数学物理方程

第一部分 2022 年真题解析

1. 针对本征值问题，证明正交性

$$(1).\begin{cases} X''(x)+\lambda X(x)=0 \\ X'(0)=0,\ X'(l)=0 \end{cases} \qquad (2).\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)\cos\alpha-X'(0)\sin\alpha=0 \\ X(l)\cos\beta+X'(l)\sin\beta=0 \end{cases}$$

证明：(1)对于 $X_1(x)$ 和 $X_2(x)$ 显然有

$$\begin{cases} X_1''(x)+\lambda_1 X_1(x)=0 \\ X_2''(x)+\lambda_2 X_2(x)=0 \end{cases}$$

第一个方程乘 $X_2(x)$ 减去第二个方程乘 $X_1(x)$ 可得

$$X_2(x)X_1''(x)-X_1(x)X_2''(x)=(\lambda_2-\lambda_1)X_1(x)X_2(x)$$

积分可得

$$\int_0^l [X_2(x)X_1''(x)-X_1(x)X_2''(x)]\mathrm{d}x=(\lambda_2-\lambda_1)\int_0^l X_1(x)X_2(x)\mathrm{d}x$$

$$[X_2(x)X_1'(x)-X_1(x)X_2'(x)]\big|_0^l=(\lambda_2-\lambda_1)\int_0^l X_1(x)X_2(x)\mathrm{d}x$$

$$X_2(l)X_1'(l)-X_1(l)X_2'(l)-X_2(0)X_1'(0)+X_1(0)X_2'(0)=(\lambda_2-\lambda_1)\int_0^l X_1(x)X_2(x)\mathrm{d}x$$

带入边界条件可得

$$(\lambda_2-\lambda_1)\int_0^l X_1(x)X_2(x)\mathrm{d}x=0\xrightarrow{\lambda_2\neq\lambda_1}\int_0^l X_1(x)X_2(x)\mathrm{d}x=0.\square$$

(2)同(1)可得

$$X_2(l)X_1'(l)-X_1(l)X_2'(l)-X_2(0)X_1'(0)+X_1(0)X_2'(0)=(\lambda_2-\lambda_1)\int_0^l X_1(x)X_2(x)\mathrm{d}x$$

$$-X_2'(l)X_1'(l)\tan\beta+X_1'(l)X_2'(l)\tan\beta-X_2'(0)X_1'(0)\tan\alpha+X_1'(0)X_2'(0)\tan\alpha=(\lambda_2-\lambda_1)\int_0^l X_1(x)X_2(x)\mathrm{d}x$$

带入边界条件可得

$$(\lambda_2-\lambda_1)\int_0^l X_1(x)X_2(x)\mathrm{d}x=0\xrightarrow{\lambda_2\neq\lambda_1}\int_0^l X_1(x)X_2(x)\mathrm{d}x=0.\square$$

2. 分离变量法求解问题

$$\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=9$$

$$u|_{x=0}=0,\ u|_{x=l}=0,\ u|_{t=0}=0,\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=0$$

解：设$u(x,t)=v(x)+w(x,t)$，则有

$$-a^2v''(x)=9$$

$$v|_{x=0}=0,\ v|_{x=l}=0$$

求解该方程可得

$$v(x)=-\frac{9}{2a^2}x^2+\frac{9l}{2a^2}x=\frac{9x(l-x)}{2a^2}$$

而又有

$$\frac{\partial^2 w}{\partial t^2}-a^2\frac{\partial^2 w}{\partial x^2}=0$$

$$w|_{x=0}=0,\ w|_{x=l}=0,\ w|_{t=0}=-v|_{t=0}=\frac{9x(x-l)}{2a^2},\ \left.\frac{\partial w}{\partial t}\right|_{t=0}=-\left.\frac{\partial v}{\partial t}\right|_{t=0}=0$$

带入一般解有

$$w(x,t)=\sum_{n=1}^{\infty}\left(A_n\sin\frac{n\pi a}{l}t+B_n\cos\frac{n\pi a}{l}t\right)\sin\frac{n\pi}{l}x$$

带入边界条件可得

$$A_n=0 \qquad\qquad B_n=-\frac{36l^2}{(2n+1)^3a^2\pi^3}$$

则有

$$w(x,t)=-\frac{36l^2}{a^2\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\cos\frac{(2n+1)\pi a}{l}t\sin\frac{(2n+1)\pi}{l}x$$

综上可得

$$u(x,t)=\frac{9x(l-x)}{2a^2}-\frac{36l^2}{a^2\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\cos\frac{(2n+1)\pi a}{l}t\sin\frac{(2n+1)\pi}{l}x.\square$$

注：

$$B_n=\frac{2}{l}\int_0^l\frac{9x(x-l)}{2a^2}\sin\frac{n\pi}{l}x\,\mathrm{d}x=\frac{9}{a^2l}\int_0^l x(x-l)\sin\frac{n\pi}{l}x\,\mathrm{d}x$$

$$=\frac{9}{a^2l}\int_0^l x(x-l)\sin\frac{n\pi}{l}x\,\mathrm{d}x=\frac{18l^2(\cos n\pi-1)}{n^3a^2\pi^3}=-\frac{36l^2}{(2n+1)^3a^2\pi^3}$$

3. 讨论下列方程类型并化为标准形式

$$y\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

解：判别式：$\triangle=b^2-ac=-y$，则有

(1)$\triangle<0 \Longleftrightarrow y>0$：该方程属于椭圆型

(2)$\triangle>0 \Longleftrightarrow y<0$：该方程属于双曲型

(3)$\triangle=0 \Longleftrightarrow y=0$：该方程属于抛物型

(这里仅仅讨论第一种情况)特征线：$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{b\pm\sqrt{b^2-ac}}{a}=\pm\frac{\sqrt{-y}}{y}=\pm\mathrm{i}\frac{1}{\sqrt{y}}$，解之可得

$$x\pm\mathrm{i}\frac{2}{3}y^{\frac{3}{2}}=C$$

作两次换元可得

$$\begin{cases}\xi=x+\mathrm{i}\frac{2}{3}y^{\frac{3}{2}}\\ \eta=x-\mathrm{i}\frac{2}{3}y^{\frac{3}{2}}\end{cases}\qquad\qquad \begin{cases}\rho=\xi+\eta=2x\\ \sigma=\mathrm{i}(\xi-\eta)=-\frac{4}{3}y^{\frac{3}{2}}\end{cases}$$

