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《计算力学》

习题解答

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目录

第一章 变分学

1.1 求泛函的极值曲线：(1)$V=\int_0^1 (y'^2+12xy)\mathrm{d}x$；(2)$V=\int_{x_0}^{x_1}(y'^2+y^2+2y\mathrm{e}^x)\mathrm{d}x$.

解：(1)因为$F(x,y,y')=y'^2+12xy$，所以有

$$\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)=12x-2y''=12x-2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$$

因此可得

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-6x=0$$

从而解得

$$y=x^3+C_1x+C_2$$

(2)因为$F(x,y,y')=y'^2+y^2+2y\mathrm{e}^x$，所以有

$$\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)=2y+2\mathrm{e}^x-2y''=2y+2\mathrm{e}^x-2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$$

因此可得

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-y=\mathrm{e}^x$$

其对应的齐次方程为

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-y=0$$

其特征方程为

$$\lambda^2-1=0\Longrightarrow\lambda_1=1，\lambda_2=-1$$

则其对应的齐次方程的通解为

$$y_1=C_1\mathrm{e}^{-x}+C_2\mathrm{e}^x$$

下求非齐次方程的一个特解. 根据方程的结构，令$y_2=C_3x\mathrm{e}^x$是其的一个特解，带入可得

$$C_3(x+2)\mathrm{e}^x-C_3x\mathrm{e}^x=2C_3\mathrm{e}^x=\mathrm{e}^x$$

解得$C_3=\frac{1}{2}$，叠加可得

$$y=y_1+y_2=C_1\mathrm{e}^{-x}+C_2\mathrm{e}^x+\frac{1}{2}x\mathrm{e}^x.\square$$

1.2 求泛函的极值曲线$V=\int_{x_0}^{x_1}(2yz-2y^2+y'^2-z'^2)\mathrm{d}x$.

解：这是含有多个自变函数的泛函方程，因为$F(x,y,z,y',z')=2yz-2y^2+y'^2-z'^2$，则有

$$\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)=2z-4y-2y''=2z-4y-2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0\cdots\cdots①$$

$$\frac{\partial F}{\partial z}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial z'}\right)=2y+2\frac{\mathrm{d}^2z}{\mathrm{d}x^2}=0\cdots\cdots②$$

将式①带入式②可得

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+y=0$$

其特征方程为

$$\lambda^4+2\lambda^2+1=0$$

解得

$$\lambda_1=\mathrm{i}，\lambda_2=-\mathrm{i}$$

于是可得

$$y=(C_1+C_2x)\sin x+(C_3+C_4x)\cos x$$

这样也就得到了

$$z=2y+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=(C_1-2C_4+C_2x)\sin x+(2C_2+C_3+C_4x)\cos x.\square$$

1.3 求泛函的极值曲线$V=\int_{x_0}^{x_1}(x^2+16y^2-y''^2)\mathrm{d}x$.

解：这是含有自变函数高阶导数的泛函方程，因为$F(x,y,y',y'')=x^2+16y^2-y''^2$，则有

$$\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial y''}\right)=32y-2y^{(4)}=32y-2\frac{\mathrm{d}^4y}{\mathrm{d}x^4}=0$$

其特征方程为

$$\lambda^4-16=(\lambda+2)(\lambda-2)(\lambda^2+4)=0$$

解得

$$\lambda_1=-2，\lambda_2=2，\lambda_3=-2\mathrm{i}，\lambda_4=2\mathrm{i}$$

于是可得

$$y=C_1\mathrm{e}^{-2x}+C_2\mathrm{e}^{2x}+C_3\sin 2x+C_4\cos 2x.\square$$

1.4 弹性梁长为l，受均布载荷q作用，其总势能$\Pi=\frac{1}{2}\int_0^l EI\left(\frac{\partial^2w}{\partial x^2}\right)^2\mathrm{d}x-\int_0^l qw\,\mathrm{d}x$，根据弹性体受力平衡时总势能为最小的原理，分别求出两端固支、简支和悬臂三种梁的挠度曲线.

解：首先求解这个泛函方程，因为$F(x,w,w',w'')=\frac{1}{2}EIw''^2-qw$，则有

$$\frac{\partial F}{\partial w}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial w'}\right)+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial w''}\right)=-q+EIw^{(4)}=-q+EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4}=0$$

解得

$$w=\frac{q}{24EI}x^4+C_1x^3+C_2x^2+C_3x+C_4$$

(1)两端固支时，则有边界条件

$$w(0)=0,\ w'(0)=0,\ w(l)=0,\ w'(l)=0$$

解得

$$w=\frac{q}{24EI}x^4-\frac{ql}{12EI}x^3+\frac{ql^2}{24EI}x^2$$

(2)两端铰支时，则有边界条件

$$w(0)=0,\ w''(0)=0,\ w(l)=0,\ w''(l)=0$$

解得

$$w=\frac{q}{24EI}x^4-\frac{ql}{12EI}x^3+\frac{ql^3}{24EI}x$$

(3)一端自由时，则有边界条件

$$w(0)=0,\ w'(0)=0,\ w''(l)=0,\ w'''(l)=0$$

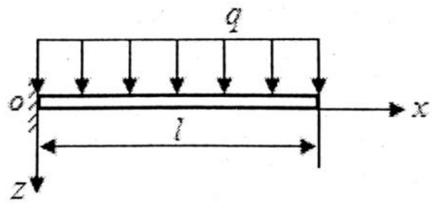
解得

$$w=\frac{q}{24EI}x^4-\frac{ql}{6EI}x^3+\frac{ql^2}{4EI}x^2.\square$$

第三章 变分问题的直接解法

3.1 基于最小势能原理，用里兹法求解图示受均布载荷悬臂梁的挠度，挠度函数选如下两种形式，并比较两种计算所得的最大挠度.

$$(1)\,w(x)=a_2x^2+a_3x^3\text{；}(2)\,w(x)=A\left(1-\cos\frac{\pi x}{2l}\right)$$



