Math 135 - Discussion Supplements for Fall 23

Contents are motivated from [1], the lecture notes by Zhou Mo. Corrected some errors in one of the exercises (2025 - thanks to Mo for pointing it out!). [1]

Contents

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1 Discussion 1

Mathematical Preliminaries for Laplace Transforms.

To begin, we shall not review theoretical concepts from 31AB/33B as you should be familiar with the concepts of differentiation, integration, and differential equations. Instead, here are some common indefinite integrals that you should know (and will be used commonly in this class):

- $\int x^a \mathrm{d}x = \frac{x^{1+a}}{1+a} + C$ for $a \neq 1$,
- $\int \frac{1}{x} \mathrm{d}x = \ln|x| + C$,
- $\int (bx+c)^a \mathrm{d}x = \frac{(bx+c)^{1+a}}{b(1+a)} + C$ for $a \neq 1$ and $b \neq 0$,
- $\int \sin(ax)\mathrm{d}x = -\frac{\cos(ax)}{a} + C$ for $a \neq 0$,
- $\int \cos(ax)\mathrm{d}x = \frac{\sin(ax)}{a} + C$ for $a \neq 0$,
- $\int e^{kx}\mathrm{d}x = \frac{e^{kx}}{k} + C$ for $k \neq 0$.

Note that k can be a complex number for the last integral in the list above.

Next, we recall the concept of a change of variable via the use of an example as follows. Observe that

$$\begin{aligned}\int_a^b f(x)\mathrm{d}x &= \int_{x=a}^{x=b} f(x)\mathrm{d}x = \int_{x-a=0}^{x-a=b-a} f(x-a+a)\mathrm{d}(x-a) = \int_{y=0}^{y=b-a} f(y+a)\mathrm{d}y \\ &= \int_0^{b-a} f(y+a)\mathrm{d}y.\end{aligned}$$

Here, we have used the substitution[2] $y = x - a$ and perform the change of variables accordingly.

Another related concept is integration by parts. For any functions $u(x)$ and $v(x)$, we have

$$\int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int u'(x)v(x)\mathrm{d}x.$$

This is useful for evaluating integrals of the form $\int e^{-px}x^n \mathrm{d}x$. For example, by setting $u(x) = x$ and $v(x) = e^{-x}$ (and hence $u'(x) = 1$ and $v'(x) = -e^{-x}$), we have

$$\int_0^1 xe^{-x}\mathrm{d}x = \frac{xe^{-x}}{-1}\Big|_{x=0}^{x=1} - \int_0^1 \frac{1\cdot e^{-x}}{-1}\mathrm{d}x = -\frac{1}{e} + \int_0^1 e^{-x}\mathrm{d}x = -\frac{1}{e} + \frac{e^{-x}}{-1}\Big|_{x=0}^{x=1} = -\frac{2}{e} + 1.$$

Moving on, one should be familiar with complex numbers and Euler's identity. In particular, the latter states that for any real number θ,

$$e^{\mathrm{i}\theta} = \sin(\theta) + \mathrm{i}\cos(\theta).$$

It is worth noting that since for a complex number $a + b\mathrm{i}$, we have $|a + b\mathrm{i}| = \sqrt{a^2 + b^2} = 1$, then

$$|e^{\mathrm{i}\theta}| = \sqrt{\sin^2(\theta) + \cos^2(\theta)} = 1.$$

The above properties will be useful for evaluating integrals of the form $\int e^{-px}\sin(kx)\mathrm{d}x$ or$\int e^{-px}\cos(kx)\mathrm{d}x$. For instance, observe that we can compute $\int_0^2 \cos(x)\mathrm{d}x$ with an alternative method using Euler's identity as follows:[3]

$$\begin{aligned}\int_0^2 \cos(x)\mathrm{d}x &= \int_0^2 \mathrm{Re}\left(e^{\mathrm{i}x}\right)\mathrm{d}x = \mathrm{Re}\left(\int_0^2 e^{\mathrm{i}x}\mathrm{d}x\right) = \mathrm{Re}\left(\frac{e^{\mathrm{i}x}}{\mathrm{i}}\Big|_{x=0}^{x=2}\right) \\ &= \mathrm{Re}\left(\frac{e^{2\mathrm{i}}-1}{\mathrm{i}}\right) = \mathrm{Re}\left(\frac{\cos(2)+\mathrm{i}\sin(2)-1}{\mathrm{i}}\right) \\ &= \mathrm{Re}\left(\sin(2) + \mathrm{i}(1-\cos(2))\right) = \sin(2).\end{aligned} \tag{1}$$

[2] Note that the function $y(x) = x - a$ is a one-to-one function; when applying a suitable substitution, a reason why it might have failed could be due to the fact that the choice of the substitution (function) is not one-to-one.

[3] Rigorously speaking, $f(x)|_{x=A}^{x=B}$ is a short-hand notation for $\lim_{x\to B} f(x) - \lim_{x\to A} f(x)$.

Here, we use the fact that $\frac{1}{\mathrm{i}} = \frac{\mathrm{i}}{\mathrm{i}^2} = \frac{\mathrm{i}}{-1} = -\mathrm{i}$.

Last but not least, we recall the concept of an improper integral as follows. We say that the integral $\int_0^\infty f(x)\mathrm{d}x$ **converges** if the limit

$$\lim_{b\to\infty}\int_0^b f(x)\mathrm{d}x$$

exists. We then denote

$$\int_0^\infty f(x)\mathrm{d}x := \lim_{b\to\infty}\int_0^b f(x)\mathrm{d}x. \tag{2}$$

To check if the improper integral converges, it suffices to check if it **converges absolutely**. We say that the improper integral $\int_0^\infty f(x)\mathrm{d}x$ **converges** if there is a function g such that $|f(x)| \le g(x)$ for all x and

$$\int_0^\infty g(x)\mathrm{d}x$$

converges. In most cases, if $f(x)$ is a non-negative function, it suffices to compute the improper integral by computing $\int_0^b (\text{something})\,\mathrm{d}x$ and taking the appropriate limits as in (2).

In the following pages, we have some worked examples to help you to consolidate these concepts.

Example 1. Let p and q be real numbers.

(i) With the help of Euler's identity, compute $\int_0^\infty e^{-px}\sin(x)\mathrm{d}x$.

(ii) Hence, compute $\int_2^\infty e^{-q(x+1)}\sin(2x-4)\mathrm{d}x$.

For what values of p and q do the integrals in (i) and (ii) converge?

Suggested Solution:

(i)

$$\begin{aligned}\int_0^\infty e^{-px}\sin(x)\mathrm{d}x &= \int_0^\infty e^{-px}\,\mathrm{Im}\left(e^{\mathrm{i}x}\right)\mathrm{d}x = \mathrm{Im}\left(\int_0^\infty e^{(\mathrm{i}-p)x}\mathrm{d}x\right)\\ &= \mathrm{Im}\left(\frac{e^{(\mathrm{i}-p)x}}{(\mathrm{i}-p)}\Big|_{x=0}^{x=\infty}\right) = \mathrm{Im}\left(\frac{1}{p-\mathrm{i}}\right) = \mathrm{Im}\left(\frac{(p+\mathrm{i})}{(p-\mathrm{i})(p+\mathrm{i})}\right)\\ &= \mathrm{Im}\left(\frac{p+\mathrm{i}}{p^2+1}\right) = \boxed{\frac{1}{p^2+1}}.\end{aligned} \tag{3}$$

In the computations above,

- We have used the fact that $(p-\mathrm{i})(p+\mathrm{i}) = p^2-\mathrm{i}^2 = p^2+1$.
- Since $|e^{(\mathrm{i}-p)x}| = |e^{\mathrm{i}x}e^{-px}| = |e^{\mathrm{i}x}|e^{-px} = e^{-px} \to 0$ as $x\to 0$ and $p>0$, we demand that $p>0$ for the improper integral to converge.

Note 1: You should not be too concerned with swapping improper integrals and taking the imaginary part of an integral. In fact, one can show using techniques in Analysis that this is valid as long as the improper integral converges.

Note 2: Rigorously speaking, we should be evaluating

$$\int_0^b e^{-px}\sin(x)\mathrm{d}x = \ldots = \frac{1-e^{-pb}(\cos(b)+p\sin(b))}{1+p^2} \tag{4}$$

and taking the limit as $b\to\infty$ to obtain $\frac{1}{1+p^2}$. Not only does this show that the improper integral converges, you can also observe that the term $e^{-pb}(\cos(b)+p\sin(b))\to 0$ as $b\to\infty$ if $p>0$. For practical purposes, the above computation would probably be fine for the purpose of this class.

(ii) By change of variables $y=2x-4$, since $x+1=\frac{2x+2}{2}=\frac{2x-4}{2}+3$, by treating $p=q/2$ we have

$$\begin{aligned}\int_2^\infty e^{-q(x+1)}\sin(2x-4)\mathrm{d}x &= \int_{x=2}^{x=\infty} e^{-q(x+1)}\sin(2x-4)\mathrm{d}x\\ &= \int_{2x-4=0}^{2x-4=\infty} e^{-p\left(\frac{2x-4}{2}+3\right)}\sin(2x-4)\mathrm{d}\left(\frac{2x-4}{2}\right)\\ &= e^{-3q}\int_{2x-4=0}^{2x-4=\infty} e^{-\frac{q}{2}(2x-4)}\sin(2x-4)\mathrm{d}\left(\frac{2x-4}{2}\right)\\ &= \frac{e^{-3q}}{2}\int_{y=0}^{y=\infty} e^{-q/2(y)}\sin(y)\mathrm{d}y\\ &= \frac{e^{-3q}}{2}\cdot\frac{1}{(q/2)^2+1} = \boxed{\frac{2e^{-3q}}{q^2+4}}.\end{aligned} \tag{5}$$

Since in (i) we demand $p>0$, then it suffices to require $q=2p>0$.

Example 2. By an appropriate substitution, for $n \geq 1$, evaluate

$$\int_0^\infty x^{n-1} e^{-x^n} \,\mathrm{d}x.$$

Suggested Solution:

Let $y = x^n$. This implies that at $x = 0$ we have $y = 0$ and at $x \to \infty$ we also have $y \to \infty$. Then, we have $\mathrm{d}y = nx^{n-1}\mathrm{d}x$ and thus

$$\int_0^\infty x^{n-1} e^{-x^n} \mathrm{d}x = \frac{1}{n} \int_0^\infty e^{-x^n} \left(nx^{n-1}\mathrm{d}x\right) = \frac{1}{n} \int_0^\infty e^{-y} \mathrm{d}y = \boxed{\frac{1}{n}}.$$

Example 3. Let p and q be real numbers.

(i) Compute $\int_1^\infty x^2 e^{-qx} \mathrm{d}x$.

(ii) Hence, compute $\int_2^\infty x^2 e^{-px} \mathrm{d}x$.

For what values of p and q do the integrals in (i) and (ii) converge?

Suggested Solution:

(i) By applying integration by parts twice, we have

$$\begin{aligned}\int_1^\infty x^2 e^{-qx} \mathrm{d}x &= \frac{x^2 e^{-qx}}{-q}\Big|_{x=1}^{x=\infty} + \int_1^\infty \frac{2}{q} x e^{-qx} \mathrm{d}x \\ &= \frac{e^{-q}}{q} + \frac{2}{q}\left(\frac{x e^{-qx}}{-q}\Big|_{x=1}^{x=\infty} + \int_1^\infty \frac{e^{-qx}}{q} \mathrm{d}x\right) \\ &= \frac{e^{-q}}{q} + \frac{2}{q}\left(\frac{e^{-q}}{q} + \frac{e^{-q}}{q^2}\right) \\ &= \boxed{\frac{e^{-q}(q^2+2q+2)}{q^3}}.\end{aligned} \tag{6}$$

(ii) By change of variables ($y = x/2$ and treating $q = 2p$ in (i)), we have

$$\begin{aligned}\int_2^\infty x^2 e^{-px} \mathrm{d}x = \int_{x=2}^{x=\infty} x^2 e^{-px} \mathrm{d}x &= \int_{x/2=1}^{x/2=\infty} \left(2 \cdot x/2\right)^2 e^{-2p(x/2)} 2\mathrm{d}(x/2) \\ &= 2\int_{y=1}^{y=\infty} (2y)^2 e^{-2py} \mathrm{d}y \\ &= 8\int_{y=1}^{y=\infty} y^2 e^{-2py} \mathrm{d}y \\ &= 8 \cdot \frac{e^{-2p}((2p)^2 + 2(2p) + 2)}{(2p)^3} = \boxed{\frac{e^{-2p}(4p^2+4p+2)}{p^3}}.\end{aligned} \tag{7}$$

In (i), we require $x^2 e^{-qx} \to 0$ as $x \to \infty$ to evaluate $\frac{x^2 e^{-qx}}{-q}\big|_{x=1}^{x=\infty}$. This implies that we require $q > 0$ for the integral to converge (so that the exponential decay goes to 0 faster than a polynomial decay). Since in (ii), $p = q/2$, equivalently, we require that $p > 0$.

2 Discussion 2

Laplace Transforms.

$$\underbrace{\mathcal{L}\underbrace{[f(x)]}_{\text{Takes a function} f(x)}(p)}_{\text{Outputs a function in } p} = \underbrace{\int_0^\infty e^{-px} f(x)\mathrm{d}x}_{\text{The formula here}} = \underbrace{F(p)}_{\text{Usually denote the output function in } p \text{ as } F(p)} \tag{8}$$

We call $\mathcal{L}$ above the **Laplace transform**, which takes in a function and outputs a function.[4] We sometimes refer to p as the Laplace variable.

Table of Laplace Transforms:

$f(x)$	$F(p) = \mathcal{L}[f(x)](p)$
1	$\frac{1}{p}$
x	$\frac{1}{p^2}$
x^n	$\frac{n!}{p^{n+1}}$
e^{ax}	$\frac{1}{p-a}$ for $p > a$
$\sin(ax)$	$\frac{a}{p^2+a^2}$
$\cos(ax)$	$\frac{p}{p^2+a^2}$
$\sinh(ax)$	$\frac{a}{p^2-a^2}$ for $p > \lvert a\rvert$
$\cosh(ax)$	$\frac{p}{p^2-a^2}$ for $p > \lvert a\rvert$
$\delta(x)$	1

Properties: For any constants α, β and appropriate functions $f(x)$ and $g(x)$,

- $\mathcal{L}[\alpha f(x) + \beta g(x)](p) = \alpha\mathcal{L}[f(x)](p) + \beta\mathcal{L}[g(x)](p)$.
- *Shifting formula:* $\mathcal{L}[e^{ax}f(x)](p) = \mathcal{L}[f(x)](p-a) = F(p-a)$.[5]
  "Exponential scaling in space shifts the Laplace variable by the negative of that amount."
- $\mathcal{L}[y'(x)](p) = p\mathcal{L}[y](p) - y(0)$. (This is an expression in p, and $y(0)$ does not depend on p.)
- $\mathcal{L}[y''(x)](p) = p^2\mathcal{L}[y](p) - py(0) - y'(0)$.

In the table of transforms above, $\delta(x)$ refers to the **delta** function. Technically speaking, it is not a function but rather, a "function" that satisfies

- $\delta(x) = 0$ for all $x \neq 0$, and
- $\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1$. (In fact, as long as the interval contains zero, then this integral evaluates to 1.)

In other words, it is an "infinite" spike at $x = 0$ such that the overall integral sums up to 1. For more details on how to approximate this function, refer to Problem 49.5 in the textbook.

Last but not least, we indicate the **inverse Laplace transform** as $\mathcal{L}^{-1}$. Roughly speaking, the inverse Laplace transform of a function $F(p)$ outputs "$?(x)$", which is obtained by asking "the Laplace transform of what gives me $F(p)$". Diagrammatically, we have:

$$\mathcal{L}^{-1}[F(p)] = \,? \quad \underbrace{\iff}_{\text{is the same as asking}} \quad F(p) = \underbrace{\mathcal{L}[?]}_{\text{The Laplace transform of what gives me } F(p)}. \tag{9}$$

For example,

$$\mathcal{L}[x] = \frac{1}{p^2} \iff \mathcal{L}^{-1}[\mathcal{L}[x](p)] = \mathcal{L}^{-1}\left[\frac{1}{p^2}\right] \iff \mathcal{L}^{-1}\left[\frac{1}{p^2}\right] = x.$$

[4] Note that sometimes the dependence on p for $\mathcal{L}[f(x)]$ is usually assumed and thus not explicitly written, or it is represented by another function $F(p)$ as seen above.

[5] Here, $F(p) = \mathcal{L}[f(x)](p)$ represents the Laplace transform of the function without the exponential factor.

Example 4. Using the shifting formula, for $a, b \in \mathbb{R}$, compute the following Laplace transforms :

(i) $\mathcal{L}[e^{bx}e^{ax}]$ for $p > a + b$,

(ii) $\mathcal{L}[e^{bx}x^n]$,

(iii) $\mathcal{L}[e^{bx}\sin(ax)]$,

(iv) $\mathcal{L}[e^{bx}\cos(ax)]$, and

(v) $\mathcal{L}[e^{bx}\delta(x)]$.

Suggested Solutions:
Recall that "Exponential scaling in space shifts the Laplace variable by the negative of that amount."

(i) $\mathcal{L}[e^{bx}e^{ax}] = \frac{1}{(p-b)-a} = \frac{1}{p-(a+b)}$

(ii) $\mathcal{L}[e^{bx}x^n] = \frac{n!}{(p-b)^{n+1}}$,

(iii) $\mathcal{L}[e^{bx}\sin(ax)] = \frac{a}{(p-b)^2+a^2}$,

(iv) $\mathcal{L}[e^{bx}\cos(ax)] = \frac{(p-b)}{(p-b)^2+a^2}$,

(v) $\mathcal{L}[e^{bx}\delta(x)] = 1$. (There is no p in the output function to shift.)

Applications and Tips for Computing (Inverse) Laplace Transforms:
To compute the inverse Laplace transform of a rational fraction, the usual strategy is to apply partial fraction decomposition. Recall that

- Linear Factors: $\frac{f(p)}{(p-a)(p-b)} = \frac{A}{p-a} + \frac{B}{p-b}$.
  (Add $\frac{\text{constant}}{\text{linear factor}}$ for each linear factor)
- Repeated Factors: $\frac{f(p)}{(p-a)^n} = \frac{A_1}{p-a} + \frac{A_2}{(p-a)^2} + \cdots + \frac{A_n}{(p-a)^n}$
  (Repeat for increasing power of $p - a$ in the denominator, with each term having just a constant above.)
- Irreducible Quadratic Factors: $\frac{f(p)}{(p-d)((p+a)^2+b)} = \frac{A}{p-d} + \frac{Bp+C}{(p+a)^2+b}$.
  (Use the same strategy as above, but for each irreducible quadratic factor in the denominator, we pick a linear factor with undetermined coefficients in the numerator.)

Note that $A, B, C, A_1, \cdots, A_n$ are constants to be determined.
Then, recall that

- $\mathcal{L}^{-1}\left[\frac{1}{p}\right] = 1$. Hence, by the shifting formula, $\mathcal{L}^{-1}\left[\frac{1}{p-a}\right] = e^{ax}$.
- $\mathcal{L}^{-1}\left[\frac{1}{p^2}\right] = x$ for repeated factors (generalizable to p^n in the denominator). Hence, by the shifting formula, $\mathcal{L}^{-1}\left[\frac{1}{(p-a)^2}\right] = xe^{ax}$.
- $\mathcal{L}^{-1}\left[\frac{b}{p^2+b^2}\right] = \sin(bx)$. Hence, by the shifting formula, $\mathcal{L}^{-1}\left[\frac{b}{(p-a)^2+b^2}\right] = e^{ax}\sin(bx)$.

Another application is in solving second-order constant coefficient ordinary differential equations (or other similar variants); ie
$$y''(x) + Ay'(x) + By(x) = C, \quad y(0) = y_0, \quad y'(0) = y_0'$$
for constants A, B, C, y_0, and y_0'.

This is done by the following steps:

1. Take Laplace transform on both sides of the equation.
   (In the process, use $\mathcal{L}[y'] = p\mathcal{L}[y] - y(0)$ and $\mathcal{L}[y''] = p^2\mathcal{L}[y] - py(0) - y'(0)$.)
2. Make Laplace transform the subject of the formula.
   (ie. $F(p) = ...$ or $\mathcal{L}[y](p) = ...$)
3. Take inverse Laplace transform on both sides to retrieve the solution $y(x)$. This is done by computing the inverse Laplace transform on the right-hand side of the result from Step 2.

We will see two examples of this in a bit.

Example 5. With the help of the table of Laplace transforms and relevant properties, evaluate

(i) $\mathcal{L}\left[e^{4x}(x+1)^2\right]$,

(ii) $\mathcal{L}^{-1}\left[\frac{1}{p+1}+\frac{1}{(p-1)^2+1}\right]$, and

(iii) $\mathcal{L}^{-1}\left[\frac{p}{(p+1)(p^2+4p+5)}\right]$.

Suggested Solutions:

(i) "Exponential scaling in space shifts the Laplace variable by the negative of that amount." Hence, it suffices to evaluate $\mathcal{L}\left[(x+1)^2\right]$. Note that

$$\begin{aligned}\mathcal{L}\left[(x+1)^2\right] &= \mathcal{L}\left[x^2+2x+1\right]\\ &= \mathcal{L}[x^2]+2\mathcal{L}[x]+\mathcal{L}[1]\\ &= \frac{2}{p^3}+\frac{2}{p^2}+\frac{1}{p}.\end{aligned} \tag{10}$$

Hence, by the shifting formula, we have

$$\mathcal{L}\left[e^{4x}(x+1)^2\right](p) = \boxed{\frac{2}{(p-4)^3}+\frac{2}{(p-4)^2}+\frac{1}{(p-4)}}. \tag{11}$$

Note that shifting the space variable does not correspond to exponential scaling in the Laplace variable! (ie the opposite of the shifting formula is not true.)

(ii) $\mathcal{L}^{-1}\left[\frac{1}{p+1}+\frac{1}{(p-1)^2+1}\right] = e^{-x}\mathcal{L}^{-1}\left[\frac{1}{p}\right]+e^{x}\mathcal{L}^{-1}\left[\frac{1}{p^2+1}\right] = \boxed{e^{-x}+e^{x}\sin(x)}$.

(iii) First, we have to consider the partial fraction decomposition as follows. Since the denominator consists of a linear factor $p+1$ and an irreducible factor $p^2+4p+5 \underbrace{=}_{\text{complete the square}} (p+2)^2+1$, we have

$$\begin{aligned}\frac{p}{(p+1)(p^2+4p+5)} &= \frac{A}{p+1}+\frac{Bp+C}{p^2+4p+5}\\ &= \frac{A(p^2+4p+5)+(Bp+C)(p+1)}{(p+1)(p^2+4p+5)}.\end{aligned} \tag{12}$$

This implies that $p \equiv A(p^2+4p+5)+(Bp+C)(p+1) = (A+B)p^2+(4A+B+C)p+(5A+C)$. Hence, we have

$$\begin{cases} A+B &= 0,\\ 4A+B+C &= 1,\\ 5A+C &= 0.\end{cases} \tag{13}$$

Solving the above system gives $A=-\frac{1}{2}, B=\frac{1}{2}$, and $C=\frac{5}{2}$. This implies that

$$\frac{p}{(p+1)(p^2+4p+5)} = -\frac{1}{2}\cdot\frac{1}{p+1}+\frac{1}{2}\cdot\frac{p+5}{(p+2)^2+1}. \tag{14}$$

Since

$$\mathcal{L}^{-1}\left[\frac{1}{p+1}\right] = e^{-x}$$

and

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{p+5}{(p+2)^2+1}\right] &= \mathcal{L}^{-1}\left[\frac{p+2}{(p+2)^2+1}\right]+3\mathcal{L}^{-1}\left[\frac{1}{(p+2)^2+1}\right]\\ &= e^{-2x}\mathcal{L}^{-1}\left[\frac{p}{p^2+1}\right]+3e^{-2x}\mathcal{L}^{-1}\left[\frac{1}{p^2+1}\right]\\ &= e^{-2x}\cos(x)+3e^{-2x}\sin(x),\end{aligned}$$

we have

$$\mathcal{L}^{-1}\left[\frac{p}{(p+1)(p^2+4p+5)}\right] = \boxed{-\frac{e^{-x}}{2}+\frac{e^{-2x}}{2}(3\sin(x)+\cos(x))}. \tag{15}$$

Example 6. (Exercise 50.4.) Find the solution of $y'' - 2ay' + a^2 y = 0$ in which the initial conditions $y(0) = y_0$ and $y'(0) = y_0'$ are left unrestricted.