求解可得

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial x}+\frac{\partial u}{\partial \sigma}\frac{\partial \sigma}{\partial x}=2\frac{\partial u}{\partial \rho}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(2\frac{\partial u}{\partial \rho}\right)=\frac{\partial}{\partial \rho}\left(2\frac{\partial u}{\partial \rho}\right)\frac{\partial \rho}{\partial x}+\frac{\partial}{\partial \sigma}\left(2\frac{\partial u}{\partial \rho}\right)\frac{\partial \sigma}{\partial x}=4\frac{\partial^2 u}{\partial \rho^2}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial y}+\frac{\partial u}{\partial \sigma}\frac{\partial \sigma}{\partial y}=-2\sqrt{y}\frac{\partial u}{\partial \sigma}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(-2\sqrt{y}\frac{\partial u}{\partial \sigma}\right)=-\frac{1}{\sqrt{y}}\frac{\partial u}{\partial \sigma}-2\sqrt{y}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \sigma}\right)=-\frac{1}{\sqrt{y}}\frac{\partial u}{\partial \sigma}+4y\frac{\partial^2 u}{\partial \sigma^2}$$

带入原方程可得

$$4y\frac{\partial^2 u}{\partial \rho^2}-\frac{1}{\sqrt{y}}\frac{\partial u}{\partial \sigma}+4y\frac{\partial^2 u}{\partial \sigma^2}=0\Longrightarrow\frac{\partial^2 u}{\partial \rho^2}+\frac{\partial^2 u}{\partial \sigma^2}+\frac{1}{3\sigma}\frac{\partial u}{\partial \sigma}=0.\square$$

4. 见书，略.

5. 利用分离变量法分离下列方程

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}=0$$

解：该方程为三元，则设$u(r,\theta,z)=v(r,\theta)Z(z)$，则有

$$\frac{1}{r}\frac{\partial}{\partial r}\left[Z(z)r\frac{\partial v}{\partial r}\right]+Z(z)\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+v(r,\theta)\frac{\mathrm{d}^2 Z(z)}{\mathrm{d}z^2}=0$$

分离可得

$$\frac{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}}{v(r,\theta)}=-\frac{\frac{\mathrm{d}^2 Z(z)}{\mathrm{d}z^2}}{Z(z)}=-\lambda$$

即

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left[r\dfrac{\partial v}{\partial r}\right]+\dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial\theta^2}+\lambda v(r,\theta)=0 \\ \dfrac{\mathrm{d}^2 Z(z)}{\mathrm{d}z^2}-\lambda Z(z)=0 \end{cases}$$

对于第一个方程，继续利用分离变量法，设$v(r,\theta)=R(r)\Theta(\theta)$，则有

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\Theta(\theta)r\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right]+R(r)\frac{1}{r^2}\frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2}+\lambda R(r)\Theta(\theta)=0$$

分离可得

$$\frac{\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right]+\lambda R(r)}{\frac{R(r)}{r^2}}=-\frac{\frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2}}{\Theta(\theta)}=-\mu$$

即

$$\begin{cases} \dfrac{1}{r}\dfrac{\mathrm{d}}{\mathrm{d}r}\left[r\dfrac{\mathrm{d}R(r)}{\mathrm{d}r}\right]+\lambda R(r)+\mu\dfrac{R(r)}{r^2}=0 \\ \dfrac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2}-\mu\Theta(\theta)=0 \end{cases}$$

综上可得

$$\begin{cases} \dfrac{1}{r}\dfrac{\mathrm{d}}{\mathrm{d}r}\left[r\dfrac{\mathrm{d}R(r)}{\mathrm{d}r}\right]+\left(\lambda+\dfrac{\mu}{r^2}\right)R(r)=0 \\ \dfrac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2}-\mu\Theta(\theta)=0 \\ \dfrac{\mathrm{d}^2 Z(z)}{\mathrm{d}z^2}-\lambda Z(z)=0 \end{cases} .\square$$

6. Fourier 推导无界弦上的自由振动

$$\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=0$$

$$u|_{t=0}=\phi(x),\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=\psi(x)$$

解：作 Fourier 变换有

$$U(k,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x,\ \Phi(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x,\ \Psi(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x$$

且有

$$\mathscr{F}\left[\frac{\partial^2 u}{\partial x^2}\right]=(\mathrm{i}k)^2\mathscr{F}[u(x,t)]=-k^2U(k,t)$$

则可得

$$\frac{\mathrm{d}^2U(k,t)}{\mathrm{d}t^2}+k^2a^2U(k,t)=0$$

$$U(k,t)|_{t=0}=\Phi(k),\ \left.\frac{\partial U(k,t)}{\partial t}\right|_{t=0}=\Psi(k)$$

解得通解为

$$U(k,t)=A\sin kat+B\cos kat$$

带入边界条件可得

$$A=\frac{\Psi(k)}{ka}\qquad\qquad B=\Phi(k)$$

则有

$$U(k,t)=\Phi(k)\cos ka+\frac{\Psi(k)}{ka}\sin kat$$

利用 Fourier 反演公式可得

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\Phi(k)\cos kat+\frac{\Psi(k)}{ka}\sin kat\right]\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k$$

对于 $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(k)\cos kat\cdot\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k$ 有

