Robustness of Control Barrier Functions for Safety Critical Control ⋆

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Abstract: Barrier functions (also called certificates) have been an important tool for the verification of hybrid systems, and have also played important roles in optimization and multi-objective control. The extension of a barrier function to a controlled system results in a control barrier function. This can be thought of as being analogous to how Sontag extended Lyapunov functions to control Lypaunov functions in order to enable controller synthesis for stabilization tasks. A control barrier function enables controller synthesis for safety requirements specified by forward invariance of a set using a Lyapunov-like condition. This paper develops several important extensions to the notion of a control barrier function. The first involves robustness under perturbations to the vector field defining the system. Input-to-State stability conditions are given that provide for forward invariance, when disturbances are present, of a "relaxation" of set rendered invariant without disturbances. A control barrier function can be combined with a control Lyapunov function in a quadratic program to achieve a control objective subject to safety guarantees. The second result of the paper gives conditions for the control law obtained by solving the quadratic program to be Lipschitz continuous and therefore to gives rise to well-defined solutions of the resulting closed-loop system.

Keywords: Barrier function, Invariant set, Quadratic program, Robustness, Continuity

1. INTRODUCTION

Lyapunov functions are used to certify stability properties of a set without calculating the exact solution of a system. In a similar manner, barrier certificates (functions) are used to verify temporal properties (such as safety, avoidance, eventuality) of a set, without the difficult task of computing the system's reachable set; see Prajna and Rantzer (2007), Prajna et al. (2007). These same references show that when the vector fields of the system are polynomial and the sets are semi-algebraic, barrier certificates can be computed by sum-of-squares optimization. In the original formulation of Prajna et al. (2007), all sublevel sets of the barrier certificate were required to be invariant because the derivative of the barrier certificate along solutions was required to be non-positive. This condition was relaxed by Kong et al. (2013) and Dai et al. (2013) so that tighter over-approximations of the reachable set could be obtained, and such that more expressive barrier certificates could be synthesized using semi-definite programming. The key idea there was to only require that a single sublevel set be invariant, namely, the set of points where the barrier certificate was non-positive.

The natural extension of barrier functions to a system with control inputs is a control barrier function (CBF), first proposed by Wieland and Allgöwer (2007); this work used the original condition of a barrier function that imposes invariance of all sublevel sets. The unification of control Lyapunov functions (CLFs) with CBFs appeared at the same conference in Romdlony and Jayawardhana (2014) and Ames et al. (2014b), using two contrasting formulations. The objective of Romdlony and Jayawardhana (2014) was to incorporate into a single feedback law the conditions required to simultaneously achieve asymptotic stability of an equilibrium point, while avoiding an unsafe set. The feedback law was constructed using Sontag's universal control formula (Sontag (1989)), provided that a "control Lyapunov barrier function" inequality could be met. Importantly, if the stabilization and safety objectives were in conflict, then no feedback law could be proposed. In contrast, the approach of Ames et al. (2014b) was to pose a feedback design problem that *mediates* the safety and stabilization requirements, in the sense that safety is

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always guaranteed, and progress toward the stabilization objective is assured when the two requirements "are not in conflict".

The essential difference between these two approaches is perhaps best understood through an example. A vehicle equipped with Adaptive Cruise Control (ACC) seeks to converge to and maintain a fixed cruising speed, as with a common cruise control system. Converging to and maintaining fixed speed is naturally expressed as asymptotic stabilization of a set. With ACC, the vehicle must in addition guarantee a safety condition, namely, when a slower moving vehicle is encountered, the controller must automatically reduce vehicle speed to maintain a guaranteed lower bound on time headway or following distance, where the distance to the leading vehicle is determined with an onboard radar. When the leading car speeds up or leaves the lane, and there is *no longer a conflict between safety and desired cruising speed*, the adaptive cruise controller automatically increases vehicle speed. The time-headway safety condition is naturally expressible as a control barrier function. In the approach of Ames et al. (2014b), a Quadratic Program (QP) mediates the two inequalities associated with the CLFs and CBFs; in particular, relaxation is used to make the stability objective a soft constraint while safety is maintained as a hard constraint. In this way, safety and stability do not need to be simultaneously satisfiable. On the other hand, the approach of Romdlony and Jayawardhana (2014) is only applicable when the two objectives can be simultaneously met.

A second, although less important, difference in the two approaches is that Romdlony and Jayawardhana (2014) used the more restrictive invariance condition of Prajna and Rantzer (2007), while Ames et al. (2014b) used the relaxed condition of Kong et al. (2013), appropriately interpreted for the type of barrier function often used in optimization, see Boyd and Vandenberghe (2004), where the barrier function is unbounded on the boundary of the allowed set, instead of vanishing on the set boundary.

The present paper builds on previous work in two important directions. First, the robustness of barrier functions and control barrier functions under model perturbation is investigated. An Input-to-State (ISS) stability property of a safe set is established when perturbations are present and the barrier function vanishes on the set boundary. The second result gives conditions that guarantee local Lipschitz continuity of the feedback law arising from the QP used to mediate safety and asymptotic convergence to a set. The analysis is based on the constraint qualification conditions along with the KKT conditions for optimality. While the result is applicable to the type of barrier function in Ames et al. (2014b), it will be stated for barrier functions used in this paper that vanish on the set boundary.

The remainder of the paper is organized as follows. Section 2 defines zeroing barrier functions and zeroing control barrier functions, and establishes a robustness property to model perturbations. Section 3 develops the conditions for the solution of the QP to be locally Lipschitz continuous in the problem data. The theory developed is illustrated in Section 4 on adaptive cruise control. Section 5 summarizes the conclusions.

Notation: The set of real, positive real and non-negative real numbers are denoted by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$, respectively. The Euclidean norm is denoted by $\|\cdot\|$. The transpose of matrix A is denoted by $A^\top$. The interior and boundary of a set $\mathcal{S}$ are denoted by $\mathrm{Int}(\mathcal{S})$ and $\partial\mathcal{S}$, respectively. The distance from x to a set $\mathcal{S}$ is denoted by $\|x\|_{\mathcal{S}} = \inf_{s\in\mathcal{S}} \|x - s\|$. For any essentially bounded function $g : \mathbb{R} \to \mathbb{R}^n$, the infinity norm of g is denoted by $\|g\|_\infty = \operatorname{ess\,sup}_{t\in\mathbb{R}} \|g(t)\|$.

