课本知识点整理

第一章

1. 外力

定义：其他物体对研究对象的作用力，分为体积力和表面力.

2. 体力

定义：分布在物体体积内的力(如重力，惯性力).可由下式定义

$$\lim_{\triangle V \to 0} \frac{\triangle \boldsymbol{F}}{\triangle V} = \boldsymbol{f}$$

其中$\triangle \boldsymbol{F}$是作用在$\triangle V$的体力，极限矢量$\boldsymbol{f}$就是该物体在P点所受体力的集度，$\boldsymbol{f}$的方向就是$\triangle \boldsymbol{F}$的极限方向.

量纲：$\mathrm{L}^{-2}\mathrm{MT}^{-2}$.

3. 面力

定义：分布在物体表面上的力(如流体压力，接触力).可由下式定义

$$\lim_{\triangle S \to 0} \frac{\triangle \boldsymbol{F}}{\triangle S} = \bar{\boldsymbol{f}}$$

其中$\triangle \boldsymbol{F}$是作用在$\triangle S$的面力，极限矢量$\bar{\boldsymbol{f}}$就是该物体在P点所受面力的集度，$\bar{\boldsymbol{f}}$的方向就是$\triangle \boldsymbol{F}$的极限方向.

量纲：$\mathrm{L}^{-1}\mathrm{MT}^{-2}$.

4. 内力

定义：物体内部不同部分之间相互作用的力. 可由下式定义

$$\lim_{\triangle A \to 0} \frac{\triangle \boldsymbol{F}}{\triangle A} = \boldsymbol{p}$$

其中$\triangle \boldsymbol{F}$是作用在$\triangle A$的内力，极限矢量$\boldsymbol{p}$就是该物体在截面mn上P点的应力，$\boldsymbol{p}$方向就是$\triangle \boldsymbol{F}$的极限方向.

性质：将截面mn上P点的应力$\boldsymbol{p}$沿截面的法向和切向方向分解即可得正应力$\boldsymbol{\sigma}$和切应力$\boldsymbol{\tau}$.

量纲：$\mathrm{L}^{-1}\mathrm{MT}^{-2}$.

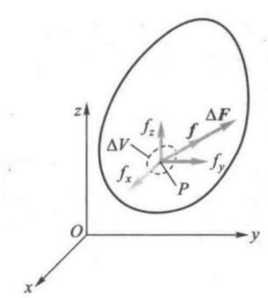

Fig 1. 体力示意图

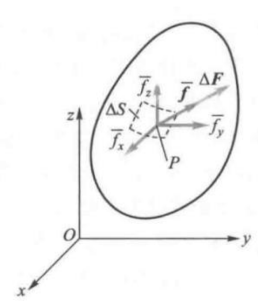

Fig 2. 面力示意图

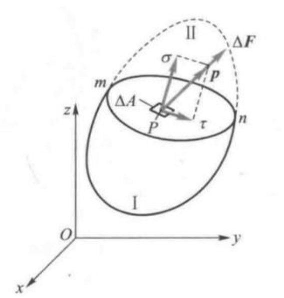

Fig 3. 应力示意图

5. 应变

定义：物体各部分线段的长度改变和两线段夹角的改变.
正应变：各线段的每单位长度的伸缩.
切应变：各线段之间直角的改变量.

6. 位移

定义：位置移动的量.

7. 基本假设

假设 1：连续性——假定物体是连续的，即整个物体的体积都被组成这个物质的介质所填满，不留下任何空隙.

假设 2：完全弹性——假定物体是完全弹性的，即物体在引起变形的外力被除去以后，能完全恢复原形而没有任何剩余变形.

假设 3：均匀性——假定物体是均匀的，即整个物体是由同一材料组成的.

假设 4：各向同性——假定物体是各向同性的，即物体的弹性在所有各个方向都相同.

假设 5：位移和应变是微小的——假定物体受力以后，整个物体所有各点的位移都远远小于物体原来的尺寸,而且应变和转角都远小于 1.

第二章

1. 平面应力问题

定义：只有平面应力分量σ_x、σ_y和τ_{xy}存在，且仅为x,y的函数的弹性力学问题.

2. 平面应变问题

定义：只有平面应变分量ε_x、ε_y和γ_{xy}存在，且仅为x,y的函数的弹性力学问题.

3. 平衡微分方程

平面问题中的平衡微分方程如下

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+f_x=0$$
$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+f_y=0$$

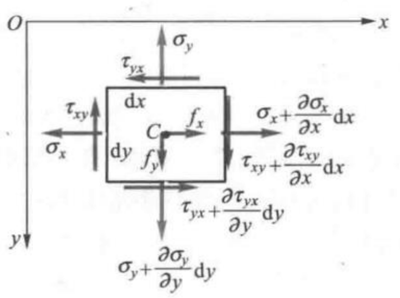

Fig 4. 正平行六面体微元示意图

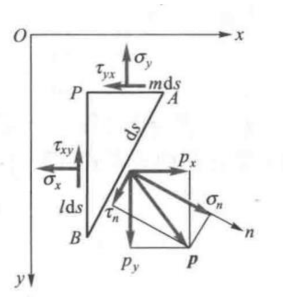

Fig 5. 一点的应力状态示意图

4. 一点的应力状态方程

$$p_x=l\sigma_x+m\tau_{xy} \qquad \sigma_n=lp_x+mp_y=l^2\sigma_x+m^2\sigma_y+2lm\tau_{xy}$$
$$p_y=m\sigma_y+l\tau_{xy} \qquad \tau_n=lp_y-mp_x=lm(\sigma_y-\sigma_x)+(l^2-m^2)\tau_{xy}$$

其中$l=\cos(n,x)$，$m=\cos(n,y)$.

推论：

证：在主应力平面上，由于切应力$\tau_n=0$，因此我们可以得到

$$p_x=l\sigma \qquad p_y=m\sigma$$

其中σ为主应力平面的主应力.带入平衡微分方程可得

$$l\sigma=l\sigma_x+m\tau_{xy} \qquad m\sigma=m\sigma_y+l\tau_{xy}$$

变形可得

$$\frac{m}{l}=\frac{\sigma-\sigma_x}{\tau_{xy}} \qquad \frac{m}{l}=\frac{\tau_{xy}}{\sigma-\sigma_y}$$

联立上式可得

$$\sigma^2-(\sigma_x+\sigma_y)\sigma+(\sigma_x\sigma_x-{\tau_{xy}}^2)=0$$

解得两个主应力分别为

$$\sigma_1=\frac{\sigma_x+\sigma_y}{2}+\frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2-4{\tau_{xy}}^2} \qquad \sigma_2=\frac{\sigma_x+\sigma_y}{2}-\frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2-4{\tau_{xy}}^2}$$

显然有

$$\sigma_1+\sigma_2=\sigma_x+\sigma_y$$

对于主应力的方向有

$$\tan\alpha_1=\frac{\sin\alpha_1}{\cos\alpha_1}=\frac{m_1}{l_1}=\frac{\sigma_1-\sigma_x}{\tau_{xy}} \qquad \tan\alpha_2=\frac{\sin\alpha_2}{\cos\alpha_2}=\frac{m_2}{l_2}=\frac{\tau_{xy}}{\sigma_2-\sigma_y}$$

其中α_1是σ_1与x轴的夹角，α_2是σ_2与x轴的夹角.显然有

$$\tan\alpha_1\cdot\tan\alpha_2=\frac{\sigma_1-\sigma_x}{\tau_{xy}}\cdot\frac{\tau_{xy}}{\sigma_2-\sigma_y}=\frac{\sigma_y-\sigma_2}{\tau_{xy}}\cdot\frac{\tau_{xy}}{\sigma_2-\sigma_y}=-1$$

即主应力σ_1的方向与σ_2的方向是互相垂直的.下求该点的最大与最小主应力.我们可以令

$$\sigma_x=\sigma_1\qquad\qquad \sigma_y=\sigma_2\qquad\qquad \tau_{xy}=0$$

则当$l\in[0,1]$时有

$$\sigma_n=l^2\sigma_1+m^2\sigma_2=l^2\sigma_1+(1-l^2)\sigma_2=l^2(\sigma_1-\sigma_2)+\sigma_2\in[\sigma_2,\sigma_1]$$

即两个主应力也就是最大的与最小的正应力.下求该点的最大与最小切应力.有

$$\tau_n=\pm lm(\sigma_2-\sigma_1)=\pm l\sqrt{1-l^2}(\sigma_2-\sigma_1)=\pm\sqrt{l^2-l^4}(\sigma_2-\sigma_1)=\pm\sqrt{\frac{1}{4}-\left(l^2-\frac{1}{2}\right)^2}(\sigma_2-\sigma_1)$$

显然可以看出当$l^2=\dfrac{1}{2}$时，τ_n为最大与最小切应力，分别为

$$\tau_1=\frac{\sigma_1-\sigma_2}{2}\qquad\qquad \tau_2=-\frac{\sigma_1-\sigma_2}{2}$$

即最大与最小切应力发生在与x轴及y轴(即主应力方向)成45°的斜面上.