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(k)\cos kat\cdot\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k&=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{2}\int_{-\infty}^{\infty}\Phi(k)\left[\mathrm{e}^{\mathrm{i}kat}+\mathrm{e}^{-\mathrm{i}kat}\right]\cdot\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k\\&=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{2}\int_{-\infty}^{\infty}\Phi(k)\left[\mathrm{e}^{\mathrm{i}k(x+at)}+\mathrm{e}^{\mathrm{i}k(x-at)}\right]\mathrm{d}k\\&=\frac{1}{2}\left\{\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty}\Phi(k)\mathrm{e}^{\mathrm{i}k(x+at)}\mathrm{d}k+\int_{-\infty}^{\infty}\Phi(k)\mathrm{e}^{\mathrm{i}k(x-at)}\mathrm{d}k\right]\right\}\\&=\frac{1}{2}\left[\phi(x+at)+\phi(x-at)\right]\end{aligned}$$

对于 $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\Psi(k)}{ka}\sin kat\cdot\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k$ 有

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\Psi(k)}{ka}\sin kat\cdot\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(k)\left[\int_0^t\cos ka\tau\,\mathrm{d}\tau\right]\cdot\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k\\&=\int_0^t\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(k)\cos ka\tau\cdot\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k\right]\mathrm{d}\tau\\&=\frac{1}{2}\int_0^t\left[\psi(x+a\tau)+\psi(x-a\tau)\right]\mathrm{d}\tau\\&=\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi\end{aligned}$$

综上可得

$$u(x,t)=\frac{1}{2}\left[\phi(x+at)+\phi(x-at)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi.\square$$

7. Poisson 第一类边值问题 Green 函数，导出下列定解问题的解

$$\begin{cases}\nabla^2u(\boldsymbol{r})=-f(\boldsymbol{r})\\u(\boldsymbol{r})|_{\Sigma}=h(\boldsymbol{r})\end{cases}$$

解：其对应的 Green 函数为

$$\begin{cases}\nabla^2 G(\boldsymbol{r};\ \boldsymbol{r}') = -\dfrac{1}{\alpha}\delta(\boldsymbol{r}-\boldsymbol{r}') \\ G(\boldsymbol{r};\ \boldsymbol{r}')|_{\Sigma} = 0\end{cases}$$

第一个方程乘$G(\boldsymbol{r};\ \boldsymbol{r}')$减去第二个方程乘$u(\boldsymbol{r})$可得

$$G(\boldsymbol{r};\ \boldsymbol{r}')\nabla^2 u(\boldsymbol{r}) - u(\boldsymbol{r})\nabla^2 G(\boldsymbol{r};\ \boldsymbol{r}') = -\left[G(\boldsymbol{r};\ \boldsymbol{r}')f(\boldsymbol{r}) - \frac{1}{\alpha}u(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}')\right]$$

积分可得

$$\iiint_V [G(\boldsymbol{r};\ \boldsymbol{r}')\nabla^2 u(\boldsymbol{r}) - u(\boldsymbol{r})\nabla^2 G(\boldsymbol{r};\ \boldsymbol{r}')]\mathrm{d}V = -\iiint_V \left[G(\boldsymbol{r};\ \boldsymbol{r}')f(\boldsymbol{r}) - \frac{1}{\alpha}u(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}')\right]\mathrm{d}V$$

$$\iint_\Sigma [G(\boldsymbol{r};\ \boldsymbol{r}')\nabla u(\boldsymbol{r}) - u(\boldsymbol{r})\nabla G(\boldsymbol{r};\ \boldsymbol{r}')]\mathrm{d}\Sigma = \frac{1}{\alpha}u(\boldsymbol{r}') - \iiint_V G(\boldsymbol{r};\ \boldsymbol{r}')f(\boldsymbol{r})\mathrm{d}V$$

则有

$$\begin{aligned} u(\boldsymbol{r}') &= \alpha\iiint_V G(\boldsymbol{r};\ \boldsymbol{r}')f(\boldsymbol{r})\mathrm{d}V + \alpha\iint_\Sigma [G(\boldsymbol{r};\ \boldsymbol{r}')|_\Sigma \nabla u(\boldsymbol{r})|_\Sigma - u(\boldsymbol{r})|_\Sigma \nabla G(\boldsymbol{r};\ \boldsymbol{r}')|_\Sigma]\mathrm{d}\Sigma \\ &= \alpha\iiint_V G(\boldsymbol{r};\ \boldsymbol{r}')f(\boldsymbol{r})\mathrm{d}V - \alpha\iint_\Sigma u(\boldsymbol{r})|_\Sigma \nabla G(\boldsymbol{r};\ \boldsymbol{r}')|_\Sigma \mathrm{d}\Sigma \\ &= \alpha\iiint_V G(\boldsymbol{r};\ \boldsymbol{r}')f(\boldsymbol{r})\mathrm{d}V - \alpha\iint_\Sigma h(\boldsymbol{r})\frac{\partial G(\boldsymbol{r};\ \boldsymbol{r}')}{\partial \boldsymbol{n}}\mathrm{d}\Sigma \end{aligned}$$

将$\boldsymbol{r}$和$\boldsymbol{r}'$对调一下可得

$$u(\boldsymbol{r}) = \alpha\iiint_V G(\boldsymbol{r}';\ \boldsymbol{r})f(\boldsymbol{r}')\mathrm{d}V' - \alpha\iint_\Sigma h(\boldsymbol{r}')\frac{\partial G(\boldsymbol{r}';\ \boldsymbol{r})}{\partial \boldsymbol{n}'}\mathrm{d}\Sigma'.\square$$

第二部分　2023 年部分作业题解析

1. (P170)第十章 T2.(1)

解：直接写出通解表达式为

$$g(t;\tau)=\begin{cases}A(\tau)\sinh kt+B(\tau)\cosh kt,\ t<\tau\\ C(\tau)\sinh kt+D(\tau)\cosh kt,\ t>\tau\end{cases}$$