Suggested Solutions: To solve the above differential equation, we follow the sequence of steps as outlined above.

1. Note that $\mathcal{L}[y'] = p\mathcal{L}[y] - y_0$ and $\mathcal{L}[y''] = p^2\mathcal{L}[y] - py_0 - y_0'$. Hence, by taking the Laplace transform on both sides of the differential equation, we have

$$\begin{aligned} \mathcal{L}[y''] - 2a\mathcal{L}[y'] + a^2\mathcal{L}[y] &= 0 \\ p^2\mathcal{L}[y] - py_0 - y_0' - 2a\left(p\mathcal{L}[y] - y_0\right) + a^2\mathcal{L}[y] &= 0 \end{aligned}$$

2. Making $\mathcal{L}[y]$ the subject of the formula, we have

$$\begin{aligned} p^2\mathcal{L}[y] - py_0 - y_0' - 2a\left(p\mathcal{L}[y] - y_0\right) + a^2\mathcal{L}[y] = 0 \\ (p^2 - 2ap + a^2)\mathcal{L}[y] &= y_0' + (p-2a)y_0 \\ (p-a)^2\mathcal{L}[y] &= y_0' + (p-2a)y_0 \\ \mathcal{L}[y] &= \frac{y_0' + (p-2a)y_0}{(p-a)^2}. \end{aligned}$$

3. It remains to evaluate the inverse Laplace transform of the expression on the right. One can either use the standard method of partial fraction decomposition or note that $y_0' + (p-2a)y_0 = y_0(p-a) + y_0' - ay_0$, and hence

$$\frac{y_0' + (p-2a)y_0}{(p-a)^2} = \frac{y_0(p-a) + y_0' - ay_0}{(p-a)^2} = \frac{y_0}{p-a} + \frac{y_0' - ay_0}{(p-a)^2}.$$

Since

$$\mathcal{L}^{-1}\left[\frac{1}{p-a}\right] = e^{ax}$$

and ("Exponential scaling in space shifts the Laplace variable by the negative of that amount.")

$$\mathcal{L}^{-1}\left[\frac{1}{(p-a)^2}\right] = e^{ax}\mathcal{L}^{-1}\left[\frac{1}{p^2}\right] = xe^{ax},$$

we then have

$$\mathcal{L}^{-1}\left[\frac{y_0' + (p-2a)y_0}{(p-a)^2}\right] = y_0 e^{ax} + (y_0' - ay_0)xe^{ax}.$$

Hence, we have

$$y(x) = \mathcal{L}^{-1}\left[\frac{y_0' + (p-2a)y_0}{(p-a)^2}\right] = \boxed{y_0 e^{ax} + (y_0' - ay_0)xe^{ax}}. \tag{16}$$

Example 7. (Exercise 50.5.)

(i) By applying $\mathcal{L}[y'](p) = p\mathcal{L}[y](p) - y(0)$, deduce that

$$\mathcal{L}\left[\int_0^x f(t)\mathrm{d}t\right](p) = \frac{F(p)}{p}, \tag{17}$$

in which

$$F(p) = \mathcal{L}[f(x)](p).$$

(ii) Hence or otherwise, verify the formula above by computing

$$\mathcal{L}^{-1}\left[\frac{1}{p(p+1)}\right] \tag{18}$$

in two different ways.

Suggested Solutions:

(i) From the above formula, we rearrange to obtain

$$\mathcal{L}[y](p) = \frac{\mathcal{L}[y'](p) + y(0)}{p}. \tag{19}$$

Now, set $y(x) = \int_0^x f(t)\mathrm{d}t$. Observe that

- $y(0) = \int_0^0 f(t)\mathrm{d}t = 0$.
- By fundamental theorem of calculus, we have $y'(x) = f(x)$.

Hence, by (19), we have

$$\mathcal{L}\left[\int_0^x f(t)\mathrm{d}t\right](p) = \frac{\mathcal{L}[y](p) + 0}{p} = \frac{F(p)}{p}. \tag{20}$$

(ii) Method 1: Partial Fractions. By partial fraction decomposition, we have

$$\frac{1}{p(p+1)} = \frac{1}{p} - \frac{1}{p+1}.$$

Since $\mathcal{L}\left[\frac{1}{p}\right] = 1$ and $\mathcal{L}\left[\frac{1}{p+1}\right] = e^{-x}$, we have

$$\mathcal{L}^{-1}\left[\frac{1}{p(p+1)}\right](x) = 1 - e^{-x}. \tag{21}$$

Method 2: Using (i). By taking the inverse Laplace transform on both sides of (20), we have

$$\int_0^x f(t)\mathrm{d}t = \mathcal{L}^{-1}\left[\frac{F(p)}{p}\right]. \tag{22}$$

Comparing with $\mathcal{L}^{-1}\left[\frac{1}{p(p+1)}\right]$, we set $\mathcal{L}[f](p) = F(p) = \frac{1}{p+1}$. Since

$$\mathcal{L}^{-1}\left[\frac{1}{p+1}\right] = e^{-x},$$

this implies that $f(x) = \mathcal{L}^{-1}\left[\frac{1}{p+1}\right] = e^{-x}$. By (22), since

$$\int_0^x f(t)\mathrm{d}t = \int_0^x e^{-t}\mathrm{d}t = 1 - e^{-x},$$

we have

$$1 - e^{-x} = \mathcal{L}^{-1}\left[\frac{1}{p(p+1)}\right]. \tag{23}$$

This gives the same answer as from the first method.

Example 8. (Exercise 50.6.) By using (17) from Example 7 or otherwise, solve the following differential equation:

$$y' + 4y + 5\int_0^x y(t)\mathrm{d}t = e^{-x}, \quad y(0) = 0. \tag{24}$$

Suggested Solutions:

Method 1: Using Example 7. By taking the Laplace transform on both sides of the differential equation, we have

$$\begin{aligned} \mathcal{L}[y'] + 4\mathcal{L}[y] + 5\mathcal{L}\left[\int_0^x y(t)\mathrm{d}t\right] &= \mathcal{L}[e^{-x}] \\ p\mathcal{L}[y] - y(0) + 4\mathcal{L}[y] + 5\left(\frac{\mathcal{L}[y]}{p}\right) &= \frac{1}{p+1} \\ \left(p + 4 + \frac{5}{p}\right)\mathcal{L}[y] &= \frac{1}{p+1} \\ \mathcal{L}[y] &= \frac{p}{(p+1)(p^2+4p+5)}. \end{aligned} \tag{25}$$

Method 2: Differentiating the differential equation. By differentiating (24) on both sides, we obtain

$$y'' + 4y' + 5y = -e^{-x}. \tag{26}$$

Since this is a second-order differential equation, we need an initial condition for $y'(0)$. This can be obtained by substituting $x = 0$ into (24). Indeed, this yields

$$y'(0) = -4y(0) - 5\int_0^0 y(t)\mathrm{d}t + e^{-0} = 1. \tag{27}$$

We now perform the usual strategy of taking the Laplace transform on both sides of (26) to obtain

$$\begin{aligned} p^2\mathcal{L}[y] - py(0) - y'(0) + 4(p\mathcal{L}[y] - y(0)) + 5\mathcal{L}[y] &= -\mathcal{L}[e^{-x}] \\ (p^2 + 4p + 5)\mathcal{L}[y] - 1 &= -\frac{1}{p+1} \\ (p^2 + 4p + 5)\mathcal{L}[y] &= \frac{p}{p+1} \\ \mathcal{L}[y] &= \frac{p}{(p+1)(p^2+4p+5)}. \end{aligned} \tag{28}$$

In both methods, we see that we obtain the same expression for $\mathcal{L}[y]$. Taking the inverse Laplace transform on both sides and applying our results from Example 5 (iii), we have

$$y(x) = \mathcal{L}^{-1}\left[\frac{p}{(p+1)(p^2+4p+5)}\right] = \boxed{-\frac{e^{-x}}{2} + \frac{e^{-2x}}{2}(3\sin(x) + \cos(x))}. \tag{29}$$

3 Discussion 3

More Properties of Laplace Transforms.

Recall the table of Laplace Transforms below:

$f(x)$	$F(p) = \mathcal{L}[f(x)](p)$
1	$\frac{1}{p}$
x	$\frac{1}{p^2}$
x^n	$\frac{n!}{p^{n+1}}$
e^{ax}	$\frac{1}{p-a}$ for $p > a$
$\sin(ax)$	$\frac{a}{p^2+a^2}$
$\cos(ax)$	$\frac{p}{p^2+a^2}$
$\sinh(ax)$	$\frac{a}{p^2-a^2}$ for $p > \lvert a\rvert$
$\cosh(ax)$	$\frac{p}{p^2-a^2}$ for $p > \lvert a\rvert$
$\delta(x)$	1

For any two given functions f and g, we define the **convolution** of f and g, denoted by $f * g$, as

$$(f * g)(x) = \int_0^x f(x-y)g(y)\mathrm{d}y. \tag{30}$$

Updated List of Properties: For any constants α, β and appropriate functions $f(x)$ and $g(x)$, we denote $F(p) = \mathcal{L}[f(x)](p)$ and $G(p) = \mathcal{L}[g(x)](p)$. Then, we have the following properties for Laplace transforms and

- Linearity: $\mathcal{L}[\alpha f(x) + \beta g(x)](p) = \alpha\mathcal{L}[f(x)](p) + \beta\mathcal{L}[g(x)](p)$.
- *Shifting formula:* $\mathcal{L}[e^{ax}f(x)](p) = F(p-a)$.
- Laplace Transform of Derivative: $\mathcal{L}[y'(x)] = pF(p) - y(0)$.
- Laplace Transform of Derivative: $\mathcal{L}[y''(x)] = p^2\mathcal{L}[y](p) - py(0) - y'(0)$.
- Laplace Transform of Integrals (See Exercise 50.5): $\mathcal{L}\left[\int_0^x f(t)\mathrm{d}t\right](p) = \frac{F(p)}{p}$.
- Derivatives of Laplace Transforms: $\mathcal{L}[(-x)^n f(x)](p) = \frac{\mathrm{d}^n}{\mathrm{d}p^n}F(p)$.
- Integrals of Laplace Transforms: $\mathcal{L}\left[\frac{f(x)}{x}\right](p) = \int_p^\infty F(s)\mathrm{d}s$.
- Convolution Theorem: $\mathcal{L}[f * g](p) = F(p)G(p)$.
  "Laplace transform of the convolution is the product of their Laplace transforms."

Recall from above that $\mathcal{L}\left[\frac{f(x)}{x}\right] = \int_0^\infty e^{-px}\frac{f(x)}{x}\mathrm{d}x = \int_p^\infty F(s)\mathrm{d}s$. By taking the limit as $p \to 0$, formally, we have

$$\int_0^\infty \frac{f(x)}{x}\mathrm{d}x = \int_0^\infty F(s)\mathrm{d}s = \int_0^\infty F(p)\mathrm{d}p. \tag{31}$$

This can be useful in evaluating integrals of the form in (31).

Convolution theorem can be useful in helping to solve some integral equations of the form:

$$y(x) = g(x) + \int_0^x f(x-t)y(t)\mathrm{d}t$$

for some given functions f and g. This is because we could rewrite the above as

$$y(x) = g(x) + (f * y)(x).$$

Taking Laplace transform yields

$$\mathcal{L}[y] = \mathcal{L}[g] + \mathcal{L}[f * y].$$

By the convolution theorem, we have

$$\mathcal{L}[y] = \mathcal{L}[g] + \mathcal{L}[f]\mathcal{L}[y].$$

We can now make $\mathcal{L}[y]$ the subject of the formula:

$$\mathcal{L}[y] = \frac{\mathcal{L}[g]}{1 - \mathcal{L}[f]}$$

and perform inverse Laplace transform accordingly assuming that $1 - \mathcal{L}[f] \neq 0$.

Example 9.

(i) Using $\mathcal{L}[(-x)^n f(x)](p) = \frac{\mathrm{d}^n}{\mathrm{d}p^n} F(p)$, evaluate $\mathcal{L}\left[x^2 e^x\right]$ for $p > 1$.

(ii) (Exercise 51.7a) By using (31), show that $f(\xi) = \int_0^\infty \frac{\sin(\xi x)}{x} \mathrm{d}x = \frac{\pi}{2}$ for all $\xi > 0$.

Suggested Solutions:

(i) Observe that since

$$\mathcal{L}[(-x)^2 f(x)] = \mathcal{L}[x^2 f(x)] = \frac{\mathrm{d}^2}{\mathrm{d}p^2} F(p)$$

and $\mathcal{L}[e^x] = \frac{1}{p-1}$ for $p > 1$, we have

$$\mathcal{L}[x^2 e^x] = \frac{\mathrm{d}^2}{\mathrm{d}p^2} \left(\frac{1}{p-1} \right) = \boxed{\frac{2}{(p-1)^3}}.$$

(ii) By applying (31), recall that

$$\mathcal{L}[\sin(\xi x)] = \frac{\xi}{p^2 + \xi^2}.$$

Hence, we have for each $\xi > 0$,

$$f(\xi) = \int_0^\infty \frac{\sin(\xi x)}{x} \mathrm{d}x = \int_0^\infty \frac{\xi}{p^2 + \xi^2} \mathrm{d}p = \arctan\left(\frac{p}{\xi}\right) \Big|_{p=0}^{p=\infty} = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

Here, we use the fact that $\arctan(0) = 0$ and $\arctan(\infty) = \pi/2$.

Example 10. Solve the following integro-differential equation:

$$y'(x) = -y(x) - \int_0^x e^{t-x} y(t)\mathrm{d}t$$

with initial condition $y(0) = 1$.

Suggested Solutions:

Observe that the above can be re-written using convolution (with $f(x) = e^{-x}$) as:

$$y'(x) = -y(x) - (f * y)(x).$$

Taking the Laplace transform on both sides of the equation, using $y(0) = 0$, $\mathcal{L}[e^{-x}] = 1/(p+1)$, $\mathcal{L}[y'] = p\mathcal{L}[y] - y(0)$, and the convolution theorem, we have

$$\begin{aligned} p\mathcal{L}[y] - y(0) &= -\mathcal{L}[y] - \mathcal{L}[f * y] \\ p\mathcal{L}[y] - 1 &= -\mathcal{L}[y] - \mathcal{L}[f]\mathcal{L}[y] \\ p\mathcal{L}[y] - 1 &= -\mathcal{L}[y] - \left(\frac{1}{p+1}\right)\mathcal{L}[y] \\ \left(p + 1 + \frac{1}{p+1}\right)\mathcal{L}[y] &= 1 \\ \mathcal{L}[y] &= \frac{p+1}{(p+1)^2+1}. \end{aligned} \tag{32}$$

By taking the inverse Laplace transform and recalling that
"Exponential scaling in space shifts the Laplace variable by the negative of that amount", we thus deduce that

$$\begin{aligned} y(x) &= \mathcal{L}^{-1}\left[\frac{(p+1)}{(p+1)^2+1}\right](x) \\ &= e^{-x}\mathcal{L}^{-1}\left[\frac{p}{p^2+1}\right](x) \\ &= \boxed{e^{-x}\cos(x)}. \end{aligned} \tag{33}$$

4 Discussion 4

Picard's Method of Successive Approximation.

We begin by considering the initial value problem of the form

$$\begin{cases} y'(x) = f(x, y(x)), \\ y(x_0) = y_0, \end{cases} \tag{34}$$

where $f(x, y)$ is an arbitrary function defined and continuous in some neighborhood of the point (x_0, y_0). Picard's method of successive approximations aims to approximate the solutions to (34) as follows. By integrating the differentiation equation in (34) starting from x_0, we have the equivalent integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\mathrm{d}t. \tag{35}$$

By viewing (35) as

$$y = G(y) \tag{36}$$

for some "operator/function/map" G, where $G(y)(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\mathrm{d}t$, if we let $y_0(x) = y_0$ and consider the iteration

$$y_{n+1} = G(y_n). \tag{37}$$

If the sequence of functions converges to some $\hat{y}$, then we take limits as n goes to infinity on both sides of (37) and assuming that we can interchange G and limits, we then have

$$\hat{y} = G(\hat{y}).$$

The above equation also implies that $\hat{y}$ (the limit of the sequence) solves (37) and hence the integral equation (35), and henceforth the initial value problem in (34).

Explicitly, the method demands that we do the following:

1. Let $y_0(x) = y_0$.
2. Compute the sequence of functions $y_1(x), y_2(x), \ldots$ using the formula

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\mathrm{d}t.$$

We shall see an example of Picard's method as mentioned above in action in the example on the next page.

Example 11. Consider the initial value problem

$$\begin{cases} y'(x) = -xy, \\ y(0) = 1, \end{cases} \tag{38}$$

for $x > 0$.

(i) Determine the exact solution to (38).

(ii) Apply Picard's method with $y_0(x) = y(0)$ to calculate $y_1(x), y_2(x)$, and $y_3(x)$.

(iii) How does your solution in (ii) compare with that in (i)?

Suggested Solutions:

(i) By separation of variables, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -xy \to \int \frac{1}{y}\mathrm{d}y = -\int x\mathrm{d}x.$$

Hence,

$$\ln|y| = -\frac{x^2}{2} + C, \quad \text{or} \quad y(x) = Ae^{-\frac{x^2}{2}}.$$

Plugging in $y(0) = 1$ gives $A = 1$. The solution to (38) is thus given by

$$\boxed{y(x) = e^{-\frac{x^2}{2}}}.$$

(ii) Let $y_0(x) = y(0) = 1$ for each $x > 0$. Note that Picard's method gives the following iteration:

$$y_{n+1}(x) = y(0) + \int_0^x f(t, y_n(t))\mathrm{d}t. \tag{39}$$

Since $f(x, y) = -xy$ and $y(0) = 1$, we have

$$y_{n+1}(x) = 1 - \int_0^x t y_n(t)\mathrm{d}t. \tag{40}$$

Hence,

$$\begin{aligned} y_1(x) &= 1 - \int_0^x t y_0(t)\mathrm{d}t = 1 - \int_0^x t\mathrm{d}t = 1 - \frac{x^2}{2} \\ y_2(x) &= 1 - \int_0^x t y_1(t)\mathrm{d}t = 1 - \int_0^x t(1 - t^2/2)\mathrm{d}t = 1 - \frac{x^2}{2} + \frac{x^4}{2 \times 4} = 1 - \frac{x^2}{2} + \frac{1}{2}\left(\frac{x^2}{2}\right)^2 \\ y_3(x) &= 1 - \int_0^x t y_2(t)\mathrm{d}t = \dots = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} \\ &= 1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2!}\left(-\frac{x^2}{2}\right)^2 + \frac{1}{3!}\left(-\frac{x^2}{2}\right)^3. \end{aligned} \tag{41}$$

(iii) One can observe by pattern recognition (or formal computations) that

$$y_n(x) = 1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2!}\left(-\frac{x^2}{2}\right)^2 + \cdots + \frac{1}{n!}\left(-\frac{x^2}{2}\right)^n.$$

Furthermore, recall that the Taylor series for e^w is given by

$$e^w = 1 + w + \frac{1}{2!}w^2 + \cdots + \frac{1}{n!}w^n + \cdots,$$

which implies that the Taylor series for $e^{-\frac{x^2}{2}}$ is given by

$$e^{-\frac{x^2}{2}} = 1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2!}\left(-\frac{x^2}{2}\right)^2 + \cdots + \frac{1}{n!}\left(-\frac{x^2}{2}\right)^n + \cdots.$$

We conclude that the n-th approximation of Picard's method corresponds to the Taylor series of the actual solution truncated up till the n-th order/term.

Existence and Uniqueness of Solutions to Initial Value Problems.

Recall the initial value problem in (34):

$$\begin{cases} y'(x) = f(x, y(x)), \\ y(x_0) = y_0, \end{cases} \tag{42}$$

where $f(x, y)$ is an arbitrary function defined and continuous in some neighborhood of the point (x_0, y_0). The following theorem guarantees the existence and uniqueness of the solution to (42).

Theorem 12. (Picard's Theorem; Theorem A.) Let $f(x, y)$ and $\frac{\partial f}{\partial y}$ be continuous functions of x and y on a closed rectangle R given by

$$R = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ c \leq y \leq d\}$$

for some real numbers $a < b$ and $c < d$. If (x_0, y_0) is any interior point of R,[a] then there exists a number $h > 0$ with the property that the initial value problem in (42) has a unique solution on the interval $|x - x_0| \leq h$.

[a] In other words, $a < x_0 < b$ and $c < y_0 < d$.

Next, we shall give a slight improvement to Picard's theorem above. Before that, we review what it means for a function to be Lipschitz below.

Definition 13. A function $F : [a, b] \to \mathbb{R}$ with $a < b$ is **Lipschitz** if there is a constant $L > 0$ such that for all $x, y \in [a, b]$, we have

$$|F(x) - F(y)| \leq L|x - y|.$$

We call L here the associated Lipschitz constant.

We are now ready to state the improvement of Picard's theorem below.

Theorem 14. (Picard's Theorem; Theorem A Modified.) Let $f(x, y)$ be continuous functions of x and y on a closed rectangle R given by

$$R = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ c \leq y \leq d\}$$

for some real numbers $a < b$ and $c < d$. Furthermore, we also demand that for any given $x \in [a, b]$, $f(x, \cdot)$ is Lipschitz with Lipschitz constant not depending on x. In other words, there exists a constant $L > 0$ such that for each $x \in [a, b], y_1, y_2 \in \mathbb{R}$,

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$$

with L not depending on x.

If (x_0, y_0) is any interior point of R, then there exists a number $h > 0$ with the property that the initial value problem in (42) has a unique solution on the interval $|x - x_0| \leq h$.

Note that Theorem A only guarantees **local** existence and uniqueness. For **global** existence and uniqueness, see Theorem B below.

Theorem 15. (Theorem B.) Let $f(x, y)$ be continuous functions of x and y on a closed strip S given by

$$S = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \ -\infty < y < \infty\}$$

for some real numbers $a < b$. In addition, we also demand that for any given $x \in [a, b]$, $f(x, \cdot)$ is Lipschitz with Lipschitz constant not depending on x. In other words, there exists a constant $L > 0$ such that for each $x \in [a, b], y_1, y_2 \in \mathbb{R}$,

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$$

with L not depending on x.

If $(x_0, y_0) \in S$, then the initial value problem (42) has a unique solution on the interval $x \in [a, b]$.