5. 几何方程

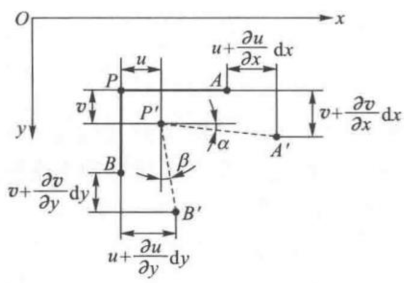

Fig 6. 刚体的位移示意图

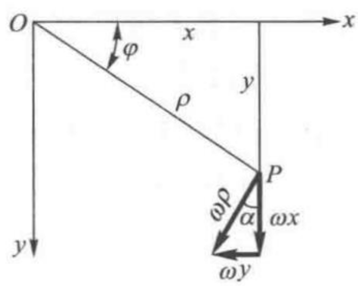

Fig 7. 位移分量示意图

5.1. 位移分量完全确定导出应变分量

$$\varepsilon_x=\frac{\partial u}{\partial x}\qquad\qquad \varepsilon_y=\frac{\partial v}{\partial y}\qquad\qquad \gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}$$

5.2. 应变分量完全确定导出位移分量

$$u=u_0-\omega y\qquad\qquad v=v_0+\omega x$$

证：不妨令$\varepsilon_x=\varepsilon_y=\gamma_{xy}=0$，带入应变分量表达式积分可得

$$u=f_1(y)\qquad\qquad v=f_2(x)\qquad\qquad f_2'(x)+f_1'(y)=0$$

显然$f_2'(x)$和$f_1'(y)$只能为常数，不妨设$f_1'(y)=-\omega$，$f_2'(x)=\omega$，积分可得

$$u = f_1(y) = u_0 - \omega y \qquad\qquad v = f_2(y) = v_0 + \omega x$$

其中u_0和v_0是物体沿x轴和y轴方向的刚体平移，ω是物体绕z轴的刚体转动.

6. 物理方程

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \mu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \mu(\sigma_x + \sigma_z)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \mu(\sigma_x + \sigma_y)\right]$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz}\,,\ \ \gamma_{zx} = \frac{1}{G}\tau_{zx}\,,\ \ \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$G = \frac{E}{2(1+\mu)}$$

其中μ为泊松比，G为切变模量，E为弹性模量(杨氏模量).由此可以导出

(1)平面应力问题的物理方程($\sigma_z = 0$，$\tau_{yz} = 0$，$\tau_{zx} = 0$)

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \mu\sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \mu\sigma_x)$$

$$\gamma_{xy} = \frac{2(1+\mu)}{E}\tau_{xy}$$

(2)平面应变问题的物理方程($\varepsilon_z = 0$，$\tau_{yz} = 0$，$\tau_{zx} = 0$)

$$\varepsilon_x = \frac{1-\mu^2}{E}\left(\sigma_x - \frac{\mu}{1-\mu}\sigma_y\right)$$

$$\varepsilon_y = \frac{1-\mu^2}{E}\left(\sigma_y - \frac{\mu}{1-\mu}\sigma_x\right)$$

$$\gamma_{xy} = \frac{2(1+\mu)}{E}\tau_{xy}$$

显然平面应力问题的物理方程通过$E = \dfrac{E}{1-\mu^2}$和$\mu = \dfrac{\mu}{1-\mu}$代换可以得到平面应变的物理方程.

7. 边界条件

定义：边界条件表示在边界上的位移与约束，或应力与面力之间的关系式.

(1)平面问题的位移边界条件

$$(u)_s = \bar{u}(s),\ (v)_s = \bar{v}(s)，在S_u上$$

其中$(u)_s$和$(v)_s$表示边界上的位移分量，$\bar{u}(s)$和$\bar{v}(s)$是在边界上已知的函数.

(2)平面问题的应力(面力)边界条件

$$\begin{cases}(l\sigma_x+m\tau_{xy})_s=\overline{f_x}(s)\\(m\sigma_y+l\tau_{xy})_s=\overline{f_y}(s)\end{cases}，在 S_\sigma 上$$

其中 $\overline{f_x}(s)$ 和 $\overline{f_y}(s)$ 是在边界 s 上的已知函数；l, m 是边界面外法线的方向余弦.

(3)平面问题的混合边界条件

8. 圣维南原理

定理：如果把物体的一小部分边界上的面力，变换为分布不同但静力等效的面力(主矢量相同，对于同一点的主矩也相同)，那么近处的应力分布将有显著的改变，但是远处所受的影响可以不计.

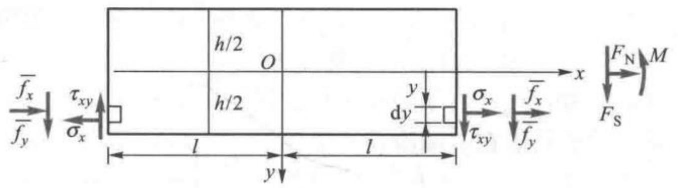

Fig 8. 部分小边界应力示意图

在 $x=l$ 的小边界上，积分边界条件可以写为

$$\begin{cases}\int_{-\frac{h}{2}}^{\frac{h}{2}}(\sigma_x)_{x=l}\mathrm{d}y=F_N\\\int_{-\frac{h}{2}}^{\frac{h}{2}}y(\sigma_x)_{x=l}\mathrm{d}y=M\\\int_{-\frac{h}{2}}^{\frac{h}{2}}(\tau_{xy})_{x=l}\mathrm{d}y=F_S\end{cases}.$$

9. 相容方程

变形协调方程或相容方程如下

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

(1)平面应力问题

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y)=-(1+\mu)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}\right)$$

(2)平面应变问题

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y)=-\frac{1}{1-\mu}\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}\right)$$

10. 应力函数

平衡微分方程如下

$$\sigma_x=\frac{\partial^2\Phi}{\partial y^2}-xf_x \qquad \sigma_y=\frac{\partial^2\Phi}{\partial x^2}-yf_y \qquad \tau_{xy}=-\frac{\partial^2\Phi}{\partial x\partial y}$$

其中Φ称为平面问题的应力函数，又称为艾里应力函数.

相容方程如下

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \Phi}{\partial y^2}-xf_x+\frac{\partial^2 \Phi}{\partial x^2}-yf_y\right)=0$$

当$f_x=f_y=0$时则为

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}\right)=0$$

$$\frac{\partial^4 \Phi}{\partial x^4}+2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2}+\frac{\partial^4 \Phi}{\partial y^4}=0$$

$$\nabla^4 \Phi=0$$

求解一个应力函数Φ，它必须满足在区域内的相容方程式，在边界上的应力边界条件式(假设全部都为应力边界条件)；在多连体中，还需满足位移单值条件.

第三章

1. 半逆解法

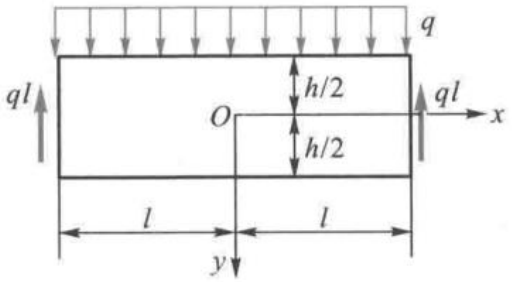

Fig 9. 简支梁受均布载荷

$$\sigma_x=f(y)$$

$$\Phi=\frac{1}{2}x^2 f(y)+xf_1(y)+f_2(y)$$

$$\begin{cases}\sigma_x=\dfrac{6q}{h^3}(l^2-x^2)y+q\dfrac{y}{h}\left(4\dfrac{y^2}{h^2}-\dfrac{3}{5}\right)\\ \sigma_y=-\dfrac{q}{2}\left(1+\dfrac{y}{h}\right)\left(1-\dfrac{2y}{h}\right)^2\\ \tau_{xy}=-\dfrac{6q}{h^3}x\left(\dfrac{h^2}{4}-y^2\right)\end{cases}$$

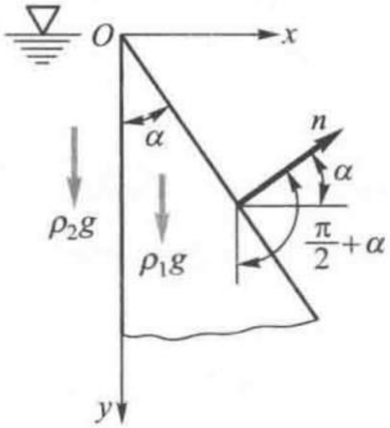

Fig 10. 楔形体受重力和液体压力

$$\Phi=ax^3+bx^2y+cxy^2+dy^3$$

$$\begin{cases}\sigma_x=-\rho_2 gy\\ \sigma_y=(\rho_1 g\cot\alpha-2\rho_2 g\cot^3\alpha)x\\ \qquad+(\rho_2 g\cot^2\alpha-\rho_1 g)y\\ \tau_{xy}=\tau_{yx}=-\rho_2 gx\cot^2\alpha\end{cases}$$

第四章

1. 极坐标下的平衡微分方程

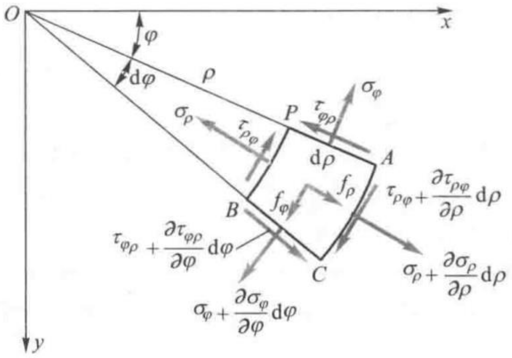

Fig 11. 极坐标微元示意图

极坐标中的平衡微分方程

$$\frac{\partial\sigma_\rho}{\partial\rho}+\frac{1}{\rho}\frac{\partial\tau_{\rho\varphi}}{\partial\varphi}+\frac{\sigma_\rho-\sigma_\varphi}{\rho}+f_\rho=0$$

$$\frac{1}{\rho}\frac{\partial\sigma_\varphi}{\partial\varphi}+\frac{\partial\tau_{\rho\varphi}}{\partial\rho}+\frac{2\tau_{\rho\varphi}}{\rho}+f_\varphi=0$$

