

# Generic Structural Stability in $2 \times 2$ Systems of Conservation Laws

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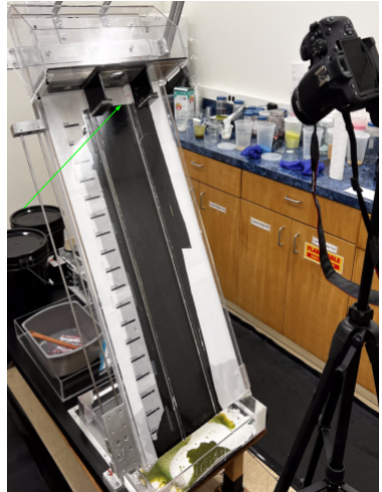


# Motivation

## Gravity-Driven Particle-Laden Flow

### Experimental Setup:

- Inclined Slope
- Negatively buoyant monodisperse particles (i.e. glass beads) mixed with oil.
- Mixture of particles and oil added with a gate before the start of the experiment.
- Release the gate to start.



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$h(x, t)$  : Height of the slurry mixture.

$\phi_0(x, t)$  :  $z$ -averaged particle volume fraction.

$x$  : Distance downstream (from the gate).

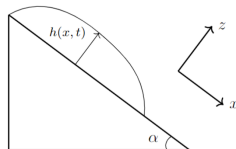
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Assumptions: Fast Equilibrium + Lubrication Assumption.



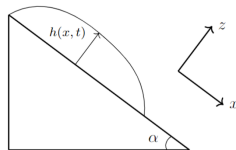
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- Conservation of suspension volume:

$$\partial_t h + \partial_x F(h, h_0) = 0.$$

- Conservation of the number of particles:

$$\partial_t (h \phi_0) + \partial_x G(h, h \phi_0) = 0.$$

- Functional form of flux functions:

$$F(h, h \phi_0) = h^3 f\left(\frac{h \phi_0}{h}\right) = h^3 f(\phi_0),$$

$$G(h, h \phi_0) = h^3 g\left(\frac{h \phi_0}{h}\right) = h^3 g(\phi_0).$$

Issue:  $f$  and  $g$  are computationally expensive to evaluate.

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$(\phi(s), \sigma(s)), s \in [0, 1]$ :

$$\begin{cases} \phi'(s) = \frac{(-B_2 + (B_2 + 1)\phi(s) + \rho_s \phi(s)^2)(\phi_m - \phi(s))}{\sigma(s)(\phi_m + (B_1 - 1)\phi(s))} H(\phi(s)) H(\phi_m - \phi(s)), \\ \sigma'(s) = -1 - \rho_s \phi(s), \\ \sigma(0) = 1 + \rho_s \phi_0, \\ \sigma(1) = 0, \end{cases}$$

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Given:

$$\begin{cases} u_t + (F_0(u, v))_x = 0, \\ v_t + (G_0(u, v))_x = 0, \end{cases}$$

with given initial data  
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yield solutions are sufficiently close in the following sense:

- $L^1$  stability of  $L^1 \cap BV$  solution with respect to flux functions, and
- Structural Stability of Riemann Problems; ie solutions with initial data of the form

$$(u, v)(0, x) = \begin{cases} (u_l, v_l) & \text{for } x < 0, \\ (u_r, v_r) & \text{for } x > 0. \end{cases}$$



# Main Result

Consider:

$$\begin{cases} u_t + (F_0(u, v))_x = 0, \\ v_t + (G_0(u, v))_x = 0, \end{cases}$$

with Riemann initial data

$$(u, v)(0, x) = \begin{cases} (u_l, v_l) & \text{for } x < 0, \\ (u_r, v_r) & \text{for } x > 0, \end{cases}$$

for  $(t, x) \in [0, \infty) \times \mathbb{R}$ ,  $U \subset \mathbb{R}^2$  open, and  $F_0, G_0 \in C^2(U)$ .

# Main Result

Assumptions:  $(F_0, G_0)$  forms a

(i) **Strictly hyperbolic** system in  $U$ :

Jacobian matrix  $J(u, v; F_0, G_0) = \begin{pmatrix} (F_0)_u & (F_0)_v \\ (G_0)_u & (G_0)_v \end{pmatrix}$  possess two distinct real eigenvalues for each  $(u, v) \in U$ .

(ii) **Genuinely non-linear** system in  $U$ : For  $k \in \{1, 2\}$

$$\nabla \underbrace{\lambda_k}_{\text{k-Eigenvalue}}(u, v; F_0, G_0) \cdot \underbrace{\mathbf{r}_k}_{\text{k-Right Eigenvector}}(u, v; F_0, G_0) \neq 0.$$

Convention:  $\lambda_1 < \lambda_2$ .

(iii) **Uni-directional** system in  $U$ : Either

- $(F_0)_v(u, v) \neq 0$  for all  $(u, v) \in U$  or
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Perturbations:

$$\begin{cases} \tilde{F} = F_0 + F_\delta, \\ \tilde{G} = G_0 + G_\delta. \end{cases}$$

## Theorem (Generic Approximation Theorem & Structural Stability.)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For almost every  $(u_l, v_l) \neq (u_r, v_r) \in U$ , consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset  $K \subset U$  containing  $(u_l, v_l)$  and  $(u_r, v_r)$  in its interior,

- 1 The unperturbed system satisfies the transversality property on  $K$ ,
- 2 There exists  $\varepsilon_1, \varepsilon_2 > 0$  such that for any perturbations  $(F_\delta, G_\delta) \in C^2(K)^2$  with  $\|(F_\delta, G_\delta)\|_{C^2(K)^2} < \varepsilon_1$ , the corresponding perturbed  $2 \times 2$  system admits a unique double-wave entropy solution with an intermediate state  $(\tilde{u}^*, \tilde{v}^*) \in \text{int}(K)$  satisfying  $\|(\tilde{u}^*, \tilde{v}^*) - (u^*, v^*)\|_2 < \varepsilon_2$ .
- 3 The perturbed system satisfies the transversality property on the same compact set  $K$ .

# Main Result

Roughly speaking, this translates to:

For a system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption, for almost every (generic) Riemann initial data, unique solutions and their structures (shock/rarefactions) are preserved upon a sufficiently small  $C^2$  perturbation to the flux functions.

Furthermore, the “amplitudes” of shock and rarefaction upon perturbation are only perturbed by a small amount.

