

《塑性力学》

习题解答

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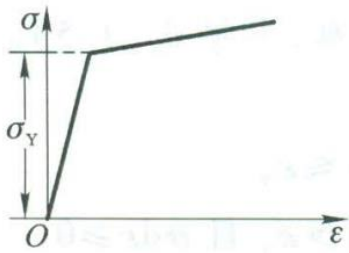
第一章 金属材料的塑性性质

1.6 对于线性随动强化模型，已知 $E_p/E = 1/100$ ：

(1) 给定应力路径为 $0 \rightarrow 1.5\sigma_Y \rightarrow 0 \rightarrow -\sigma_Y \rightarrow 0$ ，求对应的应变值；

(2) 给定应力路径为 $0 \rightarrow 51\varepsilon_Y \rightarrow 0 \rightarrow -21\varepsilon_Y \rightarrow 0$ ，求对应的应力值。

解：对于线性随动强化模型，其总的弹性范围的大小不变，即在 $[-\sigma_Y, \sigma_Y]$ 之间，长度为 $2\sigma_Y$ 。



(1) 首先分析弹性和塑性阶段：

$$0 \xrightarrow{\text{弹性}} \sigma_Y \xrightarrow{\text{塑性}} 1.5\sigma_Y \xrightarrow{\text{弹性}} 0 \xrightarrow{\text{弹性}} -0.5\sigma_Y \xrightarrow{\text{塑性}} -\sigma_Y \xrightarrow{\text{弹性}} 0$$

因为

弹性： $\Delta\sigma = E\Delta\varepsilon$ ，塑性： $\Delta\sigma = E_p\Delta\varepsilon = 0.01E\Delta\varepsilon$ ，其中 $\sigma_Y = E\varepsilon_Y$

因此可得

$$0 \xrightarrow{\text{弹性}} \varepsilon_Y \xrightarrow{\text{塑性}} 51\varepsilon_Y \xrightarrow{\text{弹性}} 49.5\varepsilon_Y \xrightarrow{\text{塑性}} 49\varepsilon_Y \xrightarrow{\text{弹性}} -\varepsilon_Y \xrightarrow{\text{塑性}} 0$$

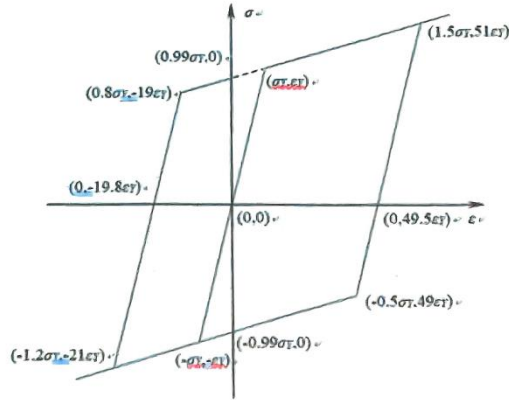
(2) 首先分析弹性和塑性阶段：

$$0 \xrightarrow{\text{弹性}} \varepsilon_Y \xrightarrow{\text{塑性}} 51\varepsilon_Y \xrightarrow{\text{弹性}} 49\varepsilon_Y \xrightarrow{\text{弹性}} 0 \xrightarrow{\text{弹性}} -21\varepsilon_Y \xrightarrow{\text{塑性}} -19.8\varepsilon_Y \xrightarrow{\text{弹性}} -19\varepsilon_Y \xrightarrow{\text{塑性}} 0$$

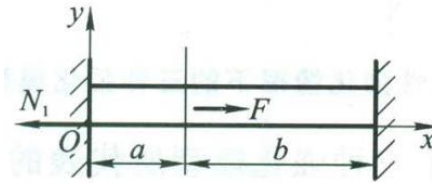
因此可得

$$0 \xrightarrow{\text{弹性}} \sigma_Y \xrightarrow{\text{塑性}} 1.5\sigma_Y \xrightarrow{\text{弹性}} -0.5\sigma_Y \xrightarrow{\text{弹性}} -0.99\sigma_Y \xrightarrow{\text{塑性}} -1.2\sigma_Y \xrightarrow{\text{弹性}} 0 \xrightarrow{\text{弹性}} 0.8\sigma_Y \xrightarrow{\text{塑性}} 0.99\sigma_Y$$

由此可得应力应变曲线图如下。□



1.7 如图所示的等截面杆，截面积为 A ，在 $x=a$ (其中 $b>a$) 处作用一逐渐增加的力 F ，求左端反力 N_1 与 F 的关系，设材料为理想弹性。



解：变形协调方程如下

$$N_1 - N_2 = F, \quad \sigma_1 - \sigma_2 = \frac{F}{A}, \quad a\varepsilon_1 + b\varepsilon_2 = 0$$

(1) 弹性阶段：有 $\varepsilon_2 = -\frac{a}{b}\varepsilon_1$ ，考虑本构关系后，有

$$\sigma_2 = E\varepsilon_2 = -\frac{a}{b}E\varepsilon_1 = -\frac{a}{b}\sigma_1$$

由于 $b>a$ ，则 2 段应力小于 1 段，1 段先进入塑性强化，带入平衡方程有

$$\sigma_1 = \frac{F/A}{1 + a/b} = \frac{Fb}{A(a+b)}, \quad N_1 = \frac{Fb}{a+b}, \quad F_1 = \frac{(a+b)N_Y}{b}$$

(2) 1 段进入塑性阶段，2 段还是弹性阶段：

$$\sigma_1 = \sigma_Y \implies N_1 = \sigma_1 A = \sigma_Y A$$

则有

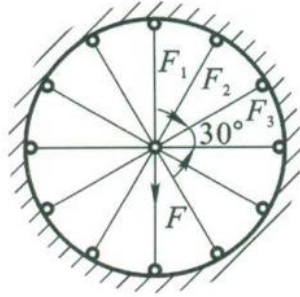
$$\sigma_Y - E\varepsilon_2 = \frac{F}{A} \implies \varepsilon_2 = \frac{\sigma_Y - \frac{F}{A}}{E} \geq -\varepsilon_Y \implies F \leq 2N_Y$$

因此有

$$N_1 = \begin{cases} \frac{Fb}{a+b}, & 0 \leq F \leq F_1 = \frac{(a+b)N_Y}{b} \\ N_Y, & F_1 \leq F \leq 2N_Y \end{cases} \quad \square$$

第二章 结构塑性性态的基本特征

2.1 图示的桁架由 12 根同样尺寸的弹塑性杆件构成，它们连接在一刚性外环上，中间受一集中力 F 的作用。求该桁架的弹性极限载荷 F_e 和塑性极限载荷 F_Y 。



解：考虑小变形，即两根水平杆不受力。因为结构先是反对称，其次是对称的，列出节点平衡方程有

$$N_1 \cdot \frac{1}{2} + N_2 \cdot \frac{\sqrt{3}}{2} + N_3 \cdot \frac{1}{2} \Rightarrow N_1 + \sqrt{3} N_2 + N_3 = \frac{1}{2} F$$

用应力表示平衡方程有

$$\sigma_1 + \sqrt{3} \sigma_2 + \sigma_3 = \frac{1}{2} \frac{F}{A_0}$$

变形协调方程为

$$\delta = \varepsilon_1 l_0 = \frac{2}{\sqrt{3}} \varepsilon_2 l_0 = 2 \varepsilon_3 l_0$$

于是有

$$\varepsilon_1 = \frac{2}{\sqrt{3}} \varepsilon_2 = 2 \varepsilon_3$$

(1) 弹性阶段

应力应变关系有

$$\sigma_1 = E \varepsilon_1, \quad \sigma_2 = E \varepsilon_2, \quad \sigma_3 = E \varepsilon_3$$

