National University of Singapore

MA1522 - Linear Algebra for Computing

Midterm Exam

(AY2025/2026 Semester 1)

Time allowed: 1 hour

<u>INSTRUCTIONS TO STUDENTS</u>

1. This assessment consists of **15** questions.
2. Submit your answers in Examplify.
3. You are allowed to use MATLAB and any offline calculator.
4. You are allowed to refer to any soft copy or hard copy notes. All soft copy notes are to be access through the device you are using for the exam.
5. You are required to answer ALL questions.
6. This is a NON-SECURE BLOCK INTERNET assessment.

1. Consider the system

$$\begin{cases} x_1 & - & ax_2 & + & (2-a)x_3 & - & (a+1)x_4 & = & -(a+1) \\ x_1 & - & ax_2 & + & ax_3 & - & x_4 & = & 1 \\ -x_1 & - & x_2 & + & (a-2)x_3 & + & (a+1)x_4 & = & 1 \\ x_1 & + & (a+2)x_2 & + & x_3 & + & (a-1)x_4 & = & 3a \\ & & (a+1)x_2 & + & (a-1)x_3 & + & ax_4 & = & 2a+1 \end{cases}$$

for some constant $a \in \mathbb{R}$. Which of the following statements are true? Select all that apply.

(a) The system will be inconsistent for two values of a.

(b) The system will have infinitely many solutions for only one value of a.

(c) The system will not have a unique solution of any value of a.

(d) The system will have infinitely many solutions for two values of a.

(e) The system will be inconsistent for only one value of a.

Solution: The system reduces to

$$\left(\begin{array}{cccc|c} 1 & -a & 2-a & -a-1 & -a-1 \\ 1 & -a & a & -1 & 1 \\ -1 & -1 & a-2 & a+1 & 1 \\ 1 & a+2 & 1 & a-1 & 3a \\ 0 & a+1 & a-1 & a & 2a+1 \end{array}\right) \xrightarrow{reduction} \left(\begin{array}{cccc|c} 1 & -a & 2-a & -a-1 & -a-1 \\ 0 & -a-1 & 0 & 0 & -a \\ 0 & 0 & a-1 & a & a+1 \\ 0 & 0 & 0 & a & a \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

One can see that if $a = \pm 1$ the system is inconsistent, and if $a = 0$, the system has infinitely many solution. Otherwise, the system has a unique solution.

2. Suppose $\mathbf{A}$ is a 3×3 matrix such that

$$\begin{pmatrix} -6 & -1 & 2 & -3 & -12 \\ 6 & 2 & 0 & 0 & 16 \\ 9 & 1 & -4 & 6 & 16 \end{pmatrix} = \mathbf{A} \begin{pmatrix} 3 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & -2 & 5 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix}.$$

Write $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Which of the following statements are true? Select all that apply.

(a) $\mathbf{A}$ is singular.

(b) $a_{11} = -2$ and $a_{12} = -1$.

(c) $\mathbf{A}$ is a product of some elementary matrices.

(d) $a_{21} = 2$ and $a_{22} = -1$.

(e) $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is the only solution to $\mathbf{Ax} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$.

Solution: By block multiplication, $\mathbf{A}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -6 & -1 & 2 \\ 6 & 2 & 0 \\ 9 & 1 & -4 \end{pmatrix}$. Note that $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ is invertible but $\begin{pmatrix} -6 & -1 & 2 \\ 6 & 2 & 0 \\ 9 & 1 & -4 \end{pmatrix}$ is not. So, $\mathbf{A}$ is not invertible, and hence (a) is true and (c) is false. By block multiplication,

$$\begin{pmatrix} -6 & -1 & 2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 3a_{11} & a_{12} & a_{12} - a_{13} \end{pmatrix} \implies a_{11} = -2,\ a_{12} = -1.$$

So, (b) is true. Similarly,

$$\begin{pmatrix} 6 & 2 & 0 \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 3a_{21} & a_{22} & a_{22} - a_{23} \end{pmatrix} \implies a_{21} = 2,\ a_{22} = 2.$$

So, (d) is false. By block multiplication

$$\mathbf{A}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}.$$

But since $\mathbf{A}$ is singular, the solution is not unique.

3. Let

$$\mathbf{A} = \begin{pmatrix} 1 & a & 2 & 0 \\ 0 & 1 & a & 3 \\ 1 & 1 & a & 1 \end{pmatrix},$$

for some constant $a \in \mathbb{R}$. Which of the following statements are true? Select all that apply.