2. 极坐标下的几何方程

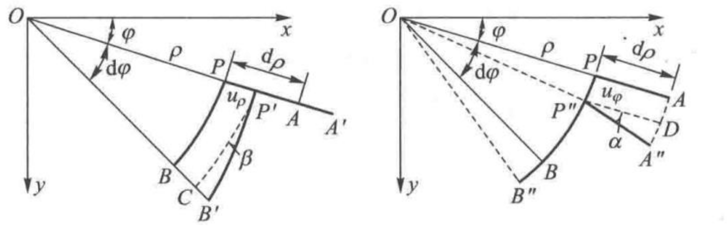

Fig 12. 极坐标微元示意图

极坐标中的几何方程

$$\varepsilon_\rho=\frac{\partial u_\rho}{\partial\rho}\qquad \varepsilon_\varphi=\frac{u_\rho}{\rho}+\frac{1}{\rho}\frac{\partial u_\varphi}{\partial\varphi}\qquad \gamma_{\rho\varphi}=\frac{1}{\rho}\frac{\partial u_\rho}{\partial\varphi}+\frac{\partial u_\varphi}{\partial\rho}-\frac{u_\varphi}{\rho}$$

3. 极坐标下的物理方程

(1)平面应力问题的物理方程($\sigma_z=0$，$\tau_{yz}=0$，$\tau_{zx}=0$)

$$\varepsilon_\rho=\frac{1}{E}(\sigma_\rho-\mu\sigma_\varphi)$$

$$\varepsilon_\varphi=\frac{1}{E}(\sigma_\varphi-\mu\sigma_\rho)$$

$$\gamma_{\rho\varphi}=\frac{2(1+\mu)}{E}\tau_{\rho\varphi}$$

(2)平面应变问题的物理方程($\varepsilon_z=0$，$\tau_{yz}=0$，$\tau_{zx}=0$)

$$\varepsilon_\rho = \frac{1-\mu^2}{E}\left(\sigma_\rho - \frac{\mu}{1-\mu}\sigma_\varphi\right)$$
$$\varepsilon_\varphi = \frac{1-\mu^2}{E}\left(\sigma_\varphi - \frac{\mu}{1-\mu}\sigma_\rho\right)$$
$$\gamma_{\rho\varphi} = \frac{2(1+\mu)}{E}\tau_{\rho\varphi}$$

显然平面应力问题的物理方程通过$E=\dfrac{E}{1-\mu^2}$和$\mu=\dfrac{\mu}{1-\mu}$代换可以得到平面应变的物理方程.

4. 极坐标下的相容方程

极坐标和直角坐标之间的关系式

$$\rho^2 = x^2 + y^2 \qquad \varphi = \arctan\frac{y}{x} \qquad x = \rho\cos\varphi \qquad y = \rho\sin\varphi$$

求导可得

$$\frac{\partial\rho}{\partial x} = \frac{x}{\rho} = \cos\varphi \qquad \frac{\partial\rho}{\partial y} = \frac{y}{\rho} = \sin\varphi$$
$$\frac{\partial\varphi}{\partial x} = -\frac{y}{\rho^2} = -\frac{\sin\varphi}{\rho} \qquad \frac{\partial\varphi}{\partial y} = \frac{x}{\rho^2} = \frac{\cos\varphi}{\rho}$$

当$f_x = f_y = 0$时，带入直角坐标的相容方程可得

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2}$$
$$\nabla^4\Phi = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right)^2\Phi = 0$$

5. 应力分量的坐标转换式

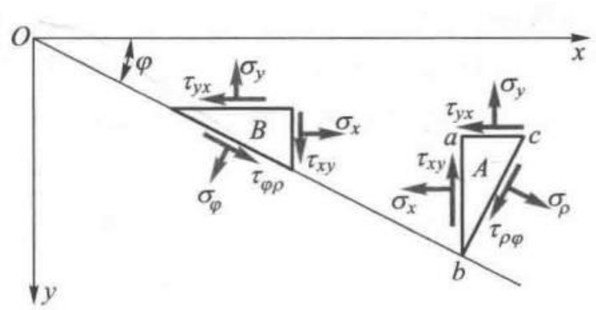

Fig 13. 三角板微元示意图

由平衡条件可以得到直角坐标向极坐标的变换式

$$\sigma_\rho = \sigma_x\cos^2\varphi + \sigma_y\sin^2\varphi + 2\tau_{xy}\sin\varphi\cos\varphi$$
$$\sigma_\varphi = \sigma_x\sin^2\varphi + \sigma_y\cos^2\varphi - 2\tau_{xy}\sin\varphi\cos\varphi$$
$$\tau_{\rho\varphi} = (\sigma_y - \sigma_x)\sin\varphi\cos\varphi + \tau_{xy}(\cos^2\varphi - \sin^2\varphi)$$

同时可以得到极坐标向直角坐标的变换式

$$\sigma_x = \sigma_\rho \cos^2\varphi + \sigma_\varphi \sin^2\varphi - 2\tau_{\rho\varphi}\sin\varphi\cos\varphi$$
$$\sigma_y = \sigma_\rho \sin^2\varphi + \sigma_\varphi \cos^2\varphi + 2\tau_{\rho\varphi}\sin\varphi\cos\varphi$$
$$\tau_{xy} = (\sigma_\rho - \sigma_\varphi)\sin\varphi\cos\varphi + \tau_{\rho\varphi}(\cos^2\varphi - \sin^2\varphi)$$

6. 轴对称应力及相应位移

在轴对称应力下，应力函数只是ρ的函数，因此有

$$\Phi = \Phi(\rho)$$

带入相容方程并求解可得

$$\Phi = A\ln\rho + B\rho^2\ln\rho + C\rho^2 + D$$

带入平衡方程求解可得轴对称应力的一般解答

$$\sigma_\rho = \frac{A}{\rho^2} + B(1 + 2\ln\rho) + 2C$$

$$\sigma_\varphi = -\frac{A}{\rho^2} + B(3 + 2\ln\rho) + 2C$$

$$\tau_{\rho\varphi} = \tau_{\varphi\rho} = 0$$

带入物理方程可得轴对称应力下位移分量的一般解答(平面应力情况)

$$u_\rho = \frac{1}{E}\left[-(1+\mu)\frac{A}{\rho} + 2(1-\mu)B\rho(\ln\rho - 1) + (1-3\mu)B\rho + 2(1-\mu)C\rho\right] + I\cos\varphi + K\sin\varphi$$

$$u_\varphi = \frac{4B\rho\varphi}{E} + H\rho - I\sin\varphi + K\cos\varphi$$

7. 圆环或圆筒受均布压力

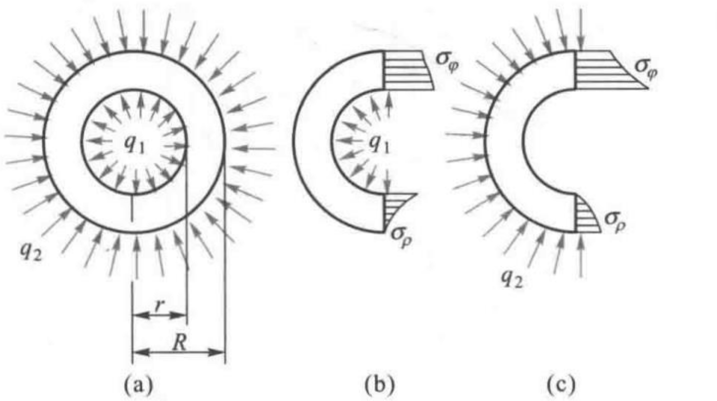

Fig 14. 圆环受均布压力示意图

有应力边界条件

$$(\tau_{\rho\varphi})_{\rho=r} = 0，\quad (\tau_{\rho\varphi})_{\rho=R} = 0$$
$$(\sigma_\rho)_{\rho=r} = -q_1，\quad (\sigma_\rho)_{\rho=R} = -q_2$$

带入轴对称应力的一般解答且令$B = 0$(排除多值函数)可得

$$\sigma_\rho = -\frac{\dfrac{R^2}{\rho^2}-1}{\dfrac{R^2}{r^2}-1}q_1 - \frac{1-\dfrac{r^2}{\rho^2}}{1-\dfrac{r^2}{R^2}}q_2$$

$$\sigma_\varphi = \frac{\dfrac{R^2}{\rho^2}+1}{\dfrac{R^2}{r^2}-1}q_1 - \frac{1+\dfrac{r^2}{\rho^2}}{1-\dfrac{r^2}{R^2}}q_2$$

显然σ_ρ总是压应力，σ_φ总是拉应力，应力分布如 Fig 14(b) 所示. 当圆环半径$R\to\infty$时可以得到具有圆孔无限大薄板(圆形孔道的无限大弹性体)解答

$$\sigma_\rho = -\frac{r^2}{\rho^2}q_1 \qquad \sigma_\varphi = \frac{r^2}{\rho^2}q_1$$

其大致应力分布 Fig 14(c) 所示.