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *Lipschitz continuous* on $I \subset \mathbb{R}^n$ if there exists a constant $L \in \mathbb{R}^+$ such that $\|f(x_2) - f(x_1)\| \leq L\|x_2 - x_1\|$ for all $x_1, x_2 \in I$, and called *locally Lipschitz continuous* at a point $x \in \mathbb{R}^n$ if there exist constants $\delta \in \mathbb{R}^+$ and $M \in \mathbb{R}^+$ such that $\|f(x) - f(x')\| \leq M\|x - x'\|$ holds for all $\|x - x'\| \leq \delta$. A continuous function $\beta_1 : [0, a) \to [0, \infty)$ for some $a > 0$ is said to belong to *class* $\mathcal{K}$ if it is strictly increasing and $\beta_1(0) = 0$. A continuous function $\beta_2 : [0, b) \times [0, \infty) \to [0, \infty)$ for some $b > 0$ is said to belong to *class* $\mathcal{KL}$, if for each fixed s, the mapping $\beta_2(r, s)$ belongs to class $\mathcal{K}$ with respect to r and for each fixed r, the mapping $\beta_2(r, s)$ is decreasing with respect to s and $\beta_2(r, s) \to 0$ as $s \to \infty$.

2. ZEROING (CONTROL) BARRIER FUNCTIONS

The barrier function and control barrier function considered in this paper are based on Kong et al. (2013), Dai et al. (2013), and Wieland and Allgöwer (2007). As in Ames et al. (2014b), the primary focus is to establish forward invariance of a given set $\mathcal{C}$, which one may interpret as an under approximation of the "initial set" and the "safe set" in previous formulations of barrier functions. The main contribution of the section is a robustness property under model perturbations.

Consider a nonlinear system on $\mathbb{R}^n$,

$$\dot{x} = f(x), \tag{1}$$

with f locally Lipschitz continuous. Denote by $x(t, x_0)$ the solution of (1) with initial condition $x_0 \in \mathbb{R}^n$. To simplify notation, the solution is also denoted by $x(t)$ whenever the initial condition does not play an important role in the discussion. The *maximal interval of existence* of $x(t, x_0)$ is denoted by $I(x_0)$. When $I(x_0) = \mathbb{R}_0^+$ for any $x_0 \in \mathbb{R}^n$, the differential equation (1) is said to be *forward complete*. A set $\mathcal{S}$ is called *forward invariant* if for every $x_0 \in \mathcal{S}$, $x(t, x_0) \in \mathcal{S}$ for all $t \in I(x_0)$.

For $\epsilon \geq 0$, define the family of closed sets $\mathcal{C}_\epsilon$ as

$$\mathcal{C}_\epsilon = \{x \in \mathbb{R}^n : h(x) \geq -\epsilon\}, \tag{2}$$

$$\partial\mathcal{C}_\epsilon = \{x \in \mathbb{R}^n : h(x) = -\epsilon\}, \tag{3}$$

$$\mathrm{Int}(\mathcal{C}_\epsilon) = \{x \in \mathbb{R}^n : h(x) > -\epsilon\}, \tag{4}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function. By construction, $\mathcal{C}_{\epsilon_1} \subset \mathcal{C}_{\epsilon_2}$ for any $\epsilon_2 > \epsilon_1 \geq 0$. For simplicity, the set $\mathcal{C}_0$ is denoted by $\mathcal{C}$.

The definition of a barrier function is made easier through an appropriate extension of the notion of class $\mathcal{K}$ function.

Definition 1. (Based on Khalil (2002)) A continuous function $\beta : (-b, a) \to (-\infty, \infty)$ for some $a, b > 0$ is said to belong to *extended class* $\mathcal{K}$ if it is strictly increasing and $\beta(0) = 0$.

2.1 Zeroing Barrier Functions

The class of barrier functions considered in this paper is defined as follows.

Definition 2. Consider a dynamical system (1) and the set $\mathcal{C}$ defined by (2)-(4) for some continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$. If there exist a locally Lipschitz extended class $\mathcal{K}$ function α and a set $\mathcal{D}$ with $\mathcal{C} \subseteq \mathcal{D} \subset \mathbb{R}^n$ such that

$$L_f h(x) \geq -\alpha(h(x)), \forall\, x \in \mathcal{D}, \tag{5}$$

then the function h is called a *zeroing barrier function (ZBF).*

Existence of a ZBF implies the forward invariance of $\mathcal{C}$, as shown by the following theorem.

Theorem 3. Given a dynamical system (1) and a set $\mathcal{C}$ defined by (2)-(4) for some continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, if h is a ZBF defined on the set $\mathcal{D}$ with $\mathcal{C} \subseteq \mathcal{D} \subset \mathbb{R}^n$, then $\mathcal{C}$ is forward invariant.

Proof. Note that for any $x \in \partial\mathcal{C}$, $L_f h(x) \geq -\alpha(h(x)) = 0$. According to Nagumo's theorem (Blanchini and Miani (2008)), the set $\mathcal{C}$ is forward invariant. □

Recall that the original barrier condition in Prajna et al. (2007) requires that $\dot{h} \geq 0$, when expressed in the notation of the present paper, which implies that all superlevel sets of h inside $\mathcal{C}$ are invariant. As in Dai et al. (2013), Kong et al. (2013) and Ames et al. (2014b), inequality (5) relaxes the conventional condition by requiring a single superlevel set of h, which is $\mathcal{C}$ itself, to be invariant.

2.2 Robustness Properties of ZBFs

In this section, the extent to which forward invariance of the set $\mathcal{C}$, asserted in Theorem 3, is robust with respect to different perturbations on the dynamics (1) is investigated. This will be accomplished by showing that existence of a ZBF implies asymptotic stability of the set $\mathcal{C}$.

Recall that a closed and forward invariant set $\mathcal{S} \subseteq \mathbb{R}^n$ is said to be locally asymptotically stable for a forward complete system (1) if there exist an open set $\mathcal{R}$ containing $\mathcal{S}$ and a class $\mathcal{KL}$ function β such that for any $x_0 \in \mathcal{R}$

$$\|x(t, x_0)\|_{\mathcal{S}} \leq \beta\left(\|x_0\|_{\mathcal{S}}, t\right). \tag{6}$$

Whenever the set $\mathcal{S}$ is compact, inequality (6) implies $I(x_0) = \mathbb{R}_0^+$ for all $x_0 \in \mathcal{R}$. Therefore, the forward completeness assumption on (1) is no longer needed. Note that asymptotic stability of $\mathcal{S}$ implies invariance of $\mathcal{S}$ as can be seen by noting that $x_0 \in \mathcal{S}$ implies $\|x_0\|_{\mathcal{S}} = 0$ and $\beta(\|x_0\|_{\mathcal{S}}, t) = 0$ which, in turn, implies $\|x(t, x_0)\|_{\mathcal{S}}=0$ and $x(t, x_0) \in \mathcal{S}$.