解：位移边界条件为

$$w(0)=0\,,\ w'(0)=0$$

(1)代入梁的总势能表达式可得

$$\begin{aligned}\Pi&=\frac{1}{2}\int_0^l EI\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2\mathrm{d}x-\int_0^l qw\,\mathrm{d}x\\&=\frac{1}{2}\int_0^l EI(2a_2+6a_3x)^2\mathrm{d}x-\int_0^l q(a_2x^2+a_3x^3)\mathrm{d}x\\&=2EIl(a_2^2+3a_2a_3l+3a_3^2l^2)-\frac{1}{12}ql^3(4a_2+3a_3l)\end{aligned}$$

由$\delta\Pi=\dfrac{\partial\Pi}{\partial a_k}\delta a_k=0$，$k=2,3$ 可得

$$\begin{cases}2EI(2a_2+3a_3l)-\dfrac{1}{3}ql^2=0\\2EI(3a_2+6a_3l)-\dfrac{1}{4}ql^2=0\end{cases}\Longrightarrow\begin{cases}a_2=\dfrac{5ql^2}{24EI}\\a_3=-\dfrac{ql}{12EI}\end{cases}$$

由此可得

$$w(x)=\frac{5ql^2}{24EI}x^2-\frac{ql}{12EI}x^3$$

最大挠度为

$$w(l)=\frac{5ql^2}{24EI}\cdot l^2-\frac{ql}{12EI}\cdot l^3=\frac{ql^4}{8EI}=0.125\frac{ql^4}{EI}$$

(2)代入梁的总势能表达式可得

$$\begin{aligned}\Pi &= \frac{1}{2}\int_0^l EI\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x - \int_0^l qw\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^l EI\left(\frac{A\pi^2}{4l^2}\cos\frac{\pi x}{2l}\right)^2 \mathrm{d}x - \int_0^l qA\left(1-\cos\frac{\pi x}{2l}\right)\mathrm{d}x \\ &= \frac{A^2\pi^4 EI}{32l^4}\int_0^l \cos^2\frac{\pi x}{2l}\,\mathrm{d}x - qAl + qA\int_0^l \cos\frac{\pi x}{2l}\,\mathrm{d}x = \frac{A^2\pi^4 EI}{64l^3} - \left(1-\frac{2}{\pi}\right)qAl\end{aligned}$$

由$\delta\Pi = \frac{\partial \Pi}{\partial A}\delta A = 0$可得

$$\frac{A\pi^4 EI}{32l^3} - \left(1-\frac{2}{\pi}\right)ql = 0 \Longrightarrow A = \frac{32}{\pi^4}\left(1-\frac{2}{\pi}\right)\frac{ql^4}{EI}$$

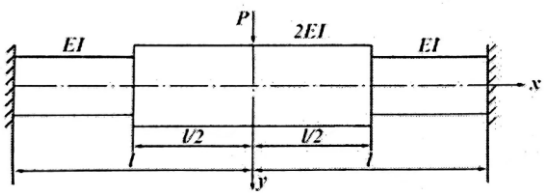
由此可得

$$w(x) = \frac{32}{\pi^4}\left(1-\frac{2}{\pi}\right)\frac{ql^4}{EI}\left(1-\cos\frac{\pi x}{2l}\right)$$

最大挠度为

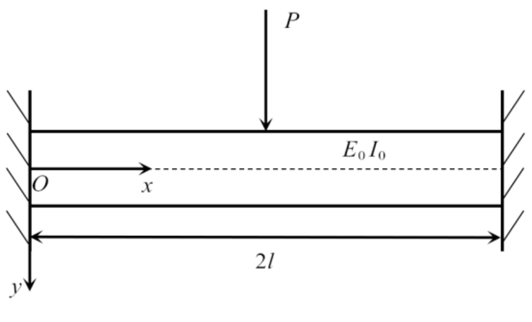
$$w(l) = \frac{32}{\pi^4}\left(1-\frac{2}{\pi}\right)\frac{ql^4}{EI}\left(1-\cos\frac{\pi}{2}\right) = 0.119375\frac{ql^4}{EI}.\square$$

3.2 应用最小势能原理，求解图示变剖面梁的挠度.(精确解$w(0)=0.02864\frac{Pl^3}{EI}$)



解：这是变截面超静定梁问题，采用等效梁方法，设等效梁的刚度为E_0I_0，长度为 $2l$. 等效挠度为

$$w_e(x) = \frac{Px^2}{24E_0I_0}(2x-3l),\ 0 \leqslant x \leqslant l$$



中点挠度为$a_1 = w_e(l) = -\frac{Pl^3}{24E_0I_0}$，则

$$w_e(x) = \frac{a_1}{l^3}(3lx^2 - 2x^3),\ w_e''(x) = \frac{6a_1}{l^3}(l-2x),\ 0 \leqslant x \leqslant l$$

等效梁的弹性势能为

$$U_1=2\cdot\frac{1}{2}\int_0^l E_0I_0\left[\frac{6a_1}{l^3}(l-2x)\right]^2\mathrm{d}x=\frac{36E_0I_0a_1^2}{l^6}\int_0^l(2x-l)^2\mathrm{d}x=\frac{12E_0I_0a_1^2}{l^3}$$

取等效梁挠度w_e做为变截面梁挠度w的第一级近似，则有

$$w(x)=w_e(x)=\frac{a_1}{l^3}(3lx^2-2x^3),\ w''(x)=\frac{6a_1}{l^3}(l-2x),\ 0\leqslant x\leqslant l$$

变截面梁的弹性势能为

$$U_2=2\cdot\frac{1}{2}\int_0^{\frac{l}{2}}EI\left[\frac{6a_1}{l^3}(l-2x)\right]^2\mathrm{d}x+2\cdot\frac{1}{2}\int_{\frac{l}{2}}^l 2EI\left[\frac{6a_1}{l^3}(l-2x)\right]^2\mathrm{d}x$$

$$=\frac{36EIa_1^2}{l^6}\int_0^{\frac{l}{2}}(2x-l)^2\mathrm{d}x+\frac{72EIa_1^2}{l^6}\int_{\frac{l}{2}}^l(2x-l)^2\mathrm{d}x=\frac{18EIa_1^2}{l^3}$$