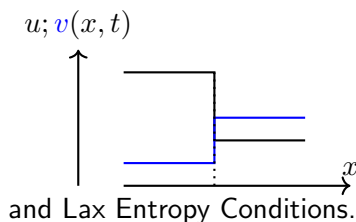
# Crash Course: $2 \times 2$ System

$$\text{General System: } \begin{cases} u_t + (F(u, v))_x = 0, \\ v_t + (G(u, v))_x = 0. \end{cases}$$

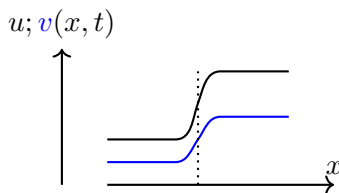
Riemann initial data:

$$(u, v)(0, x) = \begin{cases} (u_l, v_l) & \text{for } x < 0, \\ (u_r, v_r) & \text{for } x > 0, \end{cases}$$

Shock:



Rarefaction:



# Crash Course: $2 \times 2$ System

## State Space $(u, v)$ Analysis :

Given a state  $(u_l, v_l)$ ,

- (Shocks) Hugoniot loci: (Rankine-Hugoniot) All  $(u, v)$  satisfying  $\begin{pmatrix} F(u, v) - F(u_l, v_l) \\ G(u, v) - G(u_l, v_l) \end{pmatrix} = s \begin{pmatrix} u - u_l \\ v - v_l \end{pmatrix}$  for some  $s$ .

Equivalently,

$$(F(u, v) - F(u_l, v_l))(v - v_l) - (G(u, v) - G(u_l, v_l))(u - u_l) = 0.$$

Required to satisfy 1-wave Lax Entropy condition.

- 1-Rarefaction Curves: All  $(u, v)$  solving

$$\begin{cases} \frac{d}{d\lambda}(u(\lambda), v(\lambda)) = \mathbf{r}_1(u(\lambda), v(\lambda)) \\ (u(\lambda(u_l, v_l)), v(\lambda(u_l, v_l))) = (u_l, v_l) \end{cases}$$

Integral curves of the right eigenvector  $\mathbf{r}_1$ .

Solve for increasing  $\lambda$ .

# Crash Course: $2 \times 2$ System

## State Space $(u, v)$ Analysis :

Given a state  $(u_r, v_r)$ ,

- (Shocks) Hugoniot loci: (Rankine-Hugoniot) All  $(u, v)$  satisfying  $\begin{pmatrix} F(u, v) - F(u_r, v_r) \\ G(u, v) - G(u_r, v_r) \end{pmatrix} = s \begin{pmatrix} u - u_r \\ v - v_r \end{pmatrix}$  for some  $s$ .

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$$(F(u, v) - F(u_r, v_r))(v - v_r) - (G(u, v) - G(u_r, v_r))(u - u_r) = 0.$$

Required to satisfy 2-wave Lax Entropy condition.

- 2-Rarefaction Curves: All  $(u, v)$  solving

$$\begin{cases} \frac{d}{d\lambda}(u(\lambda), v(\lambda)) = \mathbf{r}_2(u(\lambda), v(\lambda)) \\ (u(\lambda(u_l, v_l)), v(\lambda(u_l, v_l))) = (u_l, v_l) \end{cases}$$

Integral curves of the right eigenvector  $\mathbf{r}_2$ .

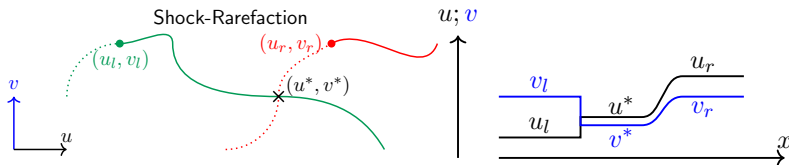
Solve for decreasing  $\lambda$ .



# Crash Course: $2 \times 2$ System

## Constructing Composite Solutions:

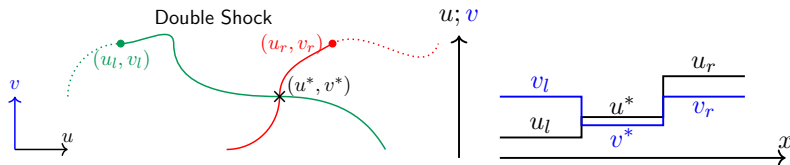
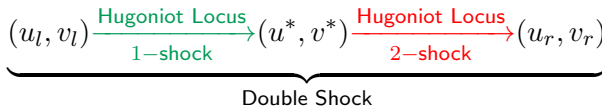
$$\underbrace{(u_l, v_l) \xrightarrow[\text{1-shock}]{\text{Hugoniot Locus}} (u^*, v^*) \xrightarrow[\text{2-rarefaction}]{\text{Rarefaction Curve}} (u_r, v_r)}_{\text{Shock-Rarefaction Solution}}$$



Hugoniot Loci     /  
Rarefaction Curve     \

# Crash Course: $2 \times 2$ System

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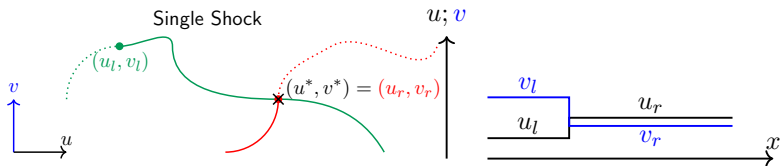


Hugoniot Loci      /  
Rarefaction Curve      \

# $2 \times 2$ System

## Unstable Case I: Single Wave Solution.

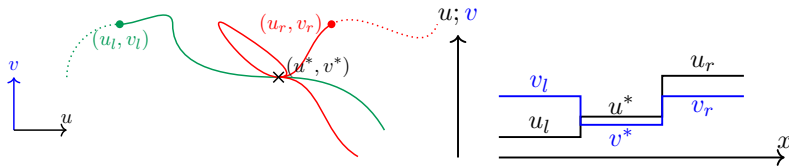
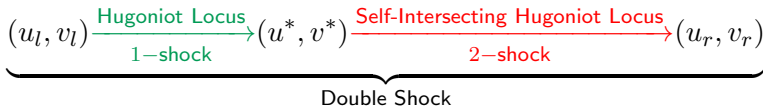
$$\underbrace{(u_l, v_l) \xrightarrow[\text{1-shock}]{\text{Hugoniot Locus}} (u^*, v^*) = (u_r, v_r)}_{\text{Single Shock}}$$



Hugoniot Loci     /  
Rarefaction Curve     ···

# $2 \times 2$ System

## Unstable Case II: Self-Intersecting Hugoniot Loci.

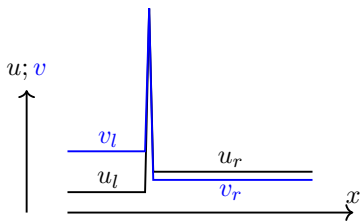
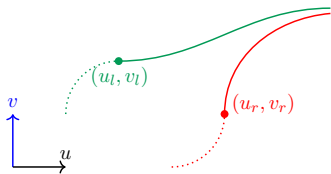


Hugoniot Loci /  
Rarefaction Curve ···

# $2 \times 2$ System

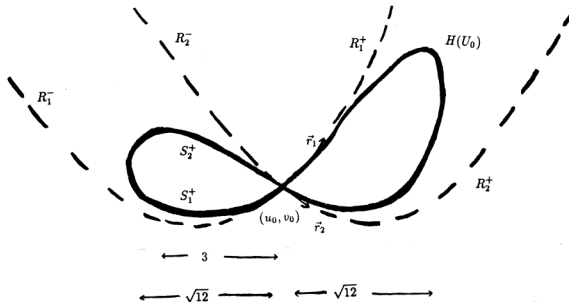
Unstable Case III: Singular ( $\delta$ ) Shock - Intersection at  $\infty$ .  
 To be interpreted in the sense of distributions.  
 (Wang and Bertozzi, 2014.)

$$\underbrace{(u_l, v_l) \xrightarrow[\text{1-shock}]{\text{Hugoniot Locus}} \infty \xleftarrow[\text{2-shock}]{\text{Hugoniot Locus}} (u_r, v_r)}_{\text{Singular Shock}}$$



## $2 \times 2$ System

Case IV: Singular Shock - Self-intersecting at given states.  
(Keyfitz and Kranzer, 1990.)