Once asymptotic stability of $\mathcal{C}$ is established, several robustness results in the literature will be used to characterize the robustness of forward invariance of the set $\mathcal{C}$. The critical observation, upon which all the results in this section rely, is that, if $\mathcal{D}$ is open, then a ZBF h induces a Lyapunov function $V_{\mathcal{C}} : \mathcal{D} \to \mathbb{R}_0^+$ defined by:

$$V_{\mathcal{C}}(x) = \begin{cases} 0, & \text{if } \ x \in \mathcal{C}, \\ -h(x), & \text{if } x \in \mathcal{D}\backslash\mathcal{C}. \end{cases} \tag{7}$$

It is easy to see that: **1)** $V_{\mathcal{C}}(x) = 0$ for $x \in \mathcal{C}$; **2)** $V_{\mathcal{C}}(x) > 0$ for $x \in \mathcal{D}\backslash\mathcal{C}$; and **3)** $L_f V_{\mathcal{C}}(x)$ satisfies the following inequality for $x \in \mathcal{D}\backslash\mathcal{C}$:

$$L_f V_{\mathcal{C}}(x) = -L_f h(x) \leq \alpha \circ h(x) = \alpha(-V_{\mathcal{C}}(x)) < 0,$$

where α is the locally Lipschitz extended class $\mathcal{K}$ function introduced in Definition 2. It thus follows from these three properties, from the fact that $V_{\mathcal{C}}$ is continuous on its domain and continuously differentiable at every point $x \in \mathcal{D}\backslash\mathcal{C}$, and from [1] Theorem 2.8 in Lin et al. (1996) that the set $\mathcal{C}$ is asymptotically stable whenever (1) is forward complete or the set $\mathcal{C}$ is compact. The preceding discussion is summarized in the following result.

Proposition 4. Let $h : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function defined on an open set $\mathcal{D} \subseteq \mathbb{R}^n$. If h is a ZBF for the dynamical system (1), then the set $\mathcal{C}$ defined by h is asymptotically stable. Moreover, the function $V_{\mathcal{C}}$ defined in (7) is a Lyapunov function.

The relationships between asymptotic stability and different robustness properties are well documented in the literature. For the reader's benefit, the following proposition paraphrases several existing results using the notation of this paper.

Proposition 5. Under the assumptions of Proposition 4 the following statements hold:

- There exist $\varepsilon \in \mathbb{R}_0^+$ and class $\mathcal{K}$ function $\sigma : [0, \varepsilon] \to \mathbb{R}_0^+$ such that for any continuous function $g_1 : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $\|g_1(x)\| \leq \sigma\left(\|x\|_{\mathcal{C}}\right)$ for $x \in \mathcal{D}\backslash\mathrm{Int}(\mathcal{C})$, the set $\mathcal{C}$ is still asymptotically stable for the system $\dot{x} = f(x) + g_1(x)$ describing the effect of a disturbance modeled by g_1 on system (1).
- There exist a constant $k \in \mathbb{R}^+$ and class $\mathcal{K}$ function γ such that the set $\mathcal{C}_{\gamma(\|g_2\|_\infty)} \subseteq \mathcal{D}$ is locally asymptotically stable for the system $\dot{x} = f(x) + g_2(t)$ describing the effect of a disturbance modeled by g_2, and satisfying $\|g_2\|_\infty \leq k$, on system (1).

The first result in Proposition 5 corresponds to Theorem 2.8 in Bacciotti and Rosier (2005). A disturbance satisfying the inequality $\|g_1(x)\| \leq \sigma\left(\|x\|_{\mathcal{C}}\right)$ is called a vanishing perturbation since its magnitude decreases as the state x approaches the set $\mathcal{C}$ and it vanishes on the boundary of $\mathcal{C}$. For this type of perturbation, the set $\mathcal{C}$ remains invariant. Moreover, even if a disturbance pushes the state into $\mathcal{D}\backslash\mathcal{C}$, the set $\mathcal{C}$ is asymptotically reached.

The second result in Proposition 5 corresponds to the observation that the system $\dot{x} = f(x) + u$ is locally input-to-state stable when u is seen as a disturbance input. In this case, the disturbance $u(t) = g_2(t)$ is called a non-vanishing perturbation and the only assumption is that it is sufficiently small, in the sense that $\|g_2\|_\infty \leq k$. Note that the "size" of the new asymptotically stable set $\mathcal{C}_{\gamma(\|g_2\|_\infty)}$, as measured by $\gamma\left(\|g_2\|_\infty\right)$, is an increasing function of the disturbance bound $\|g_2\|_\infty$. Similarly to vanishing perturbations, if a disturbance pushes the state into $\mathcal{D}\backslash\mathcal{C}_{\gamma(\|g_2\|_\infty)}$, the set $\mathcal{C}_{\gamma(\|g_2\|_\infty)}$ is asymptotically reached.

2.3 Zeroing Control Barrier Functions

Consider an affine control system of the form

[1] While Theorem 2.8 requires the function V to be smooth, V can always be smoothed as shown in Proposition 4.2 in Lin et al. (1996).

$$\dot{x} = f(x) + g(x)u, \tag{8}$$

with f and g locally Lipschitz continuous, $x \in \mathbb{R}^n$ and $u \in U \subset \mathbb{R}^m$.

Definition 6. Given a set $\mathcal{C} \subset \mathbb{R}^n$ defined by (2)-(4) for a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, the function h is called a *zeroing control barrier function (ZCBF)* defined on set $\mathcal{D}$ with $\mathcal{C} \subseteq \mathcal{D} \subset \mathbb{R}^n$, if there exists an extended class $\mathcal{K}$ function α such that

$$\inf_{u \in U} [L_f h(x) + L_g h(x)u + \alpha(h(x))] \geq 0, \ \forall x \in \mathcal{D}. \tag{9}$$

The ZCBF h is said to be locally Lipschitz continuous if α and the derivative of h are both locally Lipschitz continuous.