8. 圆孔的孔口应力集中

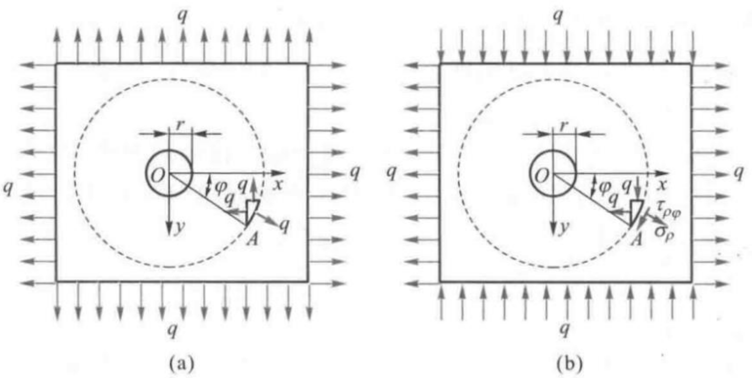

Fig 15. 矩形薄板圆孔示意图

对于 Fig15(a)有$\sigma_x = q$, $\sigma_y = q$, $\tau_{xy}=0 \Longleftrightarrow \sigma_\rho = q$, $\tau_{\rho\varphi}=0$，因此可得解答($R\to\infty$)

$$\sigma_\rho = q\left(1-\frac{r^2}{\rho^2}\right) \qquad \sigma_\varphi = q\left(1+\frac{r^2}{\rho^2}\right) \qquad \tau_{\rho\varphi}=\tau_{\varphi\rho}=0$$

对于 Fig15(b)有$\sigma_x = q$, $\sigma_y = -q$, $\tau_{xy}=0$，则在大圆弧处有

$$(\sigma_\rho)_{\rho=R} = q\cos 2\varphi \qquad (\tau_{\rho\varphi})_{\rho=R} = -q\sin 2\varphi$$

在孔边处有

$$(\sigma_\rho)_{\rho=r} = 0 \qquad (\tau_{\rho\varphi})_{\rho=r} = 0$$

因此假设应力函数$\Phi = f(\rho)\cos 2\varphi$，带入相容方程可得

$$\Phi = \cos 2\varphi\left(A\rho^4 + B\rho^2 + C + \frac{D}{\rho^2}\right)$$

带入边界条件可以求得应力分量为

$$\sigma_\rho = q\cos 2\varphi\left(1-\frac{r^2}{\rho^2}\right)\left(1-3\frac{r^2}{\rho^2}\right)$$

$$\sigma_\varphi = -q\cos 2\varphi\left(1+3\frac{r^4}{\rho^4}\right)$$

$$\tau_{\rho\varphi} = \tau_{\varphi\rho} = -q\sin 2\varphi\left(1-\frac{r^2}{\rho^2}\right)\left(1+3\frac{r^2}{\rho^2}\right)$$

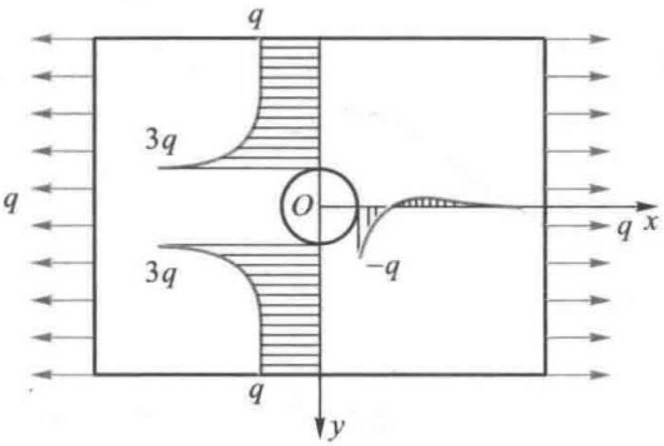

Fig 16. 矩形薄板圆孔只受拉力示意图

若矩形薄板圆孔只受拉力，则其应力分量为

$$\sigma_\rho = \frac{q}{2}\left(1-\frac{r^2}{\rho^2}\right) + \frac{q}{2}\cos 2\varphi\left(1-\frac{r^2}{\rho^2}\right)\left(1-3\frac{r^2}{\rho^2}\right)$$

$$\sigma_\varphi = \frac{q}{2}\left(1+\frac{r^2}{\rho^2}\right) - \frac{q}{2}\cos 2\varphi\left(1+3\frac{r^2}{\rho^2}\right)$$

$$\tau_{\rho\varphi} = \tau_{\varphi\rho} = -\frac{q}{2}\sin 2\varphi\left(1-\frac{r^2}{\rho^2}\right)\left(1+3\frac{r^2}{\rho^2}\right)$$

9. 半平面体在边界上受集中力

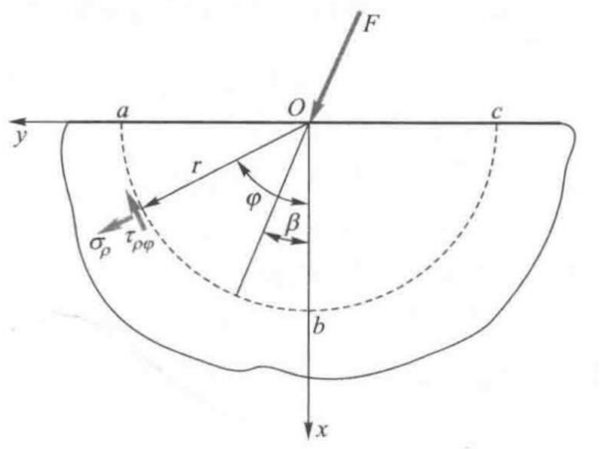

Fig 17. 半平面体在边界上受集中力示意图

设应力函数形式为$\Phi = \rho f(\varphi)$，带入相容方程可得

$$\Phi = A\rho\cos\varphi + B\rho\sin\varphi + \rho\varphi(C\cos\varphi + D\sin\varphi)$$

舍掉前面两项有

$$\Phi = \rho\varphi(C\cos\varphi + D\sin\varphi)$$

带入边界条件和平衡方程可以得到

$$\sigma_\rho = -\frac{2F}{\pi\rho}\cos(\beta-\varphi) \qquad \sigma_\varphi = 0 \qquad \tau_{\rho\varphi} = \tau_{\varphi\rho} = 0$$

当$\beta=0$时，F 垂直于直线边界，通过应力转换公式可得直角坐标中的应力分量

$$\sigma_x = -\frac{2F}{\pi}\frac{\cos^2\varphi}{\rho} \qquad \sigma_y = -\frac{2F}{\pi}\frac{\sin^2\varphi\cos\varphi}{\rho} \qquad \tau_{xy} = \tau_{yx} = -\frac{2F}{\pi}\frac{\sin\varphi\cos^2\varphi}{\rho}$$

$$\sigma_x = -\frac{2F}{\pi}\frac{x^3}{(x^2+y^2)^2} \qquad \sigma_y = -\frac{2F}{\pi}\frac{xy^2}{(x^2+y^2)^2} \qquad \tau_{xy} = \tau_{yx} = -\frac{2F}{\pi}\frac{x^2y}{(x^2+y^2)^2}$$

10. 半平面体在边界上受分布力

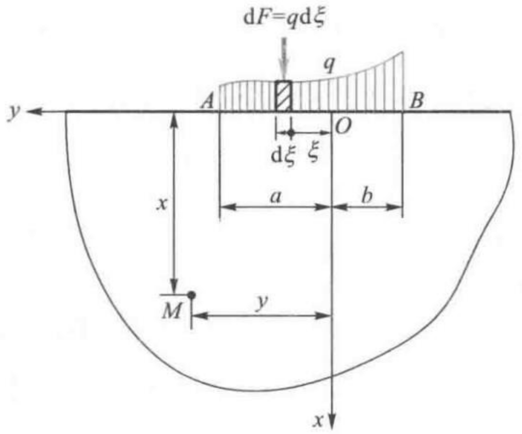

Fig 18. 半平面体在边界上受分布力示意图

微小集中力dF在 M 点引起的应力为

$$d\sigma_x = -\frac{2q\,d\xi}{\pi}\frac{x^3}{[x^2+(y-\xi)^2]^2}$$

$$d\sigma_y = -\frac{2q\,d\xi}{\pi}\frac{x(y-\xi)^2}{[x^2+(y-\xi)^2]^2}$$

$$d\tau_{xy} = -\frac{2q\,d\xi}{\pi}\frac{x^2(y-\xi)}{[x^2+(y-\xi)^2]^2}$$

积分可得

$$\sigma_x = -\frac{q}{\pi}\left[\arctan\frac{y+b}{x} - \arctan\frac{y-a}{x} + \frac{x(y+b)}{x^2+(y+b)^2} - \frac{x(y-a)}{x^2+(y-a)^2}\right]$$

$$\sigma_y = -\frac{q}{\pi}\left[\arctan\frac{y+b}{x} - \arctan\frac{y-a}{x} - \frac{x(y+b)}{x^2+(y+b)^2} + \frac{x(y-a)}{x^2+(y-a)^2}\right]$$

$$\tau_{xy} = \frac{q}{\pi}\left[\frac{x^2}{x^2+(y+b)^2} - \frac{x^2}{x^2+(y-a)^2}\right]$$

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历年考试真题

1. 平面应力的外力特征是只在板边上受有平行于板面并且不沿厚度变化的面力或约束，同时，体力也平行于板面并且不沿厚度变化.