由$\delta\Pi=\dfrac{\partial\Pi}{\partial a_1}\delta a_1=0$可得

$$\frac{24E_0I_0a_1}{l^3}-\frac{36EIa_1}{l^3}=0\Longrightarrow E_0I_0=\frac{3}{2}EI$$

于是可得变截面梁的挠度为

$$w(x)=\frac{Px^2}{72EI}(4x-6l),\ 0\leqslant x\leqslant l$$

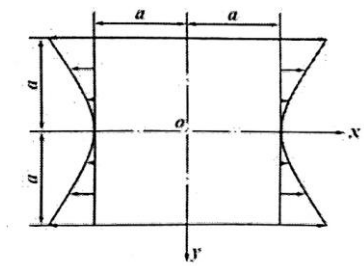
所以有

$$w(l)=\frac{P\cdot l^2}{72EI}(4\cdot l-6l)=-\frac{Pl^3}{36EI}=-0.0277778\frac{Pl^3}{EI}$$

误差为

$$\varepsilon=\left|\frac{\frac{1}{36}-\frac{11}{384}}{\frac{11}{384}}\right|\times100\%=3.03\%.\square$$

3.3 应用最小余能原理，求解图示正方形板在拉力$\sigma_x|_{x=\pm a}=q\left(\dfrac{y}{a}\right)^2$作用下的应力分布. (体力不计)



解：假设应力函数为

$$\varphi=\frac{1}{12}q\frac{y^4}{a^2}+A_1(x^2-a^2)^2(y^2-a^2)^2$$

于是有

$$\frac{\partial^2\varphi}{\partial x^2}=4A_1(3x^2-a^2)(y^2-a^2)^2,\ \frac{\partial^2\varphi}{\partial y^2}=q\left(\frac{y}{a}\right)^2+4A_1(x^2-a^2)^2(3y^2-a^2)$$

$$\frac{\partial^2\varphi}{\partial x\partial y}=16A_1xy(x^2-a^2)(y^2-a^2)$$

带入总余能表达式可得(不计体力，$\mu=0$)

$$\begin{aligned}\Pi_c&=\frac{1}{2E}\iint\left[\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2+\left(\frac{\partial^2\varphi}{\partial y^2}\right)^2+2\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y\\
&=\frac{1}{2E}\iint\left[\begin{array}{c}16A_1^2(3x^2-a^2)^2(y^2-a^2)^4+q^2\left(\frac{y}{a}\right)^4+8qA_1\left(\frac{y}{a}\right)^2(x^2-a^2)^2(3y^2-a^2)\\+16A_1^2(x^2-a^2)^4(3y^2-a^2)^2+512A_1^2x^2y^2(x^2-a^2)^2(y^2-a^2)^2\end{array}\right]\mathrm{d}x\,\mathrm{d}y\\
&=\frac{8A_1^2}{E}\int_{-a}^{a}(3x^2-a^2)^2\mathrm{d}x\int_{-a}^{a}(y^2-a^2)^4\mathrm{d}y+\frac{q^2}{2a^2E}\int_{-a}^{a}\mathrm{d}x\int_{-a}^{a}y^4\mathrm{d}y\\
&\quad+\frac{4qA_1}{a^2E}\int_{-a}^{a}(x^2-a^2)^2\mathrm{d}x\int_{-a}^{a}y^2(3y^2-a^2)\mathrm{d}y+\frac{8A_1^2}{E}\int_{-a}^{a}(x^2-a^2)^4\mathrm{d}x\int_{-a}^{a}(3y^2-a^2)^2\mathrm{d}y\\
&\quad+\frac{256A_1^2}{E}\int_{-a}^{a}x^2(x^2-a^2)^2\mathrm{d}x\int_{-a}^{a}y^2(y^2-a^2)^2\mathrm{d}y\\
&=2\cdot\frac{8A_1^2}{E}\cdot\frac{8}{5}a^5\cdot\frac{256}{315}a^9+\frac{q^2}{2a^2E}\cdot2a\cdot\frac{2}{5}a^5+\frac{4qA_1}{a^2E}\cdot\frac{16}{15}a^5\cdot\frac{8}{15}a^5+\frac{256A_1^2}{E}\cdot\frac{16}{105}a^7\cdot\frac{16}{105}a^7\\
&=\frac{32768}{1225}\frac{A_1^2a^{14}}{E}+\frac{2}{5}\frac{q^2a^4}{E}+\frac{512}{225}\frac{qA_1a^8}{E}\end{aligned}$$

由$\delta\Pi_c=\frac{\partial\Pi_c}{\partial A_1}\delta A_1=0$可得

$$\frac{65536}{1225}\frac{A_1a^{14}}{E}+\frac{512}{225}\frac{qa^8}{E}=0\Longrightarrow A_1=-\frac{49}{1152}\frac{q}{a^6}=-0.0425347\frac{q}{a^6}$$

于是有

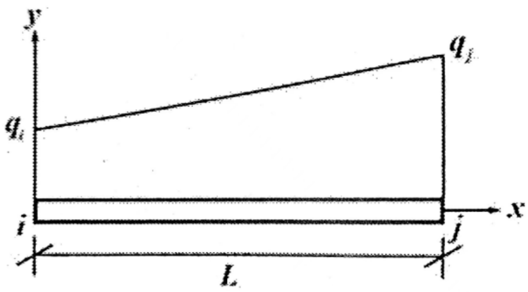
$$\varphi=\frac{1}{12}q\frac{y^4}{a^2}-\frac{49}{1152}\frac{q}{a^6}(x^2-a^2)^2(y^2-a^2)^2$$