Remarks:

- Theorem A Modified is stronger than Theorem A since $\frac{\partial f}{\partial y}$ is continuous on a closed rectangle implies that $f(x, \cdot)$ is Lipschitz (in y). This follows from the mean value theorem since for each $y_1 < y_2$, we have

  $$|f(x, y_1) - f(x, y_2)| \leq \left|\frac{\partial f}{\partial y}(x, y^*)\right| |y_2 - y_1|$$

  for some $y^* \in [y_1, y_2]$. Hence, continuity of $\frac{\partial f}{\partial y}$ on a closed rectangle implies bounded, and thus

  $$\left|\frac{\partial f}{\partial y}(x, y)\right| \leq C$$

  for some positive constant $C > 0$. In particular, $\left|\frac{\partial f}{\partial y}(x, y^*)\right| \leq C$ and thus $f(x, \cdot)$ is Lipschitz with Lipschitz constant C. The is consistent with the proof of Theorem A which only utilizes the fact that $f(x, \cdot)$ is Lipschitz, which is implied by continuity of $\frac{\partial f}{\partial y}$ on a closed rectangle.

- The above pointer also implies that

  $$\text{On a closed rectangle,} \quad \text{Differentiable} \implies \text{Lipschitz.}$$

  Hence, on some occasions, it might suffice to apply Theorem B by showing that $f(x, y)$ is differentiable with respect to y. Furthermore, the implication is not true if we are not considering a closed rectangle

  Note that the converse is not true; there are Lipschitz functions that are not differentiable. Take $f(x) = |x|$ with $x \in \mathbb{R}$ for instance; it is a Lipschitz function with Lipschitz constant 1 but it is not differentiable at $x = 0$.

- A useful tool for proving that a function is Lipschitz (or not) is the triangle inequality. It states that for any $a, b \in \mathbb{R}$,

  $$||b| - |a|| \leq |b - a| \leq |b| + |a|.$$

- If the Lipschitz condition is dropped and we only assume that $f(x, y)$ is continuous, then we have the **Peano's theorem**. The theorem only **guarantees existence, and not uniqueness**.

Example 16. Show that $f(x, y) = e^{-x}y^3$ is Lipschitz in y for $(x, y) \in [0, \infty) \times [-3, 3]$.

Suggested Solutions: In other words, we want to show that there is constant L (independent of x) such that for all $y_1, y_2 \in [-3, 3]$, we have

$$|f(x, y_1) - f(x, y_2)| \le L|y_1 - y_2|.$$

We start from the left-hand side and observe that

$$\begin{aligned}
|e^{-x}y_1^3 - e^{-x}y_2^3| &= |e^{-x}||y_1^3 - y_2^3| \\
&\le 1 \cdot |y_1^3 - y_2^3| && 0 \le e^{-x} \le 1 \text{ for all } x \ge 0 \\
&\le |(y_1 - y_2)(y_1^2 + y_1y_2 + y_2^2)| && a^3 - b^3 = (a - b)(a^2 + ab + b^2) \\
&\le \left(|y_1|^2 + |y_1||y_2| + |y_2|^2\right)|y_1 - y_2| && \text{Triangle inequality} |a + b + c| \le |a| + |b| + |c| \\
&\le \left(3^2 + 3 \times 3 + 3^2\right)|y_1 - y_2| && \text{Since } y_1, y_2 \in [-3, 3], \text{ so } |y_1|, |y_2| \le 3 \\
&\le 27|y_1 - y_2|.
\end{aligned}$$

$\square$

Example 17. (Exercise 70.7, Modified.) Consider the initial value problem given by

$$\begin{cases} y'(x) = f(y(x)), \\ y(x_0) = y_0, \end{cases} \tag{43}$$

with $f(y) = y|y|$ for $y \in \mathbb{R}$.

(i) Show that f is differentiable on $\mathbb{R}$ and compute $f'(y)$ for each $y \in \mathbb{R}$.

(ii) Hence, show that $f'(y)$ is continuous on $\mathbb{R}$.

(iii) Show that $f(y)$ is Lipschitz on $[0, 1]$, but not on $\mathbb{R}$.

(iv) Using your answer in (i), (ii), and Theorem A, deduce the set of points (x_0, y_0) for which (43) has a unique solution on some interval $|x - x_0| \leq h$.

Suggested Solutions:

(i) Observe that for $y > 0$, we have $f(y) = y^2$. For $y < 0$, we have $f(y) = -y^2$. Hence, for $y \neq 0$, we can compute $f'(y)$ directly to give

$$f'(y) = \begin{cases} 2y & \text{for } y > 0, \\ -2y & \text{for } y < 0. \end{cases}$$

It remains to show that $f'(0)$ exists and compute its value. By the definition of derivative, we want to show that the limit

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h - 0} = \lim_{h \to 0} \frac{f(h)}{h}$$

exists. Since $f(h)$ takes two different arguments depending on if $h < 0$ or $h > 0$, we shall instead show that the limit exists by showing that

$$f'(0) = \lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0^+} \frac{f(h)}{h} = \lim_{h \to 0^-} \frac{f(h)}{h}.$$

Observe that

$$\lim_{h \to 0^+} \frac{f(h)}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = \lim_{h \to 0^+} h = 0$$

and

$$\lim_{h \to 0^+} \frac{f(h)}{h} = \lim_{h \to 0^-} \frac{-h^2}{h} = \lim_{h \to 0^-} -h = 0.$$

This then implies that $f'(0) = 0$, and thus

$$f'(y) = \begin{cases} 2y & \text{for } y > 0, \\ 0 & \text{for } y = 0, \\ -2y & \text{for } y < 0. \end{cases}$$

Hence f is differentiable on $\mathbb{R}$ with the derivative given above.

(ii) With $f'(y)$ as computed in (i), to see that f' is continuous on $\mathbb{R}$, it suffices to check that it is continuous at $y = 0$. Indeed, we have

$$\lim_{y \to 0^+} f'(y) = \lim_{y \to 0^+} 2y = \lim_{y \to 0^-} f'(y) = \lim_{y \to 0^-} (-2y) = 0 = f'(0).$$

(iii) <u>Lipschitz on $[0, 1]$.</u> Take any $y_1, y_2 \in [0, 1]$. Then, we have

$$\begin{aligned} |f(y_1) - f(y_2)| &= |y_1|y_1| - y_2|y_2|| = ||y_1|(y_1 - y_2) + y_2(|y_1| - |y_2|)| \\ &\leq |y_1||y_1 - y_2| + |y_2| \, ||y_1| - |y_2|| \leq |y_1||y_1 - y_2| + |y_2| \, |y_1 - y_2| \\ &\leq (|y_1| + |y_2|)|y_1 - y_2| \leq 2|y_1 - y_2|. \end{aligned}$$

Not Lipschitz on $\mathbb{R}$. Suppose for a contradiction that it is. Then for all $y_1, y_2 \in \mathbb{R}$, $|f(y_1) - f(y_2)| = |y_1|y_1| - y_2|y_2|| \leq L|y_1 - y_2|$ for some constant $L > 0$. In particular, at $y_2 = 0$, we have

$$|y_1|y_1|| \leq L|y_1| \implies |y_1|^2 \leq L|y_1| \implies |y_1| \leq L.$$

Since this has to be true for all $y_1 \in \mathbb{R}$, we pick $y_1 = 2L$. The above tells us that $2L \leq L$, a contradiction. Hence, f is **not** Lipschitz on $\mathbb{R}$.

(iv) Since for each (x_0, y_0), the functions $f(y)$ and $f'(y)$ are continuous, we can apply Theorem A on any closed rectangle containing (x_0, y_0) in its interior and conclude that for all $(x_0, y_0) \in \mathbb{R}^2$, (43) has a unique solution on some interval $|x - x_0| \leq h$.

Example 18. (Leaky Bucket Problem; Strogatz 2.5.6 Modified.) Consider a water bucket with a hole in the bottom. Water flows out of the bucket through the hole and leaves a puddle of water on the ground. Let $h(t)$ be the height of the water remaining in the bucket at time t, and k be a positive constant. By employing relevant physical laws, one can show that the height evolves according to the following differential equation:

$$h'(t) = -k\sqrt{h(t)}.$$

Let $f(t,h) = -k\sqrt{h}$ for each $t \in \mathbb{R}$ and $h \geq 0$. We then observe that the differential equation can be written as

$$h'(t) = f(t, h(t)). \tag{44}$$

(i) Show that $f(t,h)$ is not Lipschitz in h on $\mathbb{R} \times [0, \infty)$.

(ii) Given the initial condition $h(0) = 0$, demonstrate that non-uniqueness actually happens with (44) for $t < 0$.

(iii) Briefly explain why the existence of a solution is guaranteed despite exhibiting non-uniqueness as seen in (ii).

(iv) Give a physical interpretation of the non-uniqueness phenomenon as observed in (ii).

Suggested Solutions:

(i) Suppose for a contradiction that it is. Then, there is a constant $L > 0$ such that for all $t \in \mathbb{R}, h_1, h_2 \geq 0$, we have
$$|-k\sqrt{h_1} + k\sqrt{h_2}| = |f(t,h_1) - f(t,h_2)| \leq L|h_1 - h_2|.$$
Since this is true for all $h_1, h_2 \geq 0$, we pick $h_2 = 0$. Then, this implies
$$k\sqrt{h_1} \leq L|h_1| = L\sqrt{h_1}^2 \implies \sqrt{h_1} \geq k/L.$$
Now, pick $h_1 = \frac{k^2}{4L^2}$. The above inequality then implies that
$$k/(2L) \geq k/L,$$
which is a contradiction (since k and L are all positive).

(ii) Observe that for $h' = -k\sqrt{h}$, with $h(0) = 0$, the constant solution
$$h(t) = 0$$
satisfies the initial condition and solves the ODE (since $h' = h = 0$ for each t).

On the other hand, we can employ the separation of variables as follows. Since
$$\frac{\mathrm{d}h}{\mathrm{d}t} = -kh^{1/2} \implies \int h^{-1/2}\mathrm{d}h = \int -k\mathrm{d}t \implies \frac{\sqrt{h}}{1/2} = -kt + C.$$
Hence, we have
$$h(t) = \left(C - \frac{kt}{2}\right)^2.$$
Employing the initial condition $h(0) = 0$, we will have $C = 0$. This implies that
$$h(t) = \frac{k^2t^2}{4}, \qquad \text{for } t < 0 \tag{45}$$
is yet another solution that solves the initial value problem that is not the zero function. We have thus demonstrated that there are two different solutions to the ODE. Hence, the differential equation exhibits non-uniqueness.

Remark: In fact, any function of the form

$$h(t) = \begin{cases} \frac{k^2(t-t_0)^2}{4} & \text{for } t < t_0 \\ 0 & \text{for } t_0 \le t \le 0 \end{cases} \tag{46}$$

for any $t_0 < 0$ is a solution! This implies that we have infinitely many solutions to the differential equation!!

(iii) Even though $f(t,h)$ is not Lipschitz in h for $(t,h) \in \mathbb{R} \times [0,\infty)$, it is nonetheless continuous in this same domain. **Peano's theorem** thus applies, at least on the time interval $|t| \le h$ for some $h > 0$. Hence, we must still be able to "solve" the differential equation for a small time, which implies that solutions have to exist.

(iv) Physical Interpretation: If you see an empty bucket with a puddle of water, it is not immediately obvious if the bucket has been emptied some time ago (in (46), that corresponds to t_0), or was it just emptied when you first look at the bucket (in (46), that corresponds to $t_0 = 0$ so we are looking at (45)).

5 Discussion 5

Preliminaries for Fourier Series

The main question of interest to ask is if we can write any function $f(x)$ in a trigonometric series of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)). \tag{47}$$

Before we begin with the rigorous theory, we would like to be able to perform formal computations without needing to worry about convergence of the series etc. Furthermore, for ease of computation, we will be considering a function that is defined on the closed interval $x \in [-\pi, \pi]$.

Before we begin, we shall recap a couple of trigonometric identities as follows.

- Basic Identities:
$$\begin{aligned} \sin(\theta)^2 + \cos(\theta)^2 &= 1, \\ \sin(-\theta) &= -\sin(\theta), \\ \cos(-\theta) &= \cos(\theta). \end{aligned}$$

- Angle Sum:
$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \\ \cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta. \end{aligned}$$

- Double Angle Formulas (set $\alpha = \beta$ from above):
$$\begin{aligned} \sin(2\theta) &= 2\sin\theta\cos\theta, \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta, \end{aligned}$$

- Factor Formulae:
$$\begin{aligned} \sin\theta\cos\varphi &= \frac{1}{2}\left(\sin(\theta+\varphi) + \sin(\theta-\varphi)\right), \\ \cos\theta\cos\varphi &= \frac{1}{2}\left(\cos(\theta-\varphi) + \cos(\theta+\varphi)\right), \\ \sin\theta\sin\varphi &= \frac{1}{2}\left(\cos(\theta-\varphi) - \cos(\theta+\varphi)\right). \end{aligned}$$
In particular, by setting $\theta = mx$ and $\varphi = nx$, we have
$$\begin{aligned} \sin mx\cos nx &= \frac{1}{2}\left(\sin((m+n)x) + \sin((m-n)x)\right), \\ \cos mx\cos nx &= \frac{1}{2}\left(\cos((m+n)x) + \cos((m-n)x)\right), \\ \sin mx\sin nx &= \frac{1}{2}\left(\cos((m-n)x) - \cos((m+n)x)\right). \end{aligned}$$

- Orthogonality: For non-negative integers m and n, we have
$$\begin{aligned} \int_{-\pi}^{\pi} \sin(mx)\cos(nx)\mathrm{d}x &= 0, \quad \text{for all } m \neq n \text{ and } n \geq 1, m \geq 0, \\ \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\mathrm{d}x &= 0, \quad \text{for all } m \neq n \text{ and } m, n \geq 0, \\ \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\mathrm{d}x &= 0, \quad \text{for all } m \neq n \text{ and } m, n \geq 1, \end{aligned}$$
The above formulae are evaluated with the help of factor formulae. Note that they do not cover the case when $m = n$, or if the coefficient inside the $\cos$ term is zero (and hence $\cos(0x) = 1$). These are evaluated

below (which one can check with the help of trigonometric formulas):

$$\begin{aligned}\int_{-\pi}^{\pi} \sin^2(mx)\mathrm{d}x &= \pi, \quad \text{for all } m \geq 1,\\ \int_{-\pi}^{\pi} \sin(mx)\mathrm{d}x = \int_{-\pi}^{\pi} \cos(mx)\mathrm{d}x &= 0, \quad \text{for all } m \geq 1,\\ \int_{-\pi}^{\pi} \cos^2(nx)\mathrm{d}x &= \pi, \quad \text{for all } n \geq 1,\\ \int_{-\pi}^{\pi} 1\mathrm{d}x &= 2\pi, \quad n = 0 \text{ for the above formula.}\end{aligned}$$

These formulae imply that $1, \sin(nx), \cos(nx)$ forms an "**orthogonal basis**" for some appropriate "space of functions" which we will talk more about as we go deep into the rigorous theory for the above concept.

To evaluate the coefficients in (47), we multiply both sides of the equation by $\sin(mx)$ to obtain

$$\sin(mx)f(x) = \frac{1}{2}a_0\sin(mx) + \sum_{n=1}^{\infty}(a_n\sin(mx)\cos(nx) + b_n\sin(mx)\sin(nx)). \tag{48}$$

Integrating from $-\pi$ to π and assuming that we can interchange integration and derivatives, we have

$$\int_{-\pi}^{\pi}\sin(mx)f(x)\mathrm{d}x = \frac{1}{2}a_0\int_{-\pi}^{\pi}\sin(mx)\mathrm{d}x + \sum_{n=1}^{\infty}(a_n\int_{-\pi}^{\pi}\sin(mx)\cos(nx)\mathrm{d}x + b_n\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\mathrm{d}x). \tag{49}$$

By the orthogonality formula, we see that the only integral on the right that survives is when $m = n$, and we are forced to evaluate $\int_{-\pi}^{\pi}\sin^2(mx)\mathrm{d}x = \pi$. Thus, we have

$$\begin{aligned}\pi b_m &= \int_{-\pi}^{\pi}\sin(mx)f(x)\mathrm{d}x\\ b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(mx)f(x)\mathrm{d}x \quad \text{for } m \geq 1.\end{aligned} \tag{50}$$

Using a similar idea but by multiplying with $\cos(mx)$ for $m \geq 1$ instead, we have

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(mx)f(x)\mathrm{d}x \quad \text{for } m \geq 1. \tag{51}$$

Last but not least, to evaluate a_0, we do not multiply by anything and integrate from $-\pi$ to π to obtain

$$\int_{-\pi}^{\pi}f(x)\mathrm{d}x = \frac{1}{2}a_0\int_{-\pi}^{\pi}1\mathrm{d}x + \sum_{n=1}^{\infty}(a_n\int_{-\pi}^{\pi}\cos(nx)\mathrm{d}x + b_n\int_{-\pi}^{\pi}\sin(nx)\mathrm{d}x) = \frac{1}{2}a_0\cdot 2\pi. \tag{52}$$

Hence, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\mathrm{d}x. \tag{53}$$

Thus, we see that the formula in (51) can be generalized for $m \geq 0$. Henceforth, the **Fourier coefficients** a_n and b_n for the series (also known as the **Fourier series**) in (47) are given by

$$\begin{aligned}&\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(nx)f(x) \quad \text{for } n \geq 0},\\ &\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(nx)f(x) \quad \text{for } n \geq 1}.\end{aligned} \tag{54}$$

Example 19. Find the Fourier series for the function defined by $f(x) = 1$ for all $x \in [-\pi, \pi]$.

Suggested Solutions: Using the formulas in (54), we have

- For $n \geq 1$, we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)\mathrm{d}x = \frac{1}{\pi}\frac{\sin(nx)}{n}|_{x=-\pi}^{x=\pi} = \frac{1}{n\pi}\left(\sin(n\pi) - \sin(-n\pi)\right) = 0$$

  since $\sin(n\pi) = 0$ for any integer n. The above integration does not hold for $n = 0$ since the general formula for integrating $\cos$ does not hold if the argument in it is zero (because this reduces to a constant function 1 that integrates to x, which we will see in a bit).

- For a_0, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{d}x = \frac{2\pi}{\pi} = 2.$$

- For $n \geq 1$, we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)\mathrm{d}x = 0.$$

  (This definite integral has been computed under orthogonal formulae above.)

Hence, we have

$$\begin{aligned} f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)) \\ &= \frac{1}{2}(2\pi) \\ &= 1. \end{aligned}$$

This is not surprising since $1, \sin(nx), \cos(nx)$ are orthogonal functions, and hence if we try to represent one of the basis functions with this basis, you should only be able to retrieve itself!

Example 20. Using the concept of orthogonality, deduce the Fourier series for the function

$$f(x) = \sin(x+1) \quad \text{for } x \in [-\pi, \pi]$$

without explicitly computing the Fourier coefficients in (54).

Suggested Solutions:

Observe that by the sum of angles formula, we have

$$\begin{aligned} \sin(x+1) &= \sin(x)\cos(1) + \cos(x)\sin(1) \\ f(x) &= \boxed{\sin(1)\cos(x) + \cos(1)\sin(x)}. \end{aligned}$$

Hence, the above equation is in the form of a Fourier series, with $a_1 = \sin(1), b_1 = \cos(1)$, and $a_n = b_n = 0$ for all other valid values of n.

Mathematical Preliminaries for Fourier Series

For a function $f(x)$, we say that:

- It is bounded if there is a constant M such that for all x in consideration,

$$|f(x)| \leq M. \tag{55}$$

- It is continuous at x_0 if

$$\lim_{x\to x_0^+} f(x) \text{ and } \lim_{x\to x_0^-} f(x) \text{ exist, and } \quad \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = f(x_0). \tag{56}$$

- It is differentiable at x_0 if

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \text{ exist.} \tag{57}$$

  We then define

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}. \tag{58}$$

- It is discontinuous at x_0 if it is not continuous at x_0.

There are many ways in which a function can be discontinuous (that is, (56) fails). These types of discontinuity are listed below.

1. We can have that the limits $\lim_{x\to x_0^+} f(x)$ and $\lim_{x\to x_0^-} f(x)$ exist, and are equal, but

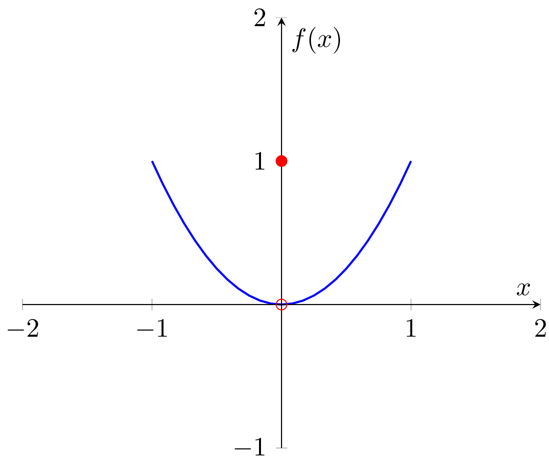
$$\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) \neq f(x_0). \tag{59}$$

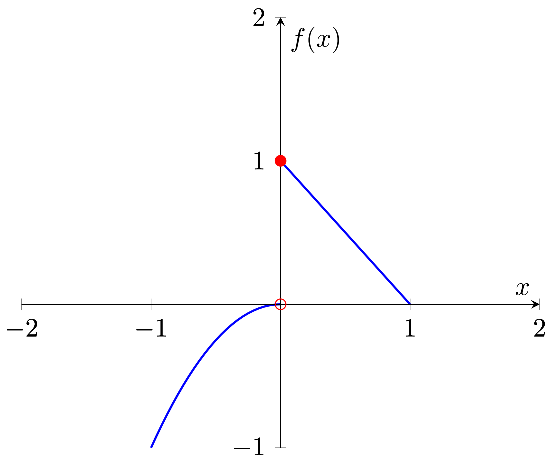
   We call this type of discontinuity a **removable discontinuity**. For convenience, we shall assume that the functions we are considering do not have removable continuity, as a finite number of these[6] will not modify the integrals in the Fourier coefficients as seen in (54).