于是可得

$$\sigma_1 = \frac{2}{\sqrt{3}} \sigma_2 = 2 \sigma_3$$

于是解得

$$\sigma_1 = \frac{1}{3 + \sqrt{3}} \cdot \frac{F}{A_0}, \quad \sigma_2 = \frac{1}{2(1 + \sqrt{3})} \cdot \frac{F}{A_0}, \quad \sigma_3 = \frac{1}{2(3 + \sqrt{3})} \cdot \frac{F}{A_0}$$

因为 $\sigma_1 > \sigma_2 > \sigma_3$ ，所以随着 F 的增加，1杆先屈服。当 $\sigma_1 = \sigma_Y$ 时，对应的外载为

$$F_e = (3 + \sqrt{3})\sigma_Y A_0$$

在 F_e 的作用下，节点的位移为

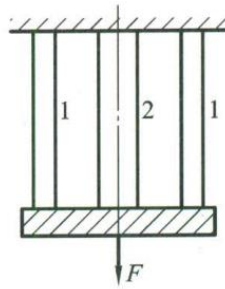
$$\delta = \varepsilon_1 l_0 = \frac{\sigma_1}{E} l_0 = \frac{\sigma_Y l_0}{E} = \delta_e$$

(2) 塑性流动阶段

因为材料是理想弹塑性的，因此 $\sigma_1 = \sigma_2 = \sigma_Y$ ，于是结构当且仅当 $\sigma_3 = \sigma_Y$ 时进入塑性流动状态，则有

$$F_Y = 2(3 + \sqrt{3})\sigma_Y A_0. \square$$

2.2 在图示的理想弹塑性材料制成的结构中，杆1(两根)和杆2的长度相等，共同承受拉力 F 的作用。设此两杆的弹性模量 E 相同，屈服极限分别为 σ_{Y1} 和 σ_{Y2} ($\sigma_{Y1} < \sigma_{Y2}$)，截面积分别为 A_1 和 A_2 ($A_1 < A_2$)。



(1) 绘出载荷与变形的关系图；

(2) 当两杆都拉至屈服后卸载到 $F = 0$ ，求此时两杆中的残余应力；

(3) 给出卸载后杆1中不再产生塑性变形的条件。

解：(1) 因为结构是对称的，列出节点平衡方程有

$$2N_1 + N_2 = F$$

用应力表示平衡方程有

$$2A_1\sigma_1 + A_2\sigma_2 = F$$

变形协调方程为

$$\delta = \varepsilon_1 l_0 = \varepsilon_2 l_0$$

于是有

$$\varepsilon_1 = \varepsilon_2$$

处于弹性阶段时，应力应变关系有

$$\sigma_1 = E\varepsilon_1, \quad \sigma_2 = E\varepsilon_2$$

于是可得

$$\sigma_1 = \sigma_2$$

于是解得

$$\sigma_1 = \sigma_2 = \frac{F}{2A_1 + A_2}$$

载荷与变形的关系式为

$$\delta = \varepsilon_1 l_0 = \frac{\sigma_1}{E} l_0 = \frac{l_0}{E} \cdot \frac{F}{2A_1 + A_2} = \frac{l_0}{(2A_1 + A_2)E} F$$

(2) 因为 $\sigma_1 = \sigma_2$ ，但是 $\sigma_{Y1} < \sigma_{Y2}$ ，所以 1 杆先进入塑性阶段，此时有 $\sigma_1 = \sigma_{Y1}$ ，则

$$\sigma_2 = \frac{F - 2A_1\sigma_{Y1}}{A_2}$$

继续加载，当 $\sigma_2 = \sigma_{Y2}$ 时，2 杆进入塑性阶段，整个结构进入塑性流动阶段，则

$$F_Y = 2A_1\sigma_{Y1} + A_2\sigma_{Y2}, \quad \delta_Y = \varepsilon_2 l_0 = \frac{\sigma_{Y2}}{E} l_0$$

可以求得该结构的弹性极限载荷为 $F_e = (2A_1 + A_2)\sigma_{Y1}$ ，因为卸载服从弹性规律，加载到 F_Y

后卸载，应力变化为

$$\Delta\sigma_1 = E\Delta\varepsilon_1, \quad \Delta\sigma_2 = E\Delta\varepsilon_2, \quad \Delta F = 2A_1\Delta\sigma_1 + A_2\Delta\sigma_2, \quad \Delta\varepsilon_1 = \Delta\varepsilon_2 = \frac{\Delta\delta}{l}$$

于是有

$$\Delta\sigma_1 = \Delta\sigma_2 = \frac{\Delta F}{2A_1 + A_2} = -\frac{2A_1\sigma_{Y1} + A_2\sigma_{Y2}}{2A_1 + A_2}$$

叠加上卸载前的应力有

$$\begin{aligned} \sigma_1^\circ &= \sigma_{Y1} + \Delta\sigma_1 = \sigma_{Y1} - \frac{2A_1\sigma_{Y1} + A_2\sigma_{Y2}}{2A_1 + A_2} = \frac{(\sigma_{Y1} - \sigma_{Y2})A_2}{2A_1 + A_2} < 0 \\ \sigma_2^\circ &= \sigma_{Y2} + \Delta\sigma_2 = \sigma_{Y2} - \frac{2A_1\sigma_{Y1} + A_2\sigma_{Y2}}{2A_1 + A_2} = \frac{2(\sigma_{Y2} - \sigma_{Y1})A_1}{2A_1 + A_2} > 0 \end{aligned}$$

(3) 1 杆不发生反向屈服的条件为

$$\sigma_1^\circ = \frac{(\sigma_{Y1} - \sigma_{Y2})A_2}{2A_1 + A_2} \geq -\sigma_{Y1} \iff 2\sigma_{Y1}(A_1 + A_2) \geq \sigma_{Y2}A_2. \square$$

2.3 在三杆桁架中材料是理想弹塑性的，保持 $F = Q$ ，求此时进行比例加载。求弹性和塑性极限状态的应力与应变，同时在塑性极限状态卸载到零时残余应力和应变状态。

解：按照 $F: Q = 1: 1$ 进行加载，单调增加，首先桁架处于弹性范围之内，则有变形协调关系

$$\varepsilon_2 = \varepsilon_1 + \varepsilon_3, \quad \sigma_2 = \sigma_1 + \sigma_3$$

同时，在平衡方程

$$\sigma_2 + \frac{(\sigma_1 + \sigma_3)}{\sqrt{2}} = \frac{F}{A_0}, \quad \frac{(\sigma_1 - \sigma_3)}{\sqrt{2}} = \frac{Q}{A_0}$$

中带入 $F = Q$ ，解得

$$\begin{cases} \sigma_1 = \frac{F}{A_0} > 0 \\ \sigma_2 = \frac{\sqrt{2}}{1 + \sqrt{2}} \cdot \frac{F}{A_0} > 0 \\ \sigma_3 = -\frac{1}{1 + \sqrt{2}} \cdot \frac{F}{A_0} < 0 \end{cases}$$

显然，三者之中的 σ_1 绝对值最大，因此当 $\sigma_1 = \sigma_Y$ 时桁架达到弹性极限状态，这时有

$$F_e = \sigma_Y A_0, \quad \sigma_1^e = \sigma_Y, \quad \sigma_2^e = \frac{\sqrt{2}}{1 + \sqrt{2}} \sigma_Y, \quad \sigma_3^e = -\frac{1}{1 + \sqrt{2}} \sigma_Y$$

$$\delta_y^e = \frac{\sqrt{2}}{1 + \sqrt{2}} \frac{\sigma_Y l_0}{E} = \frac{1}{2 + \sqrt{2}} \delta_Y, \quad \delta_x^e = \sqrt{2} \frac{\sigma_Y l_0}{E} = \frac{1}{\sqrt{2}} \delta_Y$$