If $U = \mathbb{R}^m$ and $L_g h(x) \neq 0$ for $x \in \mathcal{D}$, then the function h is always a ZCBF.

Given a ZCBF h, define the set for all $x \in \mathcal{D}$

$$K_{\text{zcbf}}(x) = \{u \in U : L_f h(x) + L_g h(x)u + \alpha(h(x)) \geq 0\}.$$

Similar to Corollary 1 in Ames et al. (2014b), the following result that guarantees the forward invariance of $\mathcal{C}$ can be given.

Corollary 7. Given a set $\mathcal{C} \subset \mathbb{R}^n$ defined by (2)-(4) for a continuously differentiable function h, if h is a ZCBF on $\mathcal{D}$, then any Lipschitz continuous controller $u : \mathcal{D} \to U$ such that $u(x) \in K_{\text{zcbf}}(x)$ will render the set $\mathcal{C}$ forward invariant.

Inspired by the pointwise minimum-norm controller in Freeman and Kokotovic (1996) for rendering a control Lyapunov function negative definite, consider a control input of minimum norm that meets the control barrier function inequality in (9). When the norm arises from an inner product, the resulting controller is the solution of a quadratic program (QP). The QP perspective is especially interesting because it allows the unification of performance and safety (Ames et al. (2014b)). Specifically, the inequality for a control Lyapunov function (CLF) can be added as an additional soft constraint via a relaxation parameter, while the control barrier function inequality is maintained as a hard constraint for guaranteed safety. The question arises, however, is such a feedback law locally Lipschitz continuous? Conditions that ensures local Lipschitz continuity will be discussed in the next section.

3. LIPSCHITZ CONTINUITY OF A QUADRATIC PROGRAM FOR SAFETY AND PERFORMANCE

The main result of this section provides sufficient conditions for a QP-based feedback controller to be locally Lipschitz continuous, as required in Corollary 1 of Ames et al. (2014b) and Corollary 7 in Subsection 2.3. It will be assumed throughout this section that $U = \mathbb{R}^m$.

3.1 Quadratic Program Only With the Control Barrier Constraint

For an affine control system (8) and a set $\mathcal{C} \subset \mathbb{R}^n$ defined by (2)-(4), consider the set of controllers $u(x) \in K_{\text{zcbf}}(x)$ meeting the control barrier function condition in (9). The controller that pointwise minimizes the Euclidean norm can be found by solving the following parameterized quadratic program

$$\begin{aligned} \mathcal{P}_1(x) : \quad & \forall\, x \in \mathcal{D}, \\ & u^*(x) = \underset{u \in \mathbb{R}^m}{\operatorname{argmin}}\ u^\top u, \\ & \text{s.t.} \quad L_g h(x)u + L_f h(x) + \alpha(h(x)) \geq 0, \end{aligned} \tag{10}$$

where $u \in \mathbb{R}^m$ is the control input and constraint (10) is the ZCBF condition shown in (9).

The following result establishes the key condition for $u^*(x)$ to be locally Lipschitz continuous: the control barrier function should be relative degree one uniformly on $\mathcal{D}$ in the sense that $L_g h$ does not vanish on $\mathcal{D}$.

Theorem 8. Assume that vector fields f and g in the control system (8) are both locally Lipschitz continuous, and that $h : \mathcal{D} \to \mathbb{R}$ is a locally Lipschitz continuous ZCBF. Suppose furthermore that the relative degree one condition, $L_g h(x) \neq 0$ for all $x \in \mathcal{D}$, holds. Then the solution, $u^*(x)$, of $\mathcal{P}_1(x)$ is locally Lipschitz continuous for $x \in \mathcal{D}$.

Proof. Because $L_g h(x) \neq 0$ for $x \in \mathcal{D}$, the linear independent constraint qualification condition is satisfied (Bertsekas (1999)). Hence, the KKT optimality conditions imply there exists $\mu(x) \geq 0$ such that $u^*(x)$ and $\mu(x)$ satisfy

$$\begin{cases} u^*(x)^\top = \mu(x) L_g h(x), \\ L_f h(x) + L_g h(x) u^*(x) + \alpha(h(x)) \geq 0, \\ \mu(x) = 0 \text{ if } L_f h(x) + L_g h(x) u^*(x) + \alpha(h(x)) > 0. \end{cases}$$

Because the objective is convex and the inequality constraints are affine, the KKT necessary conditions are also sufficient (pg. 244 in Boyd and Vandenberghe (2004)). Hence, the closed form expression for $u^*(x)$ can be derived as

$$u^*(x) = \begin{cases} 0, \quad \text{if } L_f h(x) + \alpha(h(x)) > 0, \\ -\dfrac{(L_f h(x) + \alpha(h(x))) L_g h(x)^\top}{L_g h(x) L_g h(x)^\top}, \quad \text{otherwise.} \end{cases}$$

The following facts about Lipschitz continuous functions are recalled.

Fact 1. If f_1 and f_2 are locally Lipschitz continuous on a set I, then whenever their sum, $f_1 + f_2$, or product, $f_1 f_2$, makes sense, they are each locally Lipschitz continuous on I. Furthermore, if f_3 is real valued, then in a neighborhood of any point $x \in I$ where $f_3(x) \neq 0$, the reciprocal $1/f_3$ is locally Lipschitz.

Fact 2. If f_1 is locally Lipschitz continuous on a set I_1 and f_2 is locally Lipschitz continuous on a set I_2 such that $f_1(I_1) \subset I_2$, then the composition $f_2 \circ f_1$ is locally Lipschitz continuous on I_1.

With these facts in mind, define

$$\begin{aligned} \omega_1(r) &= \begin{cases} 0, \text{ if } r > 0, \\ r, \text{ if } r \leq 0, \end{cases} \ r \in \mathbb{R}, \\ \omega_2(x) &= L_f h(x) + \alpha(h(x)),\ x \in \mathcal{D}, \\ \omega_3(x) &= -\frac{L_g h(x)^\top}{L_g h(x) L_g h(x)^\top},\ x \in \mathcal{D}. \end{aligned} \tag{11}$$

The function $\omega_1(r)$ is clearly Lipschitz continuous. Because f and g are locally Lipschitz continuous and the derivative of h is locally Lipschitz continuous, both $L_f h$ and $L_g h$ are locally Lipschitz continuous on $\mathcal{D}$ by *Fact 1*. The same fact implies that ω_2 and $L_g h L_g h^\top$ are locally Lipschitz continuous on $\mathcal{D}$. Furthermore, because $L_g h(x) \neq 0$ for $x \in \mathcal{D}$, it follows that $L_g h(x) L_g h(x)^\top \neq 0$ and thus $\omega_3(x)$ is also locally Lipschitz continuous by *Fact 1*.