带入边界条件可得

$$A(\tau)=0 \qquad B(\tau)=0$$

$$C(\tau)\sinh k\tau+D(\tau)\cosh k\tau=0 \qquad C(\tau)\cosh k\tau+D(\tau)\sinh k\tau=\frac{1}{k}$$

利用 Cramer 法则解之可得

$$C(\tau)=\frac{\begin{vmatrix}0 & \cosh k\tau\\ \frac{1}{k} & \sinh k\tau\end{vmatrix}}{\begin{vmatrix}\sinh k\tau & \cosh k\tau\\ \cosh k\tau & \sinh k\tau\end{vmatrix}}=\frac{1}{k}\cosh k\tau \qquad D(\tau)=\frac{\begin{vmatrix}\sinh k\tau & 0\\ \cosh k\tau & \frac{1}{k}\end{vmatrix}}{\begin{vmatrix}\sinh k\tau & \cosh k\tau\\ \cosh k\tau & \sinh k\tau\end{vmatrix}}=-\frac{1}{k}\sinh k\tau$$

则有

$$g(t;\tau)=\frac{1}{k}\sinh k(t-\tau),\ \ t>\tau$$

综上可得

$$g(t;\tau)=\frac{1}{k}\sinh k(t-\tau)\eta(t-\tau),\ t,\ \tau>0$$

2. (P171)第十章 T3

解：其对应的格林函数为

$$\frac{\mathrm{d}^2 g(t;\tau)}{\mathrm{d}\tau^2}+k^2 g(t;\tau)=\delta(t-\tau),\ t,\ \tau>0,\ k>0$$

$$g(t;\tau)\big|_{\tau>t}=0,\ \left.\frac{\mathrm{d}g(t;\tau)}{\mathrm{d}t}\right|_{\tau>t}=0$$

原条件变为

$$\frac{\mathrm{d}^2 y(\tau)}{\mathrm{d}\tau^2}+k^2 y(\tau)=f(\tau),\ \tau>0,\ k>0$$

$$y(0)=A,\ \left.\frac{\mathrm{d}y(\tau)}{\mathrm{d}\tau}\right|_{\tau=0}=B$$

做差可得

$$g(t;\tau)\frac{\mathrm{d}^2 y(\tau)}{\mathrm{d}\tau^2}-y(\tau)\frac{\mathrm{d}^2 g(t;\tau)}{\mathrm{d}\tau^2}=g(t;\tau)f(\tau)-y(\tau)\delta(t-\tau)$$

积分可得

$$\int_0^\infty \left[g(t;\tau)\frac{\mathrm{d}^2 y(\tau)}{\mathrm{d}\tau^2} - y(\tau)\frac{\mathrm{d}^2 g(t;\tau)}{\mathrm{d}\tau^2}\right]\mathrm{d}\tau = \int_0^\infty g(t;\tau) f(\tau)\mathrm{d}\tau - \int_0^\infty y(\tau)\delta(t-\tau)\mathrm{d}\tau$$

$$\left[g(t;\tau)\frac{\mathrm{d}y(\tau)}{\mathrm{d}\tau} - y(\tau)\frac{\mathrm{d}g(t;\tau)}{\mathrm{d}\tau}\right]\Bigg|_{\tau=0} = \int_0^t g(t;\tau) f(\tau)\mathrm{d}\tau - y(t)$$

其中蕴含了$g(t;\tau)|_{\tau>t}=0$, $\left.\dfrac{\mathrm{d}g(t;\tau)}{\mathrm{d}t}\right|_{\tau>t}=0$，解之可得

$$\begin{aligned} y(t) &= \int_0^t g(t;\tau) f(\tau)\mathrm{d}\tau + \left[g(t;\tau)\frac{\mathrm{d}y(\tau)}{\mathrm{d}\tau} - y(\tau)\frac{\mathrm{d}g(t;\tau)}{\mathrm{d}\tau}\right]\Bigg|_{\tau=0} \\ &= \int_0^t g(t;\tau) f(t)\mathrm{d}\tau + \left[Bg(t;\tau) - A\frac{\mathrm{d}g(t;\tau)}{\mathrm{d}\tau}\right]\Bigg|_{\tau=0} \end{aligned}$$

因为 Green 函数的通解为$g(t;\tau)=\dfrac{1}{k}\sin k(t-\tau)$，带入可得

$$y(t) = A\cos kt + \frac{B}{k}\sin kt + \frac{1}{k}\int_0^t \sin k(t-\tau) f(\tau)\mathrm{d}\tau.\square$$

3. (P171)第十章 T4.(1)

解：步骤同上

因为 Green 函数的通解为$g(t;\tau)=\dfrac{1}{k}\sinh k(t-\tau)$，带入可得

$$y(t) = A\cosh kt + \frac{B}{k}\sinh kt + \frac{1}{k}\int_0^t \sinh k(t-\tau) f(\tau)\mathrm{d}\tau.\square$$

注：在这两个例子中，若$y(0)=0$, $\left.\dfrac{\mathrm{d}y(\tau)}{\mathrm{d}\tau}\right|_{\tau=0}=0$，则分别有

(1) $y(t)=\dfrac{1}{k}\displaystyle\int_0^t \sin k(t-\tau) f(\tau)\mathrm{d}\tau$

(2) $y(t)=\dfrac{1}{k}\displaystyle\int_0^t \sinh k(t-\tau) f(\tau)\mathrm{d}\tau$

也就是说

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + k^2 y(t) = f(t),\ t>0,\ k>0$$

$$y(0)=0,\ \left.\frac{\mathrm{d}y(t)}{\mathrm{d}t}\right|_{t=0}=0$$

解为

$$y(t)=\frac{1}{k}\int_0^t \sin k(t-\tau) f(\tau)\mathrm{d}\tau.\square$$

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - k^2 y(t) = f(t),\ t>0,\ k>0$$

$$y(0)=0,\ \left.\frac{\mathrm{d}y(t)}{\mathrm{d}t}\right|_{t=0}=0$$

解为

$$y(t)=\frac{1}{k}\int_0^t \sinh k(t-\tau)f(\tau)\mathrm{d}\tau.\square$$

4. (P171)第十章 T5.(1)