由此可求得应力分量

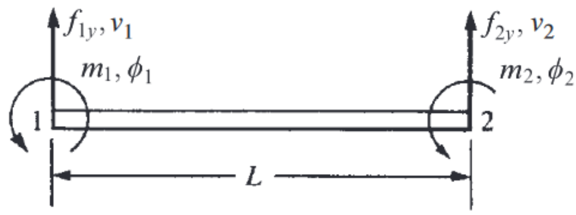
$$\begin{cases}\sigma_x=\frac{\partial^2\varphi}{\partial y^2}=q\left(\frac{y}{a}\right)^2-\frac{49}{288}q\left(\frac{x^2}{a^2}-1\right)^2\left(\frac{3y^2}{a^2}-1\right)=q\left(\frac{y}{a}\right)^2-0.170139q\left(\frac{x^2}{a^2}-1\right)^2\left(\frac{3y^2}{a^2}-1\right)\\
\sigma_y=\frac{\partial^2\varphi}{\partial x^2}=-\frac{49}{288}q\left(\frac{3x^2}{a^2}-1\right)\left(\frac{y^2}{a^2}-1\right)^2=-0.170139q\left(\frac{3x^2}{a^2}-1\right)\left(\frac{y^2}{a^2}-1\right)^2\\
\tau_{xy}=-\frac{\partial^2\varphi}{\partial x\partial y}=\frac{49}{72}q\frac{xy}{a^2}\left(\frac{x^2}{a^2}-1\right)\left(\frac{y^2}{a^2}-1\right)=0.680556q\frac{xy}{a^2}\left(\frac{x^2}{a^2}-1\right)\left(\frac{y^2}{a^2}-1\right).\square\end{cases}$$

第五章 杆系结构有限元

5.1 求图示梁单元上分布载荷的等效节点载荷列阵.



解：其静力等效节点力系如下



因为载荷$q(x)$(竖直向上)为

$$q(x)=q_i+(q_j-q_i)\frac{x}{L}=[q_i \quad q_j]\begin{bmatrix}1-\dfrac{x}{L}\\ \dfrac{x}{L}\end{bmatrix}$$

单元位移函数$v(x)$为

$$[v]=[N]\{d\}$$

$$=\left[\frac{2x^3-3x^2L+L^3}{L^3} \quad \frac{x^3L-2x^2L^2+xL^3}{L^3} \quad \frac{-2x^3+3x^2L}{L^3} \quad \frac{x^3L-x^2L^2}{L^3}\right]\begin{Bmatrix}v_1\\ \phi_1\\ v_2\\ \phi_2\end{Bmatrix}$$

于是利用功等效方法可得

$$\int_0^L q(x)v(x)\mathrm{d}x=m_1\phi_1+m_2\phi_2+f_{1y}v_1+f_{2y}v_2$$

因为

$$\int_0^L q(x)v(x)\mathrm{d}x$$

$$=\int_0^L [q_i \quad q_j]\begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{bmatrix}\begin{bmatrix} \frac{2x^3-3x^2L+L^3}{L^3} & \frac{x^3L-2x^2L^2+xL^3}{L^3} & \frac{-2x^3+3x^2L}{L^3} & \frac{x^3L-x^2L^2}{L^3} \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}\mathrm{d}x$$

$$=\int_0^L [q_i \quad q_j]\begin{bmatrix} \frac{(L-x)^3(L+2x)}{L^4} & \frac{x(L-x)^3}{L^3} & \frac{x^2(L-x)(3L-2x)}{L^4} & -\frac{x^2(L-x)^2}{L^3} \\ \frac{x(L-x)^2(L+2x)}{L^4} & \frac{x^2(L-x)^2}{L^3} & \frac{x^3(3L-2x)}{L^4} & -\frac{x^3(L-x)}{L^3} \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}\mathrm{d}x$$

$$=[q_i \quad q_j]\int_0^L \begin{bmatrix} \frac{(L-x)^3(L+2x)}{L^4} & \frac{x(L-x)^3}{L^3} & \frac{x^2(L-x)(3L-2x)}{L^4} & -\frac{x^2(L-x)^2}{L^3} \\ \frac{x(L-x)^2(L+2x)}{L^4} & \frac{x^2(L-x)^2}{L^3} & \frac{x^3(3L-2x)}{L^4} & -\frac{x^3(L-x)}{L^3} \end{bmatrix}\mathrm{d}x\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

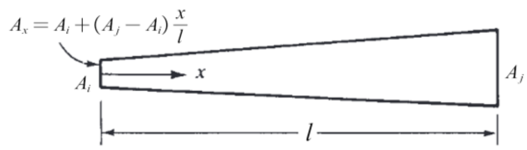
$$=[q_i \quad q_j]\begin{bmatrix} \frac{7}{20}L & \frac{1}{20}L^2 & \frac{3}{20}L & -\frac{1}{30}L^2 \\ \frac{3}{20}L & \frac{1}{30}L^2 & \frac{7}{20}L & -\frac{1}{20}L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}=\{f_{1y} \quad m_1 \quad f_{2y} \quad m_2\}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

所以可得等效节点载荷为

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix}=\begin{Bmatrix} \frac{7q_i+3q_j}{20}L \\ \frac{3q_i+2q_j}{60}L^2 \\ \frac{3q_i+7q_j}{20}L \\ -\frac{2q_i+3q_j}{60}L^2 \end{Bmatrix}.\square$$

5.2 用最小势能原理推导变截面杆的单元刚度矩阵. 设截面A_x的变化规律为

$$A_x=A_i+(A_j-A_i)\frac{x}{l}.$$



解：总势能有

$$\Pi_p=\frac{1}{2}\int_0^l [A]\{d\}^{\mathrm{T}}[B]^{\mathrm{T}}[D]^{\mathrm{T}}[B]\{d\}\mathrm{d}x-\{d\}^{\mathrm{T}}\{f\}$$

其中

$$[A]=A_x=A_i+(A_j-A_i)\frac{x}{l}=[A_i \quad A_j]\begin{bmatrix} 1-\frac{x}{l} \\ \frac{x}{l} \end{bmatrix},\ [D]=[E],\ [B]=\begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix}$$