2. Furthermore, we can also have that the limits exist but we have

$$\lim_{x\to x_0^+} f(x) \neq \lim_{x\to x_0^-} f(x). \tag{60}$$

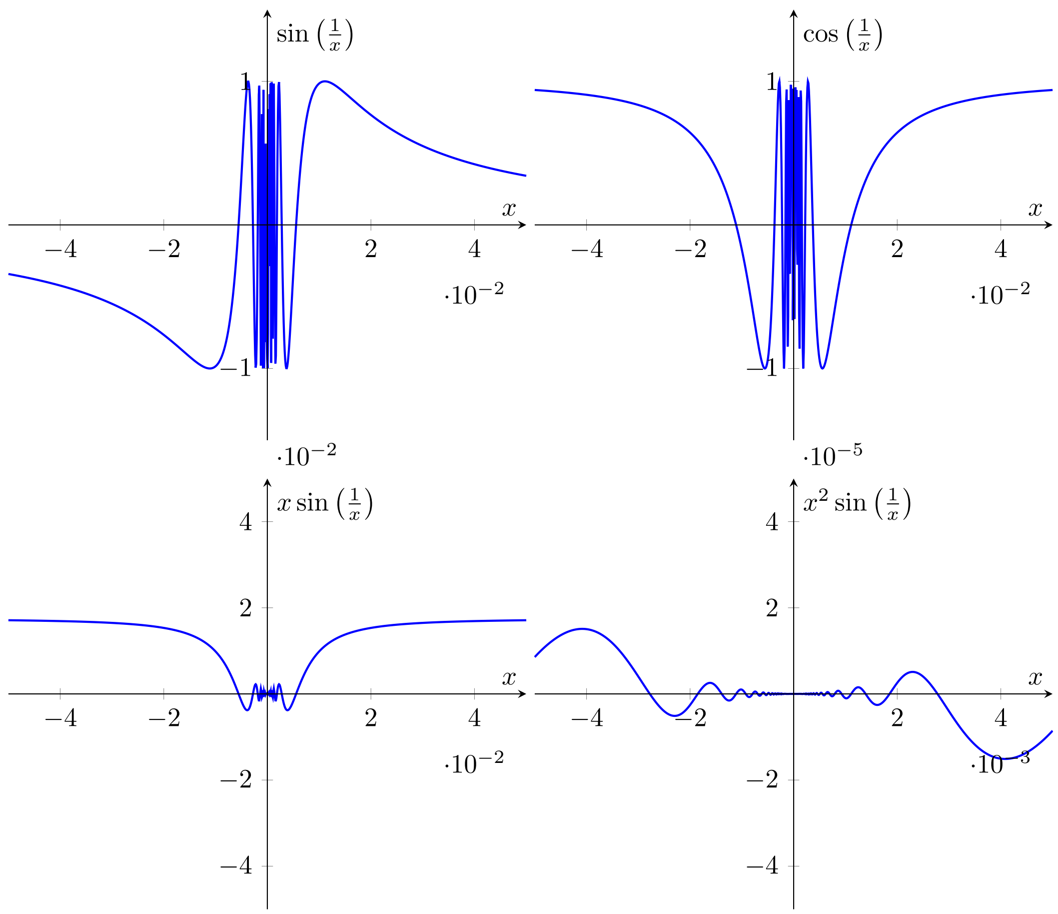
   We call this type of discontinuity a **simple/jump discontinuity**. In other words, since the left and right limits are not equal, there is a "jump" in the function corresponding to the difference in the limits from the left and right.





Filled red circle - Included; Hollow red circle - Excluded;
Left graph - Removable discontinuity at $x = 0$;
Right graph - Simple/jump discontinuity at $x = 0$.

[6]We only consider finitely many discontinuities as this is under the hypothesis of Dirichlet theorem, as we shall see in the next discussion.



Observe that from the plots above the following facts:

- $\sin\left(\frac{1}{x}\right)$ is discontinuous at $x = 0$, and this is not a simple/jump discontinuity. In fact, the limit $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exists (as the function oscillates between -1 and 1 arbitrarily fast)!

- $x\sin\left(\frac{1}{x}\right)$ is continuous at $x = 0$, since

  $$\lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = \lim_{x\to 0^-} x\sin\left(\frac{1}{x}\right) = 0.$$

  It is **not** differentiable at $x = 0$ since

  $$\lim_{h\to 0}\frac{h\sin(1/h) - 0}{h} = \lim_{h\to 0}\sin(1/h)$$

  does not exist.

- $x^2\sin\left(\frac{1}{x}\right)$ is continuous at $x = 0$. Furthermore, it is differentiable at $x = 0$ since

  $$\lim_{h\to 0}\frac{h^2\sin(1/h) - 0}{h} = \lim_{h\to 0} h\sin(1/h) = 0.$$

  However, its derivative given by

  $$f'(x) = 2x\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)$$

  is not continuous at $x = 0$ due to the highly oscillatory $\cos\left(\frac{1}{x}\right)$ term. In fact, we could also see that the discontinuity in the derivative is not simple due to the same highly oscillatory $\cos\left(\frac{1}{x}\right)$ term.

Midterm Review:
Note: The examples/exercises below are selected modified problems from the homework problems and discussion supplements (usually problems that I did not manage to go through in class for that week). These are solely done for practice purposes and are **not necessarily reflective** of the nature of questions in the midterm.

1. Solving ODEs.

- For an inhomogeneous ODE, we have

$$y(x) = y_g(x) + y_p(x),$$

  where $y_g(x)$ is the solution to the homogeneous problem (RHS = 0) and $y_p(x)$ is a particular solution to the ODE.

- For $y_g(x)$, solve characteristic polynomial for λ. For second-order ODE, the solutions will be given by

Roots are real and distinct	$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
Roots are complex	$y(x) = e^{-\text{Re}(\lambda)x}(A\cos(\text{Im}(\lambda)x) + B\sin(\text{Im}(\lambda)x))$
Roots are repeated	$y(x) = e^{\lambda x}(C_1 + C_2 x)$

- For $y_p(x)$, you have to guess this by employing the variation of parameters technique according to what is on the right-hand side of the equation.

- Know how to solve the Cauchy-Euler's equation of the form

$$ax^2 y''(x) + bxy'(x) + cy(x) = 0.$$

2. Computing Laplace Transforms - Table of Laplace Transforms and Properties

- Recall the table of Laplace Transforms below:

$f(x)$	$F(p) = \mathcal{L}[f(x)](p)$
1	$\frac{1}{p}$
x	$\frac{1}{p^2}$
x^n	$\frac{n!}{p^{n+1}}$
e^{ax}	$\frac{1}{p-a}$ for $p > a$
$\sin(ax)$	$\frac{a}{p^2+a^2}$
$\cos(ax)$	$\frac{p}{p^2+a^2}$
$\sinh(ax)$	$\frac{a}{p^2-a^2}$ for $p > \|a\|$
$\cosh(ax)$	$\frac{p}{p^2-a^2}$ for $p > \|a\|$
$\delta(x)$	1

-
  - Linearity: $\mathcal{L}[\alpha f(x) + \beta g(x)](p) = \alpha\mathcal{L}[f(x)](p) + \beta\mathcal{L}[g(x)](p)$.
  - *Shifting formula:* $\mathcal{L}[e^{ax} f(x)](p) = F(p-a)$.
  - Laplace Transform of Derivative: $\mathcal{L}[y'(x)] = pF(p) - y(0)$.
  - Laplace Transform of Derivative: $\mathcal{L}[y''(x)] = p^2\mathcal{L}[y](p) - py(0) - y'(0)$.
  - Laplace Transform of Integrals (See Exercise 50.5): $\mathcal{L}\left[\int_0^x f(t)\mathrm{d}t\right](p) = \frac{F(p)}{p}$.
  - Derivatives of Laplace Transforms: $\mathcal{L}[(-x)^n f(x)](p) = \frac{\mathrm{d}^n}{\mathrm{d}p^n} F(p)$.
  - Integrals of Laplace Transforms: $\mathcal{L}\left[\frac{f(x)}{x}\right](p) = \int_p^\infty F(s)\mathrm{d}s$.
  - Convolution Theorem: $\mathcal{L}[f * g](p) = F(p)G(p)$.
    "Laplace transform of the convolution is the product of their Laplace transforms."
  - $\mathcal{L}[y'(x)](p) = p\mathcal{L}[y](p) - y(0)$. (This is an expression in p, and $y(0)$ does not depend on p.)
  - $\mathcal{L}[y''(x)](p) = p^2\mathcal{L}[y](p) - py(0) - y'(0)$.

- For computing inverse Laplace transforms, for quadratic factors in the denominator, always check if it is factorizable (and hence do partial fraction decomposition), or if it is irreducible (in this case, look at the Laplace transforms of sin and cos).

3. Existence and Uniqueness of Solutions to ODEs.

$$\begin{cases} y'(x) = f(x, y(x)), \\ y(x_0) = y_0, \end{cases} \tag{61}$$

- Picard's Method/Iteration:
  1. Let $y_0(x) = y_0$.
  2. Compute the sequence of functions $y_1(x), y_2(x), \ldots$ using the formula
  $$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\mathrm{d}t.$$

- A function $F : [a, b] \to \mathbb{R}$ with $a < b$ is **Lipschitz** if there is a constant $L > 0$ such that for all $x, y \in [a, b]$, we have
$$|F(x) - F(y)| \leq L|x - y|.$$
We call L here the associated Lipschitz constant.

- **(Picard's Theorem; Theorem A.)** Let $f(x, y)$ and $\frac{\partial f}{\partial y}$ be continuous functions of x and y on a closed rectangle R given by
$$R = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ c \leq y \leq d\}$$
for some real numbers $a < b$ and $c < d$. If (x_0, y_0) is any interior point of R,[7] then there exists a number $h > 0$ with the property that the initial value problem in (42) has a unique solution on the interval $|x - x_0| \leq h$.

  (Picard's Theorem; Theorem A Modified.) Let $f(x, y)$ be continuous functions of x and y on a closed rectangle R given by
$$R = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ c \leq y \leq d\}$$
for some real numbers $a < b$ and $c < d$. Furthermore, we also demand that for any given $x \in [a, b]$, $f(x, \cdot)$ is Lipschitz with Lipschitz constant not depending on x. In other words, there exists a constant $L > 0$ such that for each $x \in [a, b], y_1, y_2 \in \mathbb{R}$,
$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$$
with L not depending on x. If (x_0, y_0) is any interior point of R, then there exists a number $h > 0$ with the property that the initial value problem in (42) has a unique solution on the interval $|x - x_0| \leq h$..

  (Peano's Theorem.) If f is only continuous, we can guarantee local existence, but not uniqueness. In other words, if (x_0, y_0) is any interior point of R, then there exists a number $h > 0$ with the property that the initial value problem in (42) has a solution on the interval $|x - x_0| \leq h$.

  (Theorem B.) Let $f(x, y)$ be continuous functions of x and y on a closed strip S given by
$$S = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ -\infty < y < \infty\}$$
for some real numbers $a < b$. In addition, we also demand that for any given $x \in [a, b]$, $f(x, \cdot)$ is Lipschitz with Lipschitz constant not depending on x. In other words, there exists a constant $L > 0$ such that for each $x \in [a, b], y_1, y_2 \in \mathbb{R}$,
$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$$
with L not depending on x.

  If $(x_0, y_0) \in S$, then the initial value problem (42) has a unique solution on the interval $x \in [a, b]$.

[7] In other words, $a < x_0 < b$ and $c < y_0 < d$.

- Know when to apply Theorem A (local existence) and B (global existence), and how to check if the hypothesis holds (in particular, check if a function is **Lipschitz** in y).
- Be aware of an example illustrating a lack of uniqueness or existence of solutions. This happens when the above theorems cannot be applied (See Section 70 Problem 1 and Discussion 4 Example on the Leaky Bucket problem).

Some practice problems:

Exercise 1. Solve the following initial value problems:

(i) $y''(x) + 9y(x) = \cos(x), y(0) = 1, y'(0) = 0,$

(ii) $y''(x) - y'(x) - 2y(x) = e^{3x}, y(0) = 1, y'(0) = 0,$

(iii) $x^2y''(x) - xy'(x) = 0, y(1) = 1, y'(1) = 1.$

Exercise 2. Evaluate $\mathcal{L}[xe^{10x}\sin(x)]$.
Hint: Use two properties of Laplace transforms to reduce this to one of the usual Laplace transforms.

Exercise 3. Evaluate $\mathcal{L}^{-1}\left[\frac{1}{p^2-p-2}\right]$.

Exercise 4. (Exercise 50.6.) By using the property

$$\mathcal{L}\left[\int_0^x f(t)\mathrm{d}t\right](p) = \frac{F(p)}{p},$$

solve the following differential equation:

$$y' + 4y + 5\int_0^x y(t)\mathrm{d}t = e^{-x}, \quad y(0) = 0. \tag{62}$$

Exercise 5. Solve the following integro-differential equation:

$$y'(x) = -y(x) - \int_0^x e^{t-x}y(t)\mathrm{d}t$$

with initial condition $y(0) = 1$.

Exercise 6. Consider the initial value problem

$$y' = y^3, \quad y(0) = 1.$$

(i) Starting with $y_0(x) = 1$, apply Picard's method to calculate $y_1(x)$ and $y_2(x)$.

(ii) Explain why we can expect to have a **unique** solution to the above initial value problem for $x \in [-h, h]$ for some $h > 0$.

Exercise 7. (Exercise 70.7, Modified.) Consider the initial value problem given by

$$\begin{cases} y'(x) = f(y(x)), \\ y(x_0) = y_0, \end{cases} \tag{63}$$

with $f(y) = y|y|$ for $y \in \mathbb{R}$.

(i) Show that f is differentiable on $\mathbb{R}$ and compute $f'(y)$ for each $y \in \mathbb{R}$.

(ii) Hence, show that $f'(y)$ is continuous on $\mathbb{R}$.

(iii) Show that $f(y)$ is Lipschitz on $[0, 1]$, but not on $\mathbb{R}$.

(iv) Using your answer in (i), (ii), and Theorem A, deduce the set of points (x_0, y_0) for which (63) has a unique solution on some interval $|x - x_0| \leq h$.

Answers and Hints/Partial Solutions:

- Answer to Exercise 1:

  (i) $y(x) = \frac{7}{8}\cos(3x) + \frac{1}{8}\cos(x)$.

  (ii) $y(x) = \frac{1}{4}e^{3x} + \frac{3}{4}e^{-x}$.

  (iii) $y(x) = \frac{1}{2}(1 + x^2)$.

- Answer to Exercise 2: $\frac{-2(10-p)}{((p-10)^2+1)}$. Use derivative of Laplace transform and translation property.

- Answer to Exercise 3: $\frac{1}{3}e^{2x} - \frac{1}{3}e^{-x}$.

- Answer to Exercise 4: See Discussion Supplement 2, Example 5.

- Answer to Exercise 5: See Discussion Supplement 3, Example 2.

- Answer to Exercise 6: (i) $y_1(x) = 1 + x, y_2(x) = \frac{3}{4} + \frac{(1+x)^4}{4}$.
  (ii) $f(x, y) = y^3$ is continuous and differentiable on any closed rectangle containing the point $(0, 1)$. Hence, Theorem A applies, and we thus have a unique local solution around $x = 0$.

- Answer to Exercise 7: See Discussion Supplement 4, Example 7.

6 Discussion 6

Recap: Fourier Series

The main question of interest to ask is if we can write any function $f(x)$ defined on $[-\pi, \pi)$ in a trigonometric series of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)). \tag{64}$$

The **Fourier coefficients** a_n and b_n for the series in (64) are given by

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx) f(x) \quad \text{for } n \geq 0}, \qquad \boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx) f(x) \quad \text{for } n \geq 1}. \tag{65}$$

Dirichlet Theorem, Periodic Extension, and Fourier Sine/Cosine Series.

The Dirichlet Theorem talks about the convergence of a Fourier series defined on $[-\pi, \pi)$ below:

Theorem 21. Assume that $f(x)$ is defined and bounded on $x \in [-\pi, \pi)$, and also that it has only a finite number of discontinuities and only a finite number of maxima and minima on this interval. Let $f(x)$ be defined for other values of x by the periodicity condition $f(x + 2\pi) = f(x)$. Then, the Fourier series of $f(x)$ converges to

$$\frac{1}{2}\left(f(x^-) + f(x^+)\right) \tag{66}$$

at every point x.

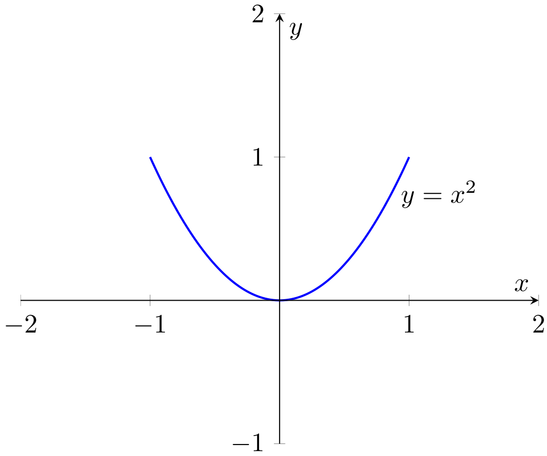
Some remarks on this include:

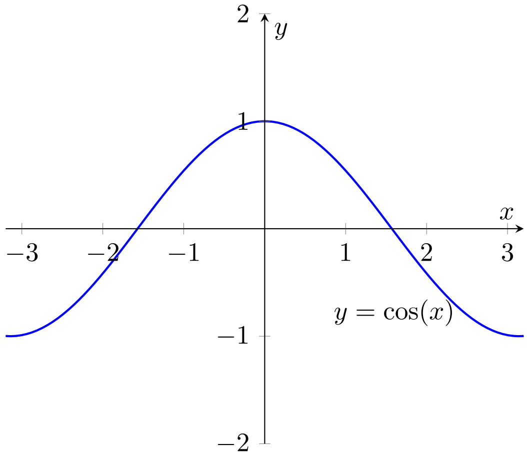
- In particular, if the function was already continuous at a given point x_0, then the Fourier series converges to
$$\frac{1}{2}(f(x^-) + f(x^+)) = \frac{1}{2}(f(x_0) + f(x_0)) = f(x_0).$$
In other words, the Fourier series converges to the value of the function that it represents at points of continuity of the function.
- On the other hand, if the function is not continuous at x_0, then the Fourier series converges to the value described in (66). Hence, if we redefine the function at x_0 to take $f(x_0) = \frac{1}{2}\left(f(x^-) + f(x^+)\right)$, then the Fourier series will converge to this value. In other words, the Fourier series will converge to $f(x)$ at every point x.

Next, we recall the definition of even and odd functions defined on $[-\pi, \pi)$ below.

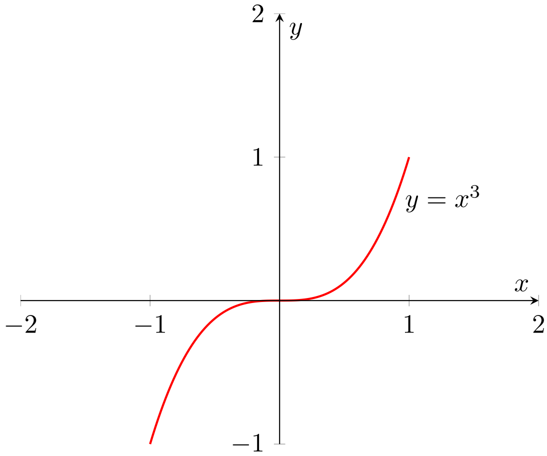
- A function $f(x)$ is even if for all x,
$$f(-x) = f(x).$$
Example: $f(x) = x^2, \cos(x)$.
- A function $f(x)$ is odd if for all x, we have
$$f(-x) = -f(x).$$
Example: $f(x) = x^3, \sin(x)$.

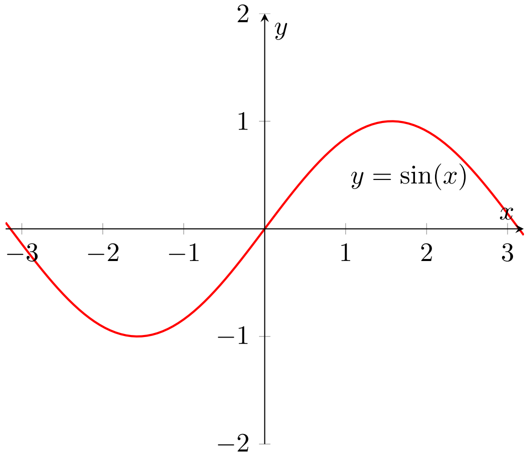
Examples of even functions (symmetrical about the y-axis):





Examples of odd functions:





Some properties of even and odd functions include:

- If $f(x)$ is even and $g(x)$ is odd, then $f(x)g(x)$ is odd.
- If $f(x)$ is even and $g(x)$ is even, then $f(x)g(x)$ is even.
- If $f(x)$ is odd and $g(x)$ is odd, then $f(x)g(x)$ is even.
- For $a > 0$ and $f(x)$ is even, we have
$$\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x.$$
- For $a > 0$ and $f(x)$ is odd, we have
$$\int_{-a}^{a} f(x)\mathrm{d}x = 0.$$

The last two properties then imply the following properties of a Fourier series of an integrable function $f(x)$ defined on $x \in [-\pi, \pi)$:

- If $f(x)$ is even, then its Fourier series only has cosine terms and the coefficients are given by
$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\mathrm{d}x, \quad \text{and } b_n = 0.$$
This is known as the **Fourier cosine series**.
- If $f(x)$ is odd, then its Fourier series only has cosine terms and the coefficients are given by
$$a_n = 0, \quad \text{and } b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\mathrm{d}x.$$
This is known as the **Fourier sine series**.

On the other hand, for functions $f(x)$ defined on $[0, \pi]$,

- We can do an even extension[8] to $[-\pi, 0]$, and compute the Fourier series for this new function on $[-\pi, \pi]$. Since the function is now even on $[-\pi, \pi]$, we have the **Fourier cosine series** defined on $[-\pi, \pi]$. This can then be used to "represent" $f(x)$ on its original interval in which it is defined, that is, $[0, \pi]$.
- We can do an odd extension[9] to $[-\pi, 0]$, and compute the Fourier series for this new function on $[-\pi, \pi]$. Since the function is now odd on $[-\pi, \pi]$, we have the **Fourier sine series** defined on $[-\pi, \pi]$. This can then be used to "represent" $f(x)$ on its original interval in which it is defined, that is, $[0, \pi]$.

[8] An even extension here means that for $x < 0$, we define $f(x) := f(-x)$. For instance, if we want to define $f(-1)$, this is defined as $f(-(-1)) = f(1)$, in which f is defined on. Note that this way of extending guarantees that the new extended function is even, and hence is called an **even extension**.

[9] An even extension here means that for $x < 0$, we define $f(x) := -f(-x)$. For instance, if we want to define $f(-1)$, this is defined as $-f(-(-1)) = -f(1)$, in which f is defined on. Note that this way of extending guarantees that the new extended function is odd, and hence is called an **odd extension**. It is worth nothing that sometimes, we have to re-define the value of 0 since for an odd function, we have $f(-0) = -f(0)$ implies $2f(0) = 0$ and thus $f(0) = 0$. This means that the original function must be redefined at $f(0)$ to be 0 for this to work.

Example 22. (Exercise 35.4, Modified.)

(i) Show that the sine series of the constant function $f(x) = \pi/4$ defined on $[0, \pi]$ is

$$\frac{\pi}{4} = \sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \cdots, \quad x \in (0, \pi). \tag{67}$$

(ii) Compute the cosine series of the same function $f(x) = \pi/4$ defined on $[0, \pi]$.

(iii) Use (67) and plug in $x = \frac{\pi}{2}$ to compute the value of the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)}.$$

(iv) Notice that when we plug in $x = \pi$ into (67), the right hand side of the equation yields $0 + 0 + 0 + \cdots = 0$ while the left hand side gives $\frac{\pi}{4}$. Explain the disparity observed above.

(v) Notice that when we plug in $x = 0$ into (67), the right hand side of the equation yields $0 + 0 + 0 + \cdots = 0$ while the left hand side gives $\frac{\pi}{4}$. Explain the disparity observed above.