解：直接写出通解为

$$g(x;\xi)=\begin{cases}A(\xi)\sinh kx+B(\xi)\cosh kx,\ x<\xi\\ C(\xi)\sinh kx+D(\xi)\cosh kx,\ x>\xi\end{cases}$$

带入边界条件可得

$$B(\xi)=0 \qquad C(\xi)\cosh k\xi+D(\xi)\sinh k\xi-A(\xi)\cosh k\xi=\frac{1}{k}$$

$$C(\xi)\sinh k+D(\xi)\cosh k=0 \qquad A(\xi)\sinh k\xi=C(\xi)\sinh k\xi+D(\xi)\cosh k\xi$$

利用 Cramer 法则解之可得

$$A(\xi)=-\frac{1}{k}\frac{\sinh k(1-\xi)}{\sinh k} \qquad B(\xi)=0$$

$$C(\xi)=\frac{1}{k}\frac{\cosh k\sinh k\xi}{\sinh k} \qquad D(\xi)=-\frac{1}{k}\sinh k\xi$$

则有

$$g(x;\xi)=\begin{cases}-\dfrac{1}{k}\dfrac{\sinh k(1-\xi)}{\sinh k}\cdot\sinh kx,\ x<\xi\\ -\dfrac{1}{k}\dfrac{\sinh k\xi\sinh k(1-x)}{\sinh k},\ x>\xi\end{cases}$$

合并为

$$g(x;\xi)=-\frac{1}{k}\frac{\sinh k(1-\xi)}{\sinh k}\cdot\sinh kx-\frac{1}{k}\left[\frac{\sinh k\xi\sinh k(1-x)}{\sinh k}-\frac{\sinh k(1-\xi)}{\sinh k}\cdot\sinh kx\right]\eta(x-\xi)$$

最终化简可得

$$g(x;\xi)=-\frac{1}{k}\frac{\sinh k(1-\xi)}{\sinh k}\cdot\sinh kx+\frac{1}{k}\sinh k(x-\xi)\eta(x-\xi).\square$$

注：部分化简过程如下

$$\frac{\sinh k\xi\sinh k(1-x)}{\sinh k}-\frac{\sinh k(1-\xi)}{\sinh k}\cdot\sinh kx=\frac{\sinh k\xi\sinh k(1-x)-\sinh kx\sinh k(1-\xi)}{\sinh k}$$

$$=\frac{\sinh k\cosh kx\sinh k\xi-\cosh k\sinh kx\sinh k\xi-\sinh k\sinh kx\cosh k\xi+\cosh k\sinh kx\sinh k\xi}{\sinh k}$$

$$=-\sinh k(x-\xi)$$

5. (P225)第十三章 T5

解：利用分离变量法可得

$$\begin{cases}X''(x)-\lambda X(x)=0\\ Y''(y)+\lambda Y(y)=0\end{cases}$$

则有

$$Y(y)=A\sin\sqrt{\lambda}\,y+B\cos\sqrt{\lambda}\,y,\ Y'(y)=\sqrt{\lambda}\left(A\cos\sqrt{\lambda}\,y-B\sin\sqrt{\lambda}\,y\right)$$

带入边界条件可得

$$A=0 \qquad B=1 \qquad \lambda_n=\left(\frac{n\pi}{b}\right)^2$$

可得

$$Y_n(y)=\cos\frac{n\pi}{b}y$$

组装可得

$$u(x,y)=Cx+D+\sum_{n=1}^{\infty}\left(C_n\sinh\frac{n\pi}{b}x+D_n\cosh\frac{n\pi}{b}x\right)\cos\frac{n\pi}{b}y$$

带入边界条件可得

$$u(0,y)=D+\sum_{n=1}^{\infty}D_n\cos\frac{n\pi}{b}y=u_0$$

$$u(a,y)=Ca+D+\sum_{n=1}^{\infty}\left(C_n\sinh\frac{n\pi a}{b}+D_n\cosh\frac{n\pi a}{b}\right)\cos\frac{n\pi}{b}y=u_0\left[3\left(\frac{y}{b}\right)^2-2\left(\frac{y}{b}\right)^3\right]$$

则有

$$D=u_0 \qquad D_n=0$$

$$C_n=-\frac{48u_0}{(2n+1)^4\pi^4\sinh\frac{(2n+1)\pi a}{b}}$$

再由$u(a,b)=u_0$可得

$$C=-\frac{u_0}{2a}$$

综上可得

$$u(x,y)=u_0\left(1-\frac{x}{2a}\right)-\frac{48u_0}{\pi^4}\sum_{n=0}^{\infty}\frac{\sinh\frac{(2n+1)\pi}{b}x}{(2n+1)^4\sinh\frac{(2n+1)\pi a}{b}}\cos\frac{(2n+1)\pi}{b}y.\square$$

注：部分计算过程如下

$$D_n=\frac{2}{b}\int_0^b(u_0-D)\cos\frac{n\pi}{b}y\,\mathrm{d}y=\frac{2(u_0-D)}{n\pi}\int_0^{n\pi}\cos u\,\mathrm{d}u=0$$

$$C_n=\frac{2}{b\sinh\frac{n\pi a}{b}}\int_0^b\left\{u_0\left[3\left(\frac{y}{b}\right)^2-2\left(\frac{y}{b}\right)^3\right]-u_0-Ca\right\}\cos\frac{n\pi}{b}y\,\mathrm{d}y$$

$$=\frac{2}{b\sinh\frac{n\pi a}{b}}\left[3u_0\int_0^b\left(\frac{y}{b}\right)^2\cos\frac{n\pi}{b}y\,\mathrm{d}y-2u_0\int_0^b\left(\frac{y}{b}\right)^3\cos\frac{n\pi}{b}y\,\mathrm{d}y-(u_0+Ca)\int_0^b\cos\frac{n\pi}{b}y\,\mathrm{d}y\right]$$