The proof is completed by noting that

$$u^*(x) = \omega_1(\omega_2(x))\omega_3(x),\ x \in \mathcal{D}.$$

Because $\omega_1(\omega_2(x))$ is locally Lipschitz continuous with respect to $x \in \mathcal{D}$ by *Fact 2*, its product with $\omega_3(x)$ is locally Lipschitz continuous by *Fact 1*, and thus $u^*(x)$ is locally Lipschitz continuous with respect to $x \in \mathcal{D}$. $\square$

Remark 9. If the objective function of $\mathcal{P}_1(x)$ is changed to $\frac{1}{2}u^\top H u + F^\top u$, where H is an $m \times m$ positive definite matrix and F is an $m \times 1$ column vector, then the solution of the modified QP is also locally Lipschitz continuous with respect to $x \in \mathcal{D}$.

3.2 Quadratic Program Incorporating both Control Barrier and Lyapunov Constraints

Suppose now that the desired performance of the system (8) can be captured by a CLF V, as in Ames et al. (2014a,b). This yields the set of control inputs that stabilize the system (8), namely

$$K_{\mathrm{clf}}(x) = \{u \in \mathbb{R}^m : L_f V(x) + L_g V(x) u < 0\}, \tag{12}$$

The minimum-norm controller of Freeman and Kokotovic chooses pointwise in x the element of $K_{\mathrm{clf}}(x)$ that minimizes the Euclidean norm. This is now combined with the control barrier function inequality.

In particular, given a CLF V and a ZCBF h with relative degree 1 in $\mathcal{D}$, the two "specifications" are combined via the following parameterized quadratic program

$$\begin{aligned}
\mathcal{P}_2(x):\quad & \forall\ x \in \mathcal{D},\\
& \mathbf{u}^*(x) = \underset{\mathbf{u}=[u^\top,\delta]^\top \in \mathbb{R}^{m+1}}{\operatorname{argmin}} \ \mathbf{u}^\top \mathbf{u}\\
& \text{s.t. } L_g V(x) u + L_f V(x) - \delta \le 0, && (13)\\
& \qquad L_g h(x) u + L_f h(x) + \alpha(h(x)) \ge 0, && (14)
\end{aligned}$$

where $u \in \mathbb{R}^m$ is the control input, δ is a relaxation parameter[2], constraint (14) is the ZCBF condition and constraint (13) is the CLF condition.

Remark 10. The QP $\mathcal{P}_2(x)$ is always feasible, because $L_g h \neq 0$ ensures that there exists u such that (14) holds, which implies that the safety guarantee can always be satisfied, while the relaxation parameter δ ensures that (13) can always be satisfied. Due to the relaxation parameter, the performance objective, such as asymptotic stabilization to an equilibrium point, may not necessarily be achieved. When the control objective and the safety guarantee are not conflicting—*and a weight is appropriately added to the objective function*—the solution will result in $\delta \approx 0$. Indeed, if the objective function is $u^\top u + k^2\delta^2$ with $k \neq 0$ the weight for δ, and $\hat{\mathbf{u}} = (\hat{u}^\top, 0)^\top$ is a feasible point for constraints (13) and (14), then the optimal solution $\mathbf{u}^* = (u^{*\top}, \delta^*)^\top$ satisfies $u^{*\top} u^* + k^2 \delta^{*2} \le \hat{u}^\top \hat{u}$, which implies that $\delta^{*2} \le \hat{u}^\top \hat{u}/k^2$. Therefore, δ^* can be made arbitrarily small if sufficiently large weight k is chosen.

[2] A weight is traditionally used on the relaxation parameter. This is taken care of after the proof of the main result.

The following theorem is the main result of this subsection.

Theorem 11. Let V be a CLF for the control system (8) with the derivative of V locally Lipschitz continuous. Assume that the vector fields f and g in the control system (8) are both locally Lipschitz continuous and that $h : \mathcal{D} \to \mathbb{R}$ is a locally Lipschitz continuous ZCBF. Suppose furthermore that the relative degree one condition, $L_g h(x) \neq 0$ for all $x \in \mathcal{D}$, holds. Then the solution, $\mathbf{u}^*(x)$, of $\mathcal{P}_2(x)$ is locally Lipschitz continuous for $x \in \mathcal{D}$.

Proof. The QP $\mathcal{P}_2(x)$ can be written equivalently as

$$\begin{aligned}
\hat{\mathcal{P}}_2(x):\quad & \forall\ x \in \mathcal{D},\\
& \mathbf{u}^*(x) = \underset{\mathbf{u}=[u^\top,\delta]^\top \in \mathbb{R}^{m+1}}{\operatorname{argmin}} \ \mathbf{u}^\top \mathbf{u}\\
& \text{s.t. } g_1(x)\mathbf{u} - c_1(x) \le 0,\\
& \qquad g_2(x)\mathbf{u} - c_2(x) \le 0,
\end{aligned}$$

where

$$\begin{aligned}
g_1(x) &= [L_g V(x), -1],\ c_1(x) = -L_f V(x),\\
g_2(x) &= [-L_g h(x), 0],\ c_2(x) = L_f h(x) + \alpha(h(x)).
\end{aligned}$$

Because $g_1(x)$ and $g_2(x)$ are linearly independent for all $x \in \mathcal{D}$, the linear independence constraint qualification is satisfied for all $x \in \mathcal{D}$. By the KKT condition, there exist $\lambda_1(x) \ge 0$, $\lambda_2(x) \ge 0$ such that $\mathbf{u}^*(x)$, $\lambda_1(x)$, $\lambda_2(x)$ satisfy

$$\begin{cases}
\mathbf{u}^*(x) = -\lambda_1(x) g_1(x)^\top - \lambda_2(x) g_2(x)^\top,\\
g_1(x)\mathbf{u}^*(x) - c_1(x) \le 0,\\
g_2(x)\mathbf{u}^*(x) - c_2(x) \le 0,\\
\lambda_1(x) = 0 \text{ if } g_1(x)\mathbf{u}^*(x) - c_1(x) < 0,\\
\lambda_2(x) = 0 \text{ if } g_2(x)\mathbf{u}^*(x) - c_2(x) < 0.
\end{cases}$$