所以有

$$\{U^*\}=\{d\}^{\mathrm{T}}[B]^{\mathrm{T}}[D]^{\mathrm{T}}[B]\{d\}=\{u_1 \quad u_2\}\begin{bmatrix}-\dfrac{1}{l}\\ \dfrac{1}{l}\end{bmatrix}[E]\begin{bmatrix}-\dfrac{1}{l} & \dfrac{1}{l}\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=\frac{E}{l^2}(u_1^2-2u_1u_2+u_2^2)$$

$$\{d\}^{\mathrm{T}}\{f\}=\{u_1 \quad u_2\}\begin{Bmatrix}f_{1x}\\f_{2x}\end{Bmatrix}=u_1f_{1x}+u_2f_{2x}$$

于是有

$$\begin{aligned}\Pi_p&=\frac{1}{2}\int_0^l[A]\{d\}^{\mathrm{T}}[B]^{\mathrm{T}}[D]^{\mathrm{T}}[B]\{d\}\mathrm{d}x-\{d\}^{\mathrm{T}}\{f\}\\
&=\frac{E}{2l^2}(u_1^2-2u_1u_2+u_2^2)\int_0^l[A_i \quad A_j]\begin{bmatrix}1-\dfrac{x}{l}\\ \dfrac{x}{l}\end{bmatrix}\mathrm{d}x-u_1f_{1x}-u_2f_{2x}\\
&=\frac{E}{2l^2}(u_1^2-2u_1u_2+u_2^2)[A_i \quad A_j]\begin{bmatrix}\dfrac{1}{2}l\\ \dfrac{1}{2}l\end{bmatrix}-u_1f_{1x}-u_2f_{2x}\\
&=\frac{E}{4l}(u_1^2-2u_1u_2+u_2^2)(A_i+A_j)-u_1f_{1x}-u_2f_{2x}\end{aligned}$$

则有

$$\frac{\partial\Pi_p}{\partial u_1}=\frac{E}{2l}(u_1-u_2)(A_i+A_j)-f_{1x},\ \frac{\partial\Pi_p}{\partial u_2}=\frac{E}{2l}(u_2-u_1)(A_i+A_j)-f_{2x}$$

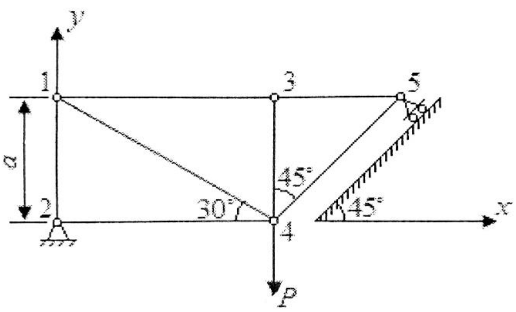
即

$$\frac{\partial\Pi_p}{\partial\{d\}}=\frac{E(A_i+A_j)}{2l}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}-\begin{Bmatrix}f_{1x}\\f_{2x}\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

即有单元刚度矩阵为

$$[k]=\frac{E(A_i+A_j)}{2l}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}.\square$$

5.3 求图示平面桁架的结构刚度矩阵，并考虑对结构如何进行约束处理.



解：对杆单元进行编号，设杆 2-1 为①，杆 1-3 为②，杆 4-1 为③，杆 2-4 为④，杆 4-3 为⑤，杆 3-5 为⑥，杆 4-5 为⑦，于是有

单元①

$$\theta^{(1)}=90°,\ \cos\theta^{(1)}=0,\ \sin\theta^{(1)}=1$$

$$[k^{(1)}]=\frac{EA}{a}\begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_1 & v_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ u_2 \\ v_2 \\ u_1 \\ v_1 \end{array}$$

单元②

$$\theta^{(2)}=0,\ \cos\theta^{(2)}=1,\ \sin\theta^{(2)}=0$$

$$[k^{(2)}]=\frac{EA}{\sqrt{3}a}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}$$

单元③

$$\theta^{(3)}=150°,\ \cos\theta^{(3)}=-\frac{\sqrt{3}}{2},\ \sin\theta^{(3)}=\frac{1}{2}$$

$$[k^{(3)}]=\frac{EA}{2a}\begin{array}{c} \begin{array}{cccc} u_4 & v_4 & u_1 & v_1 \end{array} \\ \begin{bmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \end{array} \begin{array}{c} \\ u_4 \\ v_4 \\ u_1 \\ v_1 \end{array}$$

单元④

$$\theta^{(4)}=0,\ \cos\theta^{(4)}=1,\ \sin\theta^{(4)}=0$$

$$[k^{(4)}]=\frac{EA}{\sqrt{3}a}\begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ u_2 \\ v_2 \\ u_4 \\ v_4 \end{array}$$

单元⑤

$$\theta^{(5)}=90°,\ \cos\theta^{(5)}=0,\ \sin\theta^{(5)}=1$$

$$[k^{(5)}]=\frac{EA}{a}\begin{array}{c}\begin{array}{cccc} u_4 & v_4 & u_3 & v_3 \end{array}\\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\end{array}\begin{array}{c} \\ u_4 \\ v_4 \\ u_3 \\ v_3 \end{array}$$

单元⑥

$$\theta^{(6)}=0,\ \cos\theta^{(6)}=1,\ \sin\theta^{(6)}=0$$

$$[k^{(6)}]=\frac{EA}{a}\begin{array}{c}\begin{array}{cccc} u_3 & v_3 & u_5 & v_5 \end{array}\\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{array}\begin{array}{c} \\ u_3 \\ v_3 \\ u_5 \\ v_5 \end{array}$$

单元⑦

$$\theta^{(7)}=45°,\ \cos\theta^{(7)}=\frac{\sqrt{2}}{2},\ \sin\theta^{(7)}=\frac{\sqrt{2}}{2}$$

$$[k^{(7)}]=\frac{EA}{\sqrt{2}\,a}\begin{array}{c}\begin{array}{cccc} u_4 & v_4 & u_5 & v_5 \end{array}\\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\end{array}\begin{array}{c} \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{array}$$