$$=\frac{2}{b\sinh\frac{n\pi a}{b}}\left[3u_0b\int_0^1 y^2\cos n\pi y\,\mathrm{d}y-2u_0b\int_0^1 y^3\cos n\pi y\,\mathrm{d}y-(u_0+Ca)b\int_0^1\cos n\pi y\,\mathrm{d}y\right]$$

$$=\frac{2}{b\sinh\frac{n\pi a}{b}}\left[3u_0b\cdot\frac{2n\pi\cos n\pi}{n^3\pi^3}-2u_0b\cdot\frac{3n^2\pi^2\cos n\pi+6(1-\cos n\pi)}{n^4\pi^4}\right]$$

$$=-\frac{2u_0}{\sinh\frac{n\pi a}{b}}\frac{12(1-\cos n\pi)}{n^4\pi^4}=-\frac{24u_0(1-\cos n\pi)}{n^4\pi^4\sinh\frac{n\pi a}{b}}=-\frac{48u_0}{(2n+1)^4\pi^4\sinh\frac{(2n+1)\pi a}{b}}$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}=\frac{\pi^4}{96}$$

6. (P226)第十三章 T7.(1)

解：设$u(x,y)=v(x)+w(x,y)$，则有

$$\nabla^2 v(x)=-2$$
$$v|_{x=0,a}=0$$

解之可得

$$v(x)=x(a-x)$$

而又有

$$\nabla^2 w(x,y)=0$$
$$w|_{x=0,a}=-v|_{x=0,a}=0,\ \ w|_{y=\pm\frac{b}{2}}=-v|_{y=\pm\frac{b}{2}}=-x(a-x)$$

带入一般解有

$$w(x,y)=\sum_{n=1}^{\infty}\left(C_n\sinh\frac{n\pi}{a}y+D_n\cosh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

带入边界条件可得

$$C_n=0 \qquad\qquad D_n=-\frac{8a^2}{(2n+1)^3\pi^3\cosh\frac{2n+1}{2a}\pi b}$$

则有

$$w(x,y)=-\frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\frac{\cosh\frac{2n+1}{a}\pi y}{\cosh\frac{2n+1}{2a}\pi b}\sin\frac{2n+1}{a}\pi x$$

综上可得

$$u(x,y)=x(a-x)-\frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\frac{\cosh\frac{2n+1}{a}\pi y}{\cosh\frac{2n+1}{2a}\pi b}\sin\frac{2n+1}{a}\pi x.\square$$

注：部分计算过程如下

$$w(x,y)=w(x,-y)\Longrightarrow C_n=0$$

$$D_n=-\frac{2}{a\cosh\frac{n}{2a}\pi b}\int_0^a x(a-x)\sin\frac{n}{a}\pi x\,\mathrm{d}x=\frac{2}{a\cdot\cosh\frac{n}{2a}\pi b}\cdot\frac{2a^3(\cos n\pi-1)}{n^3\pi^3}$$

$$=-\frac{2}{a\cdot\cosh\frac{2n+1}{2a}\pi b}\cdot\frac{4a^3}{(2n+1)^3\pi^3}=-\frac{8a^2}{(2n+1)^3\pi^3\cosh\frac{2n+1}{2a}\pi b}$$

7. (P226)第十三章 T7.(2)

解：步骤同上，解得

$$v(x,y)=\frac{1}{12}xy(a^3-x^3)$$

且有

$$\nabla^2 w(x,y)=0$$

$$w|_{x=0,a}=-v|_{x=0,a}=0,\ w|_{y=\pm\frac{b}{2}}=-v|_{y=\pm\frac{b}{2}}=\pm\frac{b}{24}x(a^3-x^3)$$

解之可得

$$w(x,y)=\frac{a^4b}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\frac{\sinh\frac{n\pi}{a}y}{\sinh\frac{n\pi}{2a}b}\sin\frac{n\pi}{a}x+\frac{4a^4b}{\pi^5}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\frac{\sinh\frac{2n+1}{a}\pi y}{\sinh\frac{2n+1}{2a}\pi b}\sin\frac{2n+1}{a}\pi x$$

综上可得

$$u(x,y)=\frac{1}{12}xy(a^3-x^3)+\frac{a^4b}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\frac{\sinh\frac{n\pi}{a}y}{\sinh\frac{n\pi}{2a}b}\sin\frac{n\pi}{a}x$$

$$+\frac{4a^4b}{\pi^5}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\frac{\sinh\frac{2n+1}{a}\pi y}{\sinh\frac{2n+1}{2a}\pi b}\sin\frac{2n+1}{a}\pi x.\square$$

注：部分计算过程如下

$$w(x,y)=-w(x,-y)\Longrightarrow D_n=0$$

$$C_n=\frac{2}{a\sinh\frac{n}{2a}\pi b}\int_0^a\frac{b}{24}x(a^3-x^3)\sin\frac{n}{a}\pi x\,\mathrm{d}x=\frac{a^2b}{12}\int_0^a x\sin\frac{n}{a}\pi x\,\mathrm{d}x-\frac{b}{12a}\int_0^a x^4\sin\frac{n}{a}\pi x\,\mathrm{d}x$$

$$=\frac{a^2b}{12\sinh\frac{n}{2a}\pi b}\cdot\frac{a^2\cos n\pi}{n\pi}-\frac{2}{a\cdot\sinh\frac{n}{2a}\pi b}\cdot\frac{b}{24}\cdot\frac{a^5(n^4\pi^4\cos n\pi-12n^2\pi^2\cos n\pi+24\cos n\pi-24)}{n^5\pi^5}$$