Hugoniot Loci is **not** a manifold (locally ' $\times$ ', not Euclidean).

# Regular Manifold Assumption

Recall: Hugoniot loci connects all  $(u, v)$  from a given state  $(u_g, v_g)$

$$(F(u, v) - F(u_g, v_g))(v - v_g) - (G(u, v) - G(u_g, v_g))(u - u_g) = 0.$$

Define the **Hugoniot Objective Function**:

$$H_{(u_g, v_g)} = (F(u, v) - F(u_g, v_g))(v - v_g) - (G(u, v) - G(u_g, v_g))(u - u_g).$$

Hugoniot locus is the zero level set of  $H_{(u_g, v_g)}$ .

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## Regular Manifold Assumption

The Jacobian map  $(dH_{(u_g, v_g)})_{(u, v)} : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $(D_u H_{(u_g, v_g)}(u, v) \quad D_v H_{(u_g, v_g)}(u, v))$  is surjective for each  $(u, v) \neq (u_g, v_g)$  on the Hugoniot locus.

- Always not satisfied at  $(u, v) = (u_g, v_g)$ .
- By the **Regular Value Theorem**, the Hugoniot locus restricted on  $U \setminus \{(u_g, v_g)\}$  is a  $C^1$  manifold.



# Transversality

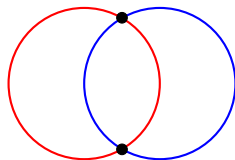
Let  $\mathcal{M}$  and  $\mathcal{N}$  be submanifolds of  $\mathbb{R}^n$ .

## Definition: Transverse Intersection

We say that  $\mathcal{M}$  and  $\mathcal{N}$  **intersects transversely** if for every  $x \in \mathcal{M} \cap \mathcal{N}$ ,

$$T_x\mathcal{M} + T_x\mathcal{N} = \mathbb{R}^n.$$

Notation:  $\mathcal{M} \pitchfork \mathcal{N}$ .



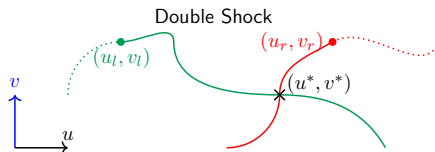
# Transversality Property

Let  $K$  be a compact subset of  $U$  containing the given left and right states  $(u_l, v_l) \neq (u_r, v_r)$ .

## Definition: Transversality Property

We say that the  $2 \times 2$  system with Riemann initial data given by  $(u_l, v_l)$  and  $(u_r, v_r)$  as left and right states satisfies the **transversality property on  $K$**  if for the “correct” curves  $\mathcal{W}_l$  (from  $(u_l, v_l)$ ) and  $\mathcal{W}_r$  (from  $(u_r, v_r)$ ) intersecting at  $(u^*, v^*) \neq (u_l, v_l)$  or  $(u_r, v_r)$ , we have

$$\mathcal{W}_l \pitchfork \mathcal{W}_r.$$



## Theorem (Generic Approximation Theorem & Structural Stability.)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For almost every  $(u_l, v_l) \neq (u_r, v_r) \in U$ , consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset  $K \subset U$  containing  $(u_l, v_l)$  and  $(u_r, v_r)$  in its interior,

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# Step I: Implicit Function Theorem on Banach Spaces

## Theorem A. (Structural Stability)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For **given states**  $(u_l, v_l) \neq (u_r, v_r) \in U$ , consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset  $K \subset U$  containing  $(u_l, v_l)$  and  $(u_r, v_r)$  in its interior,

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**Moreover,** The perturbed system satisfies the transversality property on the same compact set  $K$ .

# Step I: Implicit Function Theorem on Banach Spaces

Proof Sketch (Persistence of Existence):

- **Hugoniot Objective Function**  $H(u, v; u_g, v_g, F, G)$  given by

$$\begin{aligned} & H(u, v; u_g, v_g, F, G) \\ &= (F(u, v) - F(u_g, v_g))(v - v_g) - (G(u, v) - G(u_g, v_g))(u - u_g). \end{aligned}$$

- Hugoniot locus: All  $(u, v)$  such that  $H(u, v; u_g, v_g, F, G) = 0$ .

# Step I: Implicit Function Theorem on Banach Spaces

## Rarefaction Curves:

- Rarefaction ODEs:

$$\begin{cases} \frac{d}{d\lambda}(u(\lambda), v(\lambda)) = \mathbf{r}_k(u(\lambda), v(\lambda)) \\ (u(\lambda(u_g, v_g)), v(\lambda(u_g, v_g))) = (u_g, v_g) \end{cases}$$

- Use uni-direction assumption (iii) to normalize the 2nd component of the right eigenvector to be 1.
- Obtain a single ODE " $\frac{du}{dv} = \frac{du/d\lambda}{dv/d\lambda}$ ":

$$\begin{cases} \frac{d}{dv}u_k(v; F, G) = \Xi(u_k(v; F, G), F, G) \\ u_k(v_g) = u_g. \end{cases}$$

# Step I: Implicit Function Theorem on Banach Spaces

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- **Rarefaction Objective Function:**

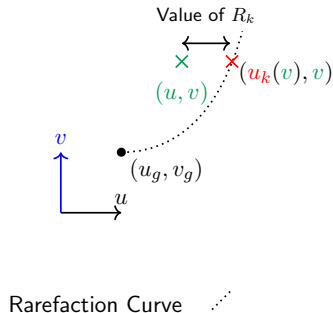
$$R_k(u, v; u_g, v_g, F, G) = u - u_k(v; u_g, v_g, F, G).$$

# Step I: Implicit Function Theorem on Banach Spaces

## Rarefaction Objective Function:

$$R_k(u, v; u_g, v_g, F, G) = u - u_k(v; u_g, v_g, F, G)$$

Interpretation: Signed Distance of  $u$ -coordinate to rarefaction curve integrated up to  $v$ .



