_0,x_0,u(\cdot)),t_0+\delta_3)-W(x_0,t_0)\\&\quad=\int_{t_0}^{t_0+\delta_3}\frac{\mathrm{d}}{\mathrm{d}t}W(\phi(t,t_0,x_0,u(\cdot)),t)\,\mathrm{d}t\\&\quad=\int_{t_0}^{t_0+\delta_3}\left(\frac{\partial W}{\partial t}(\phi(t,t_0,x_0,u(\cdot)),t)\right.\\&\qquad\left.+\frac{\partial W}{\partial x}(\phi(t,t_0,x_0,u(\cdot)),t)f(\phi(t,t_0,x_0,u(\cdot)),u(t))\right)\mathrm{d}t\\&\quad<-\theta\delta_3.\end{aligned}$$

By Lemma 1,

$$V_1(x_0,t_0)=\sup_{u(\cdot)\in\mathscr{U}_{[t_0,t_0+\delta_3]}}\left[\min\left\{\min_{t\in[t_0,t_0+\delta_3]}l(\phi(t,t_0,x_0,u(\cdot))),\right.\right.\\\left.\left.V_1(\phi(t_0+\delta_3,t_0,x_0,u(\cdot)),t_0+\delta_3)\right\}\right].$$

Therefore, there exists $u(\cdot)\in\mathscr{U}_{[t_0,t_0+\delta_3]}$ such that

$$\begin{aligned}&V_1(x_0,t_0)\\&\quad\leqslant\min\left\{\min_{t\in[t_0,t_0+\delta_3]}l(\phi(t,t_0,x_0,u(\cdot))),\right.\\&\qquad\left.V_1(\phi(t_0+\delta_3,t_0,x_0,u(\cdot)),t_0+\delta_3)\right\}+\frac{\theta\delta_3}{2}\\&\quad\leqslant V_1(\phi(t_0+\delta_3,t_0,x_0,u(\cdot)),t_0+\delta_3)+\frac{\theta\delta_3}{2}\end{aligned}$$

which is a contradiction.

Case 1.2: $\sup_{u\in U}(\partial W/\partial x)(x_0,t_0)f(x_0,u)\geqslant 0$. By Eq. (11), $(\partial W/\partial t)(x_0,t_0)<-2\theta<0$. Since W is smooth, there exists $\delta_2\in(0,\delta_1)$ such that for all $(x,t)\in\mathbb{R}^n\times(0,T)$ with $|x-x_0|^2+(t-t_0)^2<\delta_2$, $(\partial W/\partial t)(x,t)<-\theta<0$. Therefore,

$$\begin{aligned}V_1(x_0,t_0+\delta_2)-V_1(x_0,t_0)&\leqslant W(x_0,t_0+\delta_2)-W(x_0,t_0)\\&=\int_{t_0}^{t_0+\delta_2}\frac{\partial W}{\partial t}(x_0,t)\,\mathrm{d}t\\&<-\theta\delta_2.\end{aligned}$$

This contradicts Lemma 1.

Part 2. Consider an arbitrary $(x_0,t_0)\in\mathbb{R}^n\times(0,T)$ and a smooth $W:\mathbb{R}^n\times(0,T)\to\mathbb{R}$ such that V_1-W attains a local minimum at (x_0,t_0). Then, there exists $\delta_1>0$ such that for all $(x,t)\in\mathbb{R}^n\times(0,T)$ with $|x-x_0|^2+(t-t_0)^2<\delta_1$, we have that $(V_1-W)(x_0,t_0)\leqslant(V_1-W)(x,t)$. We would like to show that $(\partial W/\partial t)(x_0,t_0)+\min\{0,\sup_{u\in U}(\partial W/\partial x)(x_0,t_0)f(x_0,u)\}\leqslant 0$.