$$=\frac{a^4b(-1)^n}{12\pi n\cdot\sinh\frac{n}{2a}\pi b}-\frac{b}{12a\cdot\sinh\frac{n}{2a}\pi b}\cdot\left[\frac{a^5(n^4\pi^4-12n^2\pi^2)}{n^5\pi^5}\cos n\pi+\frac{24a^5(\cos n\pi-1)}{n^5\pi^5}\right]$$

$$=\frac{a^4b(-1)^n}{12\pi n\cdot\sinh\frac{n}{2a}\pi b}-\frac{b}{12\sinh\frac{n}{2a}\pi b}\cdot\frac{a^4(n^4\pi^4-12n^2\pi^2)}{n^5\pi^5}\cdot(-1)^n+\frac{b}{\sinh\frac{n}{2a}\pi b}\cdot\frac{2a^4(\cos n\pi-1)}{n^5\pi^5}$$

$$=-\left[-\frac{a^4b}{12\pi}\frac{1}{n\sinh\frac{n}{2a}\pi b}+\frac{a^4b}{12\pi}\frac{1}{n\cdot\sinh\frac{n}{2a}\pi b}-\frac{a^4b}{\pi^3}\frac{1}{n^3\cdot\sinh\frac{n}{2a}\pi b}\right](-1)^n+\frac{4a^4b}{\pi^5}\cdot\frac{1}{(2n+1)^5\sinh\frac{2n+1}{2a}\pi b}$$

$$=\frac{a^4b}{\pi^3}\frac{(-1)^n}{n^3\cdot\sinh\frac{n}{2a}\pi b}+\frac{4a^4b}{\pi^5}\cdot\frac{1}{(2n+1)^5\sinh\frac{2n+1}{2a}\pi b}$$

8. (P341)第十八章 T5

解：作 Fourier 变换可得

$$U(k,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x,\ \Phi(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x$$

$$\Psi(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x,\ F(k,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x,t)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x$$

方程变形为

$$\frac{\mathrm{d}^2U(k,t)}{\mathrm{d}t^2}+k^2a^2U(k,t)=F(k,t)$$

$$U(k,t)\big|_{t=0}=\Phi(k),\ \left.\frac{\partial U(k,t)}{\partial t}\right|_{t=0}=\Psi(k)$$

设$U(k,t)=v(k,t)+w(k,t)$，则有

$$\frac{\mathrm{d}^2v(k,t)}{\mathrm{d}t^2}+k^2a^2v(k,t)=F(k,t)$$

$$v(k,t)\big|_{t=0}=0,\ \left.\frac{\partial v(k,t)}{\partial t}\right|_{t=0}=0$$

解之可得(见第二部分 3)

$$v(k,t)=\frac{1}{ka}\int_0^t\sin ka(t-\tau)f(k,\tau)\mathrm{d}\tau$$

且有

$$\frac{\mathrm{d}^2 w(k,t)}{\mathrm{d}t^2} + k^2 a^2 w(k,t) = 0$$

$$w(k,t)\big|_{t=0} = \Phi(k),\ \left.\frac{\partial w(k,t)}{\partial t}\right|_{t=0} = \Psi(k)$$

解之可得(见第一部分 6)

$$w(k,t) = \frac{\Psi(k)}{ka}\sin kat + \Phi(k)\cos kat$$

可得

$$U(k,t) = \frac{\Psi(k)}{ka}\sin kat + \Phi(k)\cos kat + \frac{1}{ka}\int_0^t \sin ka(t-\tau) f(k,\tau)\mathrm{d}\tau$$

利用 Fourier 反演公式可得

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{\Psi(k)}{ka}\sin kat + \Phi(k)\cos kat + \frac{1}{ka}\int_0^t \sin ka(t-\tau) f(k,\tau)\mathrm{d}\tau\right]\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k$$

前两项解之可得

$$\frac{1}{2}[\phi(x+at)+\phi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi$$

对于 $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{ka}\int_0^t \sin ka(t-\tau) f(k,\tau)\mathrm{d}\tau\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k$ 有

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{ka}\int_0^t \sin ka(t-\tau) f(k,\tau)\mathrm{d}\tau\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k &= \int_0^t\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(k,\tau)\frac{\sin ka(t-\tau)}{ka}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k\right]\mathrm{d}\tau \\
&= \int_0^t\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(k,\tau)\left[\int_0^\tau \cos ka(t-s)\mathrm{d}s\right]\cdot\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k\right]\mathrm{d}\tau \\
&= \int_0^t\left\{\int_0^\tau\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(k,\tau)\cos ka(t-s)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k\right]\mathrm{d}s\right\}\mathrm{d}\tau \\
&= \int_0^t\left\{\frac{1}{2}\int_0^\tau [f(x+a(t-s),\tau)+f(x-a(t-s),\tau)]\mathrm{d}s\right\}\mathrm{d}\tau \\
&= \frac{1}{2a}\int_0^t\left[\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)\mathrm{d}\xi\right]\mathrm{d}\tau
\end{aligned}$$

综上可得

$$u(x,t) = \frac{1}{2}[\phi(x+at)+\phi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi + \frac{1}{2a}\int_0^t\left[\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)\mathrm{d}\xi\right]\mathrm{d}\tau.\square$$

8. 求解下列非齐次方程的定解问题

$$\nabla^2 u = -2Ax$$

$$u(0,y)=0,\ u(a,y)=0,\ u(x,0)=0,\ u(x,b)=0$$

解：设$u(x,y)=v(x)+w(x,y)$，则有

$$\nabla^2 v(x)=-2Ax$$
$$v(0,y)=0,\ v(a,y)=0$$

解之可得

$$v(x)=-\frac{1}{3}Ax(x^2-a^2)$$

且有

$$\nabla^2 w(x,y)=0$$
$$w(0,y)=0,\ w(a,y)=0,\ w(x,0)=-v(x,0),\ w(x,b)=-v(x,b)$$

带入一般解可得

$$w(x,y)=\sum_{n=1}^{\infty}\left(C_n\sinh\frac{n\pi}{a}y+D_n\cosh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