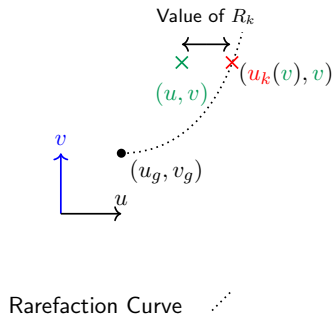


# Step I: Implicit Function Theorem on Banach Spaces

## Rarefaction Objective Function:

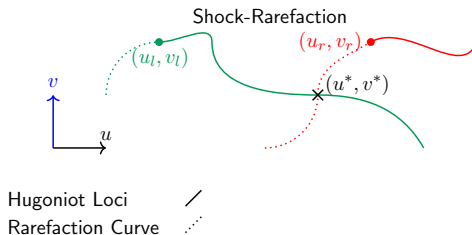
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$k$ -Rarefaction curve = Zero-level set of  $R_k$ .

# Step I: Implicit Function Theorem on Banach Spaces



Example: Unique intermediate state  $(u^*, v^*)$  and unperturbed fluxes  $(F_0, G_0)$  satisfy

$$\begin{cases} H(u^*, v^*, F_0, G_0; u_l, v_l) = 0, \\ R_2(u^*, v^*, F_0, G_0; u_r, v_r) = 0. \end{cases}$$

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Apply **Implicit Function Theorem on Banach Spaces** to

$$\mathbf{hr}(u, v, F, G) := \begin{cases} H(\mathbf{u}, \mathbf{v}, \mathbf{F}, \mathbf{G}; u_l, v_l) = 0, \\ R_2(\mathbf{u}, \mathbf{v}, \mathbf{F}, \mathbf{G}; u_r, v_r) = 0. \end{cases}$$

with  $(u, v) \in K$  and  $(F, G) \in C^2(K)^2$  to obtain a map  $\mathbf{M} : C^2(K)^2 \rightarrow K$  such that

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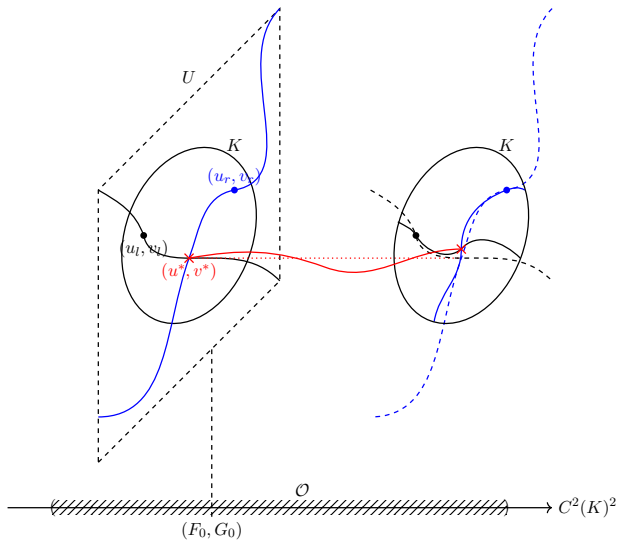
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with  $(u, v) \in K$  and  $(F, G) \in C^2(K)^2$  to obtain a map  $\mathbf{M} : C^2(K)^2 \rightarrow K$  such that

$$\begin{cases} H(\mathbf{M}(F, G), F, G; u_l, v_l) = 0, \\ R_2(\mathbf{M}(F, G), F, G; u_r, v_r) = 0. \end{cases}$$

with  $\mathbf{M}(F_0, G_0) = (u^*, v^*)$  in a  $C^2(K)^2$  neighborhood of  $(F_0, G_0)$ .

# Step I: Implicit Function Theorem on Banach Spaces



## Transition to Step II

$$\begin{cases} H(M(F, G), F, G; u_l, v_l) = 0, \\ R_2(M(F, G), F, G; u_r, v_r) = 0. \end{cases}$$

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To apply **Implicit Function Theorem on Banach Spaces**, we need to check the invertibility of a Jacobian map (matrix)

$D_{(u,v)} \mathbf{hr}(u^*, v^*, F_0, G_0) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , given by

$$\begin{pmatrix} D_u H(u^*, v^*, F_0, G_0; u_l, v_l) & D_v H(u^*, v^*, F_0, G_0; u_l, v_l) \\ D_u R_2(u^*, v^*, F_0, G_0; u_r, v_r) & D_v R_2(u^*, v^*, F_0, G_0; u_r, v_r) \end{pmatrix}.$$



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If  $\mathcal{W}_l$  = Hugoniot locus from  $(u_l, v_l)$  and  $\mathcal{W}_r$  = Rarefaction curve from  $(u_r, v_r)$ , this is equivalent to

$$\mathcal{W}_l \pitchfork \mathcal{W}_r = \mathbb{R}^2.$$

# Transition to Step II

Recall:

## Theorem A. (Structural Stability)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For **given states**  $(u_l, v_l) \neq (u_r, v_r) \in U$ , consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset  $K \subset U$  containing  $(u_l, v_l)$  and  $(u_r, v_r)$  in its interior,

**If** The unperturbed system satisfies the transversality property on  $K$ ,

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**Moreover,** The perturbed system satisfies the transversality property on the same compact set  $K$ .

# Transition to Step II

## Theorem B. (Transversality is Generic.)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For **almost every**  $(u_l, v_l) \neq (u_r, v_r) \in U$ , consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset  $K \subset U$  containing  $(u_l, v_l)$  and  $(u_r, v_r)$  in its interior, we

**Get:** The unperturbed system satisfies the transversality property on  $K$ .

# Transition to Step II

Theorem A + Theorem B = Main Theorem.

## Theorem. (Generic Approximation Theorem & Structural Stability.)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For **almost every**  $(u_l, v_l) \neq (u_r, v_r) \in U$ , consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset  $K \subset U$  containing  $(u_l, v_l)$  and  $(u_r, v_r)$  in its interior,

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## Step II: Parametric Transversality Theorems

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## Step II: Parametric Transversality Theorems

Let  $\mathcal{X}$  and  $\mathcal{Y}$  be  $C^r$  manifolds, and  $\mathcal{Z}$  be a  $C^r$  submanifold of  $\mathcal{Y}$  for  $r \geq 1$ .

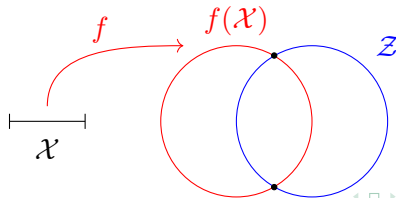
### Definition: Transversality of a Map

Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a  $C^r$  map. We say that  $f$  is **transverse** to  $\mathcal{Z}$  if for every  $a \in f^{-1}(\mathcal{Z})$ , we have

$$df(T_a\mathcal{X}) + T_{f(a)}\mathcal{Z} = T_{f(a)}\mathcal{Y}.$$

Notation:  $f \pitchfork \mathcal{Z}$ .

Intuition: “ $f(\mathcal{X}) \pitchfork \mathcal{Z}$ ”.