使用直接刚度法拼接可得整体刚度矩阵为

$$[K]=\frac{EA}{a}\begin{bmatrix} \frac{8\sqrt{3}+9}{24} & -\frac{\sqrt{3}}{8} & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & -\frac{3}{8} & \frac{\sqrt{3}}{8} & 0 & 0 \\ -\frac{\sqrt{3}}{8} & \frac{9}{8} & 0 & -1 & 0 & 0 & \frac{\sqrt{3}}{8} & -\frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{3} & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & \frac{3+\sqrt{3}}{3} & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ -\frac{3}{8} & \frac{\sqrt{3}}{8} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{8\sqrt{3}+9+6\sqrt{2}}{24} & \frac{2\sqrt{2}-\sqrt{3}}{8} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{3}}{8} & -\frac{1}{8} & 0 & 0 & 0 & -1 & \frac{2\sqrt{2}-\sqrt{3}}{8} & \frac{9+\sqrt{2}}{8} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 & -1 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{4+\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix}$$

边界约束条件为

$$u_2=v_2=0,\ v_5'=0,\ F_{4y}=-P$$

定义转换矩阵为

$$[T_5]=\begin{bmatrix} [I] & [0] & [0] & [0] & [0] \\ [0] & [I] & [0] & [0] & [0] \\ [0] & [0] & [I] & [0] & [0] \\ [0] & [0] & [0] & [I] & [0] \\ [0] & [0] & [0] & [0] & [t_5] \end{bmatrix},\ [t_5]=\begin{bmatrix} \dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \\ -\dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \end{bmatrix}$$

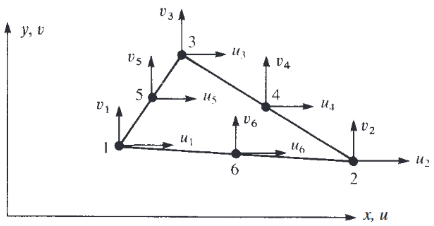
因此有

$$[K']=[T_5][K][T_5]^{\mathrm{T}}=\begin{array}{c} \begin{array}{cccc} u_1 & \cdots & u_5' & v_5' \end{array} \\ \begin{bmatrix} * & * & * & * \\ * & \ddots & \cdots & \vdots \\ * & \vdots & * & * \\ * & \cdots & * & * \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ \vdots \\ u_5' \\ v_5' \end{array}$$

在此整体刚度矩阵下便可处理节点 5 处的位移边界约束.□

第六章 弹性力学平面问题有限元

6.1 导出六节点三角形单元刚度矩阵的各子阵$[k_{ij}]$的表达式.



解：考虑六节点三角形单元，我们选取二次位函数为

$$u(x,y)=a_1+a_2x+a_3y+a_4x^2+a_5xy+a_6y^2$$
$$v(x,y)=a_7+a_8x+a_9y+a_{10}x^2+a_{11}xy+a_{12}y^2$$

于是有

$$\{\psi\}=\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}1&x&y&x^2&xy&y^2&0&0&0&0&0&0\\0&0&0&0&0&0&1&x&y&x^2&xy&y^2\end{bmatrix}\begin{bmatrix}a_1\\a_2\\\vdots\\a_{12}\end{bmatrix}=[M^*]\{a\}$$

则有

$$\{d\}=\begin{bmatrix}u_1\\v_1\\u_2\\v_2\\u_3\\v_3\\u_4\\v_4\\u_5\\v_5\\u_6\\v_6\end{bmatrix}=\begin{bmatrix}1&x_1&y_1&x_1^2&x_1y_1&y_1^2&0&0&0&0&0&0\\0&0&0&0&0&0&1&x_1&y_1&x_1^2&x_1y_1&y_1^2\\1&x_2&y_2&x_2^2&x_2y_2&y_2^2&0&0&0&0&0&0\\0&0&0&0&0&0&1&x_2&y_2&x_2^2&x_2y_2&y_2^2\\1&x_3&y_3&x_3^2&x_3y_3&y_3^2&0&0&0&0&0&0\\0&0&0&0&0&0&1&x_3&y_3&x_3^2&x_3y_3&y_3^2\\1&x_4&y_4&x_4^2&x_4y_4&y_4^2&0&0&0&0&0&0\\0&0&0&0&0&0&1&x_4&y_4&x_4^2&x_4y_4&y_4^2\\1&x_5&y_5&x_5^2&x_5y_5&y_5^2&0&0&0&0&0&0\\0&0&0&0&0&0&1&x_5&y_5&x_5^2&x_5y_5&y_5^2\\1&x_6&y_6&x_6^2&x_6y_6&y_6^2&0&0&0&0&0&0\\0&0&0&0&0&0&1&x_6&y_6&x_6^2&x_6y_6&y_6^2\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\\a_4\\a_5\\a_6\\a_7\\a_8\\a_9\\a_{10}\\a_{11}\\a_{12}\end{bmatrix}$$

因此有

$$\{a\}=\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \end{bmatrix}=\begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 \\ 1 & x_3 & y_3 & x_3^2 & x_3 y_3 & y_3^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_3 & y_3 & x_3^2 & x_3 y_3 & y_3^2 \\ 1 & x_4 & y_4 & x_4^2 & x_4 y_4 & y_4^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_4 & y_4 & x_4^2 & x_4 y_4 & y_4^2 \\ 1 & x_5 & y_5 & x_5^2 & x_5 y_5 & y_5^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_5 & y_5 & x_5^2 & x_5 y_5 & y_5^2 \\ 1 & x_6 & y_6 & x_6^2 & x_6 y_6 & y_6^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_6 & y_6 & x_6^2 & x_6 y_6 & y_6^2 \end{bmatrix}^{-1}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{bmatrix}=[X]^{-1}\{d\}$$