带入边界条件有

$$w(x,0)=\sum_{n=1}^{\infty}D_n\sin\frac{n\pi}{a}x=\frac{1}{3}Ax(x^2-a^2)$$

$$w(x,b)=\sum_{n=1}^{\infty}\left(C_n\sinh\frac{n\pi b}{a}+D_n\cosh\frac{n\pi b}{a}\right)\sin\frac{n\pi}{a}x=\frac{1}{3}Ax(x^2-a^2)$$

即

$$D_n=\frac{2A}{3a}\int_0^a x(x^2-a^2)\sin\frac{n\pi}{a}x\,\mathrm{d}x$$

$$C_n\sinh\frac{n\pi b}{a}+D_n\cosh\frac{n\pi b}{a}=\frac{2A}{3a}\int_0^a x(x^2-a^2)\sin\frac{n\pi}{a}x\,\mathrm{d}x=D_n$$

解得

$$D_n=-\frac{4Aa^3(-1)^n}{n^3\pi^3}\qquad\qquad C_n=\frac{1-\cosh\frac{n\pi b}{a}}{\sinh\frac{n\pi b}{a}}D_n=-\frac{4Aa^3(-1)^n}{n^3\pi^3}\cdot\frac{1-\cosh\frac{n\pi b}{a}}{\sinh\frac{n\pi b}{a}}$$

则有

$$\begin{aligned}w(x,y)&=-\frac{4Aa^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\left[\frac{1-\cosh\frac{n\pi b}{a}}{\sinh\frac{n\pi b}{a}}\sinh\frac{n\pi}{a}y+\cosh\frac{n\pi}{a}y\right]\sin\frac{n\pi}{a}x\\&=-\frac{8Aa^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3\sinh\frac{n\pi b}{a}}\sinh\frac{n\pi b}{2a}\cosh\frac{n\pi b}{2a}\sin\frac{n\pi}{a}x\end{aligned}$$

综上可得

$$u(x,y)=-\frac{1}{3}Ax(x^2-a^2)-\frac{8Aa^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3\sinh\frac{n\pi b}{a}}\sinh\frac{n\pi b}{2a}\cosh\frac{n\pi b}{2a}\sin\frac{n\pi}{a}x.\square$$

第三部分　部分必备结论

1. $\int_{-\infty}^{\infty} e^{-t^2} \cos 2zt \,dt = \sqrt{\pi} e^{-z^2}$.

2. $\int_0^{n\pi} x \sin x \,dx = -n\pi \cos n\pi$, $\int_0^{n\pi} x \cos x \,dx = \cos n\pi - 1$.

3. $\int_0^{n\pi} x^2 \sin x \,dx = 2(\cos n\pi - 1) - n^2\pi^2 \cos n\pi$, $\int_0^{n\pi} x^2 \cos x \,dx = 2n\pi \cos n\pi$.

4. $u(x,t)|_{x=0} = u_0 \xrightarrow{\text{Lapalce}} U(x,p)|_{x=0} = \dfrac{u_0}{p}$.

5. (1)$b^2 - ac < 0$，$\rho = \xi + \eta$，$\sigma = \mathrm{i}(\xi - \eta)$；(2)$b^2 - ac > 0$，$\rho = \xi + \eta$，$\sigma = \xi - \eta$.

6. $f^{(n)}(t) \risingdotseq p^n F(p) - \sum_{k=0}^{n-1} p^k f^{(n-1-k)}(0)$.

7. $\mathscr{F}\{f^{(n)}(t)\} = (\mathrm{i}k)^n \mathscr{F}\{f(t)\}$.

8. $\mathscr{F}[f_1(t)]\mathscr{F}[f_2(t)] = \mathscr{F}\left[\dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(\xi) f_2(t-\xi) \,d\xi\right]$.

9. $\delta(at - |x-x'|) = \begin{cases} 0 \ , |x-x'| \neq at \\ \infty \ , |x-x'| = at \end{cases}$.

10. $\eta(at - |x-x'|) = \begin{cases} 0 \ , |x-x'| > at \\ 1 \ , |x-x'| < at \end{cases}$.

11. $\mathscr{L}(1) = \dfrac{1}{p}$.

12. $$\begin{aligned} &\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)],\ \cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)] \\ &\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)],\ \sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)] \end{aligned}.$$

13. $\dfrac{dy}{dx} + p(x)y = f(x) \Longrightarrow u = \int p(x)\,dx$, $y = e^{-u}\left[C + \int f(x) e^u \,dx\right]$.

14. Green 函数在$x = \xi$处连续，其对时间的导数在$x = \xi$处有一个跃度$\dfrac{1}{p(t)}$.

15. $\dfrac{d^2 y(t)}{dt^2} - k^2 y(t) = f(t)$, $t > 0$, $k > 0$，$y(x)|_{x \to \pm\infty}$有界，则有

$$y(x) = -\frac{1}{2k} \int_{-\infty}^{\infty} e^{-k|x-\xi|} f(\xi) \,d\xi.$$

16. $\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y$, $\sinh(x-y)=\sinh x\cosh y-\cosh x\sinh y$
$\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y$, $\cosh(x-y)=\cosh x\cosh y-\sinh x\sinh y$.

17. $\cosh^2 x=1+\sinh^2 x$.

18. $e^{-\alpha p} \risingdotseq \delta(t-\alpha)$.

19. $\frac{1}{p}e^{-\alpha p} \risingdotseq \eta(t-\alpha)$.

20. $\frac{1}{p}e^{-\alpha\sqrt{p}} \risingdotseq \operatorname{erfc}\left(\frac{\alpha}{2\sqrt{t}}\right)$.

21. $\frac{1}{\sqrt{p}}e^{-\alpha\sqrt{p}} \risingdotseq \frac{1}{\sqrt{\pi t}}e^{-\frac{\alpha^2}{4t}}$.