再继续加载， $\Delta \sigma_1 = 0$ ，由增量形式的平衡方程

$$\Delta \sigma_2 + \frac{(\Delta \sigma_1 + \Delta \sigma_3)}{\sqrt{2}} = \frac{\Delta F}{A_0}, \quad \frac{(\Delta \sigma_1 - \Delta \sigma_3)}{\sqrt{2}} = \frac{\Delta Q}{A_0}$$

与 $\Delta F = \Delta Q$ 可知

$$\Delta \sigma_2 = 2 \frac{\Delta F}{A_0}, \quad \Delta \sigma_3 = -\sqrt{2} \frac{\Delta F}{A_0}$$

当 $\sigma_2 = \sigma_2^e + \Delta \sigma_2 = \sigma_Y$ 时可得

$$\Delta \sigma_2 = \frac{1}{1 + \sqrt{2}} \sigma_Y, \quad \Delta F = \frac{1}{2(1 + \sqrt{2})} \sigma_Y A_0, \quad F = Q = F_e + \Delta F = \frac{1 + \sqrt{2}}{2} \sigma_Y A_0$$

这时有

$$\sigma_1 = \sigma_2 = \sigma_Y, \quad \sigma_3 = -\frac{1}{\sqrt{2}} \sigma_Y$$

$$\delta_y = \frac{\sigma_Y l_0}{E} = \frac{1}{2} \delta_Y, \quad \delta_x = (\sqrt{2} + 1) \frac{\sigma_Y l_0}{E} = \frac{\sqrt{2} + 1}{2} \delta_Y$$

因为卸载服从弹性规律，加载到 F_Y 后卸载有

$$\Delta \sigma_1 = \frac{\Delta F}{A_0}, \quad \Delta \sigma_2 = \frac{\sqrt{2}}{1 + \sqrt{2}} \cdot \frac{\Delta F}{A_0}, \quad \Delta \sigma_3 = -\frac{1}{1 + \sqrt{2}} \cdot \frac{\Delta F}{A_0}, \quad \Delta F = -\frac{1 + \sqrt{2}}{2} \sigma_Y A_0$$

于是有

$$\Delta\sigma_1 = -\frac{1+\sqrt{2}}{2}\sigma_Y, \quad \Delta\sigma_2 = -\frac{1}{\sqrt{2}}\sigma_Y, \quad \Delta\sigma_3 = \frac{1}{2}\sigma_Y$$

叠加上卸载前的应力有

$$\sigma_1^\circ = \frac{1-\sqrt{2}}{2}\sigma_Y < 0, \quad \sigma_2^\circ = \frac{2-\sqrt{2}}{2}\sigma_Y > 0, \quad \sigma_3^\circ = \frac{1-\sqrt{2}}{2}\sigma_Y < 0. \square$$

2.4 在三杆桁架中材料是理想弹塑性的，现考察变形路径为：先加水平力 Q ，使桁架达到极限状态，然后在水平位移 δ_x 不变的情况下加上 F （此时 Q 将变化），使桁架再次进入极限状态。

试求各状态的应力与应变状态，在极限状态卸载到零时残余应力和应变状态。

解：首先桁架处于弹性范围之内，则有变形协调关系

$$\varepsilon_2 = \varepsilon_1 + \varepsilon_3, \quad \sigma_2 = \sigma_1 + \sigma_3$$

同时，有平衡方程

$$\sigma_2 + \frac{(\sigma_1 + \sigma_3)}{\sqrt{2}} = 0, \quad \frac{(\sigma_1 - \sigma_3)}{\sqrt{2}} = \frac{Q}{A_0}$$

解得

$$\begin{cases} \sigma_1 = \frac{1}{\sqrt{2}} \cdot \frac{Q}{A_0} > 0 \\ \sigma_2 = 0 \\ \sigma_3 = -\frac{1}{\sqrt{2}} \cdot \frac{Q}{A_0} < 0 \end{cases}$$

因此当 $\sigma_1 = \sigma_Y$ 时桁架达到极限状态，这时有

$$Q_Y = \sqrt{2}\sigma_Y A_0, \quad \sigma_1 = \sigma_Y, \quad \sigma_2 = 0, \quad \sigma_3 = -\sigma_Y$$

$$\delta_x = 2 \frac{\sigma_Y l_0}{E} = \delta_Y, \quad \delta_y = 0$$

然后在水平位移 δ_x 不变的情况下加上 F ，则有增量形式的平衡方程

$$\Delta\sigma_2 + \frac{(\Delta\sigma_1 + \Delta\sigma_3)}{\sqrt{2}} = \frac{\Delta F}{A_0}, \quad \frac{(\Delta\sigma_1 - \Delta\sigma_3)}{\sqrt{2}} = \frac{\Delta Q}{A_0}$$

1 杆继续拉伸，2 杆开始伸长，3 杆反向拉伸，于是

$$\Delta\sigma_1 = 0, \quad \Delta\sigma_2 = E\Delta\varepsilon_2, \quad \Delta\sigma_3 = E\Delta\varepsilon_3$$

进而求得

$$\Delta\sigma_1 = 0, \quad \Delta\sigma_2 = -2\Delta\sigma_3, \quad \Delta\sigma_3 = -\sqrt{2} \frac{\Delta Q}{A_0}, \quad \delta_y = \frac{4}{4+\sqrt{2}} \frac{\Delta F l}{EA_0}$$

叠加上初始应力可得

$$\sigma_1 = \sigma_Y, \quad \sigma_2 = \frac{4}{4 + \sqrt{2}} \frac{\Delta F}{A_0}, \quad \sigma_3 = -\sigma_Y - \sqrt{2} \frac{\Delta Q}{A_0}$$

由 $\sigma_2 = \sigma_Y$ 可得

$$\Delta F = \left(1 + \frac{\sqrt{2}}{4}\right) \sigma_Y A_0, \quad \Delta Q = -\frac{\sigma_Y A_0}{2\sqrt{2}}, \quad \delta_y = \frac{\sigma_Y}{E} l$$

此时

$$F = \left(1 + \frac{\sqrt{2}}{4}\right) \sigma_Y A_0, \quad Q = \frac{3\sqrt{2}}{4} \sigma_Y A_0$$

$$\sigma_1 = \sigma_2 = \sigma_Y, \quad \sigma_3 = -\frac{1}{2} \sigma_Y, \quad \delta_x = 2 \frac{\sigma_Y l_0}{E}, \quad \delta_y = \frac{\sigma_Y l_0}{E}$$

由于限制了节点的水平位移，桁架结构不能变为机构，仍可以继续承受载荷，在此基础上继续加载

$$\Delta \sigma_1 = 0, \quad \Delta \sigma_2 = 0, \quad \Delta \sigma_3 = E \Delta \varepsilon_3$$

进而求得

$$\Delta \sigma_1 = 0, \quad \Delta \sigma_2 = 0, \quad \Delta \sigma_3 = \sqrt{2} \frac{\Delta F}{A_0} = -\sqrt{2} \frac{\Delta Q}{A_0}, \quad \Delta \delta_y = 2\sqrt{2} \frac{\Delta F l}{EA_0}$$

叠加上初始应力可得

$$\sigma_1 = \sigma_Y, \quad \sigma_2 = \sigma_Y, \quad \sigma_3 = -\frac{1}{2} \sigma_Y - \sqrt{2} \frac{\Delta Q}{A_0}$$

