Safe Sequential Path Planning of Multi-Vehicle Systems Under Presence of Disturbances and Measurement Noise

Somil Bansal*, Mo Chen*, Jaime F. Fisac, and Claire J. Tomlin

***Abstract*—**

I. Introduction

II. Problem Formulation

Consider N vehicles whose joint dynamics described by the time-varying ordinary differential equation

$$\begin{aligned} \dot{x}_i &= f_i(t, x_i, u_i, d_i) \\ u_i &\in \mathcal{U}_i \\ d_i &\in \mathcal{D}_i \\ i &= 1, \dots, N \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$ is the state of the ith vehicle, u_i is the control and of the ith vehicle, and d_i is the disturbance experienced by the ith vehicle. In general, the physical meaning of x_i and the dynamics f_i depend on the specific dynamic model of vehicle i, and need not be the same across the different vehicles.

We assume that the control functions $u_i(\cdot), d_i(\cdot)$ are drawn from the set of measurable functions[1]. Furthermore, we assume $f_i(t, x_i, u_i, d_i)$ is bounded, Lipschitz continuous in x_i for any fixed t, u_i, d_i, and measurable in t, u_i, d_i for each x_i. Therefore given any initial state x_i^0 and any control function $u_i(\cdot)$, there exists a unique, continuous trajectory $x_i(\cdot)$ solving (1) [1].

For convenience, let $p_i \in \mathbb{R}^p$ denote the position of vehicle i; note that p_i in most practical cases would be a subset of the state x_i. Denote the rest of the states h_i, so that $x_i = (p_i, h_i)$. Under the worst case disturbance, each vehicle aims to get to some set of target states, denoted $\mathcal{T}_i \subset \mathbb{R}^{n_i}$ at some scheduled time of arrival t_{STA}. On its way to the target set $\mathcal{T}_i$, each vehicle must avoid the danger zones $\mathcal{A}_{ij}(t)$ of all other vehicles $j \neq i$ for all time. In general, the danger zone can be defined to capture any undesirable configuration between vehicle i and vehicle j. For simplicity, in this paper we define $\mathcal{A}_{ij}(t)$ as

$$\mathcal{A}_{ij}(t) = \{x_i \in \mathbb{R}^{n_i} : \|p_i - p_j(t)\| \leq R_c\}, \tag{2}$$

the interpretation of which is that a vehicle is another vehicle's danger zone if the two vehicles are within a distance of R_c apart.

The problem of driving each of the vehicles in (1) into their respective target sets $\mathcal{T}_i$ would be in general a differential game of dimension $\sum_i n_i$. Due to the exponential scaling of the complexity of the state space with the problem dimension, an optimal solution is computationally intractable.

In this paper, we impose a mild structure to the general problem in order to trade complexity for optimality: we assign a priority to each vehicle. While traveling to its target set, a vehicle may ignore the presence of lower priority vehicles, but must take full responsibility for avoiding higher priority vehicles. Such a joint path planning scheme makes intuitive and practical sense, and the priorities can be assigned, for example, on a first-come first-serve basis.

Recently, [2] described how such a sequential path planning algorithm can be implemented using a HJ reachability approach without taking into account the presence of the disturbances d_i and limited information available to each vehicle. In this paper, we extend the work in [2] to consider these practically important aspects of the problem. In particular, we answer the following inter-dependent questions that were not previously addressed:

1) How can each vehicle guarantee that it will reach its target set without getting into any danger zones, despite the disturbances it experiences?
2) How can each vehicle take into account the disturbances that other vehicles experience?
3) How should each vehicle robustly handle situations with limited information about the state and intention of other vehicles?

III. Solution via double-obstacle HJI VI and SPP

A. *Double-Obstacle Hamilton-Jacobi Variational Inequality*

Our solution method takes advantage of the double-obstacle HJ approach [3], in which one computes the reachable set $\mathcal{V}(t)$ in the presence of a time-varying target set $\mathcal{T}(t)$ and time-varying obstacles $\mathcal{G}(t)$. Mathematically, we are given a system with state z evolving according to the following ODE:

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* Both authors contributed equally to this work. All authors are with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley. {mochen72, somil, jfisac, tomlin}@eecs.berkeley.edu

[1] A function $f : X \to Y$ between two measurable spaces (X, Σ_X) and (Y, Σ_Y) is said to be measurable if the preimage of a measurable set in Y is a measurable set in X, that is: $\forall V \in \Sigma_Y, f^{-1}(V) \in \Sigma_X$, with Σ_X, Σ_Y σ-algebras on X,Y.