2. 平面应变的外力特征是受有平行于横截面而且不沿长度变化的面力或约束，同时，体力也平行于横截面而且不沿长度变化.

3. 当体力为常量时，在单连体的应力边界问题中，如果两个弹性体具有相同的边界形伏，并受到同样分布的外力，那么它们在平面应力情况下和在平面应变情况下的应力分量的分布相同.

4. 设有周边为任意形状的薄板，其表面自由并与 OXY 坐标面平行，若己知各点的位移分量为$u=-p\dfrac{1-\mu}{E}x$，$v=-p\dfrac{1-\mu}{E}y$，则板内的应力分量为$\sigma_x=-p$，$\sigma_y=-p$，$\tau_{xy}=0$.

解：因为$u=-p\dfrac{1-\mu}{E}x$，$v=-p\dfrac{1-\mu}{E}y$，所以

$$\varepsilon_x=\frac{\partial u}{\partial x}=-p\frac{1-\mu}{E}\qquad \varepsilon_y=\frac{\partial v}{\partial y}=-p\frac{1-\mu}{E}\qquad \gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=0$$

带入应力分量表达式

$$\begin{aligned}\sigma_x&=\frac{E}{1-\mu^2}(\varepsilon_x+\mu\varepsilon_y)\\&=-\frac{E(1+\mu)}{1-\mu^2}\cdot p\frac{1-\mu}{E}\\&=-p\end{aligned}\qquad \begin{aligned}\sigma_y&=\frac{E}{1-\mu^2}(\varepsilon_y+\mu\varepsilon_x)\\&=-\frac{E(1+\mu)}{1-\mu^2}\cdot p\frac{1-\mu}{E}\\&=-p\end{aligned}\qquad \tau_{xy}=\frac{E}{2(1+\mu)}\gamma_{xy}=0$$

5. 按应力求解空间问题和平面问题时，应力分量各自需要满足 6 和 1 个相容方程.

6. 平衡微分方程和应力边界条件是静力平衡条件.

7. 用应变分量表示的相容方程等价于 B.

A. 平衡微分方程

B. 几何方程

C. 物理方程

D. 几何方程和物理方程

8. 下列关于应力函数的说法，正确的是 C.

A. 应力函数与弹性体的边界条件性质相关，故应用应力函数，自然满足边界条件

B. 多项式函数自然可以作为平面问题的应力函效

C. 一次多项式应力函不产生应力，因此可以不计

D. 相同边界条件和作用载荷的平面应力和平面应变向题的应力函效不同

9. 圆筒仅受均布外压力作用时，环向最大压应力出现在内周边处.

10. 应力集中现象的特点为：(1)集中性；(2)局部性.

11. 在空间问题中导出平衡微分方程时，主要应用了连续性、小变形和均匀性假定.

12. 简要说明极坐标系下推导基本方程时所取微元体的主要特点.

解：对于圆形、楔形、扇形等物体.

13. 简述按位移求解空间问题的几种位移函数是什么性质函数及各自主要用来解决什么样的弹性力学问题.

解：位移势函数：用于无旋场；勒夫位移函数：用于轴对称问题；伽辽金位移函数：用于一般问题.

14. 为什么在轴对称应力下，得出的位移是非轴对称的？如何从数学推导和物理概念上解释这种现象？

解：因为存在刚体位移.

15. 简述虚功方程和最小势能原理.

解：虚功方程：外力在虚位移上所做的虚功等于应力在虚应变上所作的虚功.

最小势能原理：在给定的外力作用下，在满足位移边界条件的所有各组位移中，实际存在的一组位移应使总势能成为极值.

16. 在什么条件下平面应力问题和平面应变问题的 3 个应力分量与材料特性无关?

解：常体力情况下，在单连体的应力边界问题中，如果两个弹性体具有相同的边界形状并受到同样分布的外力，即使这两个弹性体的材料不相同，平面应力问题和平面应变问题的应力分量的分布也是相同的.

17. 弹性力学研究物体在外力作用下，处于弹性阶段的应力、应变和位移.

18. 在弹性力学里分析问题，要考虑静力学、几何学、和物理学三方面条件，分别建立三面方程.

19. 下列关于平面问题所受外力特点描述错误的是 D.

A. 体力分量与 Z 坐标无关

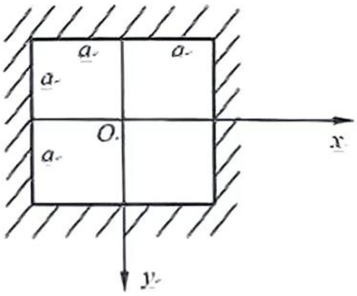
B. 面力分量与 Z 坐标无关

C. 体力、面力均等于 0

D. 体力、面力均为非 0 常数

20. 根据圣维南原理，作用在物体一小部分边界上的力可以用下列静力上等效的力代替，则仅在近处应力分布有改变，而在远处所受的影响可以忽略不计.

21. 图示正方形薄板的边长为$2a$，四边固定，只受重力的作用，不计体力. 设$\mu=0$，试取位移分量的表达式$u=A_1\left(1-\dfrac{x^2}{a^2}\right)\left(1-\dfrac{y^2}{a^2}\right)\dfrac{xy}{a^2}$，$v=B_1\left(1-\dfrac{x^2}{a^2}\right)\left(1-\dfrac{y^2}{a^2}\right)$，用里茨法求薄板的位移和应力.



解：位移分量为

$$\begin{cases} u=A_1\left(1-\dfrac{x^2}{a^2}\right)\left(1-\dfrac{y^2}{a^2}\right)\dfrac{xy}{a^2} \\ v=B_1\left(1-\dfrac{x^2}{a^2}\right)\left(1-\dfrac{y^2}{a^2}\right) \end{cases}$$

那么有

$$\frac{\partial u}{\partial x}=\frac{A_1 y}{a^2}\left(1-\frac{y^2}{a^2}\right)\left(1-\frac{3x^2}{a^2}\right) \qquad \frac{\partial u}{\partial y}=\frac{A_1 x}{a^2}\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{3y^2}{a^2}\right)$$

$$\frac{\partial v}{\partial x}=-\frac{2B_1 x}{a^2}\left(1-\frac{y^2}{a^2}\right) \qquad \frac{\partial v}{\partial y}=-\frac{2B_1 y}{a^2}\left(1-\frac{x^2}{a^2}\right)$$

带入应变能密度表达式($\mu=0$)

$$\begin{aligned} U_1&=\frac{E}{2(1-\mu^2)}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\mu\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{1-\mu}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right] \\ &=\frac{E}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right] \\ &=\frac{E}{2}\left\{\frac{A_1^2 y^2}{a^4}\left(1-\frac{y^2}{a^2}\right)^2\left(1-\frac{3x^2}{a^2}\right)^2+\frac{4B_1^2 y^2}{a^4}\left(1-\frac{x^2}{a^2}\right)^2+\frac{1}{2}\left[\frac{A_1 x}{a^2}\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{3y^2}{a^2}\right)-\frac{2B_1 x}{a^2}\left(1-\frac{y^2}{a^2}\right)\right]^2\right\} \end{aligned}$$

积分可得

$$U=4\int_0^a\int_0^a U_1\,\mathrm{d}x\,\mathrm{d}y$$

则有

$$\begin{cases} \dfrac{\partial U}{\partial A_1}=0 \\ \dfrac{\partial U}{\partial B_1}=\displaystyle\iint_A f_y v_1\,\mathrm{d}x\,\mathrm{d}y=\iint_A \rho g\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{a^2}\right)\mathrm{d}x\,\mathrm{d}y \end{cases}$$

解得

$$A_1 = \frac{175}{1066}\frac{\rho g a^2}{E}，B_1 = \frac{225}{533}\frac{\rho g a^2}{E}$$

则位移分量为

$$\begin{cases} u = \dfrac{175}{1066}\dfrac{\rho g a^2}{E}\left(1-\dfrac{x^2}{a^2}\right)\left(1-\dfrac{y^2}{a^2}\right)\dfrac{xy}{a^2} \\ v = \dfrac{225}{533}\dfrac{\rho g a^2}{E}\left(1-\dfrac{x^2}{a^2}\right)\left(1-\dfrac{y^2}{a^2}\right) \end{cases}$$

应力分量为

$$\begin{cases} \sigma_x = \dfrac{175}{1066}\left(1-\dfrac{3x^2}{a^2}\right)\left(1-\dfrac{y^2}{a^2}\right)\rho g y \\ \sigma_y = -\dfrac{450}{533}\left(1-\dfrac{x^2}{a^2}\right)\rho g y \\ \tau_{xy} = \left[-\dfrac{225}{533}\left(1-\dfrac{y^2}{a^2}\right)+\dfrac{175}{2132}\left(1-\dfrac{x^2}{a^2}\right)\left(1-\dfrac{3y^2}{a^2}\right)\right]\rho g x \end{cases}$$

22. 体力为零的单连体应力边界问题，设$\sigma_x = qy^2$，$\sigma_y = qx^2$，$\tau_{xy} = 0$，已满足边界条件，试考察它们是否为正确解答，并说明原因

解：带入相容方程

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(qx^2+qy^2) = 4q \neq 0$$

不满足相容方程，不是正确解答.