由 $\sigma_3 = \sigma_Y$ 可得

$$\Delta F = \frac{3\sqrt{2}}{4} \sigma_Y A_0, \quad \Delta Q = -\frac{3\sqrt{2}}{4} \sigma_Y A_0, \quad \Delta \delta_y = 3 \frac{\sigma_Y}{E} l$$

此时

$$F_Y = (1 + \sqrt{2}) \sigma_Y A_0, \quad Q = 0$$

$$\sigma_1 = \sigma_2 = \sigma_Y, \quad \sigma_3 = -\sigma_Y, \quad \delta_x = 2 \frac{\sigma_Y l_0}{E}, \quad \delta_y = 4 \frac{\sigma_Y l_0}{E}$$

因为卸载服从弹性规律，加载到 F_Y 后卸载有

$$\Delta \sigma_1 = \frac{1}{2} \left(\frac{\sqrt{2}}{\sqrt{2} + 1} \frac{\Delta F}{A_0} + \sqrt{2} \frac{\Delta Q}{A_0} \right), \quad \Delta \sigma_2 = \frac{\sqrt{2}}{\sqrt{2} + 1} \frac{\Delta F}{A_0}$$

$$\Delta \sigma_3 = \frac{1}{2} \left(\frac{\sqrt{2}}{\sqrt{2} + 1} \frac{\Delta F}{A_0} - \sqrt{2} \frac{\Delta Q}{A_0} \right), \quad \Delta F = (1 + \sqrt{2}) \sigma_Y A_0, \quad \Delta Q = 0$$

解得

$$\Delta \sigma_1 = -\frac{\sqrt{2}}{2} \sigma_Y, \quad \Delta \sigma_2 = \sqrt{2} \sigma_Y, \quad \Delta \sigma_3 = -\frac{\sqrt{2}}{2} \sigma_Y$$

叠加上卸载前的应力有

$$\sigma_1^\circ = \left(1 - \frac{\sqrt{2}}{2}\right)\sigma_Y > 0, \quad \sigma_2^\circ = (1 - \sqrt{2})\sigma_Y > 0, \quad \sigma_3^\circ = \left(1 - \frac{\sqrt{2}}{2}\right)\sigma_Y > 0. \square$$

第三章 应力和应变分析

3.1 物体中某一点的应力张量为

$$\sigma_{ij} = \begin{pmatrix} 10 & 0 & -10 \\ 0 & -10 & 0 \\ -10 & 0 & 10 \end{pmatrix}$$

其中应力张量各分量的单位是 Mpa, 试求主应力值及 $J_j, J'_j (j=1, 2, 3)$.

解: 写出特征行列式有

$$\begin{vmatrix} 10 - \lambda & 0 & -10 \\ 0 & -10 - \lambda & 0 \\ -10 & 0 & 10 - \lambda \end{vmatrix} = 0$$

展开可得

$$\lambda^3 - 10\lambda^2 - 200\lambda = 0$$

解得主应力大小

$$\sigma_1 = 20 \text{ Mpa}, \sigma_2 = 0 \text{ Mpa}, \sigma_3 = -10 \text{ Mpa}$$

比较系数可得

$$J_1 = 10, J_2 = 200, J_3 = 0$$

由此可得

$$J'_1 = 0, J'_2 = \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = 233.333$$

$$\sigma_m = \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{10}{3} \text{ Mpa}, J'_3 = (\sigma_1 - \sigma_m)(\sigma_2 - \sigma_m)(\sigma_3 - \sigma_m) = 740.741. \square$$

3.3 物体中某一点的应力张量为

$$\sigma_{ij} = \begin{pmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & -100 \end{pmatrix}$$

其中应力张量各分量的单位是 Mpa, 试求该点的八面体面上的总应力, 正应力 σ_8 和剪应力 τ_8 .

解: 直接写出其主应力为

$$\sigma_1 = 50 \text{ Mpa}, \sigma_2 = 50 \text{ Mpa}, \sigma_3 = -100 \text{ Mpa}$$

则八面体面上的总应力有

$$|F_8| = \sqrt{\frac{1}{3} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)} = 70.7107 \text{ Mpa}$$

正应力 σ_8 有

$$\sigma_8 = \sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = 0 \text{ Mpa}$$

剪应力 τ_8 有

$$\tau_8 = \sqrt{|F_8|^2 - \sigma_8^2} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = 70.7107 \text{ Mpa} \square$$

3.4 一薄圆管，半径为 R ，壁厚为 h ，则

(1) 受拉力 F 和内压 p 作用时，求 $\bar{\sigma}$ ；(2) 受拉力 F 和扭矩 T 作用时，求 $\bar{\tau}$ 。

解：采用柱坐标系进行计算有

(1) 在拉力方向有

$$\sigma_z = \frac{F}{A} = \frac{F}{2\pi hR}$$

在截面内有

$$\sigma_\theta = \frac{\sigma_\theta h}{h} = \frac{\int_0^{\frac{\pi}{2}} pR \sin\theta d\theta}{h} = \frac{pR}{h}, \quad \sigma_r = 0$$

因此等效正应力为

$$\begin{aligned} \bar{\sigma} &= \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \\ &= \frac{1}{\sqrt{2}} \sqrt{(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2} \\ &= \frac{1}{\sqrt{2}} \sqrt{\frac{p^2 R^2}{h^2} + \left(\frac{pR}{h} - \frac{F}{2\pi hR}\right)^2 + \frac{F^2}{4\pi^2 h^2 R^2}} \\ &= \frac{1}{\sqrt{2}h} \sqrt{2p^2 R^2 - \frac{pF}{\pi} + \frac{F^2}{2\pi^2 R^2}} \end{aligned}$$

(2) 在拉力方向有

$$\sigma_z = \frac{F}{A} = \frac{F}{2\pi hR}$$

在截面内有

$$T = \int_A \tau_{\theta z} \cdot R dA = \int_0^{2\pi} \tau_{\theta z} \cdot hR^2 d\theta \implies \tau_{\theta z} = \frac{T}{2\pi hR^2}$$

因此等效切应力为

$$\begin{aligned}
\bar{\tau} &= \frac{1}{\sqrt{6}} \sqrt{(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 + 6(\tau_{r\theta}^2 + \tau_{rz}^2 + \tau_{\theta z}^2)} \\
&= \frac{1}{\sqrt{6}} \sqrt{\frac{F^2}{2\pi^2 h^2 R^2} + 6 \cdot \frac{T^2}{4\pi^2 h^2 R^4}} \\
&= \frac{1}{\sqrt{6}} \sqrt{\frac{F^2 R^2 + 3T^2}{2\pi^2 h^2 R^4}} = \frac{\sqrt{F^2 R^2 + 3T^2}}{2\sqrt{3}\pi h R^2}. \square
\end{aligned}$$

3.5 证明下列等式

$$(1) \sigma_{ij} d\varepsilon_{ij} = s_{ij} de_{ij} + \frac{1}{3} \sigma_{kk} d\varepsilon_{jj}; \quad (2) \frac{\sqrt{s_{ij}s_{ij}}}{\sqrt{e_{kl}e_{kl}}} = \frac{2\bar{\sigma}}{3\bar{\varepsilon}}; \quad (3) J'_2 = J_2 + \frac{1}{3} J_1^2; \quad (4) \frac{\partial J'_2}{\partial \sigma_{ij}} = \frac{\partial J'_2}{\partial s_{ij}} = s_{ij}.$$