## Step II: Parametric Transversality Theorems

Typical genericity arguments utilize:

### Thom's Parametric Transversality Theorem

Let  $\mathcal{X}, \mathcal{P}$ , and  $\mathcal{Y}$  be  $C^r$  manifolds and  $\mathcal{Z}$  be a  $C^r$  submanifolds of  $\mathcal{Y}$ . Consider

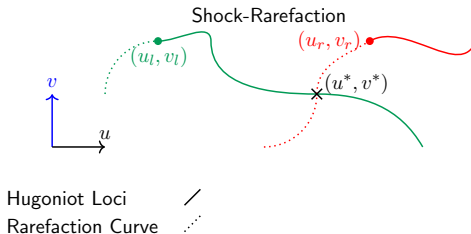
- The map  $F : \mathcal{X} \times \mathcal{P} \rightarrow \mathcal{Y}$ , and
- The associated parametric maps  $F_p : \mathcal{X} \rightarrow \mathcal{Y}$  for each  $p \in \mathcal{P}$ .

Suppose that

- 1  $r > \max\{0, \dim \mathcal{Y} + \dim \mathcal{Z} - \dim \mathcal{X}\}$ ,
- 2 The map  $(x, p) \mapsto F_p(x)$  is  $C^r$ , and
- 3  $F \pitchfork \mathcal{Z}$ .

Then, for almost every  $p \in \mathcal{P}$ ,  $F_p \pitchfork \mathcal{Z}$ .

## Step II: Parametric Transversality Theorems

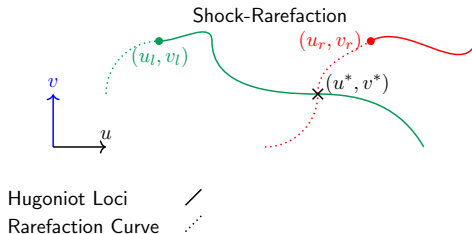


Strategy 1:  $\mathbf{hr}(\cdot; u_l, v_l, u_r, v_r) : U \rightarrow \mathbb{R}^2$  with

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## Step II: Parametric Transversality Theorems

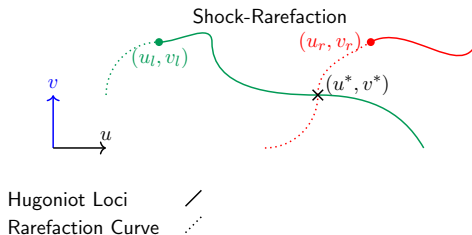


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Recall:  $\mathbf{hr}(u^*, v^*; u_l, v_l, u_r, v_r) = \mathbf{0}$ .

# Step II: Parametric Transversality Theorems



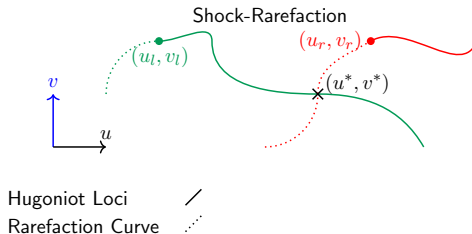
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Recall:  $\mathbf{hr}(u^*, v^*; u_l, v_l, u_r, v_r) = \mathbf{0}$ .

Hope:  $\mathbf{hr}(\cdot; u_l, v_l, u_r, v_r) \not\cap \{(0, 0)\}$  for almost every  $(u_l, v_l) \neq (u_r, v_r)$ .

# Step II: Parametric Transversality Theorems



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Conclude: At each intersection point  $\implies$  transverse intersection.

## Step II: Parametric Transversality Theorems

Define  $\Delta_{U^2} = \{(u_l, v_l, u_r, v_r) : (u_l, v_l) \neq (u_r, v_r)\}$ .

To apply **Thom's Parametric Transversality Theorem**, we need to check that

$$\mathbf{hr} : \underbrace{U}_{\text{Allowed Intersection Points}} \times \underbrace{(U^2 \setminus \Delta_{U^2})}_{\text{Parameters: Left and Right States}} \rightarrow \mathbb{R}^2$$

satisfies  $\mathbf{hr} \pitchfork \{(0, 0)\}$ .

## Step II: Parametric Transversality Theorems

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Hugoniot Loci from  $(u_g, v_g)$  are manifolds on  $U \setminus (u_g, v_g)$ .  
(i.e Keyfitz-Kranzer system.)

## Step II: Parametric Transversality Theorems

Strategy 2: Puncture the domain  $U$  at  $(u_l, v_l)$  and  $(u_r, v_r)$  for **each** given left and right states.

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Set  $U_{(u_l, v_l, u_r, v_r)} := U \setminus \{(u_l, v_l), (u_r, v_r)\}$  and define

$$ULR = \bigcup_{(u_l, v_l, u_r, v_r) \in U^2 \setminus \Delta_{U^2}} \underbrace{U_{(u_l, v_l, u_r, v_r)}}_{\text{Intersection Points}} \times \underbrace{\{(u_l, v_l, u_r, v_r)\}}_{\text{Parameters}}.$$



## Step II: Parametric Transversality Theorems

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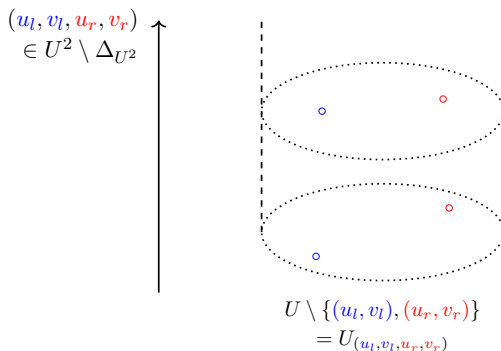
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Now, check  $\mathbf{hr} : ULR \rightarrow \mathbb{R}^2$  satisfies

$$\mathbf{hr} \pitchfork \{(0, 0)\}.$$

## Step II: Parametric Transversality Theorems



$$ULR = \bigcup_{(u_l, v_l, u_r, v_r) \in U^2 \setminus \Delta_{U^2}} U_{(u_l, v_l, u_r, v_r)} \times \{(u_l, v_l, u_r, v_r)\}.$$

$ULR$  is a 6-dimensional submanifold of  $\mathbb{R}^6$ .