Write $\mathbf{u}^*(x) = [u^*(x)^\top, \delta^*(x)]^\top$. Let $G(x) = (G_{ij}(x)) = (\langle g_i(x), g_j(x)\rangle)$, $i, j = 1, 2$ be the Gram matrix of $\{g_1(x), g_2(x)\}$. Specifically,

$$\begin{aligned}
&G_{11}(x) = L_g V(x) L_g V(x)^\top + 1,\ G_{12}(x) = -L_g V(x) L_g h(x)^\top,\\
&G_{21}(x) = -L_g h(x) L_g V(x)^\top,\ G_{22}(x) = L_g h(x) L_g h(x)^\top.
\end{aligned}$$

Because the objective is convex and the inequality constraints are affine, the KKT necessary conditions are also sufficient (pg.244 in Boyd and Vandenberghe (2004)). The closed form solution of $u^*(x), \delta^*(x)$ can therefore be expressed as

$$\begin{aligned}
u^*(x) &= -\lambda_1(x) L_g V(x)^\top + \lambda_2(x) L_g h(x)^\top,\\
\delta^*(x) &= \lambda_1(x),
\end{aligned}$$

where, dropping the argument x for compactness of notation,

$$\lambda_1(x) = \begin{cases} 0, & \text{if } G_{12}c_2 - G_{22}c_1 < 0,\\ \dfrac{G_{12}c_2 - G_{22}c_1}{G_{11}G_{22} - G_{12}G_{21}}, & \text{if } G_{12}c_2 - G_{22}c_1 \ge 0. \end{cases}$$

$$\lambda_2(x) = \begin{cases} 0, & \text{if } G_{21}c_1 - G_{11}c_2 < 0,\\ \dfrac{G_{21}c_1 - G_{11}c_2}{G_{11}G_{22} - G_{12}G_{21}}, & \text{if } G_{21}c_1 - G_{11}c_2 \ge 0. \end{cases}$$

Note that $G_{11}(x)G_{22}(x) - G_{12}(x)G_{21}(x) > 0$ for all $x \in \mathcal{D}$ due to the linear independence of $\{g_1(x), g_2(x)\}$. Using the same reasoning as in the proof of Theorem 8, the following functions are each locally Lipschitz continuous

$$\begin{aligned}
\omega_4(x) &= G_{12}(x)c_2(x) - G_{22}(x)c_1(x),\ x \in \mathcal{D},\\
\omega_5(x) &= G_{21}(x)c_1(x) - G_{11}(x)c_2(x),\ x \in \mathcal{D},\\
\omega_6(x) &= G_{11}(x)G_{22}(x) - G_{12}(x)G_{21}(x),\ x \in \mathcal{D}.
\end{aligned}$$

Moreover, $\lambda_1(x)$ and $\lambda_2(x)$ can be expressed as compositions of $\omega_1, \omega_4, \omega_5$ and ω_6, where ω_1 was defined in (11); in particular,

$$\lambda_1(x) = \omega_1(\omega_4(x))/\omega_6(x),\ x \in \mathcal{D},$$
$$\lambda_2(x) = \omega_1(\omega_5(x))/\omega_6(x),\ x \in \mathcal{D}.$$

Therefore, $\lambda_1(x), \lambda_2(x)$ are both locally Lipschitz continuous, which implies that $u^*(x), \delta^*(x)$ are locally Lipschitz continuous with respect to $x \in \mathcal{D}$. This completes the proof. □

Remark 12. If the objective function of $\mathcal{P}_2(x)$ is changed to $\frac{1}{2}\mathbf{u}^\top H\mathbf{u} + F^\top\mathbf{u}$ with H an $(m+1)\times(m+1)$ positive definite matrix and F an $(m+1)\times 1$ a column vector, then the modified QP is also locally Lipschitz continuous with respect to $x \in \mathcal{D}$.

4. EXAMPLE

In this section, the theoretical results of the paper are illustrated on adaptive cruise control (ACC). The *lead* and *following* vehicles are modeled as point-masses moving on a straight road with uncertain slope or grade (Ioannou and Chien (1993), Aström and Murray (2010)). The following vehicle is equipped with ACC, while the lead vehicle and the road act as disturbances to the following vehicle's performance objective of cruising at a given constant speed. The safety constraint is to maintain a safe following distance as specified by a time headway.

Let v_l and v_f be the velocity (in m/s) of the lead car and the following car, respectively, and D be the distance (in m) between the two vehicles. Let $x = (v_l, v_f, D)$ be the state of the system, whose dynamics can be described as

$$\begin{bmatrix} \dot{v}_l \\ \dot{v}_f \\ \dot{D} \end{bmatrix} = \underbrace{\begin{bmatrix} a_l \\ -F_r/m \\ v_l - v_f \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ g\Delta\theta \\ 0 \end{bmatrix}}_{\Delta f(x)} + \underbrace{\begin{bmatrix} 0 \\ 1/m \\ 0 \end{bmatrix}}_{\hat{g}(x)} u, \tag{15}$$

where u and m are the control input (in Newtons) and the mass (in kg) of the following car, respectively, g is the gravitational constant (in m^2/s), a_l is the acceleration (in m^2/s) of the lead car, $\Delta\theta$ is a perturbation to $\dot{v}_f$ (reflecting unmodeled road grade or aerodynamic force), and $F_r = f_0 + f_1 v_f + f_2 v_f^2$ is the aerodynamic drag term (in Newtons) with constants f_0, f_1 and f_2 determined empirically. The values of m, f_0, f_1, and f_2 are the same as those in Ames et al. (2014b).

Two constraints are imposed on the following car. The *hard* constraint requires the following car to keep a safe distance from the lead car, which can be expressed as $D/v_f \geq \tau_{des}$ with τ_{des} the desired time headway. Define the function $h = D - \tau_{des} v_f$, by which the hard constraint can be expressed as $h \geq 0$ and the set $\mathcal{C}$ can be defined by (2)-(4). The *soft* constraint requires that when adequate headway is assured, the following car achieves a desired speed v_d, which can be expressed as $v_f - v_d \to 0$, leading to the candidate CLF, $V = (v_f - v_d)^2$.

