On Three-Pursuer Guaranteed Capture in General Planar Domains

Zhengyuan Zhou, Jonathan R. Shewchuk, Haomiao Huang, and Claire J. Tomlin

***Abstract*— We study a pursuit-evasion game in which cooperating pursuers attempt to capture a single evader. We describe a strategy that guarantees that three pursuers will capture an evader of equal speed in a closed, bounded, continuous, planar domain with a finite number of obstacles. Three is the smallest bound possible; there exist domains in which an evader can evade two pursuers indefinitely. A similar result is known for graphs, but to our knowledge, this is the first such proof for continuous domains.**

I. Introduction

Consider a game between multiple, cooperating pursuers and a single evader moving with equal speed in a bounded, planar game region with an arbitrary set of obstacles. The pursuers attempt to capture the evader as quickly as possible by moving within a specified capture radius of the evader, and the evader attempts to remain free. For a particular region and a fixed number of players, is there a strategy that guarantees that the pursuers will capture the evader? How many pursuers might be required?

Solutions for pursuit-evasion games are relevant in robotics, aircraft control, security, and other applications. They have been extensively studied. Parsons considers the problem played in discrete time on graphs, and characterizes the number of pursuers required to capture an evader in a tree [1]. Aigner and Fromme show that in a finite, planar graph, three pursuers are sufficient and necessary to capture an evader [2].

These results underlie solutions to the related problem of visibility pursuit-evasion in continuous domains, where a number of searchers are attempting to bring an evader into their field of view [3], [4], [5]. Typically, the sensing is modeled as vision with unbounded range (in effect an infinite capture radius when not occluded), allowing these continuous visibility games to be transformed into discrete pursuit-evasion games and solved accordingly.

Games in continuous domains have been more directly addressed as minimax differential games, with the objective of minimizing the time required to capture the evader. Isaacs presents particular solutions for a number of specific games by solving the Hamilton–Jacobi–Isaacs (HJI) equation backwards in time from terminal conditions [6], and more games are covered in a similar manner by Basar and Older [7]. Alonso et al. [8] analytically determine upper and lower bounds on capture time for a 1 vs. 1 game played in a circle, and Kopparty and Ravishankar [9] give a pursuit strategy with bounds on capture time for games in open domains or with a small set of half-space constraints. More recently, numerical solutions to the HJI equation have been used to compute optimal strategies for games involving a small number of players with more complex dynamics or geometries [10], [11]. Generally, differential games methods have been successful at determining optimal or near-optimal strategies for either a small number of pursuers or simple game geometries.

Strategies and results for multiple pursuers in continuous planar domains of arbitrary topology have been more difficult to obtain. To our knowledge, no result has been previously presented on the number of pursuers required. Although Alonso et al. [8] hypothesize that the results of Aigner and Fromme can be adapted to the continuous game, no proof is presented for this claim and we are skeptical that a straightforward adaptation exists.

In this paper, we show that, for a general planar domain with an arbitrary topology and number of obstacles, three pursuers guarantee the capture of an evader of equal speed. We also show that in some domains, three pursuers are necessary. We briefly outline the pursuit strategy here, and discuss it in greater detail in Section IV. We first establish that one pursuer can defend a certain class of paths, in the sense that the evader is instantly captured on crossing such a path. If a defensible path separates a region into two subregions, it reduces the original game domain to a smaller game subdomain from which the evader may not escape. Each reduction of the game domain decreases the characteristic number of the game domain (a notion related to the number of obstacles) by at least one, until the final game subdomain that contains the evader is simply connected. Each such reduction requires at most two pursuers, and a third remaining pursuer suffices to capture the evader by direct pursuit, as Alexander et al. [12] show.

II. Problem Formulation

Consider a planar pursuit-evasion game in a closed, bounded domain with a finite number of obstacles. Let Ω be a closed, bounded, two-dimensional *ambient space* in which the game takes place. The boundary of Ω is a closed Jordan curve (1-manifold without boundary) of finite length (to rule out pathological structures such as fractals). Consider a finite number n of obstacles in the ambient space whose boundaries are also closed Jordan curves of finite length. Their union is

$$\Omega_{\mathrm{obs}} = \bigcup_{i=1}^{n} \Omega_{\mathrm{obs}}^{i} \tag{1}$$

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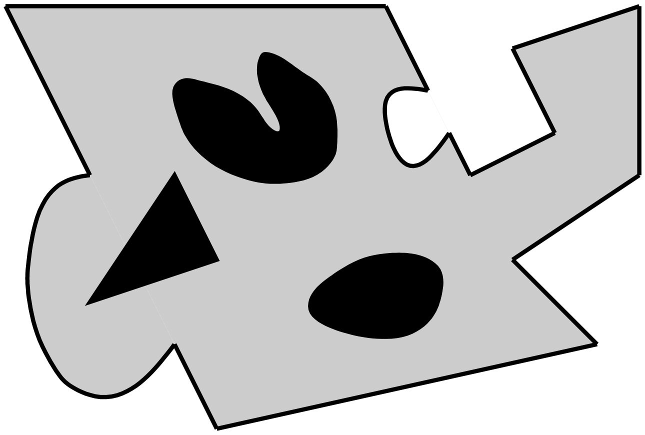

Fig. 1. A game domain. The free space is gray, and three obstacles appear in black.

where each $\Omega^i_{\rm obs} \subset \Omega$ is an open, simply-connected, nonempty point set. Typical obstacles in pursuit-evasion games are simply connected polygons (not necessarily convex), disks, and ellipsoidal disks. Figure 1 depicts a representative game domain.