则有

$$\{\psi\}=[M^*]\{a\}=[M^*][X]^{-1}\{d\}=[N]\{d\}$$

又因为

$$\{\varepsilon\}=\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}=\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \end{bmatrix}=\begin{bmatrix} a_2+2a_4 x+a_5 y \\ a_9+a_{11} x+2a_{12} y \\ a_3+a_5 x+2a_6 y+a_8+2a_{10} x+a_{11} y \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix}=[M']\{a\}$$

则有

$$\{\varepsilon\}=[M']\{a\}=[M'][X]^{-1}\{d\}=[B]\{d\}$$

于是有

$$\{\sigma\}=\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}=[D]\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}=[D]\{\varepsilon\}=[D][B]\{d\}$$

其中

$$[D]=\frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \dfrac{1-2\mu}{2} \end{bmatrix}$$

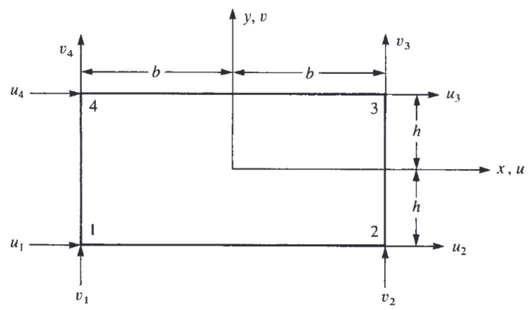
于是有

$$[k]=\iiint_V [B]^{\mathrm{T}}[D][B]\mathrm{d}V=\begin{bmatrix}[k_{11}] & [k_{12}] & [k_{13}] & [k_{14}] & [k_{15}] & [k_{16}]\\ [k_{21}] & [k_{22}] & [k_{23}] & [k_{24}] & [k_{25}] & [k_{26}]\\ [k_{31}] & [k_{32}] & [k_{33}] & [k_{34}] & [k_{35}] & [k_{36}]\\ [k_{41}] & [k_{42}] & [k_{43}] & [k_{44}] & [k_{45}] & [k_{46}]\\ [k_{51}] & [k_{52}] & [k_{53}] & [k_{54}] & [k_{55}] & [k_{56}]\\ [k_{61}] & [k_{62}] & [k_{63}] & [k_{64}] & [k_{65}] & [k_{66}]\end{bmatrix}$$

其中

$$[B]=\frac{1}{2A}\begin{bmatrix}\beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 & 0 & \beta_5 & 0 & \beta_6 & 0\\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 & 0 & \gamma_4 & 0 & \gamma_5 & 0 & \gamma_6\\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 & \gamma_4 & \beta_4 & \gamma_5 & \beta_5 & \gamma_6 & \beta_6\end{bmatrix}.\square$$

6.2 导出四节点矩形单元刚度矩阵的子阵$[k_{ij}]=\begin{bmatrix}k_1 & k_2\\ k_3 & k_4\end{bmatrix}$中各元素的表达式.



解：考虑四节点矩形单元，我们选取位函数为

$$u(x,y)=a_1+a_2x+a_3y+a_4xy$$
$$v(x,y)=a_5+a_6x+a_7y+a_8xy$$

于是有

$$\{d\}=\begin{bmatrix}u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\end{bmatrix}=\begin{bmatrix}1 & -b & -h & bh & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -b & -h & bh\\ 1 & b & -h & -bh & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & b & -h & -bh\\ 1 & b & h & bh & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & b & h & bh\\ 1 & -b & h & -bh & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -b & h & -bh\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\\ a_4\\ a_5\\ a_6\\ a_7\\ a_8\end{bmatrix}=[X]\{a\}$$

因此有

$$\{a\}=\begin{bmatrix}a_1\\ a_2\\ a_3\\ a_4\\ a_5\\ a_6\\ a_7\\ a_8\end{bmatrix}=\begin{bmatrix}1 & -b & -h & bh & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -b & -h & bh\\ 1 & b & -h & -bh & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & b & -h & -bh\\ 1 & b & h & bh & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & b & h & bh\\ 1 & -b & h & -bh & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -b & h & -bh\end{bmatrix}^{-1}\begin{bmatrix}u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\end{bmatrix}=[X]^{-1}\{d\}$$

因为有

$$u(-b,-h)=u_1,\ v(-b,-h)=v_1,\ u(b,-h)=u_2,\ v(b,-h)=v_2$$
$$u(b,h)=u_3,\ v(b,h)=v_3,\ u(-b,h)=u_4,\ v(-b,h)=v_4$$

可得

$$u(x,y)=N_1u_1+N_2u_2+N_3u_3+N_4u_4$$
$$v(x,y)=N_1v_1+N_2v_2+N_3v_3+N_4v_4$$

其中

$$N_1=\frac{(b-x)(h-y)}{4bh},\ N_2=\frac{(b+x)(h-y)}{4bh},\ N_3=\frac{(b+x)(h+y)}{4bh},\ N_4=\frac{(b-x)(h+y)}{4bh}$$

于是有

$$\{\psi\}=\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ v_4 \end{bmatrix}=[N]\{d\}$$

又因为

$$\{\varepsilon\}=\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}=\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \end{bmatrix}=\begin{bmatrix} a_2+a_4y \\ a_7+a_8x \\ a_3+a_4x+a_6+a_8y \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & y & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix}=[M']\{a\}$$

则有

$$\{\varepsilon\}=[M']\{a\}=[M'][X]^{-1}\{d\}=[B]\{d\}$$

于是有

$$\{\sigma\}=\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}=[D]\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}=[D]\{\varepsilon\}=[D][B]\{d\}$$

其中

$$[D]=\frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \dfrac{1-2\mu}{2} \end{bmatrix}$$

