Open Loop LQ Nash Derivation

David Fridovich-Keil

May 1, 2020

1 Problem Formulation

Assume we're operating in discrete time with the following linear system:

$$x_{t+1} = A_t x_t + \sum_{j=1}^{N} B_t^j u_t^j \,. \tag{1}$$

Further, assume we're in an open-loop information structure, i.e. u_t^{i*} depends upon t and perhaps also x_0 but nothing else.

Each of N players is trying to minimize the following quadratic cost:

$$J^i = \frac{1}{2} \sum_{t=1}^{T} \left[\left(x_t^T Q_t^i + 2 q_t^i \right) x_t + \sum_{j=1}^{N} \left(u_t^{jT} R_t^{ij} + 2 r_t^{ij} \right) u_t^j \right] . \tag{2}$$

For clarity to remain consistent with the indexing in Başar and Olsder, we'll index the state costs corresponding to the next time step, i.e.:

$$J^i = \frac{1}{2} \sum_{t=1}^{T} \left[\left(x_{t+1}^T Q_{t+1}^i + 2 q_{t+1}^i \right) x_{t+1} + \sum_{j=1}^{N} \left(u_t^{jT} R_t^{ij} + 2 r_t^{ij} \right) u_t^j \right] . \tag{3}$$

Here, we note that we can always just relabel things $\tilde{Q}_t^i = Q_{t+1}^i$ and likewise for q^i, such that each player's optimization problem is essentially the same except for the final time step (the initial state is fixed so it's corresponding cost is immutable in the first problem). We'll ignore this difference here and just assume we're trying to solve this problem.

2 Solution

We are looking for an *open-loop* solution, so we cannot use a state-dependent value function dynamic programming approach derived from the HJB equations. Instead, we use the Pontryagin Minimum Principle, i.e., we have the following Hamiltonian at each time step:

$$H_t^i = \frac{1}{2} \left[\left(x_{t+1}^T Q_{t+1}^i + 2 q_{t+1}^i \right) x_{t+1} + \sum_{j=1}^{N} \left(u_t^{jT} R_t^{ij} + 2 r_t^{ij} \right) u_t^j \right] \tag{4}$$

$$
\begin{aligned}
&+ p_{t+1}^{iT}\left(A_t x_t + \sum_{j=1}^{N} B_t^j u_t^j\right) \\
&= \frac{1}{2}\Big[\big((A_t x_t + \sum_{j=1}^{N} B_t^j u_t^j)^T Q_{t+1}^i + 2q_{t+1}^i\big)(A_t x_t + \sum_{j=1}^{N} B_t^j u_t^j) \qquad (5) \\
&+ \sum_{j=1}^{N}\left(u_t^{jT} R_t^{ij} + 2r_t^{ij}\right) u_t^j\Big] + p_{t+1}^{iT}\left(A_t x_t + \sum_{j=1}^{N} B_t^j u_t^j\right) .
\end{aligned}
$$

Now, the Pontryagin minimizers are the open loop controls (and states) we want. To find these minimizers, we assume the Q's and R's are PSD and set the derivative (in control) of the above expression to zero, and assume that states and costates are fixed to their optimal values. This yields:

$$0 = R_t^{ii} u_t^{i*} + r_t^{ii} + B_t^{iT} p_{t+1}^{i*} + B_t^{iT} Q_{t+1}^i x_{t+1}^* + B_t^{iT} q_{t+1}^i \tag{6}$$

$$u_t^{i*} = -R_t^{ii^{-1}}\left[B_t^{iT}(p_{t+1}^{i*} + Q_{t+1}^i x_{t+1}^* + q_{t+1}^i) + r_t^{ii}\right] . \tag{7}$$

We recall the discrete-time costate equations, which in this problem are:

$$p_t^i = A_t^T(p_{t+1}^i + Q_{t+1}^i x_{t+1} + q_{t+1}^i), p_{T+1}^i = 0 . \tag{8}$$