The controller is designed on the basis of the nominal model $\dot{x} = f(x) + \hat{g}(x)u$ corresponding to $\Delta f(x) = 0$. The hard constraint is encoded by the ZCBF condition (9) and the soft constraint by the CLF condition (12). The headway is selected as $\tau_{des} = 1.8$ following the "half the speedometer rule" (Vogel (2003)). The feedback controller $u(x)$ can then be obtained by the following QP

$$\begin{aligned} \mathbf{u}^*(x) = \underset{\mathbf{u}=[u,\delta]^\top \in \mathbb{R}^2}{\operatorname{argmin}} \ & \frac{1}{2}\mathbf{u}^T H\mathbf{u} + F^T\mathbf{u} \\ \text{s.t. } & A_{\text{clf}}\mathbf{u} \leq b_{\text{clf}}, \\ & A_{\text{zcbf}}\mathbf{u} \leq b_{\text{zcbf}}, \end{aligned}$$

where

$$H = 2\begin{bmatrix} 1/m^2 & 0 \\ 0 & p_{sc} \end{bmatrix},\ F = -2\begin{bmatrix} F_r/m^2 \\ 0 \end{bmatrix},$$

as given in Ames et al. (2014b) with p_{sc} the weight for δ,

$$A_{\text{clf}} = \left[\frac{2(v_f - v_d)}{m},\ -1\right],$$
$$b_{\text{clf}} = \frac{2(v_f - v_d)}{m}F_r - (v_f - v_d)^2,$$

and

$$A_{\text{zcbf}} = \left[-\frac{1.8}{m},\ 0\right],$$
$$b_{\text{zcbf}} = -\frac{1.8F_r}{m} - (v_l - v_f) + \alpha(h(x)).$$

According to Proposition 4, $V_\mathcal{C}$ defined in (7) equals to $1.8v_f - D$ for points outside $\mathcal{C}$ and equals to 0 for points inside $\mathcal{C}$. In the absence of perturbations, the input u arising from solutions of the QP ensures

$$L_{f+\hat{g}u}V_\mathcal{C} \leq -\kappa V_\mathcal{C},$$

where the corresponding extended class $\mathcal{K}$ function α is simply chosen as $\alpha(h) = \kappa h$ for some constant $\kappa > 0$. For the perturbed system (15), the same input u ensures

$$L_{f+\hat{g}u+\Delta f}V_\mathcal{C} \leq -\kappa V_\mathcal{C} + L_{\Delta f}V_\mathcal{C} = -\kappa V_\mathcal{C} + 1.8g\Delta\theta.$$

By choosing the class $\mathcal{K}$ function γ in Proposition 5 as $\gamma(z) = \frac{1.8g}{\kappa}z$, the set $\mathcal{C}_{\gamma(\|\Delta\theta\|_\infty)}$ is asymptotically stable. Indeed, if $x \notin \mathcal{C}_{\gamma(\|\Delta\theta\|_\infty)}$, then $h(x) = D - 1.8v_f < -\frac{1.8g}{\kappa}\|\Delta\theta\|_\infty$ and therefore,

$$\begin{aligned} L_{f+\hat{g}u+\Delta f}V_\mathcal{C} &\leq -\kappa V_\mathcal{C} + 1.8g\|\Delta\theta\|_\infty \\ &= \kappa(D - 1.8v_f) + 1.8g\|\Delta\theta\|_\infty \\ &< -\kappa\frac{1.8g}{\kappa}\|\Delta\theta\|_\infty + 1.8g\|\Delta\theta\|_\infty \\ &= 0. \end{aligned}$$

Thus, for any $x \in \mathbb{R}^3 \backslash \mathcal{C}_{\gamma(\|\Delta\theta\|_\infty)}$, $L_{f+gu+\Delta f}V_\mathcal{C}(x) < 0$, which implies that the set $\mathcal{C}_{\gamma(\|\Delta\theta\|_\infty)}$ is asymptotically stable.

Figure 1 shows the time evolution of $v_l(t)$ and $v_f(t)$, the evolution of the specification $h(x(t))$, the change of the road slope when $\kappa = 5$, the perturbation $\Delta\theta(t) = 0.1\cos(2\pi t/20)$, and the desired speed $v_d = 22$. The initial state is $v_l(0) = 20$, $v_f(0) = 18$, and $D(0) = 80$. To simplify the discussion we denote $\gamma(\|\Delta\theta\|_\infty)$ by $\gamma_{\max}$ which is $\gamma_{\max} = 0.3532$ for $\|\Delta\theta\|_\infty = 0.1$, i.e., the maximum headway distance error is 0.3532m. The top plot of Fig. 1 shows that the following car first accelerates to approximately its desired speed v_d. The vehicle then decelerates to and maintains the same final speed as the lead car in order to maintain a safe headway. Note that due to the unmeasured perturbation in road grade, the achieved tracking speeds are in a neighborhood of v_d and the lead car's final speed. The middle plot shows that the values of h are greater than -0.3525, which implies

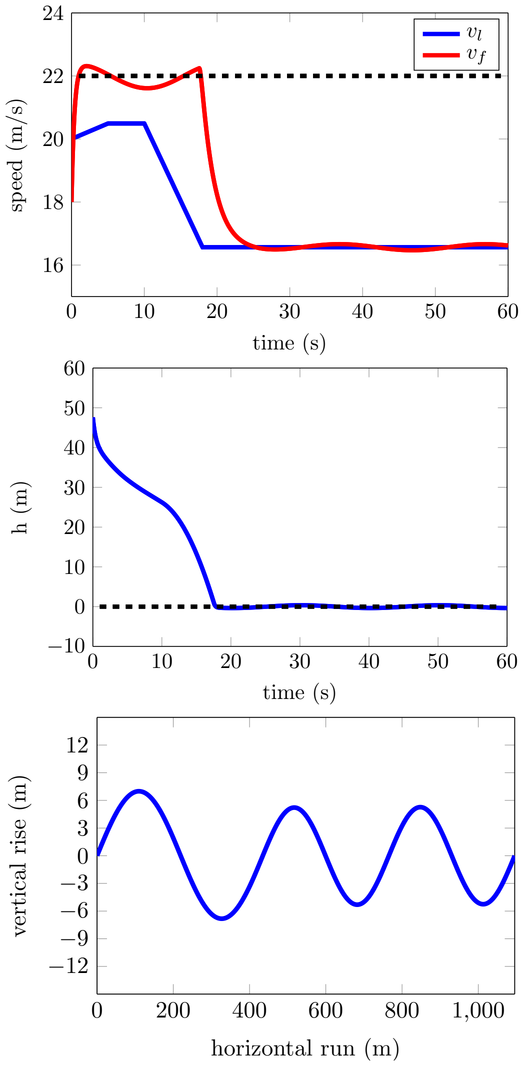

Fig. 1. Simulation results when choosing $\kappa = 5$, $\|\Delta\theta\|_\infty = 0.1$ and initial states $v_l(0) = 20$, $v_f(0) = 18$, $D(0) = 80$. (top) speed of the two cars; (middle) evolution of $h = D - 1.8v_f$; (bottom) the vertical rise of the road with respect to the horizontal run of the car.

that x is within the set $\mathcal{C}_{\gamma_{\max}}$. The bottom plot shows the vertical rise of the road with respect to the horizontal run of the car, assuming the perturbation term is exclusively interpreted as the change of road slope.