(a) $\mathbf{A}$ has a right inverse for any value of a.

(b) $\mathbf{A}$ will not have a left inverse for any value of a.

(c) $\mathbf{A}$ has a right inverse of exactly one value of a.

(d) $\mathbf{A}$ will not have a right inverse for any value of a.

(e) $\mathbf{A}$ has a left inverse for any value of a.

Solution: $\mathbf{A}$ has a right inverse if there is a $\mathbf{X}$ such that

$$\mathbf{AX} = \mathbf{I}.$$

Solve the matrix equation by block multiplication:

$$\left(\begin{array}{cccc|ccc} 1 & a & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & a & 3 & 0 & 1 & 0 \\ 1 & 1 & a & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{reduction} \left(\begin{array}{cccc|ccc} 1 & a & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & a & 3 & 0 & 1 & 0 \\ 0 & 0 & a^2-2 & 3a-2 & -1 & a-1 & 1 \end{array}\right),$$

which is consistent for any values of a. So, (a) is true and (c) and (d) is false.

$\mathbf{A}$ has a left inverse if there is a $\mathbf{X}$ such that

$$\mathbf{XA} = \mathbf{I} \iff \mathbf{A}^T\mathbf{X}^T = \mathbf{I}.$$

Solve the matrix equation by block multiplication:

$$\left(\begin{array}{ccc|cccc} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ a & 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & a & a & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{reduction}$$

$$\left(\begin{array}{ccc|cccc} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1-a & -a & 1 & 0 & 0 \\ 0 & 0 & 7 & 3a+9 & -3 & -9/2 & 3a/2+1 \\ 0 & 0 & 0 & 12/7-6a^2/7 & 6a/7-18/7 & 9a/7-6/7 & 6/7-3a^2/7 \end{array}\right),$$

which is inconsistent for any choice of a. So, (b) is true too.

4. Consider the set

$$S = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \begin{pmatrix} a & 1 & 2 \\ b & -2 & 3 \\ c & 1 & 0 \end{pmatrix} \text{ is singular} \right\}.$$

Which of the following statements are true? Select all that apply.

(a) S is a subspace and $\left\{ \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix} \right\}$ spans S.

(b) S is a subspace and $\left\{ \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix} \right\}$ spans S.

(c) S is not a subspace and $\begin{pmatrix} 16 \\ 3 \\ 6 \end{pmatrix}$ is in S.

(d) S is a subspace and $\left\{\begin{pmatrix}14\\0\\6\end{pmatrix}, \begin{pmatrix}7\\0\\3\end{pmatrix}\right\}$ spans S.

(e) S is not a subspace and $\begin{pmatrix}5\\-3\\3\end{pmatrix}$ is in S.

Solution: $\begin{pmatrix}a & 1 & 2\\ b & -2 & 3\\ c & 1 & 0\end{pmatrix}$ is singular if and only if its determinant is 0.

$$\det\left(\begin{pmatrix}a & 1 & 2\\ b & -2 & 3\\ c & 1 & 0\end{pmatrix}\right) = -3a + 2b + 7c = 0.$$

This is a homogeneous system, and so S is a subspace. Hence, (c) and (e) are false. Solving the system, we obtain

$$s\begin{pmatrix}2\\3\\0\end{pmatrix} + t\begin{pmatrix}7\\0\\3\end{pmatrix}, \quad s, t \in \mathbb{R}$$

is a general solution. So, (a) is true. Now check that

$$\text{span}\left\{\begin{pmatrix}2\\3\\0\end{pmatrix}, \begin{pmatrix}7\\0\\3\end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix}4\\-1\\2\end{pmatrix}, \begin{pmatrix}4\\6\\0\end{pmatrix}\right\} \neq \text{span}\left\{\begin{pmatrix}14\\0\\6\end{pmatrix}, \begin{pmatrix}7\\0\\3\end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix}7\\0\\3\end{pmatrix}\right\}.$$

So, (b) is true, and (d) is false.

5. Let $\mathbf{A} = \begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 4\\ 1 & -3 & -12\end{pmatrix}$ and $\mathbf{U}$ be a 3×3 matrix such that

$$\mathbf{A} = \begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 1 & -3 & 1\end{pmatrix}\mathbf{U}.$$

Which of the following statements are true? Select all that apply.