Suggested Solutions:

(i) Recall that to compute the sine series, we do an odd extension of the function. The (Fourier) (sine) series is then given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \quad \text{with} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\mathrm{d}x.$$

Plugging $f(x) = \frac{\pi}{4}$ and for $n \geq 1$, we get

$$b_n = \frac{2}{\pi}\int_0^{\pi} \frac{\pi}{4}\sin(nx)\mathrm{d}x = \frac{1}{2}\left(\frac{-\cos(nx)}{n}\right)_{x=0}^{x=\pi} = \frac{1}{2n}\left(\cos(0) - \cos(n\pi)\right).$$

Recall that $\cos(0) = 1$, and $\cos(n\pi) = (-1)^n$. Hence, we have

$$b_n = \frac{1-(-1)^n}{2n} = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

This then gives the series above in (67). In other words, we have

$$\frac{\pi}{4} = \sum_{\text{odd } n} \frac{\sin(nx)}{n} = \sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{2n-1}. \tag{68}$$

(ii) Recall that to compute the cosine series, we do an even extension of the function. The (Fourier) (cosine) series is then given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \quad \text{with} \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\mathrm{d}x.$$

Plugging $f(x) = \frac{\pi}{4}$, we get $a_0 = \frac{2}{\pi}\int_0^{\pi} \frac{\pi}{4}\mathrm{d}x = \frac{\pi}{2}$ and for $n \geq 1$,

$$a_n = \frac{2}{\pi}\int_0^{\pi} \frac{\pi}{4}\cos(nx)\mathrm{d}x = \frac{1}{2}\left(\frac{-\sin(nx)}{n}\right)_{x=0}^{x=\pi} = 0$$

since $\sin(n\pi) = 0$ for integer values of n. Hence, the (Fourier) (cosine) series is (unsurprisingly)

$$\frac{\pi}{4} = \frac{\pi}{4} + \sum_{n=1}^{\infty} 0\cos(nx).$$

This is unsurprising given the fact that the function $\frac{\pi}{4}$ remains unchanged upon an even extension, and it is one of the “basis” functions for a (full) Fourier series.

(iii) Plugging $x = \frac{\pi}{2}$ into (68) and using the fact that $\sin\left(\left(n - \frac{1}{2}\right)\pi\right) = (-1)^{n-1}$, (67) then follows.

(iv) Recall that in the process of obtaining (66), we did an odd extension up to $[-\pi, \pi)$. The function is then re-defined periodically using $f(x) = f(x + 2\pi)$. This implies that

$$f(\pi^-) = \frac{\pi}{4}$$

since we are using the function defined on $[-\pi, \pi)$, which is **not obtained from periodic extension**. On the other hand, to evaluate $f(\pi+)$, we need the function value on $[\pi, 3\pi)$, which **is defined by periodic extension**. Hence, we have

$$f(\pi^+) = f(-\pi^-) \overset{\text{odd extension}}{=} -f(\pi^-) = -\frac{\pi}{4}.$$

Hence, by **Dirichlet theorem**. the Fourier (sine) series converges to

$$\frac{1}{2}\left(f(\pi^-) + f(\pi^+)\right) = \frac{1}{2}\left(\frac{\pi}{4} - \frac{\pi}{4}\right) = 0,$$

which is consistent with what is evaluated with the Fourier (sine) series at $x = \pi$.

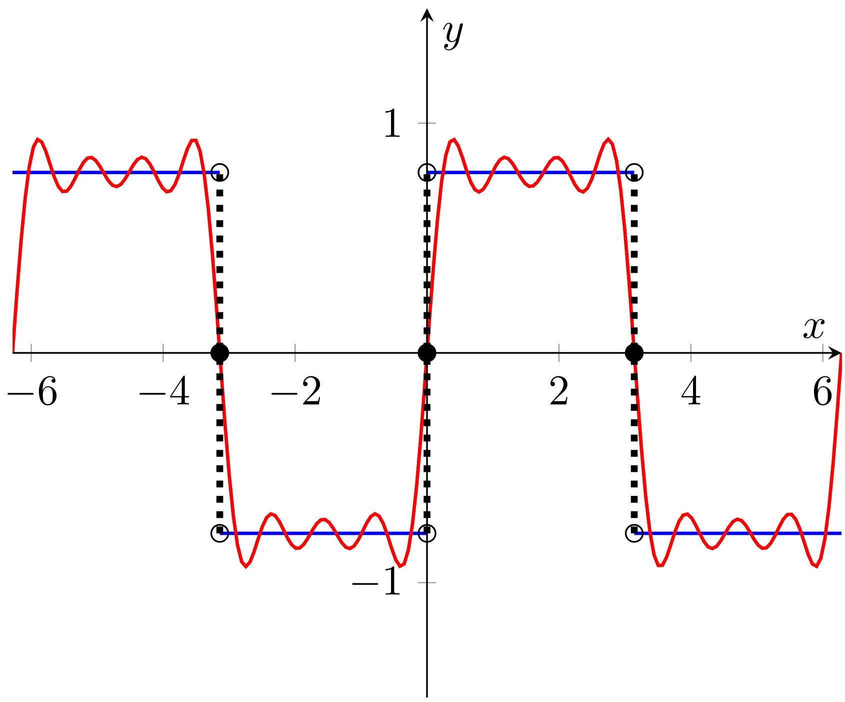
(v) This is because upon odd extension, the function should really be

$$f(x) = \begin{cases} \frac{\pi}{4} & \text{for } x \in (0, \pi], \\ 0 & \text{for } x = 0, \\ -\frac{\pi}{4} & \text{for } x \in [-\pi, 0). \end{cases}$$

Since f is discontinuous at $x = 0$, **Dirichlet theorem** says that the Fourier series will converge to

$$\frac{1}{2}\left(f(0^-) + f(0^+)\right) = \frac{1}{2}\left(-\frac{\pi}{4} + \frac{\pi}{4}\right) = 0,$$

which is consistent with what is evaluated with the Fourier (sine) series at $x = 0$.



The red curve represents the Fourier series $\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7}$, while the blue curve represents the odd extension of $f(x)$, followed by a periodic extension. Observe the points in which the Fourier sine series converges too, especially at the points of discontinuity (and see Dirichlet theorem in action!).

7 Discussion 7

Recap: Fourier Series

The main question of interest to ask is if we can write any function $f(x)$ defined on $[-\pi, \pi)$ in a trigonometric series of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right). \tag{69}$$

The **Fourier coefficients** a_n and b_n for the series in (69) are given by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx) f(x) \quad \text{for } n \geq 0, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx) f(x) \quad \text{for } n \geq 1. \end{aligned} \tag{70}$$

Furthermore, we recall the Dirichlet Theorem which talks about the convergence of a Fourier series defined on $[-\pi, \pi)$ below:

"For sufficiently nice functions" $f(x)$ defined periodically, the Fourier series of $f(x)$ converges to $\frac{1}{2}\left(f(x^-) + f(x^+)\right)$ at every point x.

Extension to Arbitrary Intervals.

Suppose instead that $f(x)$ is defined on $x \in [-L, L)$. We want to introduce a new variable t such that $t \in [-\pi, \pi)$. To do so, observe the scaling:

$$\frac{t}{\pi} = \frac{x}{L}$$

which implies that $x = \frac{Lt}{\pi}$ and hence for each $t \in [-\pi, \pi)$, we have

$$g(t) := f\left(\frac{Lt}{\pi}\right) = f(x). \tag{71}$$

Expanding $g(t)$ in the usual Fourier series, we have

$$g(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nt) + b_n \sin(nt))$$

with Fourier coefficients given in (70). Substituting $t = \frac{\pi x}{L}$ back, we then have

$$\boxed{f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)}. \tag{72}$$

with (70) turning into

$$\begin{aligned} &\boxed{a_n = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) f(x) \quad \text{for } n \geq 0}, \\ &\boxed{b_n = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) f(x) \quad \text{for } n \geq 1}. \end{aligned} \tag{73}$$

Note: The Fourier coefficients computed for $g(t)$ will **not change** when you convert the series back to $f(x)$! In fact, formulas (70) and (73) are equivalent by a change of variable!

Recommendation: Either use the formulas (72) and (73) directly or re-derive this. Do refer to the lecture notes and the textbooks on multiple examples of such computations.

Example 23. Find the Fourier series of $f(x) = x$ defined on $[-1, 1]$. Is this series valid for every $x \in [-1, 1]$? Explain your answer.

Suggested Solutions: Since x runs from -1 to 1, we define a t to run from $-\pi$ to π correspondingly by

$$\frac{x}{1} = \frac{t}{\pi}.$$

Recall in (71), we have

$$g(t) = f\left(\frac{t}{\pi}\right) = \frac{1}{\pi}t.$$

Hence, it suffices to compute the Fourier series of the "transformed" g above. By the formulas (70), we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos(nt)\mathrm{d}t = 0$$

since $g(t)\cos(nt)$ is an odd function ($g(t)$ is odd while $\cos(nt)$ is even). Furthermore, we have

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin(nt)\mathrm{d}t = \frac{2}{\pi}\int_0^{\pi}\frac{t}{\pi}\sin(nt)\mathrm{d}t = \frac{2}{\pi^2}\left(\left(-\frac{t\cos(nt)}{n}\right)\Big|_{t=0}^{t=\pi} + \frac{1}{n}\int_0^{\pi}\cos(nt)\mathrm{d}t\right) \\ &= \frac{2}{\pi^2}\left(-\frac{\pi\cos(n\pi)}{n} + \frac{1}{n}\frac{\sin(nt)}{n}\Big|_{t=0}^{t=\pi}\right) = \frac{2}{n\pi}(-1)^{n+1}. \end{aligned}$$

In the above computations, we have performed integration by parts and used the fact that $\cos(n\pi) = (-1)^n$ and $\sin(n\pi) = 0$ for all non-negative integers n. Henceforth, the Fourier series for $g(t)$ is given by

$$g(t) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin(nt).$$

Thus, the Fourier series for $f(x)$ (after substituting $t = \pi x$ back) is given by

$$\boxed{f(x) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x).} \tag{74}$$

Note that the above series is valid for $x \in (-1, 1)$. At the boundary points, i.e $x = -1$, since

$$f(-1^-) \underbrace{=}_{\text{periodicity}} f(1^+) = 1 \neq -1 = f(-1^+),$$

Dirichlet theorem tells us that the Fourier series will converge to $\frac{1}{2}(1-1) = 0$, rather than the function value which is -1 at $x = -1$. A similar argument tells us that at $x = 1$, the Fourier series also converges to 0, which is not equals to the value of the function at $x = 1$ (which should be 1). Hence, the Fourier series of $f(x)$ in (74) is only valid for $x \in (-1, 1)$ (and not at the boundary points).

Orthogonality and Inner Products for Integrable Functions.

A sequence of functions $f_n(x)$ for $n = 1, 2, \cdots$, is said to be **orthogonal** on the interval $[a, b]$ if

$$\int_a^b f_n(x)f_m(x)\mathrm{d}x \begin{cases} = 0 & \text{for } m \neq n, \\ \neq 0 & \text{for } m = n. \end{cases}$$

A sequence of functions $f_n(x)$ for $n = 1, 2, \cdots$, is said to be **orthonormal** on the interval $[a, b]$ if

$$\int_a^b f_n(x)f_m(x)\mathrm{d}x \begin{cases} = 0 & \text{for } m \neq n, \\ = 1 & \text{for } m = n. \end{cases}$$

Example: $1, \cos(x), \sin(x), \cos(2x), \cdots$ is an orthogonal sequence, while $\frac{1}{\sqrt{2\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \cdots$ is an orthonormal sequence on $[-\pi, \pi]$. To determine the scaling factor in front of say 1, suppose that we have $f(x) = A \cdot 1$ for some constant A to be determined. Recall that orthonormal implies $\int_{-\pi}^{\pi} f_n(x)^2\mathrm{d}x = \int_{-\pi}^{\pi} A^2\mathrm{d}x = 1$. This implies that $2\pi A^2 = 1$ and hence $A = \frac{1}{\sqrt{2\pi}}$. This explains why the first term of the orthonormal sequence is given by $\frac{1}{\sqrt{2\pi}}$. $\square$

Let X be the space of some functions. X can be equipped with an **inner product** $(\cdot, \cdot)$ satisfying the following properties for any $c_1, c_2 \in \mathbb{R}$ and functions $f, g \in X$:

- Linearity: $(c_1 f_1 + c_2 f_2, g) = c_1(f_1, g) + c_2(f_2, g)$.
- Symmetric: $(f, g) = (g, f)$.
- Positivity: $(f, f) \geq 0$ and $[(f, f) = 0 \implies f = 0]$.

Furthermore, we can define

- Length of a function (abstractly known as "norm") $\| \cdot \|$, as $\|f\| = \sqrt{(f, f)}$ for each $f \in X$.
- Distance between two functions (abstractly known as a "metric") $d(\cdot, \cdot)$, as $d(f, g) = \|f - g\|$ for each $f, g \in X$.

In addition, the inner product satisfies the following properties for each $f, g \in X$:

- Cauchy-Schwarz: $|(f, g)| \leq \|f\|\|g\|$.
- Triangle Inequality: $\|f + g\| \leq \|f\| + \|g\|$.
- Pythagorean Theorem: (To be proven in HW 6, Exercise 37.5.) $\|f - g\|^2 = \|f\|^2 + \|g\|^2$ if and only if f and g are orthogonal.

Example: Let $X = \mathcal{R}$, the space of integrable functions on $[a, b]$ with $a < b$.[10] We equip $\mathcal{R}$ with the inner product

$$(f, g) = \int_a^b f(x)g(x)\mathrm{d}x \quad \text{for each } f, g \in \mathcal{R}.$$

One can check that the above three properties for an inner product holds. Furthermore, the norm ("length") of an element in $\mathcal{R}$ has the expression

$$\|f\| = \sqrt{(f, f)} = \sqrt{\int_a^b f(x)^2\mathrm{d}x}.$$

For instance, if we are considering the interval $[-\pi, \pi]$, we have

$$(1, \sin(x)) = \int_{-\pi}^{\pi} \sin(x)\mathrm{d}x = 0,$$

implying that 1 is orthogonal to $\sin(x)$. Furthermore,

$$\|1\|^2 = (1, 1) = \int_{-\pi}^{\pi} 1\mathrm{d}x = 2\pi.$$

[10] We say that $f \in \mathcal{R}$ if f is integrable.

Hence, the "length" of the constant function 1 is given by $\|1\| = \sqrt{2\pi}$. Recall that in the standard Euclidean $\mathbb{R}^n$ case, to produce an orthonormal sequence from an orthogonal sequence, one just has to divide each vector by its length. This is precisely what we have done in the previous example to obtain $\frac{1}{\sqrt{2\pi}} = \frac{1}{\|1\|}$ as one of the elements in the sequence of orthonormal functions.

Furthermore, we can compute the distance between two functions as follows. Observe that

$$\begin{aligned} d(1,\sin(x))^2 = \|1-\sin(x)\|^2 &= \int_{-\pi}^{\pi}(1-\sin(x))^2\mathrm{d}x = \int_{-\pi}^{\pi} 1 - 2\sin(x) + \sin^2(x)\mathrm{d}x \\ &= 2\pi - 2\cdot 0 + \pi = 3\pi. \end{aligned}$$

Above, we use the fact that $\sin(x)$ is an odd function on $[-\pi,\pi]$, and $\int_{-\pi}^{\pi}\sin^2(x)\mathrm{d}x = \int_{-\pi}^{\pi}\frac{1-\cos(2x)}{2}\mathrm{d}x = \pi$. Observe that

$$\|\sin(x)\|^2 = (\sin(x),\sin(x)) = \int_{-\pi}^{\pi}\sin^2(x)\mathrm{d}x = \pi$$

and as computed above,

$$\|1\|^2 = 2\pi.$$

By Pythagoras theorem, since 1 and $\sin(x)$ are orthogonal, we should have

$$\|1\|^2 + \|\sin(x)\|^2 = \|1-\sin(x)\|^2$$

which is indeed as expected since $2\pi + \pi = 3\pi$. $\square$

Types of Convergence and Mean Squared Convergence of Fourier Series.

Let $p_n(x)$ be a sequence of functions attempting to approximate $f(x)$, for with both are defined on $[a,b]$. We define the **mean squared error** at each iteration n as

$$E_n = \|f-p_n\|^2 = \int_a^b (f(x)-p_n(x))^2\mathrm{d}x.$$

We say that $p_n(x)$ converges to $f(x)$:

- **Pointwise** if for each $x \in [a,b]$, $\lim_{n\to\infty} p_n(x) = f(x)$.
- **Converges in the mean** if $\lim_{n\to\infty} d(p_n,f)^2 = \lim_{n\to\infty}\|p_n - f\|^2 = \lim_{n\to} E_n = 0$.

For the remaining parts of this section, we consider an **arbitrary orthonormal** sequence of integrable functions on $[a,b]$, given by $\phi_1(x), \phi_2(x), \cdots$. The goal here is to investigate if the **generalized Fourier series**

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots \tag{75}$$

"converges" (i.e pointwise vs in the mean) to $f(x)$, with (generalized) **Fourier coefficients** given by

$$c_k = \int_a^b f(x)\phi_k(x)\mathrm{d}x. \tag{76}$$

The following theorems below will attempt to answer this as follows:

- **Theorem 1.** For each positive integer n, the n-th partial sum of the Fourier series of f, namely

$$\sum_{k=1}^{n} c_k\phi_k(x) = c_1\phi_1(x) + \cdots + c_n\phi_n(x)$$

  with c_k as the (generalized) Fourier coefficients in (76) gives a smaller mean squared error $E_n = \|f-p_n\|^2$ than any other linear combinations $p_n(x) = d_1\phi_1(x) + \cdots + d_n\phi_n(x)$. Furthermore, this minimum value is given by

$$\min E_n = \int_a^b f(x)^2\mathrm{d}x - \sum_{k=1}^{n} c_k^2.$$

- **Theorem 2.** (Bessel's Inequality.) If the numbers c_k in (76) are the Fourier coefficients of f with respect to the orthonormal basis $\{\phi_n\}$, then the series $\sum_{k=1}^{\infty} c_k^2$ converges and satisfies the **Bessel's inequality**:[11]

$$\sum_{n=1}^{\infty} c_k^2 \leq \int_a^b [f(x)]^2 \mathrm{d}x. \tag{77}$$

- **Theorem 3.** The Fourier coefficients c_k in (76) of f with respect to the orthonormal basis $\{\phi_n\}$ obeys $c_n \to 0$ as $n \to \infty$.[12]

- **Theorem 4.** The representation of f by its (generalized) Fourier series, namely

$$f(x) = c_1\phi_1(x) + \cdots + c_k\phi_k(x) + \cdots$$

is valid in the sense of mean convergence[13] if and only if Bessel's inequality in (77) becomes **Parseval's equation**:

$$\sum_{n=1}^{\infty} c_k^2 = \int_a^b [f(x)]^2 \mathrm{d}x. \tag{78}$$

- **Theorem 5.** (Conclusion: Convergence in the mean of **ordinary** Fourier Series.) If $f(x)$ is any function defined and integrable on $[-\pi, \pi]$, then $f(x)$ is represented by its ordinary Fourier series in the sense of mean convergence,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right),$$

where the a_n and b_n are ordinary Fourier coefficients of $f(x)$ in (70).

In particular, the Bessel's inequality and Parseval's equation in (77) and (78) can be applied to our **ordinary** Fourier series with coefficients in (70) to give

- **(Bessel's Inequality.)**

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 \mathrm{d}x. \tag{79}$$

- **(Parseval's Equation.)**

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 \mathrm{d}x. \tag{80}$$

Note that the Parseval's equation for our **ordinary** Fourier series can be derived directly (without looking at the generalized Fourier series and instantiating with our ordinary Fourier series), and we shall do so in one of the exercises that follows.

[11] The proof of this follows from Theorem 1, since $E_n = \|f - p_n\|^2 \geq 0$, then so is the minimum. This implies that $\int_a^b f(x)^2 \mathrm{d}x - \sum_{k=1}^{n} a_k^2 \geq 0$. We can then rearrange to obtain this theorem. For the convergence of the series, this requires some Math 131A so we shall skip it.

[12] This is a fact from Math 131A in which if a series converges, then the n-th term must go to 0 as $n \to \infty$. In this case, by Theorem 2, $c_n^2 \to 0$ and we must then have $c_n \to 0$.

[13] Convergence in the mean implies that $E_n \to 0$. For our generalized Fourier series, $E_n = \int_a^b f(x)^2 \mathrm{d}x - \sum_{k=1}^{n} c_k^2$ precisely since it **is** the minimizing series. Thus, for $E_n \to 0$, this is equivalent to the Parseval's equation.

Example 24. (Problem 37.1, Modified.) The goal of this problem is to prove the Bessel's inequality directly for the **ordinary** Fourier series. It says that for any integrable function on $[-\pi, \pi]$, its ordinary Fourier coefficients in (70) satisfy the inequality[a]

$$\frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2 \mathrm{d}x = \frac{1}{\pi}(f, f). \tag{81}$$

(i) For $n \geq 1$, define

$$s_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos(kx) + b_k \sin(kx)).$$

Show that

$$\frac{1}{\pi}(f, s_n) = \frac{1}{2}a_0^2 + \sum_{k=1}^{n}(a_k^2 + b_k^2).$$

(ii) By considering all possible products in the multiplication of $s_n(x)$ by itself, show that

$$\frac{1}{\pi}\|s_n\|^2 = \frac{1}{2}a_0^2 + \sum_{k=1}^{n}(a_k^2 + b_k^2).$$

(iii) By writing

$$\begin{aligned}
\frac{1}{\pi}\|f - s_n\|^2 &= \frac{1}{\pi}(f - s_n, f - s_n) \\
&= \frac{1}{\pi}\left((f, f) - (f, s_n) - (s_n, f) + (s_n, s_n)\right) \\
&= \frac{1}{\pi}\left(\|f\|^2 - 2(f, s_n) + \|s_n\|^2\right),
\end{aligned}$$

deduce that (81) holds.

[a] The inner product here is the standard inner product on the space of integrable functions on $[-\pi, \pi]$.