$$
\begin{aligned}
\dot{z} &= f(t, z, u, d) \\
z(0) &= z_0 \\
u &\in \mathcal{U}, d \in \mathcal{D} \\
t &\in [0, T]
\end{aligned}
\tag{3}
$$

After defining some target set $\mathcal{T}(t)$, we compute the backwards reachable set $\mathcal{V}(t)$, defined by

$$
\begin{aligned}
&\mathcal{T}(t) = \{z : \exists u \in \mathbb{U}, \forall \gamma[u] \in \Gamma, (3) \\
&\Rightarrow \exists s \in [t, T], z(s) \in \mathcal{T}(s) \wedge z(\tau) \notin \mathcal{G}(\tau) \forall \tau \in [t, s]\}
\end{aligned}
\tag{4}
$$

where $\mathbb{U}$ is the set of measurable functions satisfying control constraints at every t, and Γ is the set of non-anticipative strategies [4]. Intuitively, the reachable set is the set of states from which exists a control such that for all non-anticipative disturbances, the system is driven into the target set $\mathcal{T}(t)$ in the time horizon $[t, T]$ without first entering the obstacle set $\mathcal{G}(t)$.

Given the target set $\mathcal{T}(t)$ specified as an implicit surface function such that $\mathcal{T}(t) = \{z : l(t, z) \leq 0\}$, the reachable set can be obtained as the implicit surface function such that $\mathcal{V}(t) = \{z : V(t, z) \leq 0\}$, where $V(t, z)$ is the viscosity solution [5] to the following HJ variational inequality:

$$
\begin{aligned}
&\max \Big\{ \min \left\{ D_t V(t, z) + H\left(t, z, D_z V\right), l(t, z) - V(t, z) \right\} \\
&\qquad - g(t, z) - V(t, z) \Big\} = 0, \quad t \in [0, T] \\
&\qquad V(T, x) = \max \left\{ l(T, x), -g(T, x) \right\} \\
&\qquad H\left(t, z, p\right) = \min_{u \in \mathcal{U}} \max_{d \in \mathcal{D}} p \cdot f(t, z, u, d)
\end{aligned}
\tag{5}
$$

where $g(t, x)$ is the implicit surface function representing $\mathcal{G}(t)$: $\mathcal{G}(t) = \{z : g(t, x) \leq 0\}$. After the reachable set is computed, the optimal control can be obtained as follows:

$$
u^* = \arg \min_{u \in \mathcal{U}} \max_{d \in \mathcal{D}} H\left(t, z, D_z V\right) \tag{6}
$$

In theory, one could define the state to be the joint states of all vehicles, $z = (x_1, x_2, \ldots, x_N)$, define the dynamics (3) to follow (1), the target set $\mathcal{T}$ to correspond to the situation in which all vehicles have arrived at their targets $\mathcal{T}_i, i = 1, \ldots, N$, and the obstacle set $\mathcal{G}$ to correspond to the union of all the danger zones $\mathcal{A}_{ij}$. Then, (5) could be solved to obtain $\mathcal{V}(t)$, and then the joint optimal control would be given by (6).

However, practically, the dimensionality of the joint state z would be extremely high. In fact, for even the simplest vehicle models, solving (5) would be intractable for more than two vehicles. Therefore, we propose the sequential path planning method, which allows (5) to be solved in the state space of each vehicle, making the computation complexity scale linearly, as opposed to exponentially, with the number of vehicles.

B. Sequential Path Planning

In order to make the N-vehicle path planning problem safe and tractable, we impose a slight structure to the problem: each vehicle is assigned a priority. When planning its trajectory to its target, a higher priority vehicle can disregard the presence of a lower priority vehicle. In contrast, a lower priority vehicle must take into account the presence of all higher priority vehicles, and plan its trajectory in a way that avoids the higher priority vehicles' danger zones. For convenience, denote the vehicle with the ith highest priority as Q_i.

Optimal path planning in this setting is enabled by the HJ variational inequality which computes the backwards reachable set $\mathcal{V}_i$ from a target set $\mathcal{T}_i$ in the presence of time-varying obstacles $\mathcal{G}_i$. In the sequential path planning application, the time-varying obstacles represent regions of the state space of Q_i that must be avoided in order to ensure that Q_i does not enter any danger zones of higher priority vehicles. We present three different ways to compute $\mathcal{G}_i$, obstacles induced by higher priority vehicles in Section IV. For now, we proceed assuming $\mathcal{G}_i$ is given.