# Step II: Parametric Transversality Theorems

## Foliated Parametric Transversality Theorem

Let  $\mathcal{P}$  and  $\mathcal{Y}$  be  $C^r$  manifolds, and  $\mathcal{Z}$  be a  $C^r$  submanifold of  $\mathcal{Y}$ . Suppose that for each  $p \in \mathcal{P}$ , we consider a collection of  $C^r$  manifolds given by  $\{\mathcal{X}_p\}_{p \in \mathcal{P}}$  each with the same dimension  $\dim \mathcal{X}$ , and the following foliated set:

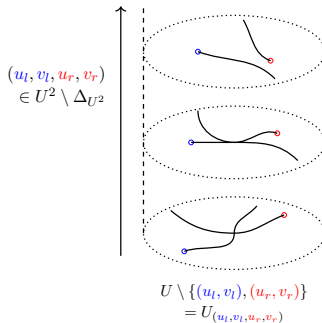
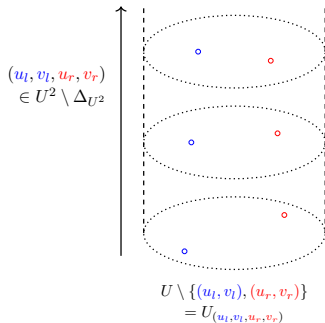
$$\mathcal{XP} := \bigcup_{p \in \mathcal{P}} \mathcal{X}_p \times \{p\}. \quad (1)$$

Consider the maps  $F : \mathcal{XP} \rightarrow \mathcal{Y}$  and the associated map  $F_p : \mathcal{X}_p \rightarrow \mathcal{Y}$  for each parameter  $p \in \mathcal{P}$ . Suppose that

1.  $r > \max\{0, \dim \mathcal{Y} + \dim \mathcal{Z} - \dim \mathcal{X}\}$ ,
2.  $\mathcal{XP}$  is a  $C^r$  manifold with dimension  $\dim \mathcal{XP} = \dim \mathcal{X} + \dim \mathcal{P}$ ,
3.  $T_{(x,p)} \mathcal{XP} = T_x \mathcal{X}_p \times T_p \mathcal{P}$  for each  $(x,p) \in \mathcal{XP}$ ,
4. The map  $(x,p) \mapsto F_p(x)$  is  $C^r$ , and
5.  $F \pitchfork \mathcal{Z}$ .

Then, for almost every  $p \in \mathcal{P}$ ,  $F_p \pitchfork \mathcal{Z}$ .

# Step II: Parametric Transversality Theorems



## Theorem (Generic Approximation Theorem & Structural Stability.)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For almost every  $(u_l, v_l) \neq (u_r, v_r) \in U$ , consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset  $K \subset U$  containing  $(u_l, v_l)$  and  $(u_r, v_r)$  in its interior,

- 1 The unperturbed system satisfies the transversality property on  $K$ ,
- 2 There exists  $\varepsilon_1, \varepsilon_2 > 0$  such that for any perturbations  $(F_\delta, G_\delta) \in C^2(K)^2$  with  $\|(F_\delta, G_\delta)\|_{C^2(K)^2} < \varepsilon_1$ , the corresponding perturbed  $2 \times 2$  system admits a unique double-wave entropy solution with an intermediate state  $(\tilde{u}^*, \tilde{v}^*) \in \text{int}(K)$  satisfying  $\|(\tilde{u}^*, \tilde{v}^*) - (u^*, v^*)\|_2 < \varepsilon_2$ .
- 3 The perturbed system satisfies the transversality property on the same compact set  $K$ .

$$\mathbf{hr}\left(\underbrace{u, v}_{\text{Double-Wave Solutions}}; \underbrace{u_l, v_l, u_r, u_r}_{\text{Generically}}, \underbrace{F, G}_{\text{Perturbation}} \rightarrow \text{Stable}\right)$$

$L^1$  Stability:

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- (Holden and Holden, 1992.)  $L^1$  stability for scalar conservation laws:

$$\|u_f(t, \cdot) - u_g(t, \cdot)\|_{L^1} \lesssim t \text{Lip}(f - g).$$

Done using the front-tracking algorithm.



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- (Bianchini and Colombo, 2002.)  $L^1$  stability for systems:

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} (t, \cdot) - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (t, \cdot) \right\|_{L^1} \lesssim C_{(F_0, G_0)} C_{(\tilde{F}, \tilde{G})} \hat{d}((F_0, G_0), (\tilde{F}, \tilde{G})).$$

Done using (semi-)standard PDE techniques on Riemann semigroup.

## Structural Stability of Riemann Problem:

- (Schechter, Marchesin, and Plohr, 1994.)  
Structurally Stable Riemann Solutions.
  - Conclusion depends on the given left and right states and transversality condition of intersecting curves that could not be checked *a priori*.
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- (Azevedo et. al., 2010 and Eschenazi et. al., 2025.)  
Topological Approach for  $2 \times 2$  systems.
  - Quadratic flux and perturbations; some work in progress.
  - Similar issue with transversality condition.
  - Done by employing desingularization methods (motivated by singularity theorem) specific to quadratic fluxes.

## Genericity for Conservation Laws:

- (Schaeffer, 1973.)

**Schaeffer Regularity Theorem** (for scalar conservation laws): For almost any  $u(0, x) \in \mathcal{S}(\mathbb{R})$ , the solution is piecewise smooth with a finite number of shock curves.

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- (Bressan, Chen, and Huang, 2024.)  
Generic Singularities for 2D Pressureless Flow.
  - $x \in \mathbb{R}^2$ , only for smooth initial data and a specific problem.

# Application I: p-system

(Wendroff, 1972 and Wendroff, 1972.)

$$\begin{cases} u_t + (p(v))_x = 0, \\ v_t - u_x = 0. \end{cases}$$

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Compressible Isentropic Flow in Lagrangian Coordinates:

- Lagrangian Coordinates  $x$
- Velocity in Lagrangian Coordinates  $u \in \mathbb{R}$
- Specific Volume  $v > 0$
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Modelling Assumptions:

- Thermodynamics :  $p'(v) < 0$  for  $v > 0$ .
- Experimental Evidence (Bethe, 1942):  $p''(v) > 0$  for  $v > 0$ .

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Assumptions:

- (i)  $p'(v) < 0$  for  $v > 0$  implies **strictly hyperbolic** system in  $(0, \infty) \times \mathbb{R}$ .
- (ii)  $p''(v) > 0$  for  $v > 0$  implies **genuinely non-linear** system in  $(0, \infty) \times \mathbb{R}$ .
- (iii)  $-1 \neq 0$  implies **uni-directional system** in  $(0, \infty) \times \mathbb{R}$ .

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- Show that  $(dH_{(u_g, v_g)})_{(u, v)} : \mathbb{R}^2 \rightarrow \mathbb{R}$  is surjective for any  $(u, v) \neq (u_g, v_g)$  on the Hugoniot locus.

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Physical Interpretation:

For a sufficiently good  $C^2$  approximation of the pressure relation (on a compact subset), unique double-wave entropy solutions are preserved.



## Application II: Gravity-Driven Particle-Laden Flow

$$\left\{ \begin{array}{l} h_t + \underbrace{\left( h^3 f \left( \frac{h\phi_0}{h} \right) \right)}_{F(h, h\phi_0)} \Big|_x = 0, \\ (h\phi_0)_t + \underbrace{\left( h^3 g \left( \frac{h\phi_0}{h} \right) \right)}_{G(h, h\phi_0)} \Big|_x = 0. \end{array} \right.$$

- $f(\phi_0), g(\phi_0), \phi_0 \in [0, \phi_m]$ .

Physical Interpretation:  $\phi_m = \text{Maximum Packing Fraction}$ .

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- $f(\phi_0), g(\phi_0), \phi_0 \in [0, \phi_m]$ .  
Physical Interpretation:  $\phi_m = \text{Maximum Packing Fraction}$ .
- If the assumptions of the theorem hold, by picking a sufficiently good approximation of  $f$  and  $g$ , hence  $F$  and  $G$ , unique double-wave entropy solutions are preserved.