Suggested Solution:

(i) Observe that

$$\begin{aligned}
\frac{1}{\pi}(f, s_n) &= \frac{1}{\pi}\left(f, \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos(kx) + b_k \sin(kx))\right) \\
&= \left(\frac{1}{2}a_0 \frac{(f, 1)}{\pi} + \sum_{k=1}^{n}\left(a_k \frac{(f, \cos(kx))}{\pi} + b_k \frac{(f, \sin(kx))}{\pi}\right)\right).
\end{aligned} \tag{82}$$

Observe that by (70) (formula for the ordinary Fourier coefficients)

$$\begin{aligned}
\frac{1}{\pi}(f, 1) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = a_0, \\
\frac{1}{\pi}(f, \cos(kx)) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\mathrm{d}x = a_k, \text{ and} \\
\frac{1}{\pi}(f, \sin(kx)) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\mathrm{d}x = b_k.
\end{aligned}$$

for $k \geq 1$. Then, (82) simplifies to

$$\frac{1}{\pi}(f, s_n) = \frac{1}{2}a_0^2 + \sum_{k=1}^{n}\left(a_k^2 + b_k^2\right). \tag{83}$$

(ii) Similar to (i), observe that

$$\begin{aligned}
&\frac{1}{\pi}(s_n, s_n)\\
&= \frac{1}{\pi}\left(\frac{1}{2}a_0 + \sum_{l=1}^{n}(a_l\cos(lx) + b_l\sin(lx)), \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k\cos(kx) + b_k\sin(kx))\right)\\
&= \frac{a_0^2}{4}\frac{(1,1)}{\pi} + \frac{1}{2}\sum_{l=1}^{n}\left(a_l\frac{(\cos(lx),1)}{\pi} + b_l\frac{(\sin(lx),1)}{\pi}\right) + \frac{1}{2}\sum_{k=1}^{n}\left(a_k\frac{(1,\cos(kx))}{\pi} + b_k\frac{(1,\sin(kx))}{\pi}\right)\\
&\quad + \sum_{l,k=1}^{n} a_la_k\frac{(\cos(lx),\cos(kx))}{\pi} + a_lb_k\frac{(\cos(lx),\sin(kx))}{\pi} + b_la_k\frac{(\sin(lx),\cos(kx))}{\pi} + b_lb_k\frac{(\sin(lx),\sin(kx))}{\pi}.
\end{aligned} \tag{84}$$

By orthogonality, we have $(\cos(lx),1) = (\sin(lx),1) = (1,\cos(kx)) = (1,\sin(kx)) = 0$ for all $l,k \geq 1$, and $(\cos(lx),\sin(kx)) = (\sin(lx),\cos(kx)) = 0$ too. Furthermore, we also have

$$\frac{(\cos(lx),\cos(kx))}{\pi} = \begin{cases} 0 & \text{if } l \neq k,\\ 1 & \text{if } l = k,\end{cases}$$

and

$$\frac{(\sin(lx),\sin(kx))}{\pi} = \begin{cases} 0 & \text{if } l \neq k,\\ 1 & \text{if } l = k,\end{cases}$$

In addition, we have $(1,1) = \int_{-\pi}^{\pi} \mathrm{d}x = 2\pi$, so $\frac{(1,1)}{\pi} = 2$. Thus, the only colored term that survives would be in green, and for $l = k$. Hence, in (84), we are only summing over $l = k = 1$ up to n in those summations, which then yields

$$\begin{aligned}
&\frac{1}{\pi}(s_n, s_n)\\
&= \frac{a_0^2}{2} + \sum_{l=k=1}^{n} a_la_k + \sum_{l=k=1}^{n} b_lb_k\\
&= \frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2).
\end{aligned} \tag{85}$$

(iii) The expansion in the question tells us that

$$\frac{1}{\pi}\|f - s_n\|^2 = \frac{1}{\pi}(\|f\|^2 - 2(f, s_n) + \|s_n\|^2).$$

Observe that from (i) and (ii) that since $\frac{1}{\pi}(f, s_n) = \frac{1}{\pi}\|s_n\|^2$, we have

$$\frac{1}{\pi}\left(-2(f,s_n) + \|s_n\|^2\right) = -\frac{(f,s_n)}{\pi} = -\frac{1}{2}a_0^2 + \sum_{k=1}^{n}\left(a_k^2 + b_k^2\right).$$

Furthermore, since $\frac{1}{\pi}\|f - s_n\|^2 \geq 0$, we have

$$0 \leq \frac{1}{\pi}\|f - s_n\|^2 = \frac{1}{\pi}(\|f\|^2 - 2(f,s_n) + \|s_n\|^2) = \frac{1}{\pi}\|f\|^2 - \left(\frac{1}{2}a_0^2 + \sum_{k=1}^{n}\left(a_k^2 + b_k^2\right)\right).$$

Since $0 \leq$ RHS, rearranging the term gives the Bessel's inequality in (81).

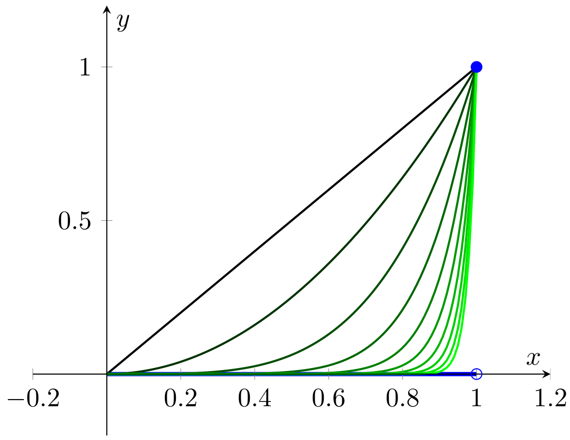
Example 25. (An Exercise on Pointwise and Mean Squared Convergence.) Let $p_n(x) = x^n$ defined on $[0,1]$ and

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1. \end{cases}$$

(i) Show that $p_n(x) \to f(x)$ pointwise.

(ii) Show that $p_n(x) \to f(x)$ in the mean (ie mean squared convergence).

Suggested Solution: Here is the picture for $f(x)$ and $p_n(x)$ for $n = 1, 2, 4, 7, 11, 16, 22, 29, 37, 46$. The curves in a mixture of black and green corresponds to $p_n(x)$, while the curve in blue corresponds to the function $f(x)$. In the diagram below, as n increases, the graph of $p_n(x)$ gets "greener" and less dark.



From the picture above, with the exception of the point $(1,1)$ which remains unchanged, at each reference point $x \in [0,1)$, the sequence $p_n(x)$ converges to 0. Furthermore, it also seems that the "difference in area squared" (a good proxy for mean squared convergence) converges to 0. With that, we shall prove these properties rigorously below.

(i) To show pointwise convergence on $[0,1]$, we have to show that for each point $x \in [0,1]$, $\lim_{n\to\infty} p_n(x) = f(x)$.

At $x = 1$, we see that $p_n(1) = 1$ for all $n \geq 1$ and $f(1) = 1$. Hence, it is easy to see that $\lim_{n\to\infty} p_n(1) = \lim_{n\to\infty} 1 = 1 = f(1)$.

At $x \in [0,1)$, we see that $p_n(x) = x^n$ for all $n \geq 1$, while $f(x) = 0$. Since $\lim_{n\to\infty} x^n = 0$ for any $x \in [0,1)$, then we have $\lim_{n\to\infty} p_n(x) = f(x)$ for any $x \in [0,1)$.

(ii) To show convergence in mean, recall by definition that we have to show that $E_n = \|p_n - f\|^2 \to 0$. Hence, it makes sense for us to compute E_n for each $n \geq 1$. Observe that

$$E_n = \|p_n - f\|^2 = \int_0^1 (p_n(x) - f(x))^2 \mathrm{d}x = \int_0^1 (x^n - 0)^2 \mathrm{d}x = \int_0^1 x^{2n} \mathrm{d}x = \frac{x^{2n+1}}{2n+1}\Big|_{x=0}^{x=1} = \frac{1}{2n+1}.$$

Note that we set $f(x) = 0$ in the integral above since changing the value of a function at an isolated point will not change the value of the integral, and $f(x)$ is just one point away from the zero function.

Hence, we have $\lim_{n\to\infty} E_n = \lim_{n\to\infty} \frac{1}{2n+1} = 0$. Thus, we have that $p_n(x) \to f(x)$ in the mean.

Example 26. (Problem 38.7 Modified.) Let $f(x) = x$ on $x \in [0, \pi]$. One can compute the Fourier sine series (as a practice, do verify this on your own!) to show that

$$x = 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \cdots\right). \tag{86}$$

By using the Parseval's equation in (80), show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Suggested Solution: From the expression above, the (ordinary) Fourier coefficients are given by

$$a_n = 0\ \forall n \geq 0, \quad \text{and} \quad b_n = \frac{2(-1)^n}{n}\ \forall n \geq 1.$$

By Parseval's equation, we have

$$\sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{n}\right)^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{\text{odd}}(x)^2 \mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 \mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \mathrm{d}x,$$

where $f_{\text{odd}}(x)$ is the odd extension of $f(x)$ (which happens to be itself) since the function f was only defined on $[0, \pi]$ and had to be extended for us to compute the Fourier sine series above.

The series on the left simplifies to $\sum_{n=1}^{\infty} \frac{4}{n^2}$, while the integral on the right yields $\frac{1}{\pi}\frac{x^3}{3}\big|_{-\pi}^{\pi} = \frac{2\pi^2}{3}$. We then have

$$\sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{2\pi^2}{3},$$

which then simplifies to the required expression.

8 Discussion 8

Partial Differential Equations (PDEs) and Boundary Value Problems.

Consider an ODE of the form
$$\mathcal{D}y + \lambda y = 0, \quad y(0) = 0, y(\pi) = 0,$$
where $\mathcal{D}$ is a differential operator. The **non-zero** solutions to the ODE above are known as **eigenfunctions** $y_n(x)$ of the ODE, with the corresponding λ_n as the **eigenvalues** of the associated differential operator $\mathcal{D}$. The following is an example on how one can compute the eigenvalues and the eigenfunctions associated with an ODE with prescribed boundary conditions.

Example 27. Find all values of $\lambda > 0$ such that

$$X''(x) = -\lambda X(x) \text{ for } x \in (0,1), \quad X(0) = X(1) = 0, \tag{87}$$

admits a non-zero solution on $x \in [0,1]$.

Suggested Solution: Recall from 33B that the general solution is given by

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x). \tag{88}$$

Using $X(0) = 0$, we have $A = 0$. We then have

$$X(x) = B\sin(\sqrt{\lambda}x). \tag{89}$$

Next, using $X(1) = 0$, we have
$$B\sin(\sqrt{\lambda}) = 0. \tag{90}$$
Thus, it is possible to have $B \neq 0$ if $\sin(\sqrt{\lambda}) = 0$. This implies that we have[14]

$$\begin{aligned} \sqrt{\lambda} &= n\pi, n \in \mathbb{Z} \\ \lambda &= n^2\pi^2, n \in \mathbb{N} \setminus \{0\} \end{aligned} \tag{91}$$

Thus, the possible values of λ such that we have a non-zero solution on $[0,1]$ are

$$\boxed{\lambda = n^2\pi^2, n \in \mathbb{N} \setminus \{0\}}. \tag{92}$$

[14] $\mathbb{N}$ for me includes 0.

We shall now proceed to attempt to solve a PDE by separation of variables. The exact details will be included in an example below. For instance, for the wave equation with speed c is given by

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = c^2 \dfrac{\partial^2 u}{\partial x^2}(x,t) & \text{in } (0,\pi)\times(0,\infty), \\ u(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ u(\pi,t) = 0 & \text{on } \{x=\pi\}\times[0,\infty), \\ u(x,0) = f(x) & \text{on } [0,\pi]\times\{t=0\}, \\ \dfrac{\partial u}{\partial t}(x,0) = 0 & \text{on } [0,\pi]\times\{t=0\}, \end{cases}$$

this was solved in class for any general $f(x)$ that is sufficiently nice. Here, we say that $u(0,t) = u(\pi,t) = 0$ refers to the **Dirichlet** boundary conditions, with **initial conditions** $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = 0$. The solution is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)\cos(nct)}$$

with

$$b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x.$$

In general, it is **not recommended** that you memorize the solution for all possible PDEs with various boundary conditions and initial conditions, and you should attempt to derive them by separation of variables. In the following example, we shall deal with the same wave equation with a more general initial condition and the **Neumann** boundary conditions (different boundary conditions) to illustrate how one can solve a PDE by separation of variables as follows.

Example 28. Consider the wave equation with Neumann boundary conditions on a bounded region $[0, L]$ for a given $L > 0$ with $c = 1$ and general initial conditions for some sufficiently well-behaved functions ϕ and ψ.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t) & \text{in } (0,L)\times(0,\infty), \\ \dfrac{\partial u}{\partial x}(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ \dfrac{\partial u}{\partial x}(L,t) = 0 & \text{on } \{x=L\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } [0,L]\times\{t=0\}, \\ \dfrac{\partial u}{\partial t}(x,0) = \psi(x) & \text{on } [0,L]\times\{t=0\}. \end{cases} \tag{93}$$

Solve this by separation of variables.

Suggested Solution:

Step 1: Look for separable solutions and derive boundary conditions.
Here, we look for **non-trivial** solutions[15] of the form

$$u(x,t) = X(x)T(t)$$

(ie separable). Using the Neumann boundary conditions, this implies that

$$\frac{\partial u}{\partial x}(0,t) = X'(0)T(t) = 0, \quad \text{and } \frac{\partial u}{\partial x}(L,t) = X'(L)T(t) = 0. \tag{94}$$

From the first equation, either $X'(0) = 0$ or $T(t) = 0$ for all $t \geq 0$. However, the latter implies that $u(x,t) = X(x)T(t) = 0$ since T is now the zero function, and we obtain a trivial (zero) "solution" (it might not even satisfy the initial conditions!) to the above PDE. Similarly, one deduces that $X'(L) = 0$. In summary,

$$X'(0) = X'(L) = 0. \tag{95}$$

Plugging this into the PDE, we get

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= X(x)T''(t) \\ \frac{\partial^2 u}{\partial x^2}(x,t) &= X''(x)T(t) \\ \frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) &= X(x)T''(t) - X''(x)T(t) = 0. \end{aligned} \tag{96}$$

Dividing both sides of the equation by $X(x)T(t)$,[16] we obtain

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} \tag{97}$$

Since the LHS of (97) only depends on x, and the RHS of (97) only depends on t, then (97) is equals to a constant. One way to understand this is that if x varies while keeping t fixed, it does not change the value on the right. This implies that it must be a constant in x for any given t. Using a similar argument, we then have

[15] "Is zero a solution" can be easily checked by substituting it into the PDE such that it satisfies the initial condition. Thus, it makes sense to just search for non-zero solutions.

[16] We did not discuss about the possibility of X and T being 0 at a point. Most books, not even Strauss, discuss this. The most convincing argument I have for you (at least, I am convinced) is that we use the argument in (98) as an intuition, and then come back to (97) and postulate that $X(x)$ are solutions to the eigenvalue problem $X''(x) = -\lambda X(x)$ for λ independent of x and t. Note that philosophically, this makes sense because when we write an ansatz/guess to the PDE, we are already restricting the functions that we are looking for to a smaller space. If it ends up not working, then it implies that either the restriction is too restrictive (say you have an ansatz $u(x,t) = 1$) or there really is no solution. Substitute this into (96), we obtain $X(x)T''(t) = -\lambda T(t)X(x)$. Since this holds for all x and t, we pick a point x^* such that $X(x^*) \neq 0$, and then divide by $X(x^*)$ on both sides to obtain $T''(t) = -\lambda T(t)$. If such a point does not exists, this implies that $X(x) = 0$ for all x, and thus $u(x,t) = X(x)T(t)$, the trivial solution.

that it is a constant in t for any given x.[17] Thus, (97) becomes

$$-\frac{X''(x)}{X(x)} = -\frac{T''(t)}{T(t)} = \lambda \tag{98}$$

where λ is a constant. In fact, this constant must be a non-negative real constant[18], since λ here is viewed as the eigenvalue to the problem $-X''(x) = \lambda X(x)$ with boundary terms $X'(0) = X'(L) = 0$.

Step 2: Solve the corresponding eigenvalue problem in X.
Now, we would like to solve the eigenvalue problem

$$\begin{cases} -X''(x) = \lambda X(x) \\ X'(0) = X'(L) = 0 \end{cases} \tag{99}$$

to obtain the corresponding eigenvalues and more importantly, eigenfunctions. If $\lambda = 0$, then $X(x) = Ax + B$. Using $X'(0) = X'(L) = 0$, we can only determine that $A = 0$. Thus, $X(x) = B$, an arbitrary constant, is an eigenfunction. In particular, the function 1 is an eigenfunction.

For $\lambda > 0$, The general solution is given by

$$\begin{aligned} X(x) &= A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \\ X'(x) &= A\sqrt{\lambda}\cos(\sqrt{\lambda}x) + B\sqrt{\lambda}\sin(\sqrt{\lambda}x). \end{aligned} \tag{100}$$

Using $X'(0) = 0$, this implies that $A\sqrt{\lambda} = 0$. Since $\lambda > 0$, this implies that $A = 0$. With $X'(x) = B\sqrt{\lambda}\sin(\sqrt{\lambda}x)$ left, we use the condition $X'(L) = 0$ to obtain

$$X'(L) = B\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0. \tag{101}$$

Note that it is now possible for this expression to be 0 with $B \neq 0$. This happens when $\sin(\sqrt{\lambda}L) = 0$, or when $\sqrt{\lambda}L = n\pi$ for $n \in \mathbb{N} \setminus \{0\}$, or

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n \in \mathbb{N} \setminus \{0\}. \tag{102}$$

The corresponding eigenfunctions (the functions attached to B in $X(x)$ since $B \neq 0$, and combining with the case when $\lambda > 0$) are

$$X_n(x) = \cos(\sqrt{\lambda_n}L) = \cos\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}, \tag{103}$$

since these functions will satisfy the boundary conditions but are non-zero functions.

Step 3: Solve the corresponding ODE in T.[19]
Going back to (98), this implies that there are only countably finitely many λ (given by λ_n above) that gives a non-zero solution. Thus, for each $n \in \mathbb{N}$, we will be solving the ODE:

$$-T_n''(t) = \lambda_n T_n(t), \tag{104}$$

where we index the function $T(t)$ by n to imply that we are solving a different ODE for different n (due to different values of λ_n). Since $\lambda_n \geq 0$, for $\lambda = 0$ (ie at $n = 0$), we obtain[20]

$$T_0(t) = \frac{A_0}{2}t + \frac{B_0}{2}. \tag{105}$$

Recall that for each n, we are solving a different ODE, so the arbitrary constants are different, and thus are indexed by n.
For $\lambda_n > 0$ (ie for $n \geq 1$), we obtain

$$\begin{aligned} T_n(t) &= A_n\sin(\sqrt{\lambda_n}t) + B_n\cos(\sqrt{\lambda_n}t) \\ &= A_n\sin\left(\frac{n\pi t}{L}\right) + B_n\cos\left(\frac{n\pi t}{L}\right). \end{aligned} \tag{106}$$

[17] If you don't buy this argument, take $\lambda(x,t) = \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}$. Note that $\frac{\partial}{\partial x}\lambda(x,t) = 0$ since $\frac{\partial}{\partial x}\lambda(x,t) = \frac{\partial}{\partial x}\frac{T''(t)}{T(t)} = 0$ is independent of t. Similarly, $\frac{\partial}{\partial t}\lambda(x,t) = \frac{\partial}{\partial t}\frac{X''(t)}{X(t)} = 0$. Thus, $\lambda_x = \lambda_t = 0$ implies that λ is a constant.

[18] You can argue this rigorously by computing the eigenvalues as in Example 27.

[20] The purpose of writing it as $\frac{B_0}{2}$ will be clear in a bit.

Step 4: Obtain general solution by linearity.

By linearity, for each n, the solution $u_n(x,t) = X_n(x)T_n(t)$ is a solution. Thus, a linear combination of these $u_n(x,t)$ is also a solution. This implies that[21]

$$\begin{aligned} u(x,t) &= \sum_{n\in\mathbb{N}} u_n(x,t) \\ &= \sum_{n\in\mathbb{N}} X_n(x)T_n(t) \\ &= (1)\left(\frac{A_0}{2}t + \frac{B_0}{2}\right) + \sum_{n\in\mathbb{N},n\geq 1} \cos\left(\frac{n\pi x}{L}\right)\left(A_n \sin\left(\frac{n\pi t}{L}\right) + B_n \cos\left(\frac{n\pi t}{L}\right)\right). \end{aligned} \tag{107}$$

Step 5: Solve for the Fourier coefficients.
First, see that (since $\sin(0) = 0, \cos(0) = 1$),

$$\phi(x) = u(x,0) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right). \tag{108}$$

Thus, the B_n are coefficients of the Fourier cosine series on the domain $[0, L]$. Using the formula in Discussion 7, we obtain

$$B_n = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\phi(x)\,\mathrm{d}x, \quad n \geq 0, \tag{109}$$

Next, take $\frac{\partial}{\partial t}$ to obtain

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \frac{A_0}{2} + \sum_{n\in\mathbb{N},n\geq 1} \cos\left(\frac{n\pi x}{L}\right)\left(\frac{A_n n\pi}{L}\cos\left(\frac{n\pi t}{L}\right) + \frac{-B_n n\pi}{L}\sin\left(\frac{n\pi t}{L}\right)\right) \\ \psi(x) = \frac{\partial u}{\partial t}(x,0) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} \underbrace{\frac{A_n n\pi}{L}}_{C_n} \cos\left(\frac{n\pi x}{L}\right). \end{aligned} \tag{110}$$

Let $C_0 = A_0$ and $C_n = \frac{A_n n\pi}{L}$, we then obtain the Fourier cosine series again (in C_n). Using the expression for Fourier series on an arbitrary domain, we get

$$C_n = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\psi(x)\,\mathrm{d}x, \quad n \geq 0, \tag{111}$$

so the A_n's are given by

$$A_n = \frac{L}{n\pi}C_n = \frac{2}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\psi(x)\,\mathrm{d}x, \quad n \geq 0. \tag{112}$$

Thus, we have

$$\boxed{u(x,t) = (1)\left(\frac{A_0}{2}t + \frac{B_0}{2}\right) + \sum_{n\in\mathbb{N},n\geq 1} \cos\left(\frac{n\pi x}{L}\right)\left(A_n \sin\left(\frac{n\pi t}{L}\right) + B_n \cos\left(\frac{n\pi t}{L}\right)\right)} \tag{113}$$

with coefficients A_n, B_n for $n \in \mathbb{N}$ given by (109) and (112).

[21] We could have written the solution as $C_0(1)(A_0 t + B_0) + \sum_{n\in\mathbb{N},n\geq 1} C_n \cos\left(\frac{n\pi x}{L}\right)\left(A_n \sin\left(\frac{n\pi t}{L}\right) + B_n \cos\left(\frac{n\pi t}{L}\right)\right)$ with arbitrary constants C_n in front, but these are absorbed in the A_n's and B_n's, so it doesn't really matter.

Following the above derivation or the derivation in the textbook/lecture notes, for the Dirichlet problem for the heat equation for $x \in [0, \pi]$ given by

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = c^2 \dfrac{\partial^2 u}{\partial x^2}(x,t) & \text{in } (0,L) \times (0,\infty), \\ u(0,t) = 0 & \text{on } \{x=0\} \times [0,\infty), \\ u(\pi,t) = 0 & \text{on } \{x=L\} \times [0,\infty), \\ u(x,0) = f(x) & \text{on } [0,\pi] \times \{t=0\}, \end{cases}$$

we have the following solution:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2c^2t} \sin(nx)}$$

with

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \mathrm{d}x.$$

As an exercise, do try to derive the above solution on your own using the steps that we did above!

Inhomogeneous Boundary Conditions for Heat Equation.

Recall that in lectures, one might be tasked to solve for $u(x,t)$

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = c^2 \dfrac{\partial^2 u}{\partial x^2}(x,t) & \text{in } (0,L) \times (0,\infty), \\ u(0,t) = C_1 & \text{on } \{x=0\} \times [0,\infty), \\ u(\pi,t) = C_2 & \text{on } \{x=L\} \times [0,\infty), \\ u(x,0) = f(x) & \text{on } [0,\pi] \times \{t=0\}, \end{cases}$$

for some constants A and B. The strategy here is to find the solution to the steady equation (ie $\frac{\partial u}{\partial t} = 0$ and assume that u does not depend on time t). For instance, if we do so, if we denote this solution as $g(x)$, then we get

$$c^2 g'' = 0$$

where $'$ here represents a derivative with respect to x. This gives $g(x) = Ax + B$. Plugging the boundary conditions $g(0) = C_1$ and $g(\pi) = C_2$, one can obtain $g(x) = C_1 + \frac{C_2 - C_1}{\pi} x$. For the original equation, we now consider the substitution

$$u(x,t) = w(x,t) + g(x).$$

This implies that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} + g'' \underbrace{=}_{\text{Since } g''=0} \frac{\partial^2 w}{\partial x^2}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial w}{\partial t}.$$

Furthermore, the boundary conditions are

$$w(0,t) = u(0,t) + g(0) = C_1 - C_1 = 0,$$

$$w(\pi,t) = u(\pi,t) + g(\pi) = C_2 - C_2 = 0,$$

and

$$w(x,0) = u(x,0) + g(x) = f(x) + g(x).$$

Hence, the resulting PDE is given by

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = c^2 \dfrac{\partial^2 w}{\partial x^2}(x,t) & \text{in } (0,L) \times (0,\infty), \\ w(0,t) = 0 & \text{on } \{x=0\} \times [0,\infty), \\ w(\pi,t) = 0 & \text{on } \{x=L\} \times [0,\infty), \\ w(x,0) = f(x) + g(x) & \text{on } [0,\pi] \times \{t=0\}. \end{cases}$$

We can now solve the above PDE by separation of variables for $w(x,t)$. The full solution to the original problem is then given by

$$u(x,t) = w(x,t) + g(x) = w(x,t) + C_1 + \frac{C_2 - C_1}{\pi}x.$$

Dirichlet Problem for a Circle and Poisson's Integral.