Begin by substituting in at time step T, i.e.,

$$p_T^{i*} = A_T^T(Q_{T+1}^i x_{T+1}^* + q_{T+1}^i) , \tag{9}$$

$$u_T^{i*} = -R_T^{ii^{-1}}\left[B_T^{iT}(Q_{T+1}^i x_{T+1}^* + q_{T+1}^i) + r_T^{ii}\right] , \tag{10}$$

Now, it is clear that we will need to know the optimal state trajectory to do this. It will certainly obey the following constraints:

$$x_{t+1}^* = A_t x_t^* + \sum_{j=1}^{N} B_t^j u_t^{j*}, x_0^* = \hat{x}_0 . \tag{11}$$

Consider the following matrices and vectors. We will subsequently show that these can be used to compute the optimal trajectories of x, u^i, and p^i:

$$\Lambda_t = I + \sum_{j=1}^{N} B_t^j R_t^{jj^{-1}} B_t^{jT} M_{t+1}^i, \tag{12}$$

$$\bar{\lambda}_t = \sum_{j=1}^{N} B_t^j R_t^{jj^{-1}} r_t^{jj}, \tag{13}$$

$$\hat{\lambda}_t = \sum_{j=1}^{N} B_t^j R_t^{jj^{-1}} B_t^{jT} \lambda_{t+1}^i, \tag{14}$$

$$\lambda_t^i = A_t^T q_{t+1}^i + q_t^i - A_t M_{t+1}^i \Lambda_t^{-1}(\hat{\lambda}_t + \bar{\lambda}_t), \text{ with } \lambda_{T+1}^i = q_{T+1}^i, \text{ and} \tag{15}$$

$$M_t^i = Q_t^i + A_t^T M_{t+1}^i \Lambda_t^{-1} A_t, \text{ with } M_{T+1}^i = Q_{T+1}^i. \tag{16}$$

Now, we shall also take:

$$u_t^{i*} = -R_t^{ii^{-1}} \left[B_t^{iT} \left(M_{t+1}^i x_{t+1}^* + \lambda_{t+1}^i \right) + r_t^{ii} \right] , \tag{17}$$

$$x_{t+1}^* = \Lambda_t^{-1} (A_t x_t^* - \hat{\lambda}_t - \bar{\lambda}_t) , \text{ and} \tag{18}$$

$$p_t^{i*} = A_t^T (M_{t+1}^i x_{t+1}^* + \lambda_{t+1}^i) , \tag{19}$$

and prove that these are correct by induction on t for each time step. To do so, we shall work from the final time toward the initial time. The base case is the final time step. We have already worked out the equation for u_T^* in (10), so let's start there. Begin by premultiplying both sides by B_T^i and summing over all $i \in 1 \dots N$. Simplifying the left side, we obtain:

$$\begin{aligned} x_{T+1}^* - A_T x_T^* &= - \sum_{i=1}^N B_T^i [R_T^{ii}]^{-1} \left(B_T^{iT} (M_{T+1}^i x_{T+1}^* + q_{T+1}^i) + r_T^{ii} \right) \\ &= (I - \Lambda_T) x_{T+1}^* - \hat{\lambda}_T - \bar{\lambda}_T . \end{aligned} \tag{20}$$

This yields the following relationship:

$$x_{T+1}^* = \Lambda_T^{-1} (A_T x_T^* - \hat{\lambda}_T - \bar{\lambda}_T) . \tag{21}$$

Recognizing that $M_{T+1}^i = Q_T^i$, we also see that

$$u_T^{i*} = -R_T^{ii^{-1}} \left[B_T^{iT} \left(M_{T+1}^i x_{T+1}^* + \lambda_{T+1}^i \right) + r_T^{ii} \right] . \tag{22}$$