23. 试验证下列应变状态是否满足相容方程，若满足，试确定各系数与物体体力之间的关系.

$$\varepsilon_x = Axy，\varepsilon_y = By^3，\gamma_{xy} = C - Dy^2，\varepsilon_z = \gamma_{yz} = \gamma_{xz} = 0$$

解：带入相容方程

$$\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2} = 0 = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

显然满足. 带入平面应变问题求出应力分量

$$\sigma_x = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}\left(\varepsilon_x+\frac{\mu}{1-\mu}\varepsilon_y\right) = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}\left(Axy+\frac{\mu}{1-\mu}By^3\right)$$

$$\sigma_y = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}\left(\varepsilon_y+\frac{\mu}{1-\mu}\varepsilon_x\right) = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}\left(By^3+\frac{\mu}{1-\mu}Axy\right)$$

$$\tau_{xy} = \frac{E}{2(1+\mu)}\gamma_{xy} = \frac{E}{2(1+\mu)}(C-Dy^2)$$

带入平衡微分方程可得

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+f_x = A\frac{E(1-\mu)y}{(1+\mu)(1-2\mu)} - D\frac{Ey}{(1+\mu)}+f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+f_y = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}\left(3By^2+\frac{\mu}{1-\mu}Ax\right)+f_y = 0$$

即得

$$f_x = \frac{Ey}{(1+\mu)}\left[D - A\frac{1-\mu}{(1-2\mu)}\right]$$

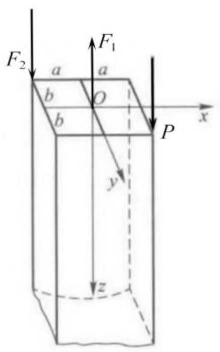
$$f_y = -\frac{E(1-\mu)}{(1+\mu)(1-2\mu)}\left(3By^2 + \frac{\mu}{1-\mu}Ax\right)$$

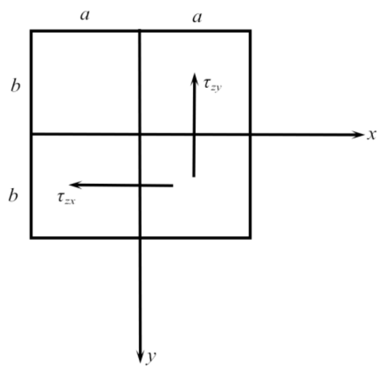
若体力为 0，则有

$$D - A\frac{1-\mu}{(1-2\mu)} = 0，3By^2 + \frac{\mu}{1-\mu}Ax = 0$$

对任意的x，y都成立，即$A = B = D = 0$，C任意.

24. 图示的弹性体为一长柱形体，试写出$Z = 0$表面上的边界条件.





解：$Z = 0$平面的法向量的投影分量$\boldsymbol{n} = (l, m, n) = (0, 0, -1)$，所以有

$$p_x = l\sigma_x + m\tau_{yx} + n\tau_{zx} = -\tau_{zx}$$

$$p_y = m\sigma_y + n\tau_{zy} + l\tau_{xy} = -\tau_{zy}$$

$$p_z = n\sigma_z + l\tau_{xz} + m\tau_{yz} = -\sigma_z$$

则有

$$\sum F_x = 0：\int_{-a}^{a}\int_{-b}^{b}(\tau_{zx})_{z=0}\mathrm{d}x\,\mathrm{d}y = 0$$

$$\sum F_y = 0：\int_{-a}^{a}\int_{-b}^{b}(\tau_{zy})_{z=0}\mathrm{d}x\,\mathrm{d}y = 0$$

$$\sum F_z = 0：\int_{-a}^{a}\int_{-b}^{b}(\sigma_z)_{z=0}\mathrm{d}x\,\mathrm{d}y = F_1 - P - F_2$$

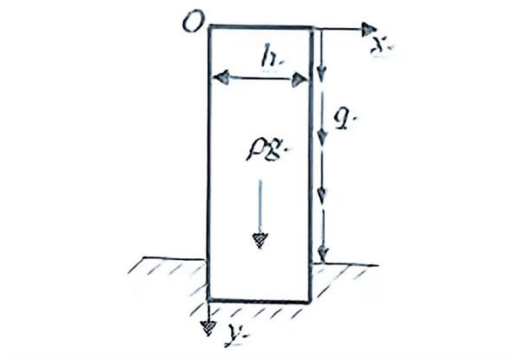
还有

$$\sum M_x = 0:\ \int_{-a}^{a}\int_{-b}^{b} y(\sigma_z)_{z=0}\mathrm{d}x\,\mathrm{d}y = (F_2 - P)b$$

$$\sum M_y = 0:\ \int_{-a}^{a}\int_{-b}^{b} x(\sigma_z)_{z=0}\mathrm{d}x\,\mathrm{d}y = (F_2 - P)a$$

$$\sum M_z = 0:\ \int_{-a}^{a}\int_{-b}^{b} [x(\tau_{zy})_{z=0} - y(\tau_{zx})_{z=0}]\mathrm{d}x\,\mathrm{d}y = 0$$

25. 设有矩形截面竖柱，其密度为ρ，在一侧面上受均布剪力q，试求应力分量.



解：在主要边界上$(x=0，x=h)$：

$$(\sigma_x)_{x=0} = 0,\ (\tau_{xy})_{x=0} = 0$$

$$(\sigma_x)_{h=0} = 0,\ (\tau_{xy})_{x=h} = q$$

在次要边界上$(y=0)$

$$\int_0^h (\sigma_y)_{y=0}\mathrm{d}x = 0,\ \int_0^h (\tau_{xy})_{y=0}\mathrm{d}x = 0,\ \int_0^h x(\sigma_y)_{y=0}\mathrm{d}x = 0$$

设应力函数形式为$\Phi = f_1(x)y + f_2(x)$，带入相容方程则有

$$f_1^{(4)}(x) = 0,\ f_2^{(4)}(x) = 0$$

则可设

$$f_1(x) = Ax^3 + Bx^2 + Cx,\ f_2(x) = Ex^3 + Fx^2$$

则有应力函数

$$\Phi = (Ax^3 + Bx^2 + Cx)y + Ex^3 + Fx^2$$

则应力分量为

$$\sigma_x = 0 \qquad \sigma_y = (6Ax + 2Bx - \rho g)y + 6Ex + 2F \qquad \tau_{xy} = -(3Ax^2 + 2Bx + C)$$

带入边界方程有

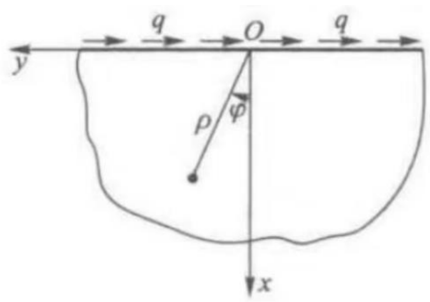
$$(\tau_{xy})_{x=0} = -C = 0\,,\ (\tau_{xy})_{x=h} = -(3Ah^2 + 2Bh + C) = q$$

$$\int_0^h (6Ex + 2F)\mathrm{d}x = 0\,,\ -\int_0^h (3Ax^2 + 2Bx + C)\mathrm{d}x = 0\,,\ \int_0^h x(6Ex + 2F)\mathrm{d}x = 0$$

解得$A = -\dfrac{q}{h^2}$，$B = \dfrac{q}{h}$，$C = 0$，$E = F = 0$，则有

$$\sigma_x = 0 \qquad \sigma_y = \left(-\frac{6q}{h^2}x + \frac{2q}{h}x - \rho g\right)y \qquad \tau_{xy} = \frac{3q}{h^2}x^2 - \frac{2q}{h}x$$

26. 如图所示半平面体表面受有均布水平力q，试用应力函数$\Phi(\rho,\varphi) = \rho^2(2B\sin 2\varphi + C\varphi)$求解应力分量.



解：(1)带入相容方程检验应力函数

$$\nabla^2\Phi = 2(2B\sin 2\varphi + C\varphi) + 2(2B\sin 2\varphi + C\varphi) - 8B\sin 2\varphi = 4C\varphi\,,\ \nabla^4\Phi = 0$$

满足.

(2)求解应力分量

$$\sigma_\rho = \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} = -4B\sin 2\varphi + 2C\varphi \qquad \sigma_\varphi = \frac{\partial^2\Phi}{\partial\rho^2} = 4B\sin 2\varphi + 2C\varphi \qquad \tau_{\rho\varphi} = -\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi}\right) = -4B\cos 2\varphi - C$$

(3)写出边界条件

$$(\tau_{\rho\varphi})_{\varphi=\pm\frac{\pi}{2}} = -q\,,\ (\sigma_\varphi)_{\varphi=\pm\frac{\pi}{2}} = 0$$

(4)带入求解方程

$$(\tau_{\rho\varphi})_{\varphi=\pm\frac{\pi}{2}} = 4B - C = -q\,,\ (\sigma_\varphi)_{\varphi=\pm\frac{\pi}{2}} = \pm\pi C = 0$$

解得$B = -\dfrac{q}{4}$，$C = 0$.