解: (1) 因为有

$$\sigma_{ij} = s_{ij} + \frac{1}{3} \sigma_{kk} \delta_{ij}$$

两边同时乘 $d\varepsilon_{ij}$ 可得

$$\sigma_{ij} d\varepsilon_{ij} = s_{ij} d\varepsilon_{ij} + \frac{1}{3} \sigma_{kk} \delta_{ij} d\varepsilon_{ij} = s_{ij} d\varepsilon_{ij} + \frac{1}{3} \sigma_{kk} d(\delta_{ij} \varepsilon_{ij}) = s_{ij} d\varepsilon_{ij} + \frac{1}{3} \sigma_{kk} d\varepsilon_{jj}$$

而

$$d\varepsilon_{ij} = d(\varepsilon_m \delta_{ij} + e_{ij}) = de_{ij}$$

因此可得

$$\sigma_{ij} d\varepsilon_{ij} = s_{ij} de_{ij} + \frac{1}{3} \sigma_{kk} d\varepsilon_{jj}.$$

(2) 因为有

$$\frac{2\bar{\sigma}}{3\bar{\varepsilon}} = \frac{2\sqrt{3J'_2}}{3\sqrt{\frac{4}{3}I'_2}} = \frac{\sqrt{J'_2}}{\sqrt{I'_2}} = \frac{\sqrt{\frac{1}{2}s_{ij}s_{ij}}}{\sqrt{\frac{1}{2}e_{kl}e_{kl}}} = \frac{\sqrt{s_{ij}s_{ij}}}{\sqrt{e_{kl}e_{kl}}}.$$

(3) 因为有

$$\begin{aligned}
J_2 + \frac{1}{3} J_1^2 &= -(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1) + \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3)^2 \\
&= -(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1) + \frac{1}{3} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1 \sigma_2 + 2\sigma_2 \sigma_3 + 2\sigma_3 \sigma_1) \\
&= \frac{1}{3} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1 \sigma_2 - \sigma_2 \sigma_3 - \sigma_3 \sigma_1)
\end{aligned}$$

所以有 $J'_2 = J_2 + \frac{1}{3} J_1^2$.

(4) 因为有

$$\begin{aligned}
 J'_2 &= \frac{1}{6} [(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 3(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{21}^2 + \sigma_{23}^2 + \sigma_{31}^2 + \sigma_{32}^2)] \\
 &= \frac{1}{2} (s_{11}^2 + s_{12}^2 + s_{13}^2 + s_{21}^2 + s_{22}^2 + s_{23}^2 + s_{31}^2 + s_{32}^2 + s_{33}^2)
 \end{aligned}$$

那么显然有

$$\frac{\partial J'_2}{\partial s_{ij}} = s_{ij}$$

而因为当 $i = j$ 时有

$$\frac{\partial J'_2}{\partial \sigma_{ii}} = \sigma_{ii} - \frac{1}{3} \sigma_{kk} = \sigma_{ii} - \sigma_m = s_{ii}$$

而当 $i \neq j$ 时有

$$\frac{\partial J'_2}{\partial \sigma_{ij}} = \sigma_{ij} = s_{ij}$$

综上可得

$$\frac{\partial J'_2}{\partial \sigma_{ij}} = \frac{\partial J'_2}{\partial s_{ij}} = s_{ij}. \square$$

第四章 屈服条件

4.2 物体中某一点的应力状态为

$$\sigma_{ij} = \begin{pmatrix} -100 & 0 & 0 \\ 0 & -200 & 0 \\ 0 & 0 & -300 \end{pmatrix}$$

其中应力张量各分量的单位是 MPa，该物体的材料在单向拉伸时的屈服极限 $\sigma_Y = 190 \text{ MPa}$ ，试用 Mises 屈服条件和 Tresca 屈服条件判断该点是处于弹性状态还是处于塑性状态。如果主应力均取为相反的负号(即 100 MPa ， 200 MPa ， 300 MPa)，那么对这点所处状态的判断有无变化？

解：主应力分别为

$$\sigma_1 = -100 \text{ MPa}, \sigma_2 = -200 \text{ MPa}, \sigma_3 = -300 \text{ MPa}$$

Tresca 屈服条件：利用公式可得

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{-100 + 300}{2} \text{ MPa} = k = 100 \text{ MPa}$$

在单向拉伸中 $k > \frac{1}{2} \sigma_Y = 95 \text{ MPa}$ ，显然该点处于塑性状态。

Mises 屈服条件：利用公式可得

$$\begin{aligned} J_2' &= \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \\ &= \frac{1}{6} [(-100 + 200)^2 + (-200 + 300)^2 + (-300 + 100)^2] = C = 10000 \end{aligned}$$

在单向拉伸中 $J_2' < \frac{1}{3} \sigma_Y^2 = \frac{1}{3} \cdot 190^2 = 12033.3$ ，显然该点处于弹性状态。

如果主应力均取为相反的负号，那么对这点所处状态的判断无变化。□

4.3 已知平面应力状态 $\sigma_x = 750 \text{ MPa}$ ， $\sigma_y = 150 \text{ MPa}$ ， $\tau_{xy} = 150 \text{ MPa}$ ，正好使材料屈服，试分别按 Mises 屈服条件和 Tresca 屈服条件计算材料单向拉伸的屈服极限 σ_Y 。

解：利用平面应力状态的主应力公式有

$$\begin{aligned} \sigma' &= \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \\ &= \left[\frac{750 + 150}{2} + \frac{1}{2} \sqrt{(750 - 150)^2 + 4 \cdot 150^2} \right] \text{ MPa} = 785.41 \text{ MPa} \\ \sigma'' &= \frac{\sigma_x + \sigma_y}{2} - \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \\ &= \left[\frac{750 + 150}{2} - \frac{1}{2} \sqrt{(750 - 150)^2 + 4 \cdot 150^2} \right] \text{ MPa} = 114.59 \text{ MPa} \\ \sigma''' &= 0 \end{aligned}$$

排序可得主应力分别为

$$\sigma_1 = \sigma' = 785.41 \text{ MPa}, \sigma_2 = \sigma'' = 114.59 \text{ MPa}, \sigma_3 = \sigma''' = 0$$

Tresca 屈服条件：利用公式可得

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{785.41 - 0}{2} \text{ MPa} = 392.705 \text{ MPa} = k$$

在单向拉伸中 $\sigma_Y = 2k = 2 \cdot 392.705 \text{ MPa} = 785.41 \text{ MPa}$.

Mises 屈服条件：利用公式可得

$$\begin{aligned} J_2' &= \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \\ &= \frac{1}{6} [(785.41 - 114.59)^2 + (114.59 - 0)^2 + (0 - 785.41)^2] = 180000 = C \end{aligned}$$

在单向拉伸中 $\sigma_Y = \sqrt{3C} = \sqrt{3 \cdot 180000} \text{ Mpa} = 734.847 \text{ Mpa}$. □.

4.4 在平面应力问题中，取 $\sigma_z = \tau_{xz} = \tau_{yz} = 0$ ，试将 Mises 屈服条件和 Tresca 屈服条件分别用 σ_x 、 σ_y 、 τ_{xy} 表示出来. (规定单向拉伸时两种屈服条件重合)

解：利用平面应力状态的主应力公式有

$$\sigma' = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}, \sigma'' = \frac{\sigma_x + \sigma_y}{2} - \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}, \sigma''' = 0$$

无法判断主应力大小，因此我们需要分情况讨论，如下

(1) 当 $\sigma' > \sigma'' > 0$ 时，排序可得主应力分别为 $\sigma_1 = \sigma'$ ， $\sigma_2 = \sigma''$ ， $\sigma_3 = 0$

Tresca 屈服条件：利用公式可得

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma'}{2} = k$$

在单向拉伸中 $\sigma_Y = 2k = 2 \cdot \frac{\sigma'}{2} = \sigma' = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$;

在纯剪切中 $\tau_Y = k = \frac{\sigma'}{2} = \frac{\sigma_x + \sigma_y}{4} + \frac{1}{4} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$.