In this section, we look at the two-dimensional Laplace equation (and possibly higher dimensions) for an unknown function $u(x,y)$ given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

One can show that in polar coordinates (r,θ), the equation is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{114}$$

By separation of variables, one can show that the solution with the boundary value $u(r=1,\theta) = f(\theta)$ is given by[22]

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

Here, the coefficients a_n and b_n are obtained by letting $r = 1$ and attempting to express $f(\theta)$ in its (full) Fourier series as follows:

$$f(\theta) = u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

The above is known as the **Dirichlet problem for a circle** for the 2D Laplace equation.

In fact, an exact solution to the Laplace equation in polar coordinates in (114) can be derived to be given by

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} f(\phi)\mathrm{d}\phi.$$

The expression above is known as the **Poisson's Integral.**

[22] Note that this involves solving an equidimensional ODE that we have went through in Supplement 2. You should try to be familiar with the derivations!

Some practice problems:

Exercise 8.

(i) Show that the sine series of the constant function $f(x) = \pi/4$ defined on $[0, \pi]$ is

$$\frac{\pi}{4} = \sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \cdots, \quad x \in (0, \pi). \tag{115}$$

(ii) Compute the cosine series of the same function $f(x) = \pi/4$ defined on $[0, \pi]$.

(iii) Use (115) and plug in $x = \frac{\pi}{2}$ to compute the value of the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)}.$$

(iv) Notice that when we plug in $x = \pi$ into (115), the right hand side of the equation yields $0 + 0 + 0 + \cdots = 0$ while the left hand side gives $\frac{\pi}{4}$. Explain the disparity observed above.

(v) Notice that when we plug in $x = 0$ into (115), the right hand side of the equation yields $0 + 0 + 0 + \cdots = 0$ while the left hand side gives $\frac{\pi}{4}$. Explain the disparity observed above.

Exercise 9.

(i) Find the Fourier series of $f(x) = x$ defined on $[-1, 1]$.

(ii) Determine the maximal subset of $[-1, 1]$ such that the Fourier series above is valid.

Exercise 10. Let

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x < \frac{\pi}{4}, \\ 0 & \text{for } \frac{\pi}{4} \le x < \frac{\pi}{2}, \\ \frac{4}{\pi}\left(\frac{3\pi}{4} - x\right) & \text{for } \frac{\pi}{2} \le x < \frac{3\pi}{4}, \\ 0 & \text{for } \frac{3\pi}{4} \le x < \pi. \end{cases}$$

Let

$$p_n(x) = \frac{a_0}{2} + a_1 \cos(x) + \cdots + a_n \cos(nx) = \sum_{k=1}^{n} \cos(kx)$$

for $x \in [0, \pi]$. Let $E_n = \int_0^\pi |f(x) - p_n(x)|^2 \mathrm{d}x$.

(i) Find the value of a_0 that minimizes E_0.

(ii) Does $p_n(x)$ converges to $f(x)$ in the mean? Briefly explain your answer.

(iii) **Without explicitly computing the general expression for** a_n, compute $\lim_{n\to\infty} a_n$.

(iv) **Without explicitly computing the general expression for** a_n, compute

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2.$$

(v) Find the maximal range of values of x in $[0, \pi]$ for which $p_n(x)$ converges to $f(x)$ pointwise.

(vi) **Without explicitly computing the general expression for** a_n, show that

$$\frac{1}{8} = \sum_{n=1}^{\infty} (-1)^n a_{2n}.$$

Hint: Consider $x = \frac{\pi}{2}$.

Exercise 11. Let $f(x) = x$ on $x \in [0, \pi]$.

(i) Show that the Fourier sine series is given by

$$x = 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \cdots\right).$$

(ii) By using the Parseval's equation, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exercise 12. Consider the wave equation for $x \in [0, \pi]$ for $c > 0$ with boundary and initial conditions given below:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = c^2 \dfrac{\partial^2 u}{\partial x^2}(x,t) & \text{in } (0,\pi) \times (0,\infty), \\ u(0,t) = 0 & \text{on } \{x = 0\} \times [0,\infty), \\ u(\pi,t) = 0 & \text{on } \{x = \pi\} \times [0,\infty), \\ u(x,0) = 0 & \text{on } [0,\pi] \times \{t = 0\}, \\ \dfrac{\partial u}{\partial t}(x,0) = f(x) & \text{on } [0,\pi] \times \{t = 0\}, \end{cases}$$

Find the general solution to the above equation by separation of variables.

Exercise 13. Recall that in polar coordinates, the 2D Laplace equation is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

with general solution (obtained by separation of variables) given by

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta)). \tag{116}$$

An alternative expression for the solution is given by the **Poisson** integral, given by

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} f(\phi)\mathrm{d}\phi. \tag{117}$$

(i) Solve the above problem with the boundary condition $u(r = 2, \theta) = 4\cos(2\theta)$.

(ii) Use your answer in (i) and (117) to deduce that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{-60\cos(2\phi)}{17 - 8\cos(\pi/2 - \phi)}\mathrm{d}\phi = -16.$$

Answers and Hints/Partial Solutions:

- Answer to Exercise 8: See Supplement 6 Example 2.
- Answer to Exercise 9: See Supplement 7 Example 1.
- Answer to Exercise 10:
  (i) By Theorem 1 in Supplement 7 (or the equivalent theorem number in the textbook), this is minimized if a_0 is the Fourier coefficient of the Fourier **cosine** series used to represent $f(x)$. Hence, we have

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\mathrm{d}x = \frac{2}{\pi}\int_0^{\frac{\pi}{4}} 1\mathrm{d}x + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{4}{\pi}\left(\frac{3\pi}{4} - x\right)\mathrm{d}x = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

(ii) By Theorem 5 in Supplement 7, Fourier series converges in the mean to the function that it attempts to represent.

(iii) By Theorem 3 in Supplement 7, any Fourier series will have its coefficients decaying to zero. This implies that $a_n \to 0$.

(iv) By Parseval's equation, this is equivalent to $\frac{1}{\pi}\int_{-\pi}^{\pi} f_{\text{even}}(x)^2 \mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} f(x)^2 \mathrm{d}x.$. Here, we are looking at the **even** extension of $f(x)$ (since we are looking at the **Fourier cosine series** as suggested by the form of $p_n(x)$). Since

$$\int_0^{\pi} f(x)^2 \mathrm{d}x = \int_0^{\frac{\pi}{4}} 1^2 \mathrm{d}x + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{16}{\pi^2}\left(\frac{3\pi}{4} - x\right)^2 \mathrm{d}x = \frac{\pi}{4} + \frac{\pi}{12} = \frac{\pi}{3}.$$

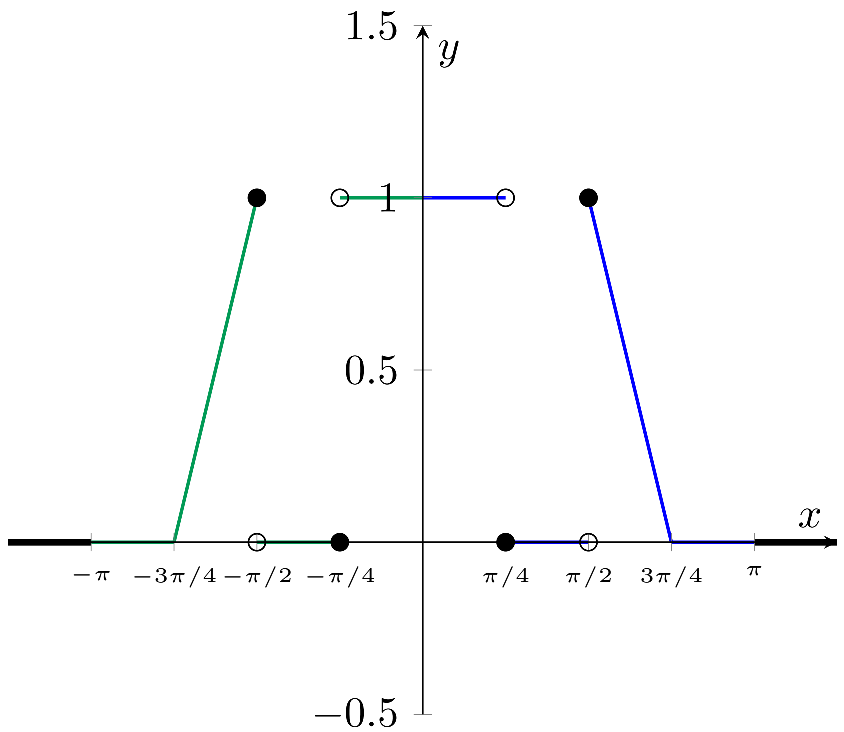
Thus, we have

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 = \frac{2}{\pi} \times \frac{\pi}{3} = \frac{2}{3}.$$

(v) Pointwise convergence has been discussed when we were talking about Dirichlet theorem. Henceforth, it suffices to look at the points of discontinuity of the **even** extension of $f(x)$. This is illustrated in the diagram below, with the blue curve representing $f(x)$, the green curve represents the even extension of $f(x)$ to $[-\pi, \pi]$, and the black curve representing periodic extension of the resulting function. As we can see, for $x \in [0, \pi]$, the function is only discontinuous at $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$. Thus, the Fourier cosine series converges pointwise to $f(x)$ for all $x \in [0, \pi] \backslash \{\frac{\pi}{4}, \frac{\pi}{2}\} = [0, \frac{\pi}{4}) \cup (\frac{\pi}{4}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.

(vi) Apply Dirichlet theorem at $x = \frac{\pi}{2}$. Since $\frac{a_0}{2} = \frac{3}{8}$, bring it to the left of the equation to obtain $\frac{1}{2} - \frac{3}{8} = \frac{1}{8}$ on the left hand side of the equality. Since $\cos\left(\frac{n\pi}{2}\right) = 0$ for odd values of n, and $\cos\left(\frac{2n\pi}{2}\right) = (-1)^{n+1}$, we then have

$$\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}\right) = \sum_{n=1}^{\infty} a_{2n} \cos(n\pi) = \sum_{n=1}^{\infty} (-1)^n a_{2n}.$$



- Answer to Exercise 11: See Supplement 7 Example 4.

- Answer to Exercise 12:
Let $u(x,t) = X(x)T(t)$. The boundary and initial conditions translate to $X(0) = X(\pi) = T(0) = 0$. By using the PDE, one should arrive at

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -\lambda.$$

Next, solve the eigenvalue problem $X'' = -\lambda X$ for $X(0) = X(\pi) = 0$ to obtain $X_n(x) = \sin(nx)$ for $\lambda_n = n^2$ for $n \geq 0$. Plugging these back to obtain the ODE for T as

$$T_n'' = -n^2c^2T_n$$

with $T'(0) = 0$. This gives the general solution to the ODE is given by

$$T_n(t) = B\sin(nct).$$

The general solution to the PDE is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_nX_n(x)T_n(t) = \sum_{n=1}^{\infty} b_n\sin(nx)\sin(nct)}.$$

By substituting $\frac{\partial u}{\partial t}(x,0) = f(x)$, we have

$$f(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} ncb_n\sin(nx)\cos(nc\times 0) = \sum_{n=1}^{\infty} \underbrace{cnb_n}_{c_n}\sin(nx).$$

This is just the Fourier sine series of $f(x)$ on $[0,\pi]$. Hence, we have $c_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\mathrm{d}x$ and thus

$$\boxed{b_n = \frac{c_n}{cn} = \frac{2}{cn\pi}\int_0^\pi f(x)\sin(nx)\mathrm{d}x}.$$

- Answer to Exercise 13:
  (i) Plugging this into (116), we have

  $$4\cos(2\theta) = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} 2^n(a_n\cos(n\theta) + b_n\sin(n\theta)).$$

  By orthogonality of the Fourier series, it is easy to see that $b_n = 0$ for all $n \geq 1$, $a_n = 0$ for all $n \geq 0$ except $n = 2$, and for $n = 2$, we have $4\cos(2\theta) = 2^2a_2\cos(2\theta)$ which implies that $a_2 = 1$. Plugging these Fourier coefficients back to (116), we have

  $$u(r,\theta) = r^2\cos(2\theta)$$

  as the **full** solution.

  (ii) Using $r = 4$ and $\theta = \pi/2$, observe that $u(4,\pi/2) = 16\cos(\pi) = -16$. On the other hand, the LHS of the expression to show is obtained by substituting $r = 4$ and $\theta = \pi/2$ into the expression in (117). Here, note that the Fourier series for $f(\theta)$ converges pointwise at $\theta = \pi/2$ since $f(\theta) = 4\cos(2\theta)$ is continuous for every $\theta \in [-\pi,\pi)$.

9 Discussion 9

Sturm-Liouville Problems.

Here, we summarize all the key terms and give a brief description on each of these terms:

- The sequence of functions $\{y_n(x)\}$ is **orthogonal** on $[a, b]$ if

$$\int_a^b y_m(x)y_n(x)\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \alpha_n > 0 & \text{if } m = n. \end{cases}$$

- The sequence of functions $\{\phi_n(x)\}$ is **orthonormal** on $[a, b]$ if

$$\int_a^b \phi_m(x)\phi_n(x)\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

  Recall that if we set can construct this from an orthogonal sequence by considering the **normalization**:

$$\phi_n(x) = \frac{y_n(x)}{\|y_n\|} = \frac{y_n(x)}{(y_n, y_n)^{\frac{1}{2}}}.$$

- The sequence of functions $\{y_n(x)\}$ is **orthogonal with respect to the weight function** $q(x)$ on $[a, b]$ if

$$\int_a^b y_m(x)y_n(x)q(x)\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \alpha_n > 0 & \text{if } m = n. \end{cases}$$

- The sequence of functions $\{y_n(x)\}$ is **orthonormal with respect to the weight function** $q(x)$ on $[a, b]$ if

$$\int_a^b y_m(x)y_n(x)q(x)\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

Consider a differential equation for $x \in [a, b]$ of the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + [\lambda q(x) + r(x)]y = 0. \tag{118}$$

with the **homogeneous boundary conditions**

$$\begin{cases} c_1 y(a) + c_2 y'(a) = 0 \\ d_1 y(b) + d_2 y'(b) = 0 \end{cases}$$

with c_1 or $c_2 \neq 0$, and d_1 or $d_2 \neq 0$, then this is known as a **Sturm-Liouville problem**.

- Under certain conditions (ie $p(x) > 0$ and $q(x) > 0$), there are countably many solutions to the Sturm-Liouville problem above. For each value of λ, we can find a corresponding solution y that solves (118). Index λ and y by n (since there are only countably many) with $\lambda_1 < \lambda_2 \cdots$, and we have:

  λ_n : **Eigenvalue**,
  $y_n(x)$: **Eigenfunction** associated with the eigenvalue above.

By the above definitions, we must have for each n, the eigenvalue λ_n and eigenfunction $y_n(x)$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y_n}{\mathrm{d}x}\right) + [\lambda_n q(x) + r(x)]y_n = 0.$$

One can then show that

$$(\lambda_m - \lambda_n)\int_a^b q(x)y_m(x)y_n(x)\mathrm{d}x = p(b)W(b) - p(a)W(b),$$

where

- The **Wronskian determinant** of two solutions $y_n(x)$ and $y_m(x)$ is given by

$$W(x) = \begin{vmatrix} y_m(x) & y_m'(x) \\ y_n(x) & y_n'(x) \end{vmatrix} = y_m(x)y_n'(x) - y_n(x)y_m'(x).$$

By the homogeneous boundary conditions, $W(a) = W(b) = 0$ (see details in lecture notes/textbook), and hence

$$(\lambda_m - \lambda_n)\int_a^b q(x)y_m(x)y_n(x)\mathrm{d}x = 0.$$

Observe that

$$\lambda_m \neq \lambda_n \implies \int_a^b q(x)y_m(x)y_n(x)\mathrm{d}x = 0.$$

Hence, **eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function** $q(x)$.

- For any given function $f(x)$ on $[a, b]$, one can conduct a generalized eigenfunction expansion given by

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x),$$

with $y_n(x)$ as the (orthogonal) eigenfunctions for a Sturm-Liouville problem (with respect to $q(x)$). Hence, by multiplying $q(x)y_m(x)$ on both sides and integrating from a to b, one can show that

$$\boxed{a_n = \frac{(f, y_n)_q}{(y_n, y_n)_q}}$$

are the coefficients of the generalized eigenfunction expansion.[23]

[23] Here, we denote $(f, g)_q = \int_a^b f(x)g(x)q(x)\mathrm{d}x$ as the generalized inner product.

Example 29. Consider the following ODE:

$$y''(x) + \lambda x y(x) = 0$$

for $x \in [0, 1]$ with boundary conditions $y(0) = y(1) = 0$. Note that the above ODE has countably many eigenvalues λ_n and eigenfunctions $y_n(x)$, though the exact solutions might not be easy to compute.

Now, suppose that we would like to compute the eigenfunction expansion of $f(x) = x$ for $x \in [0, 1]$ with respect to the eigenfunctions for the ODE above, given by

$$f(x) = a_1 y_1(x) + a_2 y_2(x) + \cdots .$$

Write down the general expression of a_n. If you use the inner product notation $(\cdot, \cdot)$, you must indicate the appropriate weight function.

Suggested Solution: This is of the form of a Sturm-Liouville problem, with $p(x) \equiv 1, q(x) = x, r(x) = 0$ and homogeneous boundary conditions. Hence, coefficients a_n are given by

$$a_n = \frac{(x, f_n)}{(f_n, f_n)} = \frac{\int_0^1 x^2 f_n(x) \mathrm{d}x}{\int_0^1 x f_n^2(x) \mathrm{d}x}$$

since here, we have

$$(f, g) = \int_0^1 \underbrace{x}_{q(x)} f(x) g(x) \mathrm{d}x,$$

with x as the weight function (in which we have read it off from $q(x) = x$).

Exact Forms, Adjoints, and Self-Adjoint ODEs.

- The differential equation
$$P(x)y'' + Q(x)y' + R(x)y = 0$$
is **exact** if and only if it can be written as
$$(P(x)y')' + (S(x)y)' = 0$$
for some $S(x)$.

- The differential equation
$$P(x)y'' + Q(x)y' + R(x)y = 0$$
is **exact** if and only
$$P''(x) - Q'(x) + R(x) = 0.$$
(Here, $S = Q - P'$ and $S' = R$ in the previous definition.)

Exact equations can be solved by first integrating to obtain
$$P(x)y' + S(x)y = C$$
and solve this by an integrating factor.

If
$$P(x)y'' + Q(x)y' + R(x)y = 0$$
is not exact, we mutiply by $\mu(x)$ to obtain
$$\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0$$
and hope that with an appropriate choice of $\mu(x)$, it is now exact.

- One can show that $\mu(x)$ can be obtained by solving
$$P(x)\mu'' + (2P'(x) - Q(x))\mu' + (P''(x) - Q'(x) + R(x))\mu = 0,$$
This is known as the **adjoint equation.**

- An adjoint equation is **self-adjoint** if its adjoint equation is itself.

- From Problem 43.6 (and also in the lecture notes), a problem is self-adjoint if and only if $P'(x) = Q(x)$. Now, we have
$$P(x)y'' + P'(x)y' + R(x)y = 0$$
and hence
$$(P(x)y')' + R(x)y = 0,$$
and the equation is thus in a Sturm-Liouville form (or the **standard form of a self-adjoint equation**).

- From Problem 43.7, the general form
$$P(x)y'' + Q(x)y' + R(x)y = 0$$
can be made into a self-adjoint equation if we multiply both sides by
$$\frac{1}{P}e^{\int Q/P\,\mathrm{d}x}.$$

Example 30. The Legendre's equation is given by

$$(1-x^2)y'' - 2xy' + \lambda y = 0,$$

with boundary conditions $y(0) = y(1) = 0$ and λ as part of an eigenvalue problem.

(i) Show that the above equation is self-adjoint.

(ii) Write the Legendre's equation in its self-adjoint form.

(iii) Write down a function $k(x)$ such that for each pair of eigenfunctions $y_n(x)$ and $y_m(x)$ corresponding to different eigenvalues, we have

$$\int_0^1 k(x)y_n(x)y_m(x)\mathrm{d}x = 0.$$

Suggested Solution:

(i) Using the result from Problem 43.6, it is self-adjoint if $P' = Q$. Indeed, we see that since $P(x) = 1 - x^2$ and $Q(x) = -2x$, then we have $P'(x) = Q(x)$.

(ii) Based on the result from Problem 43.6, we can write the above equation as

$$(P(x)y')' + R(x)y = 0,$$

or

$$((1-x^2)y')' + \lambda y = 0.$$

(iii) Since the self-adjoint form is in a Sturm-Liouville form with homogeneous boundary conditions, we read off the weight function to be $q(x) = 1$. In the context of the question above, it suffices to pick $k(x)$ to be weight function. Hence, we pick $k(x) = 1$ for all $x \in [0, 1]$.

10 Discussion 10

Euler(-Lagrange)'s Equation for Extremals.

The purpose of this section is to solve the following optimization problem:

$$\begin{aligned} &\text{minimize/maximize} \quad && I(y) = \int_{x_1}^{x_2} f(x, y(x), y'(x))\mathrm{d}x \\ &\text{subject to} && y(x_1) = y_1,\ y(x_2) = y_2. \end{aligned}$$

Suppose that such a unique optimizer exists (which we shall call it $y(x)$). Then, it must satisfy the **Euler(-Lagrange) Equation** given by

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0. \tag{119}$$

There are three cases to consider as mentioned in the textbook/lecture notes. However, I would like to approach this in the sense that we will only rely on (119). To solve the aforementioned problems, we perform the following sequence of steps:

1. Identify $f(x, y, y')$ and compute $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial y'}$, and hence $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)$.
2. Plug the above results into the Euler's Equation and the differential equation for $y(x)$. Note that it is subjected to the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

Nonetheless, for completeness, we will list out the three different cases as mentioned in the textbook below (c listed below represents an arbitrary constant):

(A) $f(y')$ only (x and y missing). Then, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$ and thus

$$\frac{\partial f}{\partial y'} = c.$$

(B) $f(x, y')$ only (y missing). Similarly, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$ and thus

$$\frac{\partial f}{\partial y'} = c.$$

(C) $f(y, y')$ only (x missing). One can show that the Euler equation reduces to

$$\frac{\partial f}{\partial y'}y' - f = c.$$

We shall see two examples of this below.

Example 31. Find the solution to the optimization problem

$$\begin{aligned} \text{minimize} \quad & \int_0^1 xy(x) + (y'(x))^2 \mathrm{d}x \\ \text{subject to} \quad & y(0) = 0,\ y(1) = 1. \end{aligned}$$

You may assume that the solution to the Euler's equation would corresponds to the minimizer of the functional $\int_0^1 xy(x) + (y'(x))^2 \mathrm{d}x$. Furthermore, you need to find both the function $y(x)$ that optimizes the functional and the value of the functional evaluated at this function.