Assume, for the sake of contradiction, that for some $\theta>0$,

$$\frac{\partial W}{\partial t}(x_0,t_0)+\min\left\{0,\sup_{u\in U}\frac{\partial W}{\partial x}(x_0,t_0)f(x_0,u)\right\}>2\theta.$$

Therefore, there exists $\hat{u}\in U$ such that $(\partial W/\partial t)(x_0,t_0)+\min\{0,(\partial W/\partial x)(x_0,t_0)f(x_0,\hat{u})\}>2\theta>0$. By smoothness of W, there exists $\delta_2 \in (0, \delta_1)$ such that for all $(x,t) \in \mathbb{R}^n \times (0,T)$ with $|x - x_0|^2 + (t - t_0)^2 < \delta_2$

$$\frac{\partial W}{\partial t}(x,t) + \min\left\{0, \frac{\partial W}{\partial x}(x,t) f(x,\hat{u})\right\} > \theta > 0. \tag{12}$$

By continuity of the solution with respect to t, there exists $\delta_3 > 0$ such that $|\phi(t,t_0,x_0,\hat{u}) - x_0|^2 + (t - t_0)^2 < \delta_2$ for all $t \in [t_0, t_0 + \delta_3]$. Therefore, for all $t \in [t_0, t_0 + \delta_3]$,

$$\begin{aligned}
&V_1(\phi(t,t_0,x_0,\hat{u}),t) - V_1(x_0,t_0) \\
&\geqslant W(\phi(t,t_0,x_0,\hat{u}),t) - W(x_0,t_0) \\
&= \int_{t_0}^{t} \left(\frac{\partial W}{\partial t}(\phi(\tau,t_0,x_0,\hat{u}),\tau) \right.\\
&\quad \left. + \frac{\partial W}{\partial x}(\phi(\tau,t_0,x_0,\hat{u}),\tau)\, f(\phi(\tau,t_0,x_0,\hat{u}),\hat{u}) \right) \mathrm{d}\tau \\
&\geqslant \int_{t_0}^{t} \left(\frac{\partial W}{\partial t}(\phi(\tau,t_0,x_0,\hat{u}),\tau) \right.\\
&\quad \left. + \min\left\{0, \frac{\partial W}{\partial x}(\phi(\tau,t_0,x_0,\hat{u}),\tau)\, f(\phi(\tau,t_0,x_0,\hat{u}),\hat{u})\right\} \right) \mathrm{d}\tau \\
&> \theta(t - t_0).
\end{aligned}$$

In particular,

$$V_1(\phi(t_0 + \delta_3, t_0, x_0, \hat{u}), t_0 + \delta_3) - V_1(x_0,t_0) > \theta\delta_3. \tag{13}$$

By Lemma 1,

$$\begin{aligned}
V_1(x_0,t_0) \geqslant \min\{&\min_{t\in[t_0,t_0+\delta_3]} l(\phi(t,t_0,x_0,\hat{u})),\\
&V_1(\phi(t_0+\delta_3,t_0,x_0,\hat{u}),t_0+\delta_3)\}.
\end{aligned}$$

Case 2.1: $V_1(\phi(t_0 + \delta_3, t_0, x_0, \hat{u}), t_0 + \delta_3) \leqslant \min_{t\in[t_0,t_0+\delta_3]} l(\phi(t,t_0,x_0,\hat{u}))$. Then $V_1(x_0,t_0) \geqslant V_1(\phi(t_0+\delta_3,t_0,x_0,\hat{u}),t_0+\delta_3)$ and therefore $V_1(\phi(t_0 + \delta_3,t_0,x_0,\hat{u}),t_0 + \delta_3) - V_1(x_0,t_0) \leqslant 0$. This contradicts Eq. (13).

Case 2.2: $V_1(\phi(t_0 + \delta_3, t_0, x_0, \hat{u}), t_0 + \delta_3) > \min_{t\in[t_0,t_0+\delta_3]} l(\phi(t,t_0,x_0,\hat{u}))$. Then

$$V_1(x_0,t_0) \geqslant \min_{t\in[t_0,t_0+\delta_3]} l(\phi(t,t_0,x_0,\hat{u})).$$

Recall that for all $t \in [t_0, t_0 + \delta_3]$ with $t > t_0$

$$V_1(\phi(t,t_0,x_0,\hat{u}),t) - V_1(x_0,t_0) \geqslant \theta(t - t_0) > 0 \tag{14}$$