Mises 屈服条件：利用公式可得

$$\begin{aligned} J_2' &= \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = \frac{1}{6} [(\sigma' - \sigma'')^2 + (\sigma'' - 0)^2 + (0 - \sigma')^2] \\ &= \frac{1}{6} \left[\left(\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \right)^2 + 2 \cdot \left(\frac{\sigma_x + \sigma_y}{2} \right)^2 + 2 \cdot \left(\frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \right)^2 \right] \\ &= \frac{1}{6} \left[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 + \frac{(\sigma_x + \sigma_y)^2}{2} + \frac{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}{2} \right] \\ &= \frac{1}{3} (\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2) = C \end{aligned}$$

在单向拉伸中 $\sigma_Y = \sqrt{3C} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2}$;

在纯剪切中 $\tau_Y = \sqrt{C} = \frac{1}{\sqrt{3}} \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2}$.

(2) 当 $\sigma' > 0 > \sigma''$ 时, 排序可得主应力分别为 $\sigma_1 = \sigma'$, $\sigma_2 = 0$, $\sigma_3 = \sigma''$

Tresca 屈服条件: 利用公式可得

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma' - \sigma''}{2} = k$$

在单向拉伸中 $\sigma_Y = 2k = 2 \cdot \frac{\sigma' - \sigma''}{2} = \sigma' - \sigma'' = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$;

在纯剪切中 $\tau_Y = k = \frac{\sigma' - \sigma''}{2} = \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$.

Mises 屈服条件: 利用公式可得

$$\begin{aligned} J_2' &= \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = \frac{1}{6} [(\sigma' - 0)^2 + (0 - \sigma'')^2 + (\sigma'' - \sigma')^2] \\ &= \frac{1}{3} (\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2) = C \end{aligned}$$

在单向拉伸中 $\sigma_Y = \sqrt{3C} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2}$;

在纯剪切中 $\tau_Y = \sqrt{C} = \frac{1}{\sqrt{3}} \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2}$.

(3) 当 $0 > \sigma' > \sigma''$ 时, 排序可得主应力分别为 $\sigma_1 = 0$, $\sigma_2 = \sigma'$, $\sigma_3 = \sigma''$

Tresca 屈服条件: 利用公式可得

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{0 - \sigma''}{2} = -\frac{\sigma''}{2} = k$$

在单向拉伸中 $\sigma_Y = 2k = -2 \cdot \frac{\sigma''}{2} = -\sigma'' = -\frac{\sigma_x + \sigma_y}{2} + \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$;

在纯剪切中 $\tau_Y = k = -\frac{\sigma''}{2} = -\frac{\sigma_x + \sigma_y}{4} + \frac{1}{4} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$.

Mises 屈服条件: 利用公式可得

$$\begin{aligned} J_2' &= \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = \frac{1}{6} [(0 - \sigma')^2 + (\sigma' - \sigma'')^2 + (\sigma'' - 0)^2] \\ &= \frac{1}{3} (\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2) = C \end{aligned}$$

在单向拉伸中 $\sigma_Y = \sqrt{3C} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2}$;

在纯剪切中 $\tau_Y = \sqrt{C} = \frac{1}{\sqrt{3}} \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2}$. □.

4.5 在平面应变问题中, 取 $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$ 及泊松比 $\nu = \frac{1}{2}$, 试将 Mises 屈服条件和 Tresca

屈服条件分别用 σ_x 、 σ_y 、 τ_{xy} 表示出来. (规定单向拉伸时两种屈服条件重合)

解: 利用平面应变状态的主应力公式有

$$\sigma' = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}, \quad \sigma'' = \frac{\sigma_x + \sigma_y}{2} - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

利用广义胡克定律可得

$$\sigma_z = \nu(\sigma_x + \sigma_y) = \frac{\sigma_x + \sigma_y}{2}$$

排序可得主应力分别为

$$\sigma_1 = \sigma', \quad \sigma_2 = \sigma_z, \quad \sigma_3 = \sigma''$$

Tresca 屈服条件：利用公式可得

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma' - \sigma''}{2} = k$$

在单向拉伸中 $\sigma_Y = 2k = 2 \cdot \frac{\sigma' - \sigma''}{2} = \sigma' - \sigma'' = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$;

在纯剪切中 $\tau_Y = k = \frac{\sigma' - \sigma''}{2} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$.

Mises 屈服条件：利用公式可得

$$\begin{aligned} J_2' &= \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = \frac{1}{6}[(\sigma' - \sigma_z)^2 + (\sigma_z - \sigma'')^2 + (\sigma'' - \sigma')^2] \\ &= \frac{1}{6}[(\sigma')^2 - 2\sigma'\sigma_z + \sigma_z^2 + \sigma_z^2 - 2\sigma''\sigma_z + (\sigma'')^2 + (\sigma'')^2 - 2\sigma'\sigma'' + (\sigma')^2] \\ &= \frac{1}{3}[(\sigma')^2 + (\sigma'')^2 + \sigma_z^2 - (\sigma' + \sigma'')\sigma_z - \sigma'\sigma''] \\ &= \frac{1}{3} \left\{ 2\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + 2\left(\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right)^2 + \left(\frac{\sigma_x + \sigma_y}{2}\right)^2 \right. \\ &\quad \left. - (\sigma_x + \sigma_y) \cdot \left(\frac{\sigma_x + \sigma_y}{2}\right) - \left[\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 - \left(\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right)^2\right] \right\} \\ &= \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} = C \end{aligned}$$

在单向拉伸中 $\sigma_Y = \sqrt{3}C = \frac{\sqrt{3}}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$;

在纯剪切中 $\tau_Y = \sqrt{C} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$. □.

4.6 一薄壁圆管，半径为 R ，壁厚为 h ，承受内压 p 作用，泊松比 $\nu = \frac{1}{2}$ ，讨论以下三种情况：

(1) 管的两端是固定的；(2) 管的两端是自由的；(3) 管的两端是封闭的。

分别对 Mises 屈服条件和 Tresca 屈服条件，求 p 多大时管子达到屈服。(规定纯剪切时两种屈服条件重合，剪切屈服应力为 τ_Y)

解：这是平面应变问题，采用柱坐标系进行计算有

(1) 在薄壁圆管横截面上有

$$\sigma_{\theta} = \frac{\sigma_{\theta} h}{h} = \frac{\int_0^{\frac{\pi}{2}} pR \sin \theta d\theta}{h} = \frac{pR}{h}, \quad \sigma_r = 0$$

利用广义胡克定律有

$$\sigma_z = \nu(\sigma_r + \sigma_{\theta}) = \frac{\sigma_{\theta}}{2} = \frac{pR}{2h}$$

排序可得主应力分别为

$$\sigma_1 = \sigma_{\theta}, \quad \sigma_2 = \sigma_z, \quad \sigma_3 = \sigma_r$$

Tresca 屈服条件：利用公式可得

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_{\theta} - 0}{2} = \frac{\sigma_{\theta}}{2} = k$$

在纯剪切中 $\tau_Y = k = \frac{\sigma_{\theta}}{2} = \frac{pR}{2h}$ ，因此当 $p = \frac{2h\tau_Y}{R}$ 达到剪切屈服条件。