Pursuers and an evader move in a *free space*

$$\Omega_{\rm free} = \Omega \setminus \Omega_{\rm obs}, \tag{2}$$

which is a closed, connected point set. The boundary of $\Omega_{\rm free}$ is $\partial\Omega_{\rm free} = \partial\Omega \cup \bigcup_{i=1}^{n} \partial\Omega^i_{\rm obs}$.

In this game, a number of *pursuers* pursue a fleeing *evader* whose goal is to evade capture for as long as possible, ideally forever. The pursuers and the evader are called *players*, and all the players are restricted to the free space $\Omega_{\rm free}$. The goal of the pursuers is to follow a strategy that guarantees capture with as few pursuers as possible, if one exists. The pursuers and the evader all have the same, finite speed. Therefore, we assume without loss of generality all players have unit speed. Each pursuer has a *capture radius* of $r > 0$; the evader is considered captured if a pursuer comes within a disk of radius r centered on the evader.

All players' controls can be general functions. In Section VII, we show that piecewise continuous controls suffice for three pursuers to guarantee capture of an evader with any control. Thus, the pursuit strategy can succeed in more practical circumstances.

III. Defending a Shortest Path

In our pursuit strategy, pursuers choose paths that subdivide the domain and defend them so that the evader cannot cross them.

Definition 1: A path $s \subset \Omega_{\rm free}$ is *1-pursuer defendable* if there is a strategy by which a pursuer can position himself on s in finite time, and thereafter guarantee that he will coincide with the evader at any time the evader moves onto s. Thus, after the pursuer is correctly positioned, the evader cannot cross s without being captured. A pursuer that is correctly positioned and following this strategy is said to be *defending* s.

For example, every straight line segment in the free space is 1-pursuer defendable. The defense strategy for a line segment requires the pursuer to stand at the orthogonal projection of the evader onto the segment, or at an endpoint of the segment if the evader's projection lies beyond that endpoint. Pursuers who defend line segments do not suffice to capture an evader in a maze-like domain, but the notion of projecting the evader onto a path generalizes to any path that is the shortest path between its endpoints.

Let $s(x, y) \subset \Omega_{\rm free}$ denote the shortest path in the free space between two points $x, y \in \Omega_{\rm free}$, and let $d(x, y)$ denote the length of the path $s(x, y)$. Every such shortest path is 1-pursuer defendable. Our strategy for defending a shortest path $s(x, y)$ takes advantage of the level sets of the *distance function* $d(x, \cdot)$ for a fixed x.

Definition 2: Given a closed game domain $\Omega_{\rm free}$, a point $z \in \Omega_{\rm free}$, and a real value $\alpha \geq 0$, the *α-level set* of the distance function $d(z, \cdot)$ is

$$\mathcal{A}(z; \alpha) = \{p \in \Omega_{\rm free} \mid d(z, p) = \alpha\}, \tag{3}$$

i.e. the set of all points whose shortest path to z has length α.

The level sets of a distance function $d(x, \cdot)$ are piecewise smooth curves. A shortest path $d(x, y)$ is a path of steepest descent from y to x through the field $d(x, \cdot)$, and in the interior of the free space (and wherever else it has a well-defined tangent line) it is orthogonal to the level sets.

Let P be a pursuer defending $s(x, y)$. Let $P(t)$ and $E(t)$ be the pursuer's position and evader's position at time t respectively. P's strategy is first to move onto the path $s(x, y)$, then to try to stay on the same level set as the evader, subject to the constraint that P must stay on $s(x, y)$.

Therefore, let $\alpha(t) = d(x, E(t))$ and define the *image* $i(t) \in s(x, y)$ of the evader at time t to be

$$i(t) = \begin{cases} s(x, y) \cap \mathcal{A}(x; \alpha(t)), & \alpha(t) \leq d(x, y), \\ y, & \alpha(t) > d(x, y). \end{cases} \tag{4}$$

If P is on the path $s(x, y)$, its strategy at time t is move along the path toward $i(t)$, if P is not already there.

Lemma 1: For any two points $x, y \in \Omega_{\rm free}$, let $s(x, y) \subset \Omega_{\rm free}$ be a shortest path connecting x and y in $\Omega_{\rm free}$. Then $s(x, y)$ is 1-pursuer defendable.

Proof: A pursuer can move to a point on $s(x, y)$ in finite time, as the domain Ω is bounded. He can then move to the image i of an evader in finite time, regardless of the motion of i, because the path is acyclic.