Suggested Solution: Note that this falls in neither of the three cases. Nonetheless, you could work out the corresponding Euler's equation from first principle. Indeed, since $f(x, y, y') = xy - (y')^2$, we have

- $\frac{\partial f}{\partial y} = x$, $\frac{\partial f}{\partial y'} = 2y'$, and hence $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = \frac{\mathrm{d}}{\mathrm{d}x}(2y') = 2y''$.
- Euler's equation yields
$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = x - 2y'' = 0.$$
This implies that $y'' = \frac{x}{2}$.
- Integrating twice, we obtain
$$y(x) = \frac{x^3}{12} + Ax + B.$$
- Using the boundary conditions $y(0) = 0, y(1) = 1,$ we have $B = 0$ and $A = \frac{11}{12}$.

Henceforth, the optimizer (minimizer) is given by

$$\boxed{y(x) = \frac{x^3}{12} + \frac{11}{12}x}.$$

The value of the functional is given by

$$\int_0^1 x\left(\frac{x^3}{12} + \frac{11}{12}x\right) + \left(\frac{x^2}{4} + \frac{11}{12}\right)^2 \mathrm{d}x = \boxed{\frac{239}{180}}.$$

Example 32. (Snell's Law.) Consider a light ray entering from a optically less dense medium to an optically denser medium. Suppose the path of a light ray is parameterized by $(x, y(x))$. In ray optics, it is known that the ray follows the path of shortest optical length, defined by

$$L = \int_{x_0}^{x_1} \eta(x)\sqrt{1+(y'(x))^2}\mathrm{d}x$$

for a path starting from (x_0, y_0) and ending at (x_1, y_1), with a given position-dependent refractive index denoted by $\eta(x)$.

(i) With the help of Euler's equation, write down a differential equation in $y(x)$ satisfied by the light ray.

(ii) It is known at along any path, the angle that it makes with the horizontal $\theta(x)$ is given by

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = \tan(\theta(x)).$$

Use this to show that the quantity $n(x)\sin(\theta(x))$ is constant for any x along the path.

As a conclusion, if light travels from a medium with optical density η_1 at an angle of θ_1 with respect to the horizontal, into a medium with optical density η_2 at an angle θ_2 with respect to the horizontal, we must have

$$\eta_1 \sin(\theta_1) = \eta_2 \sin(\theta_2).$$

Suggested Solution:

(i) Note that this falls under Case (B).[24] Hence, for $f(x, y, y') = \eta(x)\sqrt{1+(y')^2}$ we compute

$$\frac{\partial f}{\partial y'} = \frac{\eta(x)y'(x)}{\sqrt{1+(y'(x))^2}}.$$

By the Euler's equation, we thus have that the minimizing path $y(x)$ must satisfy

$$\frac{\eta(x)y'(x)}{\sqrt{1+(y'(x))^2}} = c$$

with $y(x_0) = y_0$ and $y(x_1) = y_1$.

(ii) If $y' = \tan(\theta)$, then we have

$$\frac{\eta(x)\tan\theta(x)}{\sqrt{1+\tan^2(\theta(x))}} = \eta(x)\sin(\theta(x)) = c.$$

Here, we have used the fact that $1+\tan^2(\theta) = \sec^2(\theta)$ and $\frac{\tan(\theta)}{\sec(\theta)} = \sin(\theta)$.

[24] Alternatively, you can always try to compute the Euler's equation directly and convince yourself that you will obtain the same solution.

Lagrange Multipliers

The purpose of this section is to solve the following optimization problem:

$$\begin{array}{ll} \text{minimize/maximize} & f(x_1, x_2, \cdots, x_n) \\ \text{subject to} & g(x_1, x_2, \cdots, x_n) = 0. \end{array}$$

Here, we implicitly assume that only one additional condition was imposed. (This idea can be easily generalized to that with n constraints.)
To find the point(s) that optimizes the function $f(x_1, \cdots, x_n)$, we instead consider the following unconstrained optimization problem with an additional variable:

$$\text{minimize/maximize} \quad \mathcal{L}(x_1, \cdots, x_n; \lambda) = f(x_1, x_2, \cdots, x_n) - \lambda g(x_1, x_2, \cdots, x_n).$$

As per an unconstrained optimization problem, we find the extremal points by solving

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x_1} = 0, \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial x_n} = 0, \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 0. \end{cases}$$

Suppose that upon solving the above system, we obtain an extremal point $(x_1^*, \cdots, x_n^*)$. To determine if this is a maximum or a minimum point for the objective function f, we could use physical/logical arguments to deduce if a extremal point is a minimizer or a maximizer.

We shall look at a concrete example below.

Example 33. (Exercise 68.3(a)) Find the point on the plane $ax + by + cz = d$ with $d \neq 0$[a] that is nearest to the origin.

[a]This implies that the plane does not contain the origin.

Suggested Solution:

Here, we would like to minimize the distance (or equivalently, the distance squared) between any point on the plane (x, y, z) (with $ax+by+cz = d$) and the origin $(0, 0, 0)$. Thus, we have the following optimization problem:

$$\begin{aligned} &\text{minimize} && f(x, y, z) = x^2 + y^2 + z^2 \\ &\text{subject to} && g(x, y, z) = ax + by + cz - d = 0. \end{aligned}$$

The equivalent unconstrained optimization problem is given by

$$\text{minimize} \quad \mathcal{L}(x, y, z; \lambda) = x^2 + y^2 + z^2 + \lambda\,(ax + by + cz - d)\,.$$

To do so, we first compute

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 2x + \lambda a, \\ \frac{\partial \mathcal{L}}{\partial y} = 2y + \lambda b, \\ \frac{\partial \mathcal{L}}{\partial z} = 2z + \lambda c, \\ \frac{\partial \mathcal{L}}{\partial \lambda} = ax + by + cz - d. \end{cases}$$

At the extremal point, we must have $\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial z} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0$. The first three equations imply that

$$x = -\frac{\lambda a}{2}, y = -\frac{\lambda b}{2}, z = -\frac{\lambda c}{2}.$$

Plugging these into $\frac{\partial \mathcal{L}}{\partial \lambda} = 0$, we have

$$-\frac{\lambda}{2}(a^2 + b^2 + c^2) - d = 0.$$

This implies that[25]

$$\lambda = -\frac{2d}{a^2 + b^2 + c^2}.$$

Henceforth, the corresponding extremal point is given by

$$x = \frac{ad}{a^2 + b^2 + c^2}, y = \frac{bd}{a^2 + b^2 + c^2}, z = \frac{cd}{a^2 + b^2 + c^2}.$$

To see that this is a minimum point, we can argue that it is possible to obtain ∞ for the objective function if we pick points in the plane sufficiently far away from the origin. Hence, the corresponding critical point that we have found must corresponds to a local minimum.

[25]Note that $a^2 + b^2 + c^2 > 0$ since if that is not the case, then $a = b = c = 0$, which implies that the equation of the plane is given by $0 = d$, a contradiction.

11 Revision Problems for Finals

Math 135 - Revision Problems for Finals.

Revision Problems for Finals
Note: The exercises below are selected modified problems from the homework problems/midterms/quizzes /discussion supplements (usually problems that I did not manage to go through in class for that week). These are solely done for practice purposes and are **not necessarily reflective** of the nature of questions in the final.

Exercise 14. Solve the following initial value problems:

(i) $y'(x) + xy(x) = x^2 + 1, y(0) = 0,$

(ii) $y''(x) + 4y(x) = \cos(x), y(0) = 1, y'(0) = 0,$

(iii) $y''(x) - y'(x) - 6y(x) = e^x, y(0) = 1, y'(0) = 0,$

(iv) $x^2y''(x) - 2y(x) = 0, y(1) = 1, y'(1) = 1.$

Exercise 15. With the help of the table of Laplace transforms and relevant properties, evaluate

(i) $\mathcal{L}\left[e^{4x}(x+1)^2\right]$,

(ii) $\mathcal{L}^{-1}\left[\frac{1}{p+1} + \frac{1}{(p-1)^2+1}\right]$, and

(iii) $\mathcal{L}^{-1}\left[\frac{p}{(p+1)(p^2+4p+5)}\right]$.

Exercise 16. Solve the following integro-differential equation:

$$y'(x) = -y(x) - \int_0^x e^{t-x} y(t) \mathrm{d}t$$

with initial condition $y(0) = 1$.

Exercise 17. Consider the initial value problem

$$y' = e^y, \quad y(0) = 1.$$

(i) Starting with $y_0(x) = 1$, apply Picard's method to calculate $y_1(x)$ and $y_2(x)$.

(ii) Explain why we can expect to have a **unique** solution to the above initial value problem for $x \in [-h, h]$ for some $h > 0$.

Exercise 18. Let

$$f(x) = \begin{cases} x & \text{for } 0 < x \leq 1, \\ \frac{1}{2} & \text{for } x = 0, \\ -x & \text{for } -1 \leq x < 0. \end{cases}$$

(i) Find the Fourier series of $f(x)$ defined above.

(ii) Determine the maximal subset of $[-1, 1]$ such that the Fourier series above is valid.

Exercise 19. Let

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x < \frac{\pi}{4}, \\ 0 & \text{for } \frac{\pi}{4} \le x < \frac{\pi}{2}, \\ \frac{4}{\pi}\left(\frac{3\pi}{4} - x\right) & \text{for } \frac{\pi}{2} \le x < \frac{3\pi}{4}, \\ 0 & \text{for } \frac{3\pi}{4} \le x < \pi. \end{cases}$$

Let

$$p_n(x) = b_1 \sin(x) + \cdots + b_n \sin(nx) = \sum_{k=1}^{n} b_k \sin(kx)$$

for $x \in [0, \pi]$. Let $E_n = \int_0^\pi |f(x) - p_n(x)|^2 \mathrm{d}x$.

(i) Find the value of b_1 that minimizes E_1.

(ii) Does $p_n(x)$ converges to $f(x)$ in the mean? Briefly explain your answer.

(iii) Find the maximal range of values of x in $[0, \pi]$ for which $p_n(x)$ converges to $f(x)$ pointwise.

Exercise 20. Let

$$f(x) = \begin{cases} x + 1 & \text{for } 0 \le x \le \pi, \\ 1 & \text{for } -\pi \le x < 0. \end{cases}$$

(i) Compute the Fourier series of $f(x)$.

(ii) By evaluating your answer in (i) at $x = \pi$, deduce the value of the series

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Exercise 21. Consider the wave equation for $x \in [0, \pi]$ for $c > 0$ with boundary and initial conditions given below:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = c^2 \dfrac{\partial^2 u}{\partial x^2}(x,t) & \text{in } (0,\pi) \times (0,\infty), \\ u(0,t) = 0 & \text{on } \{x = 0\} \times [0,\infty), \\ u(\pi,t) = 0 & \text{on } \{x = \pi\} \times [0,\infty), \\ u(x,0) = 0 & \text{on } [0,\pi] \times \{t = 0\}, \\ \dfrac{\partial u}{\partial t}(x,0) = f(x) & \text{on } [0,\pi] \times \{t = 0\}, \end{cases}$$

Find the general solution to the above equation by separation of variables.

Exercise 22. Recall that in polar coordinates, the 2D Laplace equation is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

with general solution (obtained by separation of variables) given by

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta)). \tag{120}$$

An alternative expression for the solution is given by the **Poisson** integral, given by

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} f(\phi)\mathrm{d}\phi. \tag{121}$$

(i) Solve the above problem for $u(r,\theta)$ with the boundary condition $f(\theta) = 4\cos(2\theta)$. Hint: Use (120).

(ii) Use your answer in (i) and (121) to deduce that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{60\cos(2\phi)}{17-8\cos(\pi/2-\phi)}\mathrm{d}\phi = -\frac{1}{4}.$$

Exercise 23.

(i) Let A be an arbitrary real number. Compute the Fourier sine series of the function $f(x) = A$ for $x \in [0,\pi]$.

(ii) Let A be an arbitrary real number. Compute the Fourier cosine series of the function $f(x) = A$ for $x \in [0,\pi]$.

(iii) Hence or otherwise, find the solution to the inhomogeneous heat equation (with $c > 0$) below:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) & \text{in } (0,L) \times (0,\infty), \\ u(0,t) = \pi^2 & \text{on } \{x=0\} \times [0,\infty), \\ u(\pi,t) = \pi^2 & \text{on } \{x=L\} \times [0,\infty), \\ u(x,0) = \pi^3 & \text{on } [0,\pi] \times \{t=0\}. \end{cases}$$

Exercise 24. Consider the following equation:

$$(1-2x^2)y'' - 4xy' + \lambda xy = 0,$$

with boundary conditions $y(0) = y(1) = 0$ and λ as part of an eigenvalue problem.

(i) Write down the adjoint equation for the equation above.

(ii) Is the equation above in Strum-Liouville form? Explain your answer.

(iii) Write down a function $k(x)$ such that for each pair of eigenfunctions $y_n(x)$ and $y_m(x)$ corresponding to different eigenvalues, we have

$$\int_0^1 k(x)y_n(x)y_m(x)\mathrm{d}x = 0.$$

Exercise 25. Find the solution to the optimization problem

$$\begin{aligned} \text{minimize} \quad & \int_0^1 y^2(x) - (y'(x))^2 \mathrm{d}x \\ \text{subject to} \quad & y(0) = 0,\ y(\pi) = 0. \end{aligned}$$

You may assume that the solution to the Euler's equation would corresponds to the minimizer of the functional $\int_0^1 y^2(x) - (y'(x))^2 \mathrm{d}x$. Furthermore, you need to find both the function $y(x)$ that optimizes the functional and the value of the functional evaluated at this function.

Exercise 26. Find the solution to the optimization problem

$$\begin{aligned} \text{maximize} \quad & (x-2)^2 + (y-1)^2 + (z-2)^2 \\ \text{subject to} \quad & x^2 + y^2 + z^2 = 1. \end{aligned}$$

Remark: This corresponds to the point on the sphere $x^2 + y^2 + z^2 = 1$ furthest away from the point $(2, 1, 2)$.

Answers and Hints/Partial Solutions:

- Answer to Exercise 14:
  (i) x or $e^{-\frac{x^2}{2}}\int_0^x e^{\frac{s^2}{2}}(1+s^2)\mathrm{d}s$ (they are equivalent, since one can show using integrating by parts that $\int_0^x e^{\frac{s^2}{2}}(1+s^2)\mathrm{d}s = xe^{\frac{x^2}{2}}$, and keeping your answer in the second form is sufficient for this case).
  (ii) $\frac{1}{3}(\cos(x)+2\cos(2x))$,
  (iii) $\frac{1}{6}\left(4e^{-2x}-e^x+3e^{3x}\right)$,
  (iv) $\frac{1+2x^3}{3x}$.
- Answer to Exercise 15: See Supplement 2 Example 2.
- Answer to Exercise 16: See Supplement 3 Example 2.
- Answer to Exercise 17:
  (i) $y_1(x)=1+ex, y_2(x)=e^{ex}$.
  (ii) $f(x,y)=e^y$ and $\frac{\partial f}{\partial y}=e^y$ are continuous on any closed rectangle containing the point $(0,1)$. Hence, Theorem A applies, and we thus have a unique local solution around $x=0$.
- Answer to Exercise 18:
  (i) With the exception of the isolated point, this is a Fourier cosine series on $[0,1]$. The Fourier series is given by
  $$\frac{1}{2}-\sum_{n=0}^{\infty}\frac{4}{\pi^2(2n+1)^2}\cos((2n+1)x).$$
  (ii) By Dirichlet Theorem, it converges to $x\in[-1,1]\setminus\{0\}$ since it should converge to 0 but $f(0)=\frac{1}{2}$.
- Answer to Exercise 19:
  (i) $b_1=\frac{2}{\pi}\int_0^\pi f(x)\sin(x)\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi/4}\sin(x)\mathrm{d}x+\frac{2}{\pi}\int_{\pi/2}^{3\pi/4}\frac{4}{\pi}\left(\frac{3\pi}{4}-x\right)\sin(x)\mathrm{d}x=\frac{(2-\sqrt{2})(4+\pi)}{\pi^2}$.
  (ii) Yes. Fourier series converges to the function that it represents in the mean.
  (iii) $[0,\pi]\setminus\left\{0,\frac{\pi}{4},\frac{\pi}{2}\right\}$. (Consider the odd extension of the graph, so this is almost equivalent to the graph in Supplement 8. However, since $f(0)=1$ but the Fourier sine series converges to 0 at $x=0$, then $x=0$ is also excluded.)
- Answer to Exercise 20: This corresponds to Midterm 2 Problem 2 Rephrased.
  (i) $1+\frac{\pi}{4}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos((2n-1)x)}{(2n-1)^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$.
  (ii) Use Dirichlet's Theorem at $x=\pi$ to deduce that $1+\frac{\pi}{2}=1+\frac{\pi}{4}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$. The required expression will then follow by algebraic manipulation.
- Answer to Exercise 21:
  Let $u(x,t)=X(x)T(t)$. The boundary and initial conditions translate to $X(0)=X(\pi)=T(0)=0$. By using the PDE, one should arrive at
  $$\frac{X''}{X}=\frac{1}{c^2}\frac{T''}{T}=-\lambda.$$
  Next, solve the eigenvalue problem $X''=-\lambda X$ for $X(0)=X(\pi)=0$ to obtain $X_n(x)=\sin(nx)$ for $\lambda_n=n^2$ for $n\geq 0$. Plugging these back to obtain the ODE for T as
  $$T_n''=-n^2c^2T_n$$
  with $T'(0)=0$. This gives the general solution to the ODE is given by
  $$T_n(t)=B\sin(nct).$$
  The general solution to the PDE is given by
  $$\boxed{u(x,t)=\sum_{n=1}^{\infty}b_nX_n(x)T_n(t)=\sum_{n=1}^{\infty}b_n\sin(nx)\sin(nct)}.$$

By substituting $\frac{\partial u}{\partial t}(x,0) = f(x)$, we have

$$f(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} ncb_n \sin(nx)\cos(nc \times 0) = \sum_{n=1}^{\infty} \underbrace{cnb_n}_{c_n} \sin(nx).$$

This is just the Fourier sine series of $f(x)$ on $[0,\pi]$. Hence, we have $c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\mathrm{d}x$ and thus

$$\boxed{b_n = \frac{c_n}{cn} = \frac{2}{cn\pi}\int_0^{\pi} f(x)\sin(nx)\mathrm{d}x}.$$

- Answer to Exercise 22:
  (i) Plugging this into (120) at the boundary $r = 1$, we have

  $$4\cos(2\theta) = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(n\theta) + b_n\sin(n\theta)).$$

  By orthogonality of the Fourier series, it is easy to see that $b_n = 0$ for all $n \geq 1$, $a_n = 0$ for all $n \geq 0$ except $n = 2$, and for $n = 2$, we have $4\cos(2\theta) = a_2\cos(2\theta)$ which implies that $a_2 = 4$. Plugging these Fourier coefficients back to (120), we have

  $$u(r,\theta) = 4r^2\cos(2\theta)$$

  as the **full** solution.

  (ii) Using $r = 1/4$ and $\theta = \pi/2$, observe that $u(1/4,\pi/2) = 4\left(\frac{1}{4}\right)^2\cos(\pi) = -\frac{1}{4}$. On the other hand, by plugging the value of $r = 1/4$ to the LHS of the expression, we have

  $$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\frac{15}{16}\cdot 4\cos(2\phi)}{\frac{17}{16} - \frac{1}{2}\cos(\pi/2-\phi)}\mathrm{d}\phi = -\frac{1}{4}.$$

  or by simplifying, we have

  $$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{60\cos(2\phi)}{17 - 8\cos(\pi/2-\phi)}\mathrm{d}\phi = -\frac{1}{4}.$$

  Here, note that the Fourier series for $f(\theta)$ converges pointwise at $\theta = \pi/2$ since $f(\theta) = 4\cos(2\theta)$ is continuous for every $\theta \in [-\pi,\pi)$.

- Answer to Exercise 23:
  (i) $b_n = \frac{2}{\pi}\int_0^{\pi} A\sin(nx)\mathrm{d}x = \frac{2A(1-(-1)^n)}{n\pi}$. Hence, we have

  $$f(x) = \sum_{n=1}^{\infty}\frac{4A}{(2n-1)\pi}\sin((2n-1)x).$$

  (ii) $a_n = \frac{2}{\pi}\int_0^{\pi} A\cos(nx)\mathrm{d}x = \frac{A\sin(n\pi)}{n} = 0$ for $n > 1$, while $a_0 = 2A$. Hence, we have

  $$f(x) = A.$$

  (iii) Stationary state solution $g(x) = \pi^2$ for all $x \in [0,\pi]$. Let $w(x,t) = u(x,t) - g(x)$. Hence, $w(x,t)$ solves

  $$\begin{cases} \frac{\partial w}{\partial t}(x,t) = c^2\frac{\partial^2 w}{\partial x^2}(x,t) & \text{in } (0,L)\times(0,\infty), \\ w(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ w(\pi,t) = 0 & \text{on } \{x=L\}\times[0,\infty), \\ w(x,0) = \pi^3 - \pi^2 & \text{on } [0,\pi]\times\{t=0\}. \end{cases}$$

  The general solution for w is given by

  $$w(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2c^2t}\sin(nx).$$

Plugging the initial data in, we have

$$\pi^3 - \pi^2 = \sum_{n=1}^{\infty} b_n \sin(nx).$$

Hence, the coefficients b_n are obtained from (i), with $A = \pi^3 - \pi^2$. Thus, we have

$$w(x,t) = \sum_{n=1}^{\infty} \frac{4(\pi^3 - \pi^2)}{(2n-1)\pi} e^{-(2n-1)^2 c^2 t} \sin((2n-1)x).$$

Henceforth, the solution $u(x,t)$ is given by

$$u(x,t) = w(x,t) + g(x) = \sum_{n=1}^{\infty} \frac{4(\pi^2 - \pi)}{(2n-1)} e^{-(2n-1)^2 c^2 t} \sin((2n-1)x) + \pi^2.$$

- Answer to Exercise 24:
  (i) $(1-2x^2)\mu'' - 4x\mu' + \lambda x\mu = 0$.
  (ii) Yes, as it is a self-adjoint equation. Self-adjoint equations can always be made into S-L form. In particular, we have $((1-2x^2)y')' + \lambda xy = 0$.
  (iii) Reading off the equation from the previous part for q, we have $k(x) = q(x) = x$.

- Answer to Exercise 25: You can either use the fact that this corresponds to Case C or compute the Euler's equation directly to obtain
  $$y'' + y = 0, y(0) = y(\pi) = 0.$$
  The solution to this is given by $y(x) = \sin(x)$. The corresponding value of the functional is given by
  $$\int_0^{\pi} \sin^2(x) - \cos^2(x)\mathrm{d}x = 0.$$

- Answer to Exercise 26: You will get two different values of λ; $\lambda = 4$ or $\lambda = -2$. For $\lambda = -2$, the corresponding point is $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ with distance 2. For $\lambda = 4$, the corresponding point is $\left(-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$ with distance 4. Henceforth, the point on the sphere that maximizes its distance from $(2,1,2)$ is $\boxed{\left(-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)}$.
  This is supported by our physical intuition, as these points are antipodal points on the sphere, with one of the points maximizing distance and the other minimizing distance. Alternatively, one could say that by Extreme Value Theorem, since the sphere is closed and bounded, the extreme value must be attained somewhere, and it suffices to compare the value of the distance at these two extremal points.

References

[1] George F. Simmons. *Differential Equations with Applications and Historial Notes*. CRC Press, 3rd edition, 2016.