(5)写出应力分量

$$\sigma_\rho = q\sin 2\varphi \qquad \sigma_\varphi = -q\sin 2\varphi \qquad \tau_{\rho\varphi} = q\cos 2\varphi$$

27. 体力为零的单连体应力边界问题，设$\sigma_x = q\dfrac{x}{a}$，$\sigma_y = q\dfrac{y}{b}$，$\tau_{xy} = -q\left(\dfrac{x}{a}+\dfrac{y}{b}\right)$，已满足边界条件，试考察它们是否为正确解答，并说明原因.

解：(1)平衡微分方程：

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=\frac{q}{a}-\frac{q}{a}=0,\ \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=-\frac{q}{a}+\frac{q}{a}=0$$

满足.

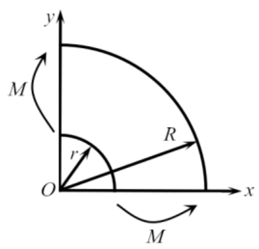
(2)相容方程：

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y)=0$$

满足.

(3)位移单值条件：满足.

28. 平面曲杆受力如图，按极坐标写出应力分量应该满足的边界条件.



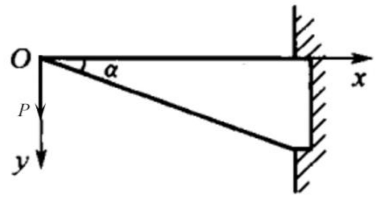
解：设极轴与x正方向的夹角为φ，则

$$(\sigma_\rho)_{\rho=r}=0,\ (\tau_{\rho\varphi})_{\rho=r}=0,\ (\sigma_\rho)_{\rho=R}=0,\ (\tau_{\rho\varphi})_{\rho=R}=0$$

$$(\sigma_\varphi)_{\varphi=0}=0,\ (\tau_{\rho\varphi})_{\varphi=0}=0,\ \int_r^R \rho(\sigma_\varphi)_{\varphi=0}\mathrm{d}\rho=-M$$

$$(\sigma_\varphi)_{\varphi=\frac{\pi}{2}}=0,\ (\tau_{\rho\varphi})_{\varphi=\frac{\pi}{2}}=0,\ \int_r^R \rho(\sigma_\varphi)_{\varphi=\frac{\pi}{2}}\mathrm{d}\rho=-M$$

29. 楔形体在楔顶受集中力作用(如图所示)，按极坐标写出楔形体侧面的边界条件和楔形体端应满足的条件.



解：侧面对应的法向量为$\boldsymbol{n}=(l,m)=\left(\cos\left(\dfrac{\pi}{2}+\alpha\right),\cos\alpha\right)=(-\sin\alpha,\cos\alpha)$，则

在下边界：

$$(\sigma_\varphi)_{\varphi=\alpha}=0\,,\ (\tau_{\rho\varphi})_{\varphi=\alpha}=0$$

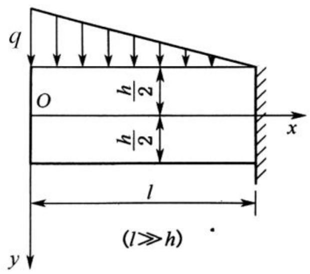
在上边界：

$$(\sigma_\varphi)_{\varphi=0}=0\,,\ (\tau_{\rho\varphi})_{\varphi=0}=0$$

考虑静力平衡条件：

$$\int_0^\alpha \sigma_\rho\cos\varphi\rho\,\mathrm{d}\varphi=0\,,\ \int_0^\alpha \sigma_\rho\sin\varphi\rho\,\mathrm{d}\varphi=-P$$

30. 图示悬臂梁受线性分布载荷作用，最大集度为q，体力不记，$l\gg h$，试求解应力分量.



解：写出边界条件

主要边界：

$$(\sigma_y)_{y=\frac{h}{2}}=0\,,\ (\tau_{xy})_{y=\frac{h}{2}}=0\,,\ (\sigma_y)_{y=-\frac{h}{2}}=-q\left(1-\frac{x}{l}\right),\ (\tau_{xy})_{y=\frac{h}{2}}=0$$

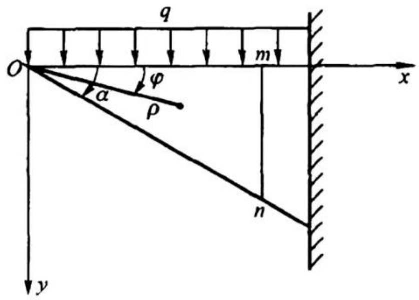
次要边界：

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}(\sigma_x)_{x=0}\mathrm{d}y=0\,,\ \int_{-\frac{h}{2}}^{\frac{h}{2}}(\tau_{xy})_{x=0}\mathrm{d}y=0\,,\ \int_{-\frac{h}{2}}^{\frac{h}{2}}y(\sigma_x)_{x=0}\mathrm{d}y=0$$

31. 图示三角形悬臂梁，在上边界$y=0$受到均布压力q的作用，试用下列应力函数

$$\Phi=C[\rho^2(\alpha-\varphi)+\rho^2\sin\varphi\cos\varphi-\rho^2\cos^2\varphi\tan\alpha]$$

求解其应力分量



解：(1)验证相容方程

$$\frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \varphi^2}=C[-2\sin 2\varphi+2\cos 2\varphi\tan\alpha]$$

$$\frac{\partial^2 \Phi}{\partial \rho^2}=2C\left[(\alpha-\varphi)+\frac{1}{2}\sin 2\varphi-\cos^2\varphi\tan\alpha\right]$$

$$\frac{1}{\rho}\frac{\partial \Phi}{\partial \rho}=2C\left[(\alpha-\varphi)+\frac{1}{2}\sin 2\varphi-\cos^2\varphi\tan\alpha\right]$$

带入可得

$$\nabla^2\Phi=4C\left[(\alpha-\varphi)+\frac{1}{2}\sin 2\varphi-\cos^2\varphi\tan\alpha\right]+C[-2\sin 2\varphi+2\cos 2\varphi\tan\alpha]$$

$$\nabla^4\Phi=\frac{4C[-2\sin 2\varphi+2\cos 2\varphi\tan\alpha]+C[8\sin 2\varphi-8\cos 2\varphi\tan\alpha]}{\rho^2}=0$$

满足.

(2)写出应力分量

$$\sigma_\rho=\frac{1}{\rho}\frac{\partial \Phi}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \varphi^2}=C[2(\alpha-\varphi)-\sin 2\varphi-2\cos^2\varphi\tan\alpha+2\cos 2\varphi\tan\alpha]$$

$$\sigma_\varphi=\frac{\partial^2 \Phi}{\partial \rho^2}=2C\left[(\alpha-\varphi)+\frac{1}{2}\sin 2\varphi-\cos^2\varphi\tan\alpha\right]$$

$$\tau_{\rho\varphi}=-\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \Phi}{\partial \varphi}\right)=-C[-1+\cos 2\varphi+\sin 2\varphi\tan\alpha]$$

(3)带入边界条件

$$(\sigma_\varphi)_{\varphi=0}=2C(\alpha-\tan\alpha)=-q$$

解得

$$C=\frac{q}{2(\tan\alpha-\alpha)}$$

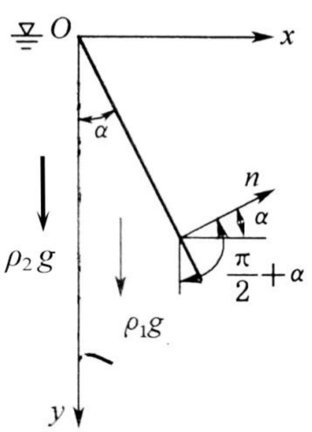
(4)求解应力分量

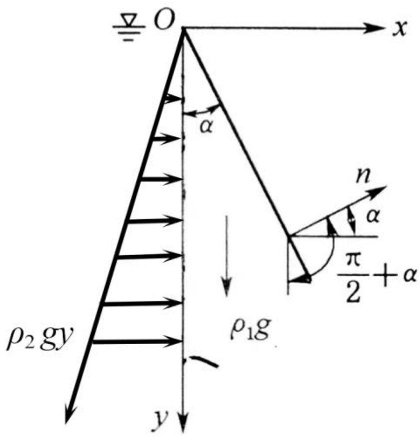
$$\sigma_\rho=\frac{1}{\rho}\frac{\partial \Phi}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \varphi^2}=\frac{q}{\tan\alpha-\alpha}(\alpha-\varphi-\sin\varphi\cos\varphi-\sin^2\varphi\tan\alpha)$$

$$\sigma_\varphi=\frac{\partial^2 \Phi}{\partial \rho^2}=\frac{q}{\tan\alpha-\alpha}(\alpha-\varphi+\sin\varphi\cos\varphi-\cos^2\varphi\tan\alpha)$$

$$\tau_{\rho\varphi}=-\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \Phi}{\partial \varphi}\right)=\frac{q}{\tan\alpha-\alpha}(\sin^2\varphi-\sin\varphi\cos\varphi\tan\alpha)$$

31. 设有楔形体，其形状及参数如图，求其应力分量.