By the following reasoning, the image $i(t)$ cannot move faster than unit speed on the path $s(x, y)$. The distance function $d(\cdot, \cdot)$ satisfies the triangle inequality: for any two points $p, q \in \Omega_{\rm free}$,

$$d(x, p) \leq d(x, q) + d(p, q). \tag{5}$$

This implies that where the gradient $\nabla d(x, \cdot)$ is defined, its magnitude nowhere exceeds one. For any $\Delta t > 0$,

$$|d(x, E(t + \Delta t)) - d(x, E(t))| \leq d(E(t + \Delta t), E(t)) \leq \Delta t. \tag{6}$$

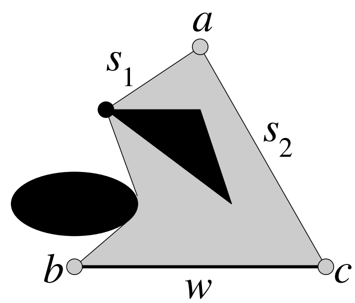

Fig. 2. An example of the triangle configuration, a game subdomain of a larger game domain (not shown). The free space is gray, and two obstacles appear in black. The paths s_1 and s_2 are shortest paths from a to b and c, respectively.

The second inequality follows because the evader has unit speed. The image $i(t)$ is defined so that $d(x, i(t)) = \min\{d(x, E(t)), d(x, y)\}$, so

$$|d(x, i(t + \Delta t)) - d(x, i(t))| \leq \Delta t. \tag{7}$$

Therefore, the image i of the evader cannot travel a distance greater than Δt along $s(x, y)$ in Δt time. This holds true for any positive Δt, even infinitesimal. It follows that i never moves faster than unit speed, even instantaneously, and once a pursuer coincides with i, he can stay with i indefinitely. ■

IV. A Guaranteed Capture Strategy for Three Pursuers

In our strategy, pursuers defend paths that subdivide the game domain into subdomains. While two pursuers prevent the evader from escaping a game subdomain, the third pursuer subdivides it further, eventually freeing one of the first two pursuers to do the same in turn.

Definition 3: A closed point set Ψ is a *game subdomain* of a game domain $\Omega = \Omega_{\text{free}} \cup \Omega_{\text{obs}}$ if its boundary $\partial\Psi$ is a Jordan curve of finite length with $\partial\Psi \subset \Omega_{\text{free}}$. Observe that this implies that $\Psi \subseteq \Omega$.

A game subdomain inherits the obstacles

$$\Psi_{\text{obs}} = \bigcup_{\Omega^i_{\text{obs}} \subseteq \Psi} \Omega^i_{\text{obs}}$$

and the free space $\Psi_{\text{free}} = \Psi \setminus \Psi_{\text{obs}} \subseteq \Omega_{\text{free}}$.

A typical example of a game subdomain is obtained by choosing a path that connects two distinct points on $\partial\Omega$, thereby dividing Ω into two or more pieces, each of which contains some subset of the original obstacles. Each piece is a game subdomain of Ω, and a domain in its own right.

The pursuers defend subdomains of a particular configuration, illustrated in Figure 2.

Definition 4: Let $\Omega = \Omega_{\text{free}} \cup \Omega_{\text{obs}}$ be a game domain, and let $\Psi = \Psi_{\text{free}} \cup \Psi_{\text{obs}}$ be a game subdomain of Ω. Ψ is of the *triangle configuration* if there are three points a, b, and c on $\partial\Psi$ that subdivide $\partial\Psi$ into three paths s_1, s_2, and w that are disjoint except at their endpoints, such that s_1 is a shortest path from a to b in Ψ, s_2 is a shortest path from a to c in Ψ, and w is a natural boundary path included in $\partial\Omega_{\text{free}}$.

If the evader is in a game subdomain of the triangle configuration and two pursuers are defending s_1 and s_2, then by Lemma 1, s_1 and s_2 are 1-pursuer defendable, even if they are not shortest paths in Ω; it suffices that they are shortest paths in Ψ. (If the evader could escape Ψ, those paths might no longer be defendable.) A circumstance can arise in which b and c are the same point, and there are two different shortest paths from that point to a. In that case, w is a single point.

Definition 5: A path s *touches* an obstacle Ψ^i_{obs} if $s \cap \partial\Psi^i_{\text{obs}}$ is not empty. A *touching point* is a point in the intersection. A path w is a *natural boundary path* if w is a connected subset of $\partial\Omega_{\text{free}}$.

The intersection of a path s with an obstacle can be one point, a path, or a union of paths and points.

The third pursuer subdivides a subdomain by choosing a shortest path within the subdomain and defending it. To ensure that this shortest path is not identical to a path defended by the other two pursuers, we modify the free space to remove points where a defended path touches an obstacle. We thereby connect the obstacles to the infinite space outside Ψ.

Definition 6: The *blocked free space* of a subdomain Ψ_{free} whose boundary includes the defended paths $s_1, s_2 \subset \partial\Psi$ is

$$\Psi_{\text{block}} = \Psi_{\text{free}} \setminus (\partial\Psi_{\text{obs}} \cap (s_1 \cup s_2)). \tag{8}$$

In general, Ψ_{block} is a neither closed nor open point set. In Figure 2, Ψ_{block} lacks one point that Ψ_{free} possesses: the point where the triangular obstacle intersects s_1. The removal of this point prevents any new shortest path from a to b from taking the same path as s_1.