(a) $\mathbf{U}$ is obtained from $\mathbf{A}$ via the following row operations, $\mathbf{U} \xrightarrow{R_3-3R_2} \xrightarrow{R_3+R_1} \xrightarrow{R_2+2R_1} \mathbf{A}$.

(b) $\mathbf{v} \in \mathbb{R}^3$ is a solution of $\mathbf{Ux} = \mathbf{0}$ if it is a solution to $\mathbf{Ax} = \mathbf{0}$.

(c) $\mathbf{U} = \begin{pmatrix}1 & 0 & 0\\ -28 & 13 & 4\\ 91 & -39 & -12\end{pmatrix}$

(d) $\mathbf{A}$ is obtained from $\mathbf{U}$ via the following row operations, $\mathbf{A} \xrightarrow{R_2+2R_1} \xrightarrow{R_3+R_1} \xrightarrow{R_3-3R_2} \mathbf{U}$.

(e) $\begin{pmatrix}3\\2\\0\end{pmatrix}$ is a solution to the system $\mathbf{U}\mathbf{x} = \begin{pmatrix}3\\8\\-3\end{pmatrix}$.

Solution: Since $\begin{pmatrix}1&0&0\\2&1&0\\1&-3&1\end{pmatrix}$ is invertible,

$$\mathbf{U} = \begin{pmatrix}1&0&0\\2&1&0\\1&-3&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&0\\2&1&4\\1&-3&-12\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&4\\0&0&0\end{pmatrix}.$$

So, (c) is false. Observe that

$$\begin{pmatrix}1&0&0\\2&1&0\\1&-3&1\end{pmatrix} = \begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&-3&1\end{pmatrix}.$$

So,

$$\mathbf{A} = \begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&-3&1\end{pmatrix}\mathbf{U}.$$

So, (a) is correct, and (d) is false. Now since $\mathbf{A}$ and $\mathbf{U}$ are row equivalent, $\mathbf{A}\mathbf{v} = \mathbf{0}$ if and only if $\mathbf{U}\mathbf{v} = \mathbf{0}$. So, (b) is true. Finally,

$$\begin{pmatrix}1&0&0\\0&1&4\\0&0&0\end{pmatrix}\begin{pmatrix}3\\2\\0\end{pmatrix} = \begin{pmatrix}3\\2\\0\end{pmatrix} \neq \begin{pmatrix}3\\8\\-3\end{pmatrix}.$$

6. Let

$$\mathbf{u}_1 = \begin{pmatrix}1\\0\\1\\2\end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix}0\\1\\-1\\1\end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix}1\\-2\\3\\0\end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix}1\\-1\\2\\1\end{pmatrix}.$$

Which of the following statements are true? Select all that apply.

(a) $\mathbf{v}_2 \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$

(b) $\mathbf{u}_2 \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$

(c) $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$

(d) $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

(e) $\mathbf{u}_2 \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$

Solution: This is just a straightforward checking of each options.

7. It is given that $\mathbf{A}$ is a 4×4 matrix such that

$$\mathbf{A}\begin{pmatrix}-2\\1\\1\\1\end{pmatrix} = \begin{pmatrix}7\\5\\-3\\7\end{pmatrix}, \quad \mathbf{A}\begin{pmatrix}-1\\0\\1\\1\end{pmatrix} = \begin{pmatrix}7\\5\\-3\\7\end{pmatrix} \quad \text{and} \quad \mathbf{A}\begin{pmatrix}2\\0\\2\\-3\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

Which of the following vectors are solutions to the system $\mathbf{A}\mathbf{x} = \mathbf{0}$?

(a) $\begin{pmatrix}1\\1\\2\\-3\end{pmatrix}$

(b) $\begin{pmatrix}0\\2\\2\\-3\end{pmatrix}$

(c) $\begin{pmatrix}0\\1\\3\\-2\end{pmatrix}$

(d) $\begin{pmatrix}1\\0\\3\\-2\end{pmatrix}$

(e) $\begin{pmatrix}1\\2\\1\\-3\end{pmatrix}$

Solution: Subtracting the first two given relations gives

$$\mathbf{A}\left(\begin{pmatrix}-2\\1\\1\\1\end{pmatrix} - \begin{pmatrix}-1\\0\\1\\1\end{pmatrix}\right) = \begin{pmatrix}7\\5\\-3\\7\end{pmatrix} - \begin{pmatrix}7\\5\\-3\\7\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

hence

$$\mathbf{A}\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

Together with $\mathbf{A}\begin{pmatrix}2\\0\\2\\-3\end{pmatrix} = \mathbf{0}$, we know that any linear combination of $\begin{pmatrix}2\\0\\2\\-3\end{pmatrix}$ and $\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$ is a solution to the homogeneous system. Check that only the vector in (a) and (b) is a linear combination of these two vectors.