Mises 屈服条件：利用公式可得

$$\begin{aligned} J_2' &= \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \\ &= \frac{1}{6} \left[\left(\frac{pR}{h} - \frac{pR}{2h} \right)^2 + \left(\frac{pR}{2h} - 0 \right)^2 + \left(0 - \frac{pR}{h} \right)^2 \right] = \frac{p^2 R^2}{4h^2} = C \end{aligned}$$

在纯剪切中 $\tau_Y^2 = C = \frac{p^2 R^2}{4h^2}$ ，因此当 $p = \frac{2h\tau_Y}{R}$ 达到剪切屈服条件。

(2) 对于两端是自由的薄壁圆管有

$$\sigma_{\theta} = \frac{\sigma_{\theta} h}{h} = \frac{\int_0^{\frac{\pi}{2}} pR \sin \theta d\theta}{h} = \frac{pR}{h}, \quad \sigma_r = 0, \quad \sigma_z = 0$$

排序可得主应力分别为

$$\sigma_1 = \sigma_{\theta}, \quad \sigma_2 = \sigma_z, \quad \sigma_3 = \sigma_r$$

Tresca 屈服条件：利用公式可得

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_{\theta} - 0}{2} = \frac{\sigma_{\theta}}{2} = k$$

在纯剪切中 $\tau_Y = k = \frac{\sigma_{\theta}}{2} = \frac{pR}{2h}$ ，因此当 $p = \frac{2h\tau_Y}{R}$ 达到剪切屈服条件。

Mises 屈服条件：利用公式可得

$$\begin{aligned} J_2' &= \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \\ &= \frac{1}{6} \left[\left(\frac{pR}{h} - 0 \right)^2 + \left(0 - \frac{pR}{h} \right)^2 \right] = \frac{p^2 R^2}{3h^2} = C \end{aligned}$$

在纯剪切中 $\tau_Y^2 = C = \frac{p^2 R^2}{3h^2}$ ，因此当 $p = \frac{\sqrt{3}h\tau_Y}{R}$ 达到剪切屈服条件。

(3) 对于两端是封闭的薄壁圆管有

$$\sigma_{\theta} = \frac{\sigma_{\theta} h}{h} = \frac{\int_0^{\frac{\pi}{2}} p R \sin \theta d\theta}{h} = \frac{pR}{h}, \quad \sigma_r = 0, \quad \sigma_z = \frac{p \pi R^2}{2 \pi R h} = \frac{pR}{2h}$$

排序可得主应力分别为

$$\sigma_1 = \sigma_{\theta}, \quad \sigma_2 = \sigma_z, \quad \sigma_3 = \sigma_r$$

Tresca 屈服条件: 利用公式可得

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_{\theta} - 0}{2} = \frac{\sigma_{\theta}}{2} = k$$

在纯剪切中 $\tau_Y = k = \frac{\sigma_{\theta}}{2} = \frac{pR}{2h}$, 因此当 $p = \frac{2h\tau_Y}{R}$ 达到剪切屈服条件.

Mises 屈服条件: 利用公式可得

$$\begin{aligned} J_2' &= \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \\ &= \frac{1}{6} \left[\left(\frac{pR}{h} - \frac{pR}{2h} \right)^2 + \left(\frac{pR}{2h} - 0 \right)^2 + \left(0 - \frac{pR}{h} \right)^2 \right] = \frac{p^2 R^2}{4h^2} = C \end{aligned}$$

在纯剪切中 $\tau_Y^2 = C = \frac{p^2 R^2}{4h^2}$, 因此当 $p = \frac{2h\tau_Y}{R}$ 达到剪切屈服条件. □

4.8 一封闭球形薄壳受内压作用, 写出 Mises 屈服条件和 Tresca 屈服条件.

解: 采用球坐标系求解, 取一块面积微元, 那么有

$$dA = R d\theta \cdot R \cos \theta \cdot d\varphi = R^2 \cos \theta d\theta d\varphi$$

于是有

$$\sigma_{\varphi} = \sigma_{\theta} = \frac{\int_A p \sin \theta dA}{2 \pi R h} = \frac{\int_0^{2\pi} \int_0^{\frac{\pi}{2}} p R^2 \sin \theta \cos \theta d\theta d\varphi}{2 \pi R h} = \frac{pR}{2h}, \quad \sigma_r = 0$$

排序可得主应力分别为

$$\sigma_1 = \sigma_2 = \frac{pR}{2h}, \quad \sigma_3 = 0$$

Tresca 屈服条件: 利用公式可得

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{pR}{4h} = k$$

在单向拉伸中 $\sigma_Y = 2k = 2 \cdot \frac{pR}{4h} = \frac{pR}{2h}$;

在纯剪切中 $\tau_Y = k = \frac{pR}{4h}$.

Mises 屈服条件: 利用公式可得

$$J_2' = \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$= \frac{1}{6} \left[\left(\frac{pR}{2h} - \frac{pR}{2h} \right)^2 + \left(\frac{pR}{2h} - 0 \right)^2 + \left(0 - \frac{pR}{2h} \right)^2 \right] = \frac{p^2 R^2}{12h^2} = C$$

在单向拉伸中 $\sigma_Y = \sqrt{3C} = \frac{pR}{2h}$;

在纯剪切中 $\tau_Y = \sqrt{C} = \frac{1}{2\sqrt{3}} \frac{pR}{h} = 0.288675 \frac{pR}{h}$. □.

第五章 塑性本构关系

5.1 在以下情况下按 Mises 屈服条件写出塑性应变增量之比:

(1) 单向拉伸轴应力状态, $\sigma_1 = \sigma_Y$;

(2) 双向压缩应力状态, $\sigma_1 = 0, \sigma_2 = \sigma_3 = -\sigma_Y$;

(3) 纯剪切应力状态, $\tau = \frac{\sigma_Y}{\sqrt{3}}$.

解: (1) 因为 $\sigma_1 = \sigma_Y$, 那么可得

$$s_1 = \sigma_1 - \sigma_m = \frac{2}{3}\sigma_Y, \quad s_2 = \sigma_2 - \sigma_m = -\frac{1}{3}\sigma_Y, \quad s_3 = \sigma_3 - \sigma_m = -\frac{1}{3}\sigma_Y$$

利用 Mises 屈服条件有

$$\frac{d\varepsilon_1}{s_1} = \frac{d\varepsilon_2}{s_2} = \frac{d\varepsilon_3}{s_3}$$

所以可得

$$d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = s_1 : s_2 : s_3 = 2 : -1 : -1$$

(2) 因为 $\sigma_1 = 0, \sigma_2 = \sigma_3 = -\sigma_Y$, 那么可得

$$s_1 = \sigma_1 - \sigma_m = \frac{2}{3}\sigma_Y, \quad s_2 = \sigma_2 - \sigma_m = -\frac{1}{3}\sigma_Y, \quad s_3 = \sigma_3 - \sigma_m = -\frac{1}{3}\sigma_Y$$

利用 Mises 屈服条件有

$$\frac{d\varepsilon_1}{s_1} = \frac{d\varepsilon_2}{s_2} = \frac{d\varepsilon_3}{s_3}$$

所以可得

$$d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = s_1 : s_2 : s_3 = 2 : -1 : -1$$

(3) 因为 $\tau = \frac{\sigma_Y}{\sqrt{3}}$, 那么可得

$$\sigma_1 = \frac{\sigma_Y}{\sqrt{3}}, \quad \sigma_2 = 0, \quad \sigma_3 = -\frac{\sigma_Y}{\sqrt{3}}$$

$$s_1 = \sigma_1 - \sigma_m = \frac{1}{\sqrt{3}}\sigma_Y, \quad s_2 = \sigma_2 - \sigma_m = 0, \quad s_3 = \sigma_3 - \sigma_m = -\frac{1}{\sqrt{3}}\sigma_Y$$