Our strategy repeatedly subdivides a game subdomain of the triangle configuration so the evader is trapped in a smaller game subdomain of the triangle configuration, each time reducing the number of obstacles or increasing the number of obstacles touched by the defended paths. Suppose the pursuers P_1 and P_2 are defending s_1 and s_2, respectively. We consider four cases.

1) The subdomain Ψ encloses no obstacle.
2) The blocked free space Ψ_{block} is separated by obstacles into two or more connected components, as illustrated in Figure 3.
3) Ψ_{block} is connected, and one or more obstacles in Ψ touch s_1, s_2, or both, as in Figures 4 and 5.
4) Ψ_{block} is connected, and no obstacle in Ψ touches s_1 or s_2, as in Figure 6. (As the figure shows, obstacles outside Ψ do not count.)

In case 1, $\Psi_{\text{free}} = \Psi$ is simply connected and, following the result presented in reference [12], P_3 can capture the evader by direct pursuit, where P_3 instantaneously minimizes minimizes the geodesic distance to the evader.

In case 2, the evader is trapped either in the connected component that contains a, labeled Region I in Figure 3, or another connection component like Region II. In the former case, we replace the subdomain with Region I, which is a smaller subdomain also of the triangle configuration and has at least one less obstacle, and iterate.

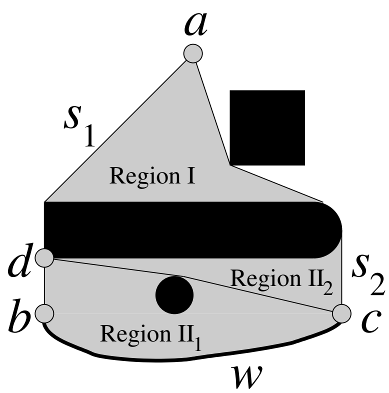

Fig. 3. A subdomain of the triangle configuration whose blocked free space has more than one connected component, corresponding to case 2.

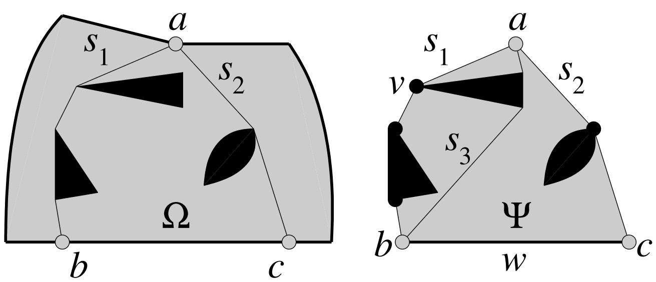

Fig. 4. At left, a subdomain of the triangle configuration embedded in a larger domain Ω, corresponding to case 3. At right, the subdomain is bounded by s_1, s_2, and a natural boundary path w. Points are removed from the blocked free space where the obstacles meet s_1 and s_2. The shortest path s_3 between a and b in the block free space must take a different route than s_1. In this example, s_3 touches only an obstacle that s_1 already touches, but the new subdomains have lower characteristic numbers.

If the evader is in a connected component that does not contain a, like Region II, the pursuer P_3 moves to defend the shortest path between c and d in Region II, which is a connected component of the blocked free space. After a finite amount of time, P_3 will reach the evader's image on $s(c,d)$ and be defending that path. At that time, the evader is in Region II_1 or Region II_2. Either one is a subdomain of the triangle configuration, has one less obstacle, and frees one of the pursuers P_1 or P_2 to defend a new path.

In case 3, suppose that s_1 touches at least one obstacle. We find the shortest path s_3 between a and b in Ψ_{block}, as illustrated in Figure 4 and Figure 5. In Section V, we show (Lemma 2) that s_3 touches at least one obstacle in Ψ_{obs}; call it Ψ^i_{obs}. Although s_3 is probably not a shortest path in Ψ_{free} (as s_1 is), it is a shortest path in Ψ_{block}, and therefore is 1-pursuer defendable so long as the evader is trapped in Ψ_{block}, which is clearly true in this case, since the evader cannot reach any point on s_1 and s_2. We direct P_3 to defend s_3. Once P_3 reaches the evader's image on s_3 and is defending s_3, the evader cannot cross s_3, which splits Ψ into two subdomains of the triangle configuration and frees one of the pursuers P_1 or P_2 to defend a new path.

In case 4, if there is no obstacle in the subdomain, P_3 can capture the evader by direct pursuit. Otherwise, in Section V we show (Theorem 1) that we can choose a point d on w so that the shortest path s_3 between a and d in Ψ_{free} touches an obstacle, as illustrated in Figure 6. The path s_3 splits Ψ into two subdomains of the triangle configuration and frees one of the pursuers P_1 or P_2.

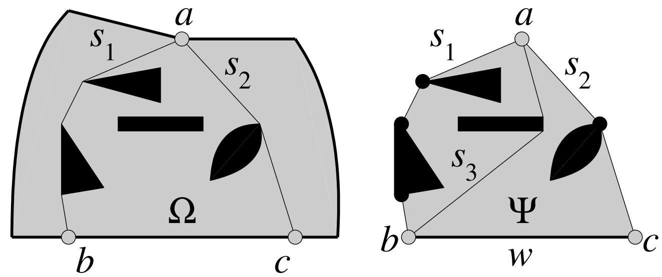

Fig. 5. An alternate illustration of case 3. This subdomain differs from Figure 4 in that s_3 touches an obstacle not touched by s_1 or s_2.