解：(1)其等价于右图，则其边界条件为：

左侧$l=-1$，$m=0$：

$$(\sigma_x)_{x=0}=-\rho_2 gy,\ (\gamma_{xy})_{=0}=0$$

右侧$l=\cos\alpha$，$m=\cos\left(\frac{\pi}{2}+\alpha\right)=-\sin\alpha$：

$$(l\sigma_x+m\tau_{xy})_{x=y\tan\alpha}=0,\ (l\tau_{xy}+m\sigma_y)_{x=y\tan\alpha}=0$$

(2)设其应力函数$\Phi=ax^3+bx^2y+cxy^2+dy^3$，则

(3)写出应力分量：

$$\sigma_x=2cx+6dy,\ \sigma_y=2by+6ax-\rho_1 gy,\ \tau_{xy}=-2bx-2cy$$

(4)带入边界条件求解：

$$(\sigma_x)_{x=0}=6dy=-\rho_2 gy,\ (\gamma_{xy})_{=0}=-2cy=0$$

解得$c=0$，$d=-\frac{1}{6}\rho_2 g$；

$$-\frac{1}{2}\rho_2 g\cos\alpha+b\tan\alpha\sin\alpha=0,\ -4b-6a\tan\alpha+\rho_1 g=0$$

解得$b=\frac{1}{2}\rho_2 g\cot^2\alpha$，$a=-\frac{1}{3}\rho_2 g\cot^3\alpha+\frac{1}{6}\rho_1 g\cot\alpha$.

(5)写出应力分量：

$$\sigma_x=-\rho_2 gy,\ \sigma_y=(\rho_2 g\cot^2\alpha-\rho_1 g)y+(\rho_1 g\cot\alpha-2\rho_2 g\cot^3\alpha)x,\ \tau_{xy}=-\rho_2 gx\cot^2\alpha$$

考前必背公式

1. 极坐标中的基本方程和直角坐标中的基本方程

基本方程	极坐标系	直角坐标系
平衡微分方程	$\frac{\partial \sigma_\rho}{\partial \rho}+\frac{1}{\rho}\frac{\partial \tau_{\rho\varphi}}{\partial \varphi}+\frac{\sigma_\rho-\sigma_\varphi}{\rho}+f_\rho=0$ $\frac{1}{\rho}\frac{\partial \sigma_\varphi}{\partial \varphi}+\frac{\partial \tau_{\rho\varphi}}{\partial \rho}+\frac{2\tau_{\rho\varphi}}{\rho}+f_\varphi=0$	$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+f_x=0$ $\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+f_y=0$
几何方程	$\varepsilon_\rho=\frac{\partial u_\rho}{\partial \rho}$ $\varepsilon_\varphi=\frac{1}{\rho}\frac{\partial u_\varphi}{\partial \varphi}+\frac{u_\rho}{\rho}$ $\gamma_{\rho\varphi}=\frac{1}{\rho}\frac{\partial u_\rho}{\partial \varphi}+\frac{\partial u_\varphi}{\partial \rho}-\frac{u_\rho}{\rho}$	$\varepsilon_x=\frac{\partial u}{\partial x}$ $\varepsilon_y=\frac{\partial v}{\partial y}$ $\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$
物理方程 (平面应力问题)	$\varepsilon_\rho=\frac{1}{E}(\sigma_\rho-\mu\sigma_\varphi)$ $\varepsilon_\varphi=\frac{1}{E}(\sigma_\varphi-\mu\sigma_\rho)$ $\gamma_{\rho\varphi}=\frac{1}{G}\tau_{\rho\varphi}=\frac{2(1+\mu)}{E}\tau_{\rho\varphi}$	$\varepsilon_x=\frac{1}{E}(\sigma_x-\mu\sigma_y)$ $\varepsilon_y=\frac{1}{E}(\sigma_y-\mu\sigma_x)$ $\gamma_{xy}=\frac{1}{G}\tau_{xy}=\frac{2(1+\mu)}{E}\tau_{xy}$
物理方程 (平面应力问题)	$\sigma_\rho=\frac{E}{1-\mu^2}(\varepsilon_\rho+\mu\varepsilon_\varphi)$ $\sigma_\varphi=\frac{E}{1-\mu^2}(\varepsilon_\varphi+\mu\varepsilon_\rho)$ $\tau_{\rho\varphi}=G\gamma_{\rho\varphi}=\frac{E}{2(1+\mu)}\gamma_{\rho\varphi}$	$\sigma_x=\frac{E}{1-\mu^2}(\varepsilon_x+\mu\varepsilon_y)$ $\sigma_y=\frac{E}{1-\mu^2}(\varepsilon_y+\mu\varepsilon_x)$ $\tau_{xy}=G\gamma_{xy}=\frac{E}{2(1+\mu)}\gamma_{xy}$
	平面应变问题$E\longrightarrow\frac{E}{1-\mu^2}$，$\mu\longrightarrow\frac{\mu}{1-\mu}$	
相容方程	$\left(\frac{\partial^2}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}\right)^2\Phi=0$	$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\Phi=0$
应力函数	$\sigma_\rho=\frac{1}{\rho}\frac{\partial \Phi}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \varphi^2}$ $\sigma_\varphi=\frac{\partial^2 \Phi}{\partial \rho^2}$ $\tau_{\rho\varphi}=-\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \Phi}{\partial \varphi}\right)$	$\sigma_x=\frac{\partial^2 \Phi}{\partial y^2}-xf_x$ $\sigma_y=\frac{\partial^2 \Phi}{\partial x^2}-yf_y$ $\tau_{xy}=-\frac{\partial^2 \Phi}{\partial x\partial y}$

2. 直角坐标边界条件

$$(l\sigma_x+m\tau_{xy})_{S_\sigma}=\left(\overline{f}_x\right)_{S_\sigma}$$

$$(l\tau_{xy}+m\sigma_y)_{S_\sigma}=\left(\overline{f}_x\right)_{S_\sigma}$$

$$l=\cos(\boldsymbol{n},\boldsymbol{x}),\ m=\cos(\boldsymbol{n},\boldsymbol{y})$$

3. 应力状态方程

$$\begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \begin{bmatrix} l \\ m \end{bmatrix},\ \begin{bmatrix} \sigma_n \\ \tau_n \end{bmatrix} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \begin{bmatrix} l \\ m \end{bmatrix}$$

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} l & m & n \\ m & n & l \\ n & l & m \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$\sigma_N = \begin{bmatrix} l \\ m \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \begin{bmatrix} l \\ m \end{bmatrix},\ \tau_N = \sqrt{p_N^2 - \sigma_N^2}$$

$$\sigma_N = \begin{bmatrix} l \\ m \\ n \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix},\ \tau_N = \sqrt{p_N^2 - \sigma_N^2}$$

4. 平面问题主应力

$$\left.\begin{matrix} \sigma_1 \\ \sigma_2 \end{matrix}\right\} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2},\ \tan\alpha_1 = \frac{\sigma_1 - \sigma_x}{\tau_{xy}}$$

5. 直角坐标变形协调方程(相容方程，平面应力问题)

$$\frac{\partial^2 \varepsilon_x}{\partial^2 y} + \frac{\partial^2 \varepsilon_y}{\partial^2 x} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right)(\sigma_x + \sigma_y) = -(1+\mu)\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)$$

6. 直角坐标与极坐标坐标变换公式

$$\begin{bmatrix} \sigma_\rho & \tau_{\rho\varphi} \\ \tau_{\varphi\rho} & \sigma_\varphi \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{\mathrm{T}}$$

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_\rho & \tau_{\rho\varphi} \\ \tau_{\varphi\rho} & \sigma_\varphi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{\mathrm{T}}$$

7. 复变函数表示的应力函数

$$U = \mathrm{Re}\left[\bar{z}\varphi_1(z) + \theta_1(z)\right]$$

$$\sigma_x + \sigma_y = 4\,\mathrm{Re}\,\varphi_1'(z)$$

$$\sigma_y - \sigma_x + 2\mathrm{i}\tau_{xy} = 2\left[\bar{z}\varphi_1''(z) + \psi_1'(z)\right],\ \psi_1'(z) = \theta_1''(z)$$

8. 应变能密度公式

$$U_1 = \frac{E}{2(1-\mu^2)}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2\mu\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{1-\mu}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right]$$

$$U = \iint_A U_1 \,\mathrm{d}x\,\mathrm{d}y$$

9. 瑞利里茨法

$$\begin{cases} \dfrac{\partial U}{\partial A_m} = \iint\limits_A f_x u_m \mathrm{d}x\,\mathrm{d}y + \int_{S_\sigma} \overline{f_x} u_m \mathrm{d}s \\ \dfrac{\partial U}{\partial B_m} = \iint\limits_A f_y u_m \mathrm{d}x\,\mathrm{d}y + \int_{S_\sigma} \overline{f_y} u_m \mathrm{d}s \end{cases}，(m=1,2,\ldots)$$

10. 伽辽金法

$$\begin{cases} \iint_A \left[\dfrac{E}{1-\mu^2}\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{1-\mu}{2}\dfrac{\partial^2 u}{\partial y^2} + \dfrac{1+\mu}{2}\dfrac{\partial^2 v}{\partial x \partial y}\right) + f_x\right] u_m \mathrm{d}x\,\mathrm{d}y = 0 \\ \iint_A \left[\dfrac{E}{1-\mu^2}\left(\dfrac{\partial^2 v}{\partial y^2} + \dfrac{1-\mu}{2}\dfrac{\partial^2 v}{\partial x^2} + \dfrac{1+\mu}{2}\dfrac{\partial^2 u}{\partial x \partial y}\right) + f_y\right] v_m \mathrm{d}x\,\mathrm{d}y = 0 \end{cases}，(m=1,2,\ldots)$$