8. Let

$$S = \left\{ \mathbf{u}_1 = \begin{pmatrix}1\\a\\1\\a+2\end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix}a\\1\\1\\a+2\end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix}1\\1\\a\\a+2\end{pmatrix} \right\},$$

and

$$T = \left\{ \mathbf{v}_1 = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\}.$$

Which of the following statements are true? Select all that apply.

(a) $\text{span}(T) \subseteq \text{span}(S)$ for infinitely many values of a.

(b) $\text{span}(S)$ is a strict subset of $\text{span}(T)$ for exactly two values of a.

(c) $\text{span}(T)$ will not be a subset of $\text{span}(S)$ for any value of a.

(d) $\text{span}(S) = \text{span}(T)$ for exactly two values of a.

(e) $\text{span}(S) = \text{span}(T)$ if $(a-1)^2(a+2) = 0$.

Solution:

$$\left(\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \;\middle|\; \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \right) = \left(\begin{array}{ccc|ccc} 1 & a & 1 & 1 & 0 & 0 \\ a & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & a & 0 & 0 & 1 \\ a+2 & a+2 & a+2 & 1 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{reduction} \left(\begin{array}{ccc|ccc} 1 & a & 1 & 1 & 0 & 0 \\ 0 & 1-a & a-1 & -1 & 0 & 1 \\ 0 & 0 & -a^2-a+2 & 1 & 1 & -a-1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

which is consistent if and only if $-a^2-a+2 = -(a-1)(a+2) \neq 0$. So, $\text{span}(T) \subseteq \text{span}(S)$ for $a \neq 1, -2$, and hence, (a) is true. On the other hand,

$$\left(\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \;\middle|\; \mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \right) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & a & 1 \\ 0 & 1 & 0 & a & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & a \\ 1 & 1 & 1 & a+2 & a+2 & a+2 \end{array}\right)$$

$$\xrightarrow{reduction} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & a & 1 \\ 0 & 1 & 0 & a & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & a \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

is consistent for any value of a. So, $\text{span}(S) \subseteq \text{span}(T)$ for any values of a.

9. Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \ ax + by + cz = d \right\}$$

for some constants a, b, c, d. It is given that V contains the set

$$\text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

If $a + b + c + d = 5$, then a=[blank], b =[blank], c =[blank] and d =[blank].

Solution: Since V contains a span, it must contain the origin. Thus $d = 0$. Now substitute the vectors into the equation,

$$\begin{cases} a(1) + b(-1) + c(0) = 0 \implies a - b = 0 \\ a(2) + b(1) + c(-1) = 0 \implies 2a + b - c = 0 \\ a(3) + b(0) + c(-1) = 0 \implies 3a - c = 0 \end{cases}$$

Together with $a + b + c = 5$, the solution of the system is

$$a = 1, \quad b = 1, \quad c = 3.$$

10. Let $\mathbb{R}^{n,n}$ denoted the set of $n \times n$ matrices. It is known that $\mathbb{R}^{n,n}$ is a vector space. Which of the following sets is a subspace? Select all that apply.

    (a) $\{\mathbf{A} \in \mathbb{R}^{n,n} \mid \mathbf{A}^T = \mathbf{A}\}$.

    (b) $\{\mathbf{A} \in \mathbb{R}^{n,n} \mid \mathbf{A}^T = -\mathbf{A}\}$.

    (c) $\{\mathbf{A} \in \mathbb{R}^{n,n} \mid \mathbf{AB} = \mathbf{BA}\}$ for a fixed matrix $\mathbf{B} \in \mathbb{R}^{n,n}$.

    (d) $\{\mathbf{A} \in \mathbb{R}^{n,n} \mid \mathbf{A}$ is invertible$\}$.

    (e) $\{\mathbf{A} \in \mathbb{R}^{n,n} \mid \mathbf{A}$ is singular$\}$.

11. Let $\mathbf{A}$ be a 4×4 real matrix that is **not a scalar matrix**. Suppose that $\mathbf{A}^2 + 2\mathbf{A} - 3\mathbf{I} = 0$. Then which of the following is invertible? Select all that apply.