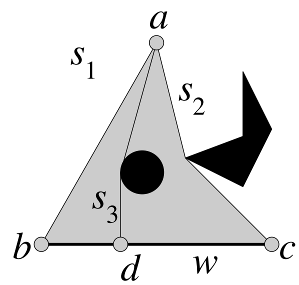

Fig. 6. An illustration of case 4. Neither $s1$ nor $s2$ touch an obstacle, and the blocked free space is the same as the free space. The point d is chosen on w so that the shortest path between a and d touches an obstacle.

To bootstrap the pursuit, we simply choose a point $a = b = c$ on the boundary of the ambient space Ω as a degenerated triangle configuration.

V. A Proof of Guaranteed Capture By Three Pursuers

We begin by showing how to construct shortest paths whose endpoints lie on the boundary of the ambient space, and are guaranteed to touch an obstacle.

Let $s \subset \Omega_{\text{free}}$ be a shortest path connecting two points $x, y \in \Omega_{\text{free}}$. A well-known fact about shortest paths is that s is straight except where it touches the boundary of Ω_{free}. This fact holds because every point $p \in s$ in the interior of Ω_{free} is the center of an open disk $D \subseteq \Omega_{\text{free}}$, and $s \cap D$ must be a straight line segment; otherwise, we could find a shorter path by replacing $s \cap D$ with one.

We wish to consider paths that are *locally shortest* in the sense that they cannot be made shorter by a small perturbation, although it might be possible to find a shorter path by taking a different route around the obstacles. A path s is *taut* if there is an $\epsilon > 0$ such that no point $p \in s$ has an open neighborhood in s of diameter less than ϵ that could be shortened by replacing it with a straight line segment included in Ω_{free}. If a taut path were a piece of string, pulling its ends would not change it. Every shortest path is taut, but not every taut path is a globally shortest path.

A point $p \in s$ is a *turning point* if no open neighborhood of p in s is a straight line segment. Turning points can be *curve points* at which there is a well-defined line tangent to s though s is not straight, or *corners* at which the line tangent to s is not defined. If we suppose s is *directed* from x to y, then every turning point is either a left turn or a right turn. At a left turn, a taut path must touch a component of $\mathbb{R}^2 \setminus \Omega_{\text{free}}$ on its left side, and at a right turn it must touch a component of $\mathbb{R}^2 \setminus \Omega_{\text{free}}$ on its right side.

Lemma 2: Let $s_1, s_2 \subset \Omega_{\text{free}}$ be two distinct taut paths connecting two points $x, y \in \Omega_{\text{free}}$, with s_2 longer than s_1 or equally long. Then $s_1 \cup s_2$ encloses an obstacle in Ω_{free}, and s_2 touches it.

Proof: We call a closed Jordan curve a loop. Because s_1 and s_2 are distinct, $s_1 \cup s_2$ includes one or more loops; moreover, in at least one of these loops the portion of s_2 in the loop is as least as long as that of s_1. At least one turning point $p \in s_2$ on that loop has curvature toward the region enclosed by the loop. Therefore, p lies on the boundary of a component of $\mathbb{R}^2 \setminus \Omega_{\text{free}}$ enclosed by the loop. This component is bounded and therefore is an obstacle. ∎

Corollary 1: If Ω_{free} has no obstacles, then any two points $x, y \in \Omega_{\text{free}}$ are connected by exactly one taut path, which is the unique shortest path connecting them.

Theorem 1: Let $\Omega = \Omega_{\text{free}} \cup \Omega_{\text{obs}}$ be a game domain, with Ω a topological disk, and suppose Ω_{obs} is nonempty. Then for every point $x \in \Omega_{\text{free}}$, there exists a point y on the boundary of Ω such that every taut path $s \subset \Omega_{\text{free}}$ connecting x and y touches an obstacle.

Moreover, the following procedure finds such a point y. Let q be any point in the closure of Ω_{obs}. Let $s_\Omega(x, q) \subset \Omega$ be a taut path (which is therefore the shortest path) from x to q in Ω. (We emphasize that this is a path in Ω, not in Ω_{free}.) If q lies on the boundary of Ω, let $y = q$ and $s_\Omega(x, y) = s_\Omega(x, q)$; otherwise, observe that a sufficiently small neighborhood of q in $s_\Omega(x, q)$ is a straight line segment, and let $s_\Omega(x, y)$ be the path found by extending $s_\Omega(x, q)$ from q along a straight line until it strikes a point y on the boundary of Ω.

Proof: Because $s_\Omega(x, q)$ is a taut path in Ω, $s_\Omega(x, y)$ is also a taut path in Ω. As Ω has no obstacles, $s_\Omega(x, y)$ is a shortest path by Corollary 1, and no other path connecting x to y in Ω is equally short.