(in fact, $V_1(\phi(\cdot,t_0,x_0,\hat{u}),\cdot)$ is monotone increasing as a function of $t \in [t_0,t_0+\delta_3]$). By Lemma 1, for all $t \in [t_0,t_0+\delta_3]$, $l(\phi(t,t_0,x_0,\hat{u})) \geqslant V_1(\phi(t,t_0,x_0,\hat{u}),t) \geqslant V_1(x_0,t_0)$. Hence,

$$V_1(x_0,t_0) = \min_{t\in[t_0,t_0+\delta_3]} l(\phi(t,t_0,x_0,\hat{u})).$$

The minimum occurs at $t = t_0$ and the minimizer is unique. If this were not the case, then there would exist $\tau \in [t_0, t_0 + \delta_3]$ with $\tau > t_0$ such that $\min_{t\in[t_0,t_0+\delta_3]} l(\phi(t,t_0,x_0,\hat{u})) = l(\phi(\tau,t_0,x_0,\hat{u}))$. Then,

$$V_1(x_0,t_0) = l(\phi(\tau,t_0,x_0,\hat{u})) \geqslant V_1(\phi(\tau,t_0,x_0,\hat{u}),\tau),$$

which would contradict Eq. (14). Therefore,

$$V_1(x_0,t_0) = \min_{t\in[t_0,t_0+\delta_3]} l(\phi(t,t_0,x_0,\hat{u})) = l(x_0).$$

By Lemma 1, $V_1(x_0,t_0) \leqslant V_1(x_0,t_0+\delta_3) \leqslant l(x_0)$. Therefore, $V_1(x_0,t_0+\delta_3) = l(x_0) = V_1(x_0,t_0)$. However,

$$\begin{aligned}
&V_1(x_0,t_0+\delta_3) - V_1(x_0,t_0) \\
&\quad\geqslant W(x_0,t_0+\delta_3) - W(x_0,t_0) \\
&\quad= \int_{t_0}^{t_0+\delta_3} \frac{\partial W}{\partial t}(x_0,t)\,\mathrm{d}t \\
&\quad\geqslant \int_{t_0}^{t_0+\delta_3} \left(\theta - \min\left\{0, \frac{\partial W}{\partial x}(x_0,t)\, f(x_0,\hat{u})\right\}\right) \mathrm{d}t \\
&\quad\geqslant \theta\delta_3.
\end{aligned}$$

This contradiction completes the proof. □

4. Flight level control: a numerical case study

To illustrate the results of Section 3, we consider the problem of maintaining an aircraft at a desired flight level. Commercial aircraft at cruising altitudes are typically assigned a flight level by Air Traffic Control (ATC). The flight levels are separated by a few hundred feet (e.g. 500 or 1000, depending on altitude and the type of airspace). Air traffic moves in different directions at different flight levels (north to south in one level, east to west in another, etc.). This arrangement is desirable because it greatly simplifies the task of ATC: the problem of ensuring aircraft separation, which is normally three dimensional, can most of the time be decomposed to a number of two dimensional (or even one dimensional) problems.

Changes in the flight level happen occasionally and have to be cleared by ATC. At all other times the aircraft have to ensure that they remain within certain bounds (e.g. ±500 ft) of their assigned level. At the same time, they also have to maintain certain limits on their speed, flight path angle, acceleration, etc. imposed by limitations of the engine and airframe, passenger comfort requirements, or to avoid dangerous situations such as aerodynamic stall. In this section we formulate a SUPMIN optimal control problem that allows us to address such constraints.

4.1. Aircraft model

We restrict our attention to the movement of the aircraft in the vertical plane and describe the motion using a point mass model. Such models are commonly used in ATC research (see, for example, Lygeros et al., 1999 and Nuic, 2000).

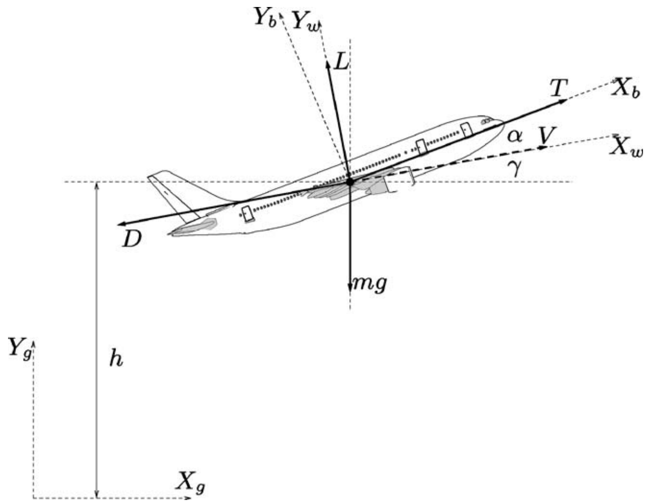

Fig. 1. Coordinate frames and forces for the aircraft model.

They are fairly simple, but still capture the essential features of aircraft flight. The analysis presented here extends to three dimensions the aerodynamic envelope protection problem studied in Lygeros et al. (1999).

Three coordinate frames are used to describe the motion of the aircraft: the ground frame ($X_g - Y_g$), the body frame ($X_b - Y_b$) and the wind frame ($X_w - Y_w$). The angles of rotation between the frames are denoted by θ (ground to body frame, known as the *pitch angle*), γ (ground to wind frame, known as the *flight path angle*) and α (wind to body frame, known as the *angle of attack*). $V \in \mathbb{R}$ denotes the speed of the aircraft (aligned with the positive X_w direction) and h its altitude. Fig. 1 shows the different forces applied to the aircraft: its weight (mg, acting in the negative Y_g direction), the aerodynamic lift (L, acting in the positive Y_w direction), the aerodynamic drag (D, acting in the negative X_w direction) and the thrust exerted by the engine (T, acting in the positive X_b direction).

A force balance leads to the following equations of motion

$$m\dot{V} = T\cos(\alpha) - D - mg\sin(\gamma),$$

$$mV\dot{\gamma} = L + T\sin(\alpha) - mg\cos(\gamma).$$

From basic aerodynamics, the lift and drag can be approximated by

$$L = \frac{S\rho V^2}{2}(C_0 + C_1\alpha),$$

$$D = \frac{S\rho V^2}{2}(B_0 + B_1\alpha + B_2\alpha^2),$$

where B_i and C_i are (dimension-less) aerodynamic coefficients, S is wing surface area and ρ is air density.

A three state model with $x_1 = V$, $x_2 = \gamma$ and $x_3 = h$ suffices for our purposes. The system is controlled by two inputs, the thrust, $u_1 = T$, and the angle of attack,[4] $u_2 = \alpha$. We assume rectangular bounds on the inputs, $u \in U = [T_{\min}, T_{\max}] \times [\alpha_{\min}, \alpha_{\max}]$. After a small angle approximation on α the equations of motion become

$$\dot{x} = \begin{bmatrix} -\dfrac{S\rho B_0}{2m}x_1^2 - g\sin(x_2) \\ \dfrac{S\rho C_0}{2m}x_1 - g\dfrac{\cos(x_2)}{x_1} \\ x_1\sin(x_2) \end{bmatrix} + \begin{bmatrix} \dfrac{1}{m} \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} -\dfrac{S\rho}{2m}x_1^2(B_1u_2 + B_2u_2^2) \\ \dfrac{S\rho C_1}{2m}x_1u_2 \\ 0 \end{bmatrix}.$$

4.2. Cost function and optimal controls

For safety reasons, certain combinations of speed and flight path angle should be avoided, because they may result in aerodynamic stall. Part of the task of the Flight Management System (FMS) is therefore to keep V and γ within a safe "aerodynamic envelope". Following Lygeros et al. (1999), we consider a simplified rectangular envelope; improvements that can be introduced to make the envelope more realistic are discussed in Tomlin et al. (2000). We require that $V_{\min} \leqslant x_1 \leqslant V_{\max}$ and $\gamma_{\min} \leqslant x_2 \leqslant \gamma_{\max}$, for some $0 < V_{\min} \leqslant V_{\max}$ and $\gamma_{\min} \leqslant \gamma_{\max}$. In addition, to ensure that the aircraft does not stray away from its flight level we require that $h_{\min} \leqslant x_3 \leqslant h_{\max}$ for some $h_{\min} \leqslant h_{\max}$. We set[5] $K = [V_{\min}, V_{\max}] \times [\gamma_{\min}, \gamma_{\max}] \times [h_{\min}, h_{\max}]$.

To encode these constraints as a cost in a SUPMIN problem we define a function $l(\cdot): \mathbb{R}^3 \to \mathbb{R}$ by

$$\begin{aligned} l(x) = \min\{&x_1 - V_{\min}, V_{\max} - x_1 \\ &x_2 - \gamma_{\min}, \gamma_{\max} - x_2, \\ &x_3 - h_{\min}, h_{\max} - x_3\}. \end{aligned}$$

Notice that $l(x) \geqslant 0$ for $x \in K$ and $l(x) < 0$ for $x \notin K$. Clearly, l is Lipschitz continuous. To keep l bounded (and since we are only interested in the behavior around the set K) we "saturate" the function l outside the set $[V_{\min} - \delta V, V_{\max} + \delta V] \times [\gamma_{\min} - \delta\gamma, \gamma_{\max} + \delta\gamma] \times [h_{\min} - \delta h, h_{\max} + \delta h]$ for some $\delta V, \delta\gamma, \delta h > 0$.

The problem is now in a form that we can apply the results of Section 3. The quantity to be maximized in the

[4] In practice, one can only control the second derivative of the angle using the aerodynamic surfaces. We ignore this complication here.

[5] Strictly speaking, to follow the development on Section 2.2 one needs to assume that the set K is open. It is easy to see, however, that allowing K to be closed makes no difference in this case.

Hamiltonian of Eq. (10) in this case is quadratic in u,

$$\frac{p_1}{m}u_1 - \frac{p_1 S\rho x_1^2}{2m}(B_1 u_2 + B_2 u_2^2) + \frac{p_2 S\rho C_1 x_1}{2m}u_2.$$

The maximizers, $\hat{u}_1$ and $\hat{u}_2$ depend on the sign of p_1. Recall that $x_1 > 0$ and let $\hat{p} = (p_2 C_1 - p_1 B_1 x_1)/2p_1 B_2 x_1$. Then

- If $p_1 < 0$ then $\hat{u}_1 = T_{\min}$ and
$$\hat{u}_2 = \begin{cases} \alpha_{\min} & \text{if } \hat{p} > (\alpha_{\min} + \alpha_{\max})/2, \\ \{\alpha_{\min}, \alpha_{\max}\} & \text{if } \hat{p} = (\alpha_{\min} + \alpha_{\max})/2, \\ \alpha_{\max} & \text{if } \hat{p} < (\alpha_{\min} + \alpha_{\max})/2. \end{cases}$$
- If $p_1 = 0$ then $\hat{u}_1 = [T_{\min}, T_{\max}]$ and
$$\hat{u}_2 = \begin{cases} \alpha_{\min} & \text{if } p_2 < 0, \\ [\alpha_{\min}, \alpha_{\max}] & \text{if } p_2 = 0, \\ \alpha_{\max} & \text{if } p_2 > 0. \end{cases}$$
- If $p_1 > 0$ then $\hat{u}_1 = T_{\max}$ and
$$\hat{u}_2 = \begin{cases} \alpha_{\min} & \text{if } \hat{p} \leqslant \alpha_{\min}, \\ \hat{p} & \text{if } \alpha_{\min} < \hat{p} < \alpha_{\max}, \\ \alpha_{\max} & \text{if } \hat{p} \geqslant \alpha_{\max}. \end{cases}$$

The singularities (where the maximizer is not unique) play very little role in the numerical computation and so will not be investigated further here; a more thorough treatment (for the two dimensional case with state x_1 and x_2) can be found in Lygeros et al. (1999).

4.3. Numerical results

The resulting optimal Hamiltonian was coded in a numerical tool developed at Stanford University (Mitchell et al., 2001; Mitchell & Tomlin, 2000) for computing viscosity solutions to Hamilton–Jacobi equations using the algorithms of Osher and Sethian (1988) and Sethian (1996). The results are shown in Fig. 2 for a B727 aircraft cruising at 35,000 ft. The aerodynamic parameters used (adapted from Seube, Moite, & Leitmann, 2002) were $B_0 = 0.07351$, $B_1 = -1.5\times10^{-3}$, $B_2 = 6.1\times10^{-4}$, $C_0 = 0.1667$ and $C_1 = 0.109$. The remaining parameters (adapted from Eurocontrol's BADA database (Nuic, 2000)) were $m = 74\cdot10^3$ kg, $g = 9.81$ m/s^2, $S = 158$ m^2, $\rho = 0.3804$ kg/m^3, $\alpha_{\min} = 0°$, $\alpha_{\max} = 16°$, $T_{\min} = 34,386N$ ("descent thrust"), and $T_{\max} = 53,973N$. The parameters used in the function l were $V_{\min} = 207$ m/s ("low-speed buffeting limit"), $V_{\max} = 260$ m/s, $\gamma_{\min} = -10°$, $\gamma_{\max} = 10°$, $h_{\min} = -150$ m, $h_{\max} = 150$ m, $\delta V = 5$ m/s, $\delta\gamma = 2.5°$, $\delta h = 10$ m. For a $100\times100\times100$ grid the computation took 2415 seconds on an Athlon 1.4 processor running SuSE Linux.

Fig. 2 shows the level set $\text{Viab}(0, K) = \{x \in \mathbb{R}^3 \mid V_1(x, 0) \geqslant 0\}$ for two different values of the horizon, $T = 1.0$ and 2.5 s. As expected from Lemma 1, these sets are nested (the level set "shrinks" as T increases). For $T \approx 2.5$ s the shrinking stops; the level sets for values $T \geqslant 2.5$ s are all the same. This means that for all states included in the level set for $T = 2.5$ s one can find control actions to keep them in this level set (and hence in the desired safety envelope) for ever. This does not imply that the B727 can be stabilized in 2.5 s. It simply means that all states outside this level set are doomed to leave the safe envelope after at most 2.5 s. Notice that these states correspond to large values of flight path angle, where only a few seconds are required to cover 150 m vertically.

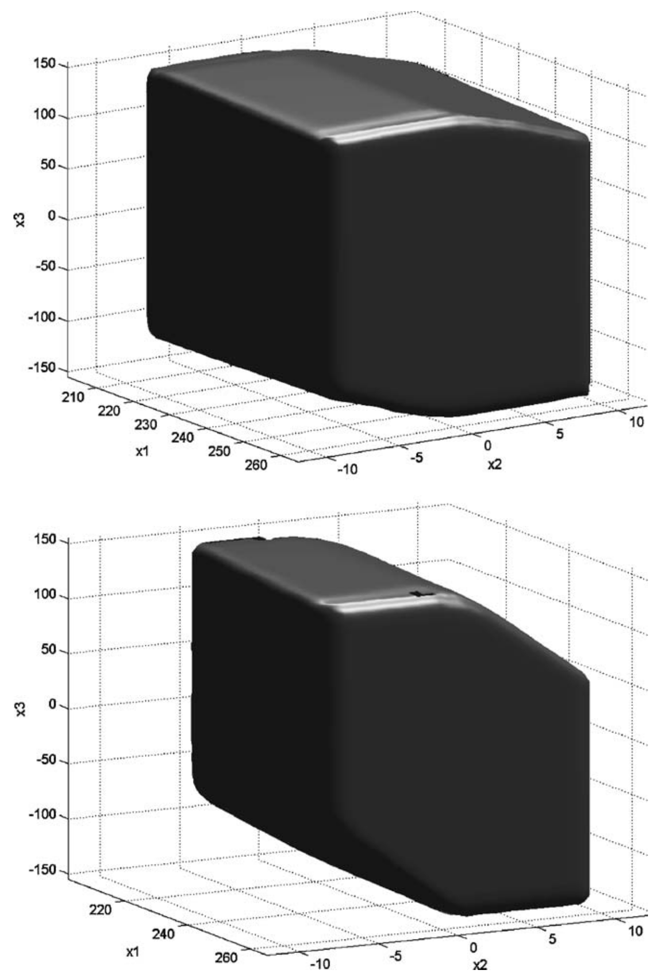

Fig. 2. Two level sets of the value function $V_1(x, 0)$, for $T = 1$ s (top) and $T = 2.5$ s (bottom).

The general shape of the level sets suggests that certain states (e.g. combining high altitude with high-flight path angle, low speed with high flight path angle, etc.) are unsafe and should be avoided. If the aircraft ever gets to such a state, then, whatever the FMS does from then on, it will sooner or later violate the flight envelope requirements. If the initial condition is inside the level set, however, unsafe states can be avoided by applying the optimal controls of Section 4.2 whenever the state trajectory hits the boundary of the level set (see Oishi et al. (2001) for practical problems associated with such a control strategy).

5. Solution of the InfMin Problem

We conclude by stating the corresponding results for the INFMIN problem. All proofs are omitted in the interest of space, the reader is referred to Lygeros (2002) for more details. We start again by showing that V_2 has appropriate continuity properties and satisfies an appropriate version of the optimality principle.

Lemma 5. *For all* $(x,t)\in\mathbb{R}^n\times[0,T]$ *and all* $h\in[0,T-t]$, $V_2(x,t)\leqslant V_2(x,t+h)$ *and* $V_2(x,T)=l(x)$. *Moreover,*

$$V_2(x,t)=\inf_{u(\cdot)\in\mathscr{U}_{[t,t+h]}}\min\{\min_{\tau\in[t,t+h]} l(\phi(\tau,t,x,u(\cdot))),\; V_2(\phi(t+h,t,x,u(\cdot)),t+h)\}.$$

There exists a constant $C>0$ *such that* $|V_2(x,t)|\leqslant C$ *and* $|V_2(x,t)-V_2(\hat{x},\hat{t})|\leqslant C(|x-\hat{x}|+|t-\hat{t}|)$ *for all* $(x,t),(\hat{x},\hat{t})\in\mathbb{R}^n\times[0,T]$.

Lemma 5 can be used to show the following.

Theorem 2. V_2 *is the unique bounded and uniformly continuous viscosity solution to the terminal value problem*

$$\frac{\partial V}{\partial t}(x,t)+\min\left\{0,\inf_{u\in U} p^{\mathrm{T}} f(x,u)\right\}=0$$

over $(x,t)\in\mathbb{R}^n\times[0,T]$ *with* $V(x,T)=l(x)$.

Finally, the following corollary is a direct consequence of Theorem 2, Proposition 3 and the uniqueness of viscosity solutions.

Corollary 1. *Let* V_3 *be the unique bounded, uniformly continuous viscosity solution of the Hamilton–Jacobi–Bellman equation* (7). *Then the function* $V_2(x,t)=\min_{\tau\in[t,T]}V_3(x,\tau)$ *is the unique bounded, uniformly continuous viscosity solution to the terminal value problem* (6).

It appears that this fact is true more generally, even in cases where an integral as well as a terminal cost is present.

6. Concluding remarks

The results presented in this paper establish a relation between the viscosity solutions of Hamilton–Jacobi partial differential equations (5) and (6) and reachability computations. The results provide a theoretical foundation for extending the use of numerical algorithms developed for the approximation of viscosity solutions to partial differential equations (Osher & Sethian, 1988; Sethian, 1996) to viability and invariance computations (Mitchell et al., 2001; Mitchell & Tomlin, 2000). The form of Eqs. (5) and (6) (in particular the fact that they are in standard Hamilton–Jacobi form, the Hamiltonians are continuous, and the boundary condition are particularly simple) make this approach especially attractive from the point of view of numerical computations.

Reachability and invariance can also be approached using tools from viability theory. Viability theory methods (Aubin, 1991) have recently been extended from continuous systems to a broad class of hybrid systems known as impulse differential inclusions (Aubin, Lygeros, Quincampoix, Sastry, & Seube, 2002). Current research concentrates on relating the results discussed here to the viability theory formulation.

Acknowledgements

The author would like to thank M. Quincampoix and M. Smith for bringing the work in references Barron and Ishii (1989), Barron (1990), and Fialho and Georgiou (1999) to his attention and S. Sastry and C. Tomlin for stimulating discussions that motivated the present work. He is grateful to I. Mitchell and C. Tomlin for providing access to the numerical tools of Mitchell et al. (2001) and Mitchell and Tomlin (2000) and to R. Gondhalekar and I. Kitsios for coding the flight control example in these tools. He would like to thank Eurocontrol Experimental Centre for providing access to aircraft parameters through their BADA database (Nuic, 2000). Research was supported by the EPSRC under grant GR/R51575/01 and by the European Commission under project HYBRIDGE, IST-2001-32460.

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