Likewise, we can substitute into (8) to recover

$$p_T^{i*} = A_T^T \left(M_{T+1}^i x_{T+1}^* + q_{T+1}^i \right) . \tag{23}$$

We have thus verified (17)-(19) and $p_t^{i*} = A_t^T \left(M_{t+1}^i x_{t+1}^* + q_{t+1}^i \right)$ for the base case where $t = T$. Let us assume that (17)-(19) hold up to time $t+1$ and verify for time t. We begin by substituting this costate relation into (7):

$$\begin{aligned} u_t^{i*} &= -R_t^{ii^{-1}} \left[B_t^{iT} (p_{t+1}^{i*} + Q_{t+1}^i x_{t+1}^* + q_{t+1}^i) + r_t^{ii} \right] \\ &= -R_t^{ii^{-1}} \left[B_t^{iT} \left(A_{t+1}^T (M_{t+2}^i x_{t+2}^* + q_{t+2}^i) + Q_{t+1}^i x_{t+1}^* + q_{t+1}^i \right) + r_t^{ii} \right] \\ &= -R_t^{ii^{-1}} \left[B_t^{iT} \left(A_{t+1}^T (M_{t+2}^i \Lambda_{t+1}^{-1} (A_{t+1} x_{t+1}^* - \hat{\lambda}_{t+1} - \bar{\lambda}_{t+1}) + q_{t+2}^i) + Q_{t+1}^i x_{t+1}^* + q_{t+1}^i \right) + r_t^{ii} \right] \\ &= -R_t^{ii^{-1}} \left[B_t^{iT} \left(M_{t+1}^i x_{t+1}^* - A_{t+1} M_{t+2}^i \Lambda_{t+1}^{-1} \hat{\lambda}_{t+1} - A_{t+1} M_{t+2}^i \Lambda_{t+1}^{-1} \bar{\lambda}_{t+1} + A_{t+1}^T q_{t+2}^i + q_{t+1}^i \right) + r_t^{ii} \right] \\ &= -R_t^{ii^{-1}} \left[B_t^{iT} (M_{t+1}^i x_{t+1}^* + \lambda_{t+1}^i) + r_t^{ii} \right] . \end{aligned}$$

As before, we continue by premultiplying both sides by B_t^i and summing over all $i \in 1 \dots N$:

$$x_{t+1}^* - A_t x_t^* = - \sum_{i=1}^N B_t^i R_t^{ii^{-1}} \left[B_t^{iT} (M_{t+1}^i x_{t+1}^* + \lambda_{t+1}^i) + r_t^{ii} \right] \tag{24}$$

$$= (I - \Lambda_t)x^*_{t+1} - \hat{\lambda}_t - \bar{\lambda}_t \tag{25}$$

$$\implies x^*_{t+1} = \Lambda_t^{-1}(A_t x^*_t - \hat{\lambda}_t - \bar{\lambda}_t)\,. \tag{26}$$

We have thus verified (17) and (18). To obtain (19), we suppose that it holds up to $t+1$ and verify it for t by substitution into (8):

$$p^{i*}_t = A_t^T(p^{i*}_{t+1} + Q^i_{t+1}x^*_{t+1} + q^i_{t+1}) \tag{27}$$

$$= A_t^T\Big(A_{t+1}^T\big(M^i_{t+2}\Lambda^{-1}_{t+1}(A_{t+1}x^*_{t+1} - \hat{\lambda}_{t+1} - \bar{\lambda}_{t+1}) + q^i_{t+2}\big) + Q^i_{t+1}x^*_{t+1} + q^i_{t+1}\Big) \tag{28}$$

$$= A_t^T\Big(M^i_{t+1}x^*_{t+1} + A_{t+1}^T\big(- M^i_{t+2}\Lambda^{-1}_{t+1}(\hat{\lambda}_{t+1} + \bar{\lambda}_{t+1}) + q^i_{t+2}\big) + q^i_{t+1}\Big) \tag{29}$$

$$= A_t^T\Big(M^i_{t+1}x^*_{t+1} + \lambda^i_{t+1}\Big) \tag{30}$$

which agrees with (19), and completes the induction.