Let $s \subset \Omega_{\text{free}}$ be a taut path connecting x and y in Ω_{free}. If $q \in s$, it immediately follows that s touches an obstacle; otherwise, s must differ from $s_\Omega(x, y)$, and therefore must be longer than $s_\Omega(x, y)$. The union $s \cup s_\Omega(x, y)$ includes one or more loops; in one of these loops, the portion provided by s is longer than the portion provided by $s_\Omega(x, y)$. Therefore, some turning point $p \in s$ on that loop has curvature toward the region enclosed by the loop. As s is taut, p lies on the boundary of a component of Ω_{obs}. ∎

We argue that our strategy is guaranteed to make progress toward having the final game subdomain to be a simply connected region of the triangle configuration. Our argument is by induction on the characteristic number of a subdomain, defined as follows.

Definition 7: The *characteristic number* $\chi(\Psi)$ of a game subdomain Ψ is the number of obstacles in Ψ plus the number of obstacles in Ψ that neither s_1 nor s_2 touches. (Thus, the latter obstacles are counted twice.)

For example, in Figure 3, $\chi(\Psi) = 3$, because the subdomain encloses one obstacle that s_1 and s_2 touch and one obstacle that neither touch.

The base case of the induction occurs when the characteristic number of a game subdomain is zero, in which case the game subdomain is a simply connected and the third pursuer can capture the evader by direct pursuit.

Proposition 1: Let $\Psi = \Psi_{\text{free}} \cup \Psi_{\text{obs}}$ be a game subdomain with respect to the game domain $\Omega = \Omega_{\text{free}} \cup \Omega_{\text{obs}}$. Let Ψ be of the triangle configuration. Suppose that at some point in time the evader is in Ψ, one pursuer P_1 is defending s_1, a second pursuer P_2 is defending s_2, and a third pursuer is available. Then there is a strategy that guarantees that the three pursuers capture the evader.

Proof: In case 1, $\chi = 0$ and P_3 captures the evader by direct pursuit. In case 2, the old subdomain gets seperated by at least one obstacle into multiple connected components. And the new subdomain does not enclose those obstacles so χ decreases by at least one. See Figure 2. In case 4, the new path s_3 touches an obstacle that neither s_1 nor s_2 touches, decreasing χ by at least one. See Figure 4.

In case 3, without loss of generality, we assume at least one obstacle touches s_1. There are two possibilities.

The first possibility is that s_3 touches an obstacle which touches either s_1 or s_2(if s_2 has any touching obstacle). Without loss of generality, we assume s_3 touches s_1. See figure 4. In this case, if the evader is in the left region(enclosed by s_1 and s_3), then we are back to case 2. If the evader is in the right region(enclosed by s_3, w and s_2), then the right region lacks at least one obstacle that touches s_1. Therefore, χ decreases by at least one.

The second possibility is that s_3 touches an obstacle which touches neither s_1 nor s_2. See Figure 5. In this case, either the left region includes the obstacle touched by s_3(This is the case shown in Figure 5) or the right region includes the obstacle. If the left region includes the obstacle, then either the evader is in the left region or in the region. If the evader is in the left region, since now the obstacle touches s_3, the χ decreases by at least 1. If the evader is in the right region, since now the obstacle is out of the right region, the χ decreases by at least 2. The same argument applies to the case that the right region includes the obstacle. ∎

This proposition establishes the main result.

Theorem 2: Given a game domain $\Omega = \Omega_{\text{free}} \cup \Omega_{\text{obs}}$, three pursuers suffice to capture the evader.

VI. The Necessity of Three Pursuers

In this section we show that three pursuers are necessary for an arbitrary game domain via an example, in which the evader can evade two pursuers indefinitely regardless of the controls chosen by the two pursuers. Consider the following game domain Ω in Figure 7. The free space in this domain consists of the straight and wiggly black lines, representing tunnels. All tunnels, straight or wiggling, have the same

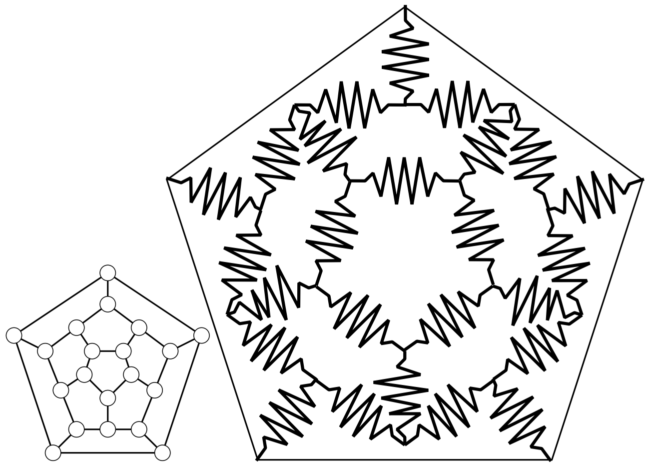

Fig. 7. On the left is the discrete dodecahedron graph. On the right is a domain with the graph of a dodecahedron. Each tunnel between two degree-three junctions has the same travel time via a optimal path. Internal structure of the tunnels and the junctions not shown.

travel time equal to 1, assuming an optimal path is taken from the center of a junction to that of a neighboring junction. In this example, every junction is a circular shape that has radius equal to some small number compare to 1. We construct a control for the evader the following.