利用 Mises 屈服条件有

$$\frac{d\varepsilon_1}{s_1} = \frac{d\varepsilon_2}{s_2} = \frac{d\varepsilon_3}{s_3}$$

所以可得

$$d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = s_1 : s_2 : s_3 = 1 : 0 : 1. \square$$

5.3 已知一长封闭圆筒, 平均半径为 R , 壁厚为 h , 承受内压 p 的作用而产生塑性变形, 材

料是各向同性的，如果忽略弹性应变，试求周向、轴向和径向应变的比例。

解：对于两端是封闭的薄壁圆管有

$$\sigma_{\theta} = \frac{\sigma_{\theta} h}{h} = \frac{\int_0^{\frac{\pi}{2}} p R \sin \theta d\theta}{h} = \frac{p R}{h}, \quad \sigma_z = \nu(\sigma_r + \sigma_{\theta}) = \frac{p R}{2h}, \quad \sigma_r = 0$$

那么可得

$$s_{\theta} = \sigma_{\theta} - \sigma_m = \frac{p R}{2h}, \quad s_z = \sigma_z - \sigma_m = 0, \quad s_r = \sigma_r - \sigma_m = -\frac{p R}{2h}$$

利用 Mises 屈服条件有

$$\frac{d\varepsilon_{\theta}}{s_{\theta}} = \frac{d\varepsilon_z}{s_z} = \frac{d\varepsilon_r}{s_r}$$

所以可得

$$d\varepsilon_{\theta} : d\varepsilon_z : d\varepsilon_r = s_{\theta} : s_z : s_r = 1 : 0 : -1. \square$$

5.4 已知某圆筒承受拉应力 $\sigma_z = \frac{1}{2}\sigma_Y$ 及扭矩的作用，若采用 Mises 屈服条件，试求屈服时扭转应力为多大？并求出此时塑性应变增量的比值。

解：因为

$$\sigma_r = 0, \quad \sigma_{\theta} = 0, \quad \sigma_z = \frac{1}{2}\sigma_Y, \quad \tau_{\theta z} = \frac{1}{2}\sigma_Y$$

那么可得

$$s_{\theta} = \sigma_{\theta} - \sigma_m = -\frac{1}{6}\sigma_Y, \quad s_z = \sigma_z - \sigma_m = \frac{1}{3}\sigma_Y, \quad s_r = \sigma_r - \sigma_m = -\frac{1}{6}\sigma_Y$$

利用 Mises 屈服条件有

$$\frac{d\varepsilon_{\theta}}{s_{\theta}} = \frac{d\varepsilon_z}{s_z} = \frac{d\varepsilon_r}{s_r} = \frac{d\varepsilon_{\theta z}}{\tau_{\theta z}}$$

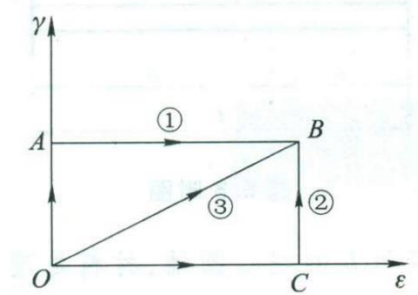
所以可得

$$d\varepsilon_{\theta} : d\varepsilon_z : d\varepsilon_r : d\varepsilon_{\theta z} = s_{\theta} : s_z : s_r : \tau_{\theta z} = -1 : 2 : -1 : 3. \square$$

第六章 简单的弹塑性问题

6.1 考虑 6.2 节中薄壁圆管承受拉伸和扭转，对图示的变形路径① OAB ，② OCB 和③ OB ，

求从 O 到达 B ，所对应的量纲为一的应力 σ 和 τ 。(用增量理论和全量理论求解)



解：设壁厚为 h ，筒的内外平均半径为 R ，显然有

$$\sigma_z = \frac{F}{2\pi R h}, \quad \tau_{\theta z} = \frac{T}{2\pi R^2 h}$$

其余应力均为 0，那么在弹性阶段有

$$\sigma = \varepsilon, \quad \tau = \gamma$$

进入塑性阶段以后，Mises 屈服条件经过量纲归一化后可化为

$$\sigma^2 + \tau^2 = 1$$

增量理论：

对于理想弹塑性材料，增量本构方程是 Prandtl-Reuss 关系，于是有

$$\frac{d\sigma}{d\varepsilon} = \sqrt{1-\sigma^2} \left(\sqrt{1-\sigma^2} - \sigma \frac{d\gamma}{d\varepsilon} \right), \quad \frac{d\tau}{d\gamma} = \sqrt{1-\tau^2} \left(\sqrt{1-\tau^2} - \tau \frac{d\varepsilon}{d\gamma} \right)$$

设在 B 点有 $\varepsilon = \varepsilon_0$, $\gamma = \gamma_0$ ，则对于路径② OCB 在 BC 段有

$$\gamma = \frac{1}{2} \ln \left(\frac{1+\tau}{1-\tau} \right)$$

解出来

$$\tau = \frac{e^{2\gamma} - 1}{e^{2\gamma} + 1} = \tanh \gamma = \tanh \gamma_0, \quad \sigma = \sqrt{1-\tau^2} = \sqrt{1-\tanh^2 \gamma} = \frac{1}{\cosh \gamma} = \frac{1}{\cosh \gamma_0}$$

则对于路径① OAB 在 AB 段有

$$\varepsilon = \frac{1}{2} \ln \left(\frac{1+\sigma}{1-\sigma} \right)$$

解出来

$$\sigma = \frac{e^{2\varepsilon} - 1}{e^{2\varepsilon} + 1} = \tanh \varepsilon = \tanh \varepsilon_0, \quad \tau = \sqrt{1 - \sigma^2} = \sqrt{1 - \tanh^2 \varepsilon} = \frac{1}{\cosh \varepsilon} = \frac{1}{\cosh \varepsilon_0}$$

对于比例路径③ OB , 其上有 $d\gamma = \frac{\gamma_0}{\varepsilon_0} d\varepsilon$, 在到达 B 点时, 材料刚达到屈服, 同时满足 $\tau = \frac{\gamma_0}{\varepsilon_0} \sigma$

和 $\sigma^2 + \tau^2 = 1$, 由此可得 B 点时的应力为

$$\tau = \sqrt{\frac{\gamma_0^2}{\varepsilon_0^2 + \gamma_0^2}} = \frac{\gamma_0}{\sqrt{\varepsilon_0^2 + \gamma_0^2}}, \quad \sigma = \sqrt{\frac{\varepsilon_0^2}{\varepsilon_0^2 + \gamma_0^2}} = \frac{\varepsilon_0}{\sqrt{\varepsilon_0^2 + \gamma_0^2}}$$

全量理论:

不考虑加载路径的影响, 直接考虑最终状态, 即用 B 点的应变 $\varepsilon = \varepsilon_0$, $\gamma = \gamma_0$ 代入下式

$$\tau = \frac{\gamma}{\sqrt{\varepsilon^2 + \gamma^2}}, \quad \sigma = \frac{\varepsilon}{\sqrt{\varepsilon^2 + \gamma^2}}$$

可得

$$\tau = \frac{\gamma_0}{\sqrt{\varepsilon_0^2 + \gamma_0^2}}, \quad \sigma = \frac{\varepsilon_0}{\sqrt{\varepsilon_0^2 + \gamma_0^2}}. \square$$