# Application II-1: Interpolating Flux Functions

Algorithm:

- 1 Place a grid on  $\phi_0 = [0, \phi_m]$  with  $\phi_m = 0.610$ , say step size  $\Delta\phi_0 = 0.001$ .
- 2 Solve the nonlinear ODE for  $\phi_0 = 0.001i$  for  $i = 1, \dots, 610$  to obtain  $f(\phi_0)$  and  $g(\phi_0)$ .
- 3 Obtain  $f(\phi_0)$  and  $g(\phi_0)$  by interpolation.
- 4 Obtain  $f'(\phi_0)$  and  $g'(\phi_0)$  by interpolation too (if needed).

Global Error for  $f = \|f - f_{\text{int}}\|_{C^1(K)} \lesssim o(\Delta\phi_0)$

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Interpretation:

The solutions exhibit structural stability for a sufficiently small grid size, with solutions converging to the original system as grid size goes to 0.

## Application II-2: Gravity-Driven Particle-Laden Flow

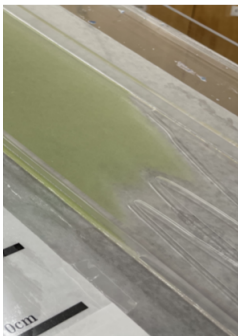
Fix  $\alpha = 25^\circ$ .  $\phi_0 \in [0, \phi_m]$ ,

$\phi_m = 0.61$ : Maximum packing fraction.

$\phi_c \approx 0.503$ : Phase transition from settled to ridged.

Settled:  $\phi_0 < \phi_c$ .

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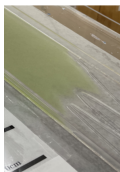
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Polynomial Approximations:

$$f(\phi_0) = \begin{cases} \sum_{j=1}^{10} \beta_{f,j}^S (\phi_c - \phi_0)^{j-1} & \text{for } \phi_0 < \phi_c, \\ \sum_{j=1}^{10} \beta_{f,j}^R (\phi_0 - \phi_c)^{j-1} & \text{for } \phi_0 > \phi_c, \end{cases}$$

and

$$g(\phi_0) = \begin{cases} \sum_{j=1}^{10} \beta_{g,j}^S (\phi_c - \phi_0)^{j-1} & \text{for } \phi_0 < \phi_c, \\ \sum_{j=1}^{10} \beta_{g,j}^R (\phi_0 - \phi_c)^{j-1} & \text{for } \phi_0 > \phi_c. \end{cases}$$

$$\beta_f = \operatorname{argmin}_{\beta_f \in \mathbb{R}^{20}} \|\mathbf{f} - \mathbf{X}\beta_f\|_2^2 + \lambda \|\mathbf{f}' - \mathbf{X}'\beta_f\|_2^2$$

subject to the assumptions above (similar for  $g$ ).

Physical “Constraints”:

- (I) :  $f, f', f'', g, g'$ , and  $g''$  are continuous at  $\phi_c$ ,
- (II) :  $f(0) = \frac{\mu_l}{3}, g(0) = 0$ ,
- (III) : Values of  $f(\phi_c)$  and  $g(\phi_c)$ ,
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Compare:

- $\lambda = 0$  (not fitting for derivatives) and
- $\lambda = 0.03$  (fitting for derivatives, obtained via leave-one-out cross validation).



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subject to the assumptions above (similar for  $g$ ). Sampled Data

Points:

- A couple of points close to  $\phi_m$ ,
- A couple of points close to  $\phi_c$ ,
- A couple of sparse points,
- Points are in triplets to provide derivative information at the middle point.

Optimization Algorithm:

- Quadratic program with linear equality constraints.
- Determine  $\lambda$  by using a leave-one-out cross validation algorithm.

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Asymptotically,

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- $\beta < 1$  if  $\alpha > 70.309^\circ$
- $\beta \in (1, 2)$  if  $\alpha \in (27.895^\circ, 70.309^\circ)$
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It can be “numerically verified” that  $f(\phi_0)$  and  $g(\phi_0)$  are

- $C^2$  across  $\phi_0 = \phi_c$  for  $\alpha = 17^\circ$ .
- $C^1$  only across  $\phi_0 = \phi_c$  for  $\alpha = 30^\circ, 60^\circ, 80^\circ$ .

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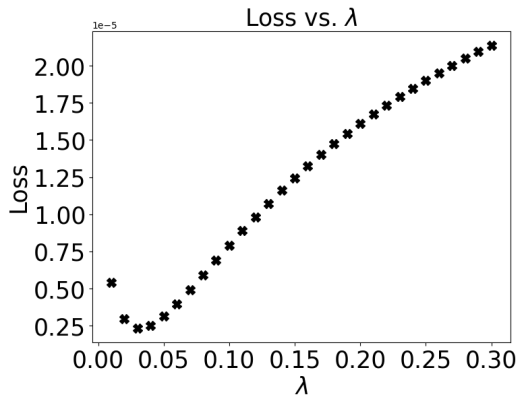
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Furthermore, most parts of the proof suggest that the above argument might work with the Sobolev Space  $W^{2,\infty}(K)$  (i.e “derivatives are Lipschitz continuous”).

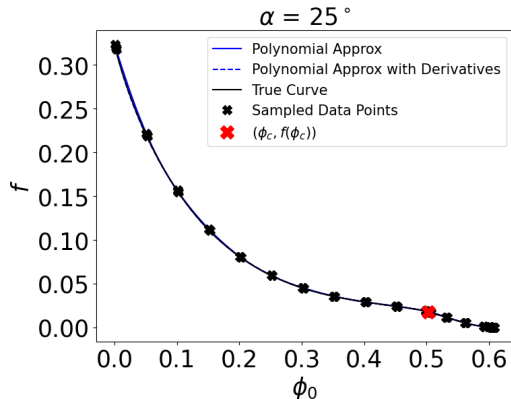
## Application II-2: Gravity-Driven Particle-Laden Flow

Optimal  $\lambda$  from leave-one-out cross validation:



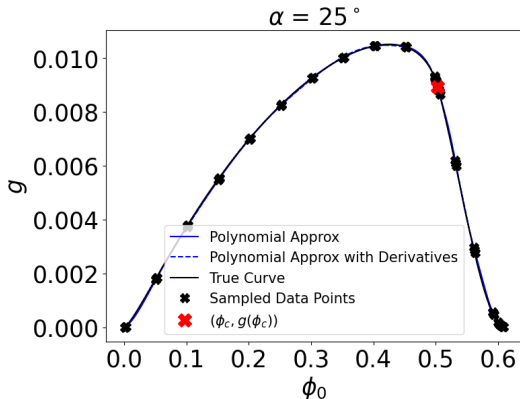
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Quality of Approximation - Flux Function  $f$ :