To obtain the optimal control for to reach the target we adapt (5) to Q_i and solve the following PDE:

$$
\begin{aligned}
&\max \Big\{ \min \left\{ D_t V_i(t, x_i) + H_i\left(t, x_i, D_{x_i} V\right), l_i(t, x_i) - V_i(t, x_i) \right\} \\
&\qquad - g_i(t, x_i) - V_i(t, x_i) \Big\} = 0, \quad t \in [t_0, t_{\text{STA}}] \\
&\qquad V_i(t_{\text{STA}}, x_i) = \max \left\{ l_i(x_i), -g_i(t_{\text{STA}}, x_i) \right\}
\end{aligned}
\tag{7}
$$

$H_i\,(t$

Here, the target set $\mathcal{T}_i$, obstacle set $\mathcal{G}_i$, and backwards reachable set $\mathcal{V}_i$ are related to $l_i(x_i), g_i(t, x_i), V_i(t, x_i)$ as follows:

$$
\begin{aligned}
\mathcal{T}_i &= \{x_i : l_i(x_i) \leq 0\} \\
\mathcal{G}_i &= \{x_i : g_i(t, x_i) \leq 0\} \\
\mathcal{V}_i &= \{x_i : V_i(t, x_i) \leq 0\}
\end{aligned}
\tag{8}
$$

From the reachable set, the optimal control for vehicle Q_i is then given as

$$
u_i^* = \arg \min_{u_i \in \mathcal{U}_i} H_i\left(t, x_i, D_{x_i} V_i\right) \tag{9}
$$

IV. Obstacle Generation

Obstacles can be generated in many different ways depending on the assumptions made about the information the vehicles have about each other.

A. Method 1: Centralized Planner

If there is a centralized planner directly controlling each of the N vehicles, the control law of each vehicle can be enforced. In this case, lower priority vehicles can safely assume that higher priority vehicles are applying the enforced control law. One possible control law that a higher priority vehicle Q_j can be assumed to be using is the control law $u_j^*(x_j)$ given by (9), which takes each vehicle to the target in the optimal way according to the value function $V_j(t, x_j)$.

From the perspective of a lower priority vehicle Q_i, a higher priority vehicle $Q_j, j < i$ induces an time-varying obstacle that represent the positions that could possibly be within the capture radius R_c of Q_j given that Q_j is executing the feedback controller $u_j^*(x_j)$. Determining this obstacle involves first solving a forward reachability problem, which computes the forward reachable set of Q_j starting from its initial state $x_j(t_0)$ at initial time t_0, denoted $\mathcal{W}_j(t)$ and defined as follows:

$$\begin{aligned}\mathcal{W}_j(t) = \{y \in \mathbb{R}^{n_j} : \dot{x}_j = f_j(x_j, u_j^*(x), d_j) \Rightarrow \\ \forall d_j \in \mathcal{D}_j, \exists s \in [t_0, t], x_j(s) = y\}\end{aligned} \tag{10}$$

Conveniently, $\mathcal{W}_j(t)$ can also be computed using (5) with the functions l and g chosen to be such that $\mathcal{T} = \{x_j(t_0)\}$ and $\mathcal{G} = \emptyset$. In practice, when there is uncertainty in the initial state of Q_j, we set $\mathcal{T}$ to be a small region around $x_j(t_0)$.

The forward reachable set $\mathcal{W}_j(t)$ represents the set of possible states of a higher priority vehicle Q_j given the worst case disturbance $d_j(\cdot)$ and given that Q_j uses the feedback controller $u_j^*(x)$. In order for a lower priority vehicle Q_i to guarantee that it does not go within a distance of R_c to Q_j, Q_i must stay a distance of at least R_c away from the set of positions in the set $\mathcal{W}_j(t)$ for all possible values of the non-position states. This gives the obstacle induced by a higher priority vehicle Q_j for a lower priority vehicle Q_i as follows:

$$\mathcal{O}_i^j(t) = \{x_i : \text{dist}(p_i, \mathcal{P}_j(t)) \leq R_c\} \tag{11}$$

where the dist function represents the minimum distance from a point to a set, and the set $\mathcal{P}_j(t)$ is the set of positions in the forward reachable set $\mathcal{W}_j(t)$, disregarding the non-position dimensions h_j:

$$\mathcal{P}_j(t) = \{p : \exists h_j, (p, h_j) \in \mathcal{W}_j(t)\} \tag{12}$$

Finally, the time-varying obstacles needed to solve (7) is just the union of all obstacles induced by higher priority vehicles:

$$\mathcal{G}_i(t) = \bigcup_{j=1}^{i-1} \mathcal{O}_i^j(t) \tag{13}$$

B. Method 2

C. Method 3

V. Numerical Simulations

VI. Conclusions and Future Work

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