    (a) $\mathbf{A}$.

    (b) $\mathbf{A} + \mathbf{I}$.

    (c) $\mathbf{A} + 2\mathbf{I}$.

    (d) $\mathbf{A} - \mathbf{I}$.

    (e) $\mathbf{A} + 3\mathbf{I}$.

Solution: For any real number a, we have

$$(\mathbf{A}+a\mathbf{I})(\mathbf{A}+(2-a)\mathbf{I}) = \mathbf{A}^2+2\mathbf{A}+a(2-a)\mathbf{I} = (3+a(2-a))\mathbf{I},$$

where the last equality follows from the assumption that $\mathbf{A}^2+2\mathbf{A}=3\mathbf{I}$. Thus, if a is not a root of $3+x(2-x)=-x^2+2x+3$, we see that $\mathbf{A}+a\mathbf{I}$ is invertible with inverse $\frac{1}{3+a(2-a)}(\mathbf{A}+(2-a)\mathbf{I})$, so (a), (b), (c) are correct.

It remains to consider the case that a is a root of $-x^2+2x+3$, i.e. $a=-1$ or 3. In these cases, note that we already have

$$0=\mathbf{A}^2+2\mathbf{A}-3\mathbf{I}=(\mathbf{A}-\mathbf{I})(\mathbf{A}+3\mathbf{I}).$$

Due to the assumption that $\mathbf{A}$ is not a scalar matrix, we see that $\mathbf{A}-\mathbf{I}$ and $\mathbf{A}+3\mathbf{I}$ are not zero matrices, so they must have a nonzero column and a nonzero row. Then any nonzero column of $\mathbf{A}+3\mathbf{I}$ is a non-trivial solution to the homogeneous equations

$$(\mathbf{A}-\mathbf{I})\mathbf{x}=\mathbf{0}.$$

Thus, $\mathbf{A}-\mathbf{I}$ cannot be invertible. Similarly, if $\begin{pmatrix} a & b & c & d\end{pmatrix}$ is a nonzero row of $\mathbf{A}-\mathbf{I}$, then taking transpose, we see that

$$(\mathbf{A}+3\mathbf{I})^T\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

Thus, $(\mathbf{A}+3\mathbf{I})^T$ is not invertible. Hence, $\mathbf{A}+3\mathbf{I}$ is also not invertible.

12. Suppose that

$$\begin{pmatrix}1&2&1\\1&3&4\\2&3&-1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}6\\11\\7\end{pmatrix},\quad \begin{pmatrix}0&-2&4\\1&1&-1\\1&-3&7\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\2\\2\end{pmatrix}.$$

Then a=[blank], b =[blank], and c =[blank]. Put x if there is not enough information.

Solution: We solve the two systems of linear equations:

$$\left(\begin{array}{ccc|c}1&2&1&6\\1&3&4&11\\2&3&-1&7\end{array}\right)\xrightarrow{RREF}\left(\begin{array}{ccc|c}1&0&-5&-4\\0&1&3&5\\0&0&0&0\end{array}\right)$$
$$\left(\begin{array}{ccc|c}0&-2&4&0\\1&1&-1&2\\1&-3&7&2\end{array}\right)\xrightarrow{RREF}\left(\begin{array}{ccc|c}1&0&1&2\\0&1&-2&0\\0&0&0&0\end{array}\right).$$

Thus, the general solutions are

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -4+5t \\ 5-3t \\ t \end{pmatrix} =: \mathbf{v}(t), \quad \text{and} \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2-s \\ 2s \\ s \end{pmatrix} =: \mathbf{u}(s),$$

for $t, s \in \mathbb{R}$ respectively. The common solution of the two systems is

$$\{\mathbf{v}(t) \mid \mathbf{v}(t) = \mathbf{u}(s) \text{ for some } s \in \mathbb{R}\}.$$

Thus, we solve

$$\begin{cases} -4+5t & = 2-s \\ 5-3t & = 2s \\ t & = s \end{cases} \implies t = s = 1.$$

Then we obtain $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

13. Let $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a vector in $\mathbb{R}^3$. Given that

$$||\mathbf{v} - \mathbf{v}_1|| = 5, \ ||\mathbf{v} - \mathbf{v}_2|| = 6, \ ||\mathbf{v} - \mathbf{v}_3|| = 7, \ ||\mathbf{v} - \mathbf{v}_4|| = 9,$$

where

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 4 \\ 4 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix}.$$

Then $a =$[blank], $b =$[blank], and $c =$[blank]. Put x if there is not enough information.