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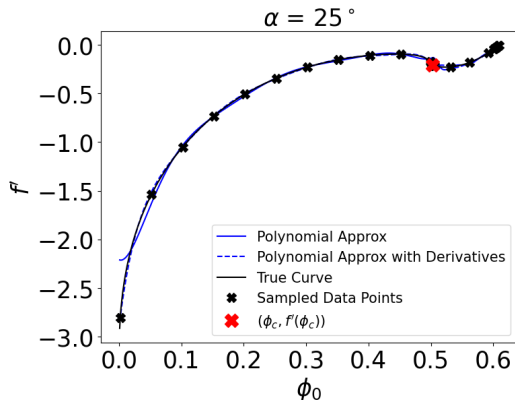
Quality of Approximation - Flux Function  $g$ :





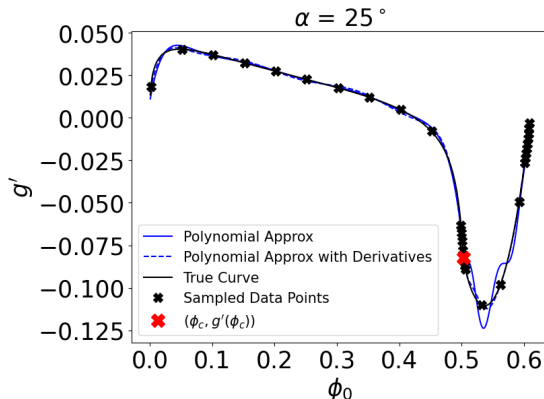
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Quality of Approximation - Derivative of Flux Function  $f'$ :



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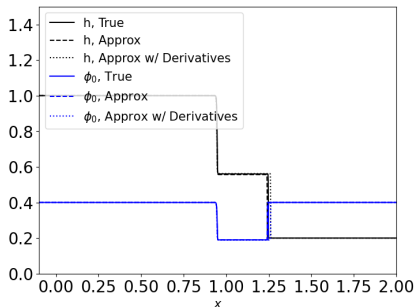


# Application II-2: Gravity-Driven Particle-Laden Flow

Riemann Initial Data:

$$(h, \phi_0)(0, x) = \begin{cases} (1, 0.4) & \text{for } x > 0, \\ (0.2, 0.4) & \text{for } x < 0. \end{cases}$$

Solution for  $(h, \phi_0)(30, x)$ :

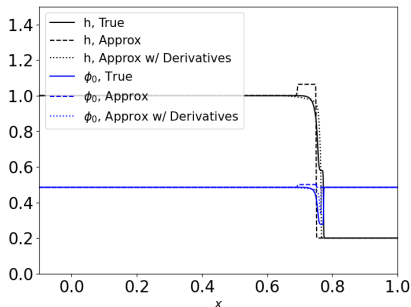


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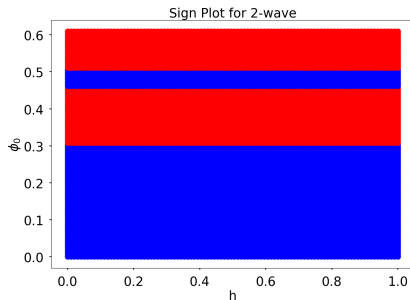
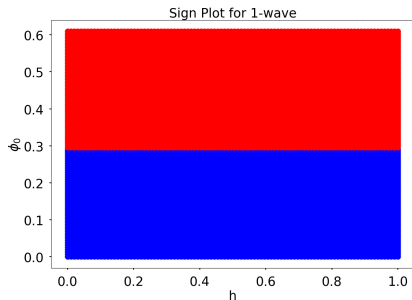
$$(h, \phi_0)(0, x) = \begin{cases} (1, 0.485) & \text{for } x > 0, \\ (0.2, 0.485) & \text{for } x < 0. \end{cases}$$

Solution for  $(h, \phi_0)(30, x)$ :



# Application II-2: Gravity-Driven Particle-Laden Flow

## Violating Genuine Nonlinearity



## Application II-2: Approximating Flux Functions

Computational Time for PDE Simulations,

$\Delta x = 0.001, \Delta t = 0.0005, t = 30, x \in [-0.1, 4],$

- Interpolation: 45s.
- Vectorized Polynomial Approximation: 984s.

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- 1 Generate sparse grid points.
- 2 Fit polynomials to  $f$  and  $g$ .
- 3 Pre-evaluate polynomials on a specified grid.
- 4 For any evaluation of  $f$  and  $g$  (especially in PDE simulations), perform numerical interpolation.

## Application II-3: Comparing Models

Lubrication Assumption gives the **same system of conservation laws** for both diffusive flux and shear-induced migration models:

$$\begin{cases} h_t + \left( h^3 f \left( \frac{h\phi_0}{h} \right) \right)_x = 0, \\ (h\phi_0)_t + \left( h^3 g \left( \frac{h\phi_0}{h} \right) \right)_x = 0. \end{cases}$$

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Observation: If the flux functions from different models are sufficiently close, solutions to the Riemann problems are sufficiently close!

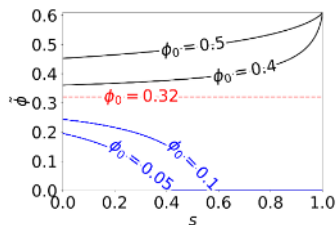
## Application II-3: Comparing Models

- Reference: A comparative study of dynamic models for gravity-driven particle-laden flows. (Lee W.P. et. al, 2025.)
- Authors: 2023 REU students, S.C. Burnett, L. Ding, A. L. Bertozzi.
- Accepted for publication in *Applied Mathematics Letters*, 2025.

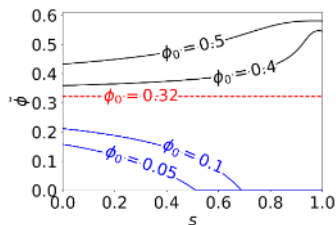
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$\alpha = 50^\circ$ , Equilibrium Profile - I.



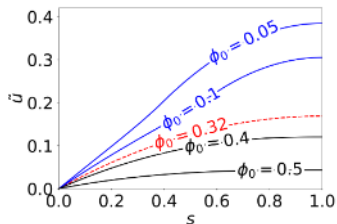
(a)



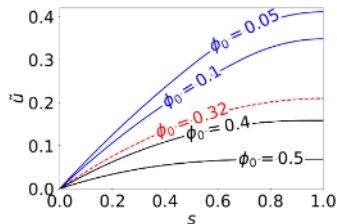
(b)

## Application II-3: Comparing Models

$\alpha = 50^\circ$ , Equilibrium Profile - II.



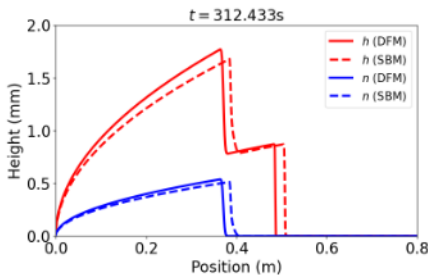
(c)