Let the initial position of the evader be at the center of a junction.The two pursuers $P1$ and $P2$ are at positions both more than distance r from the evader. The following is a strategy for the evader.

1) The evader shall remain stationary until at least one pursuer comes close within a predefined distance $\epsilon > r$ of the evader.

2) If both pursuers come within ϵ of the evader, then the evader travels through a tunnel that does not contain any pursuer via the shortest path to the center of the other end. The evader can always find such a tunnel since each end is joined to three tunnels. After the evader reaches the center of the other end, repeat step 1.

3) If only $P1$ comes close to the evader, then the evader has two tunnels to choose between. The evader then commits to the tunnel which, upon arriving the other end, would maximize the distance between the current position of $P2$ and itself, with $P2$ knowing which tunnel the evader chooses as soon as the evader commits to a tunnel .

Proposition 2: The preceding strategy enables the evader to evade indefinitely.

Proof: At the beginning of the game, the evader is not captured. If neither of the two pursuers comes within ϵ of the evader, the evader is safe sitting at the center. This establishes that the evader is safe as long as it is stationary at a junction.

If both pursuers approach the evader, then the evader is also safe while moving through the remaining tunnel to the next junction, as the distance between the evader and either pursuer will be larger than r throughout. When the evader reaches the center of the junction of the tunnel, it is still not captured. At this point, for capture purposes, this is equivalent to returning to the initial state, with the evader un-captured at the center of a junction.

If only one pursuer P_1 approaches the evader, then since every cycle is of length five, no matter where the other pursuer P_2 moves, the evader will be able to choose a tunnel such that when it reaches junction at the other end of the tunnel, P_2 will be at least 0.5 away. Again the evader is not captured when upon reaching the other junction. Therefore, by the same argument, the evader can continue to remain un-captured indefinitely. ■

VII. A Piecewise Continuous Control For Defending A Path

Lemma 1 establishes that after a pursuer is defending $s(x, y)$, the evader will coincide with the pursuer's position on entering the path $s(x, y)$ and will therefore be captured. However, the control given there can be a highly discontinuous function depending on how the evader moves. Such an intricate control is not implementable in reality and is therefore rendered useless in real world applications. We note that Lemma 1 asserts a stronger condition than what is required since the pursuer does not need to capture the evader pointwise when the evader enters the path . Below is a relaxed definition.

Definition 8: A path $s \subset \Omega_{\text{free}}$ is *1-pursuer weakly defendable* if there is a strategy by which a pursuer can position himself on s in finite time, and thereafter guarantee that he will catch the evader at any time the evader moves onto s.

Lemma 3: For any two points $x, y \in \Omega_{\text{free}}$, let $s(x, y) \subset \Omega_{\text{free}}$ be a shortest path connecting x and y in Ω_{free}. Then $s(x, y)$ is 1-pursuer weakly defendable. Furthermore, a pursuer can succeed in defending $s(x, y)$ by using a piecewise continuous control.

The high level idea remains roughly the same as before except in this case, we partition time into intervals of a carefully chosen length and construct a continuous control for each time interval. In the proof below, we use the same notation as that in the proof for Lemma 1.

Proof: Choose $\delta = r/2$. At time t, let k be the unique non-negative integer such that the following holds:

$$k\delta \leq t < (k+1)\delta. \tag{9}$$

1) If $d(P(k\delta), i(k\delta)) \leq \delta$, then P stays stationary at t. That is, if at the beginning ($t = k\delta$) of the time interval $k\delta \leq t < (k+1)\delta$, the distance between the pursuer and the image of the evader is no larger than δ, then P stays stationary for the entire time interval defined by 9.
2) Otherwise, P at time t shall head towards $i(k\delta)$ along $s(x, y)$. That is, if at the beginning($t = k\delta$) of the time