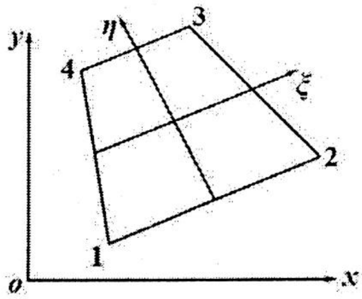
于是有

$$[k]=\int_{-b}^{b}\int_{-h}^{h}[B]^{\mathrm{T}}[D][B]t\,\mathrm{d}x\,\mathrm{d}y=\begin{bmatrix}[k_{11}] & [k_{12}] & [k_{13}] & [k_{14}]\\ [k_{21}] & [k_{22}] & [k_{23}] & [k_{24}]\\ [k_{31}] & [k_{32}] & [k_{33}] & [k_{34}]\\ [k_{41}] & [k_{42}] & [k_{43}] & [k_{44}]\end{bmatrix}$$

其中

$$[B]=\frac{1}{4bh}\begin{bmatrix}y-h & 0 & -y+h & 0 & y+h & 0 & -y-h & 0\\ 0 & x-b & 0 & -x-b & 0 & x+b & 0 & -x+b\\ x-b & y-h & -x-b & -y+h & x+b & y+h & -x+b & -y-h\end{bmatrix}.\square$$

6.3 计算图所示四节点等参单元在$\xi=\dfrac{1}{2}$，$\eta=\dfrac{1}{2}$处的插值函数的偏导数$\dfrac{\partial N_1}{\partial x}$，$\dfrac{\partial N_2}{\partial y}$. 节点坐标为：1(20, 10)，2(80, 40)，3(50, 80)，4(10, 70).



解：因为有

$$\begin{aligned}x&=N_1x_1+N_2x_2+N_3x_3+N_4x_4\\&=\frac{1}{4}[(1-\xi)(1-\eta)x_1+(1+\xi)(1-\eta)x_2+(1+\xi)(1+\eta)x_3+(1-\xi)(1+\eta)x_4]\\y&=N_1y_1+N_2y_2+N_3y_3+N_4y_4\\&=\frac{1}{4}[(1-\xi)(1-\eta)y_1+(1+\xi)(1-\eta)y_2+(1+\xi)(1+\eta)y_3+(1-\xi)(1+\eta)y_4]\end{aligned}$$

利用链式法则可得

$$\begin{cases}\dfrac{\partial}{\partial\xi}=\dfrac{\partial}{\partial x}\dfrac{\partial x}{\partial\xi}+\dfrac{\partial}{\partial y}\dfrac{\partial y}{\partial\xi}\\ \dfrac{\partial}{\partial\eta}=\dfrac{\partial}{\partial x}\dfrac{\partial x}{\partial\eta}+\dfrac{\partial}{\partial y}\dfrac{\partial y}{\partial\eta}\end{cases}$$

由 Cramer 法则解得

$$\frac{\partial}{\partial x}=\frac{\begin{vmatrix}\frac{\partial}{\partial\xi} & \frac{\partial y}{\partial\xi}\\ \frac{\partial}{\partial\eta} & \frac{\partial y}{\partial\eta}\end{vmatrix}}{\begin{vmatrix}\frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi}\\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta}\end{vmatrix}}=\frac{1}{|[J]|}\left[\frac{\partial}{\partial\xi}\frac{\partial y}{\partial\eta}-\frac{\partial}{\partial\eta}\frac{\partial y}{\partial\xi}\right],\quad \frac{\partial}{\partial y}=\frac{\begin{vmatrix}\frac{\partial x}{\partial\xi} & \frac{\partial}{\partial\xi}\\ \frac{\partial x}{\partial\eta} & \frac{\partial}{\partial\eta}\end{vmatrix}}{\begin{vmatrix}\frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi}\\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta}\end{vmatrix}}=\frac{1}{|[J]|}\left[\frac{\partial}{\partial\eta}\frac{\partial x}{\partial\xi}-\frac{\partial}{\partial\xi}\frac{\partial x}{\partial\eta}\right]$$

而又有

$$\frac{\partial x}{\partial \xi}=\frac{1}{4}[(\eta-1)x_1+(1-\eta)x_2+(1+\eta)x_3-(1+\eta)x_4]=22.5$$

$$\frac{\partial x}{\partial \eta}=\frac{1}{4}[(\xi-1)x_1-(1+\xi)x_2+(1+\xi)x_3+(1-\xi)x_4]=-12.5$$

$$\frac{\partial y}{\partial \xi}=\frac{1}{4}[(\eta-1)y_1+(1-\eta)y_2+(1+\eta)y_3-(1+\eta)y_4]=7.5$$

$$\frac{\partial y}{\partial \eta}=\frac{1}{4}[(\xi-1)y_1-(1+\xi)y_2+(1+\xi)y_3+(1-\xi)y_4]=22.5$$

$$|[J]|=\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta}-\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}=22.5\cdot 22.5-(-12.5)\cdot 7.5=600$$

且有

$$\frac{\partial N_1}{\partial \xi}=\frac{1}{4}(\eta-1)=-\frac{1}{8},\ \frac{\partial N_1}{\partial \eta}=\frac{1}{4}(\xi-1)=-\frac{1}{8}$$

$$\frac{\partial N_2}{\partial \xi}=\frac{1}{4}(1-\eta)=\frac{1}{8},\ \frac{\partial N_2}{\partial \eta}=-\frac{1}{4}(\xi+1)=-\frac{3}{8}$$

所以有

$$\frac{\partial N_1}{\partial x}=\frac{1}{|[J]|}\left[\frac{\partial N_1}{\partial \xi}\frac{\partial y}{\partial \eta}-\frac{\partial N_1}{\partial \eta}\frac{\partial y}{\partial \xi}\right]=\frac{1}{600}\cdot\left[-\frac{1}{8}\cdot 22.5-\left(-\frac{1}{8}\right)\cdot 7.5\right]=-0.003125$$

$$\frac{\partial N_2}{\partial y}=\frac{1}{|[J]|}\left[\frac{\partial N_2}{\partial \eta}\frac{\partial x}{\partial \xi}-\frac{\partial N_2}{\partial \xi}\frac{\partial x}{\partial \eta}\right]=\frac{1}{600}\cdot\left[\frac{1}{8}\cdot 22.5-\left(-\frac{3}{8}\right)\cdot(-12.5)\right]=-0.003125.\square$$