Figure 2 shows the quantities $-\min h$, the amount the safety condition is violated in meters, $\gamma_{\max} + \min h$, the tightness of the error bound in meters, and $\min u/mg$, the braking effort in fractions of g, as κ ranges from 1 to 10 (rate of convergence back to safe set) and $\|\Delta\theta\|_\infty$ ranges from 10% to 40% (road grade perturbation), where

$$\min h := \min_{0 \le t \le 60} h(x(t))$$
$$\min u := \min_{0 \le t \le 60} u(t).$$

Note that larger κ means a stricter barrier function condition, while larger $\|\Delta\theta\|_\infty$ means more uncertainty in the dynamics. The evolution of the road grade perturbation is given by $\Delta\theta(t) = 0.1K\cos(2\pi t/20)$ for a constant $K > 0$, which implies $\|\Delta\theta\|_\infty = 0.1K$. The top plot shows that $-\min h$ increases as κ decreases or $\|\Delta\theta\|_\infty$ increases, which is intuitive because with a weaker barrier function condition or larger perturbations, the specification $h > 0$ is more likely to be violated. The middle plot shows that the discrepancies between $\gamma_{\max}$ and $\min h$ are positive, which implies that x is always within the set $\mathcal{C}_{\gamma_{\max}}$ as Proposition 4 guarantees. The bottom plot shows that the magnitude of the braking force u increases as $\|\Delta\theta\|_\infty$ or κ increases.

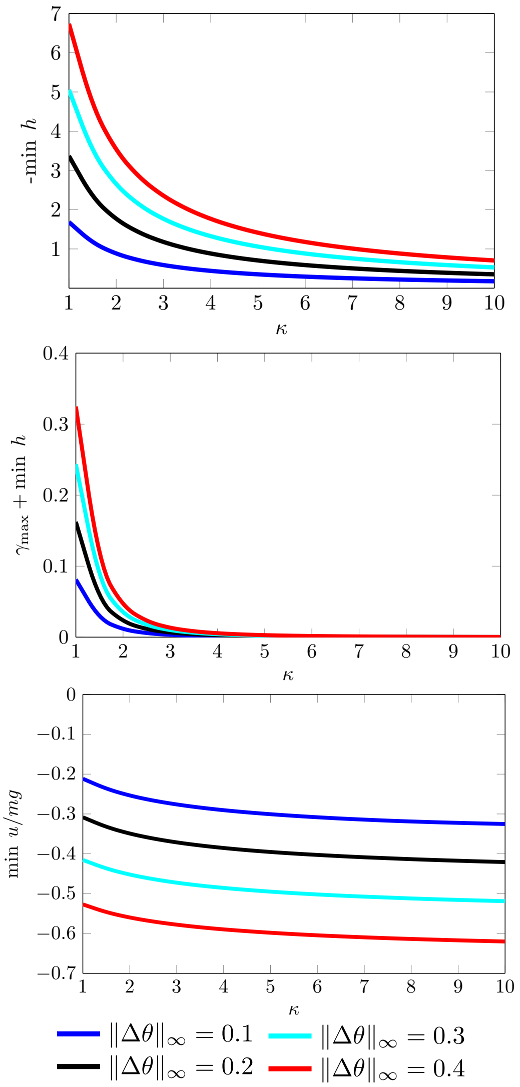

Fig. 2. Tradeoff analysis in terms of road grade uncertainty and speed of convergence to the safe set. (top) $-\min h$ increases as κ decreases and $\|\Delta\theta\|_\infty$ increases; (middle) positiveness of the discrepancies of $\gamma_{\max}$ and $\min h$ implies x is within the set $\mathcal{C}_{\gamma_{\max}}$; (bottom) the magnitude of the braking force u increases as $\|\Delta\theta\|_\infty$ and κ increases. According to the Guinness Book of World Records, the steepest street in the world is Baldwin Street in New Zealand, with a grade of 38%.

5. CONCLUSIONS

This paper defined (control) zeroing barrier functions for a given set and investigated their robustness properties under model perturbations. In particular, when the barrier function was designed to be negative on the complement of the closure of a safe set, and its derivative along solutions of the model was positive, a Lypaunov analysis showed that the set was automatically locally asymptotically stable. This led to various Input-to-State Stability (ISS) results in the presence of model perturbations. For this result to hold, it was important to consider barrier functions that vanish on the set boundary (i.e., zeroing barrier functions) rather than barrier functions that tend to infinity on the set boundary (i.e. reciprocal barrier functions). The reason is that "there are two sides of zero" and only "one side of infinity." More formally speaking, if a perturbation (or model error) makes it impossible to satisfy the invariance condition for a reciprocal barrier function, then the solution of the model must cease to exist because the control input must become unbounded as well; see Sect. III.B of Ames et al. (2014b), eqn. (CBF). On the other hand, if a perturbation (or model error) makes it impossible to satisfy the invariance condition for a zeroing barrier function, then the solution can cross the set boundary without the control input becoming unbounded.

A second result presented conditions that guarantee local Lipschitz continuity of the solution of a Quadratic Program (QP) that mediates safety (represented as a control barrier function (CBF)) and a control objective (represented as a control Lyapunov function (CLF)). A uniform relative degree condition on the CBF and relaxation of the inequality required for a CLF were shown to provide local Lipschitz continuity of the resulting feedback control law, and hence local existence and uniqueness of solutions of the associated closed-loop system. This result is applicable to both types of barrier functions.

Future studies will consider control zeroing barrier functions with constraints on the inputs, as in Ames et al. (2014b). There are many interesting open questions on existence, computation, and composition, as well as applications to systems of greater complexity than ACC.

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