Solution: The conditions are

$$\begin{cases} (a+2)^2 + & (b-4)^2 + & (c-4)^2 & = & 25 \\ (a-6)^2 + & (b+3)^2 + & (c-2)^2 & = & 36 \\ (a-8)^2 + & (b-4)^2 + & (c-6)^2 & = & 49 \\ (a+2)^2 + & (b-0)^2 + & (c+4)^2 & = & 81 \end{cases} \tag{$*$}$$

Regard the above equations as a system of linear equations with variables a^2, b^2, c^2, a, b, c, then the augmented matrix is

$$\left(\begin{array}{cccccc|c} 1 & 1 & 1 & 4 & -8 & -8 & -11 \\ 1 & 1 & 1 & -12 & 6 & -4 & -13 \\ 1 & 1 & 1 & -16 & -8 & -12 & -67 \\ 1 & 1 & 1 & 4 & 0 & 8 & 61 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 21 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right).$$

Thus, the only possible solution is $a = 2, b = 1, c = 4$. It is a straightforward verification that this is indeed a solution to the system of equations $(*)$.

14. Suppose that $\mathbf{A}$ is a 4×4 matrix such that

$$\mathbf{A}\begin{pmatrix}1 & 0 & 1 & 1\\ 1 & 1 & -1 & 2\\ 2 & 0 & 1 & 0\\ 0 & -1 & 1 & -3\end{pmatrix} = \begin{pmatrix}5 & 2 & 0 & 5\\ 17 & 4 & 3 & 9\\ -5 & -6 & 4 & -17\\ -19 & 6 & -14 & 21\end{pmatrix}.$$

Suppose that

$$\mathbf{A}\begin{pmatrix}0\\4\\1\\-3\end{pmatrix} = \begin{pmatrix}a\\b\\c\\d\end{pmatrix}.$$

Then a=[blank], b =[blank], c =[blank] and d =[blank]. Put x if there is not enough information.

Solution: We first solve

$$\begin{pmatrix}1 & 0 & 1 & 1\\ 1 & 1 & -1 & 2\\ 2 & 0 & 1 & 0\\ 0 & -1 & 1 & -3\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}0\\4\\1\\-3\end{pmatrix}.$$

Since

$$\left(\begin{array}{cccc|c}1 & 0 & 1 & 1 & 0\\ 1 & 1 & -1 & 2 & 4\\ 2 & 0 & 1 & 0 & 1\\ 0 & -1 & 1 & -3 & -3\end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cccc|c}1 & 0 & 0 & -1 & 1\\ 0 & 1 & 0 & 5 & 2\\ 0 & 0 & 1 & 2 & -1\\ 0 & 0 & 0 & 0 & 0\end{array}\right),$$

the general solution is

$$x = 1 + t, \quad y = 2 - 5t, \quad z = -1 - 2t, \quad w = t.$$

Then

$$\begin{aligned}\mathbf{A}\begin{pmatrix}0\\4\\1\\-3\end{pmatrix} &= \mathbf{A}\begin{pmatrix}1 & 0 & 1 & 1\\ 1 & 1 & -1 & 2\\ 2 & 0 & 1 & 0\\ 0 & -1 & 1 & -3\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}\\ &= \begin{pmatrix}5 & 2 & 0 & 5\\ 17 & 4 & 3 & 9\\ -5 & -6 & 4 & -17\\ -19 & 6 & -14 & 21\end{pmatrix}\begin{pmatrix}1+t\\2-5t\\-1-2t\\t\end{pmatrix} = \begin{pmatrix}9\\22\\-21\\7\end{pmatrix}.\end{aligned}$$

Thus,

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 9 \\ 22 \\ -21 \\ 7 \end{pmatrix}.$$

15. Let $f(x)$ be a cubic polynomial. Suppose that $f(0) = 2$, $f(1) = 10$, $f(2) = 34$ and $f(3) = 80$. Then $f(10) =$[blank].

    Solution: Write $f(x) = ax^3 + bx^2 + cx + d$. Then we obtain a system of linear equations in a, b, c, d with augmented matrix given by

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 10 \\ 8 & 4 & 2 & 1 & 34 \\ 27 & 9 & 3 & 1 & 80 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right).$$

    Thus, $f(x) = x^3 + 5x^2 + 2x + 2x$, and $f(10) = 1522$.