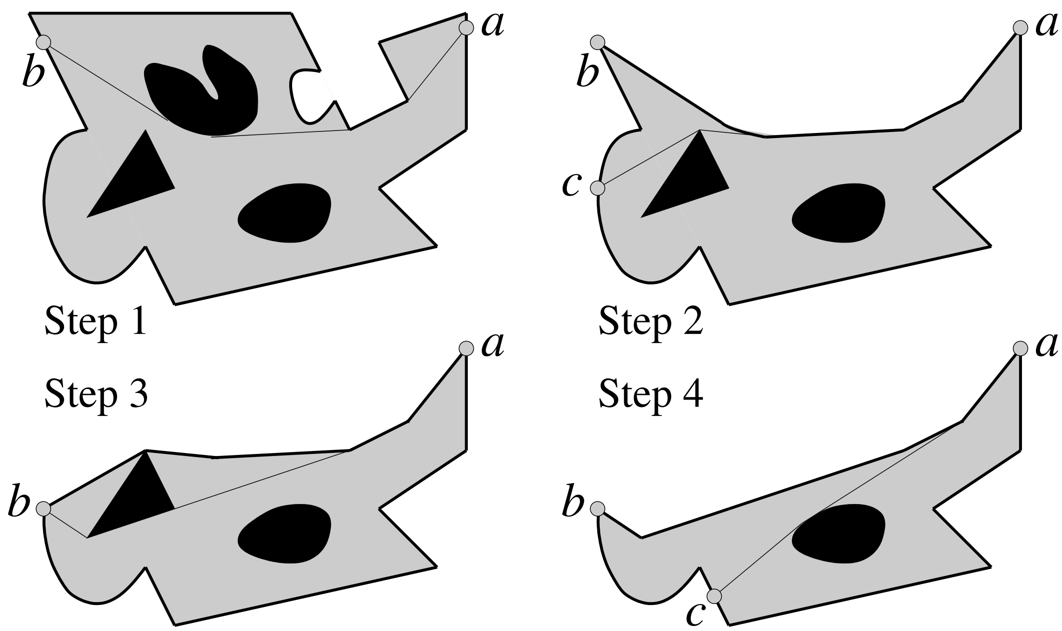

Fig. 8. An example showing the proposed pursuit algorithm used in a general game domain.

interval defined by 9, the distance between the pursuer and the image of the evader is larger than δ, then P shall head towards $i(k\delta)$ by traveling along $s(x, y)$ for this entire time interval.

We note that in the second case, P shall never stop at any point during the time interval, since by assumption we have $d(P(k\delta), i(k\delta)) > \delta$ to begin with. Therefore, in δ time, P will only be able to travel δ. In other words, P certainly will not be able to reach $i(k\delta)$ during any point of the time interval. It is also easily seen that this control is piecewise continuous.

To show that this strategy enables $s(x, y)$ to be 1-pursuer weakly defendable, we first prove that if at $t = k\delta, d(P(k\delta), i(k\delta)) \leq 2\delta$, then we will have

$$d(P(t), i(t)) \leq 2\delta \tag{10}$$

for all $t \geq k\delta$. We classify the two possibilities:

1) If $d(P(t), i(t)) \leq \delta$ at $t = k\delta$, then by our devised control strategy, P remains stationary throughout the time interval $k\delta \leq t < (k+1)\delta$. But since the evader can travel at most δ within this time interval, we are again guaranteed with the inequality 10 at $t = (k+1)\delta$ and throughout the time interval $k\delta \leq t < (k+1)\delta$.
2) If $\delta < d(P(t), i(t)) \leq 2\delta$ at $t = k\delta$, then by our control P will travel towards $i(k\delta)$ along $s(x, y)$ throughout the time interval $k\delta \leq t < (k+1)\delta$ and $i(k\delta)$ is at most 2δ away from $P(k\delta)$. We have shown in the proof of Lemma 1 that during the time interval between $k\delta$ and $(k+1)\delta$, $i(t)$ can travel at most δ. But during this same time interval $P(t)$ will travel δ since p is traveling along the shortest path nonstop. Therefore, their distance is still within 2δ at $t = (k+1)\delta$ and throughout the time interval $k\delta \leq t < (k+1)\delta$.

By induction on k, we have inequality 10 for all $t \geq k\delta$. And by the same argument used in the proof of Lemma 1, we conclude that it takes a finite amount of time for P to get within distance 2δ of the evader's image i on $s(x, y)$. Consequently, if after P manages to achieve the condition given in the inequality 10 and if e ever crosses $s(x, y)$, the euclidean distance between $P(t)$ and $i(t)$ is less than or equal to $d(P(t), i(t))$, which in turn is no larger than 2*δ = r. This establishes that e will be instantaneously captured should he enter $s(x, y)$. Hence, $s(x, y)$ is 1-pursuer weakly defendable. ∎

VIII. Demonstration of the Capture Strategy

In this section we demonstrate how our control strategy may be applied to a general game domain. The example we use is the domain shown in Figure 8. In each step, P_1 and/or P_2 act to partition the game domain into subdomains, one of which contains the evader. If the evader is in a simply connected subdomain, it is captured by P_3 using direct pursuit. Otherwise the strategy is iterated until the evader is contained within a simply connected subdomain. In Step 1, a defendable path is established between points a and b by P_1, partitioning the domain into three subdomains: two simply connected regions and a third containing obstacles. Note that this effectively removes the large irregular obstacle from the game, as it is contained entirely within a subdomain where the free space is simply connected. In Step 2, P_2 establishes a defendable path between a and a point c, selected according to the procedure outlined in Theorem 1 and freeing P_1. In Step 3, a second path between a and b established by P_1 partitions the resulting subdomain, removing the triangular obstacle. If the evader is not in one of the simply connected regions, P_2 is free to establish a path between a and a new point c, removing the ellipsoidal obstacle, as in Step 4.

IX. Conclusions and Future Work

Our results show that three players are sufficient and sometimes necessary to capture an evader in a general planar domain, a conclusion of practical consequence for multi-agent systems in pursuit-evasion tasks. The strategy proposed for the proof provides some insight toward synthesizing computable, implementable control strategies for these agents, which is the most important avenue of future work. What the optimal pursuit strategy may be in terms of capture time is also an open question.

An interesting open question is whether there is a guaranteed pursuit strategy for domains in three dimensions or higher, for which our path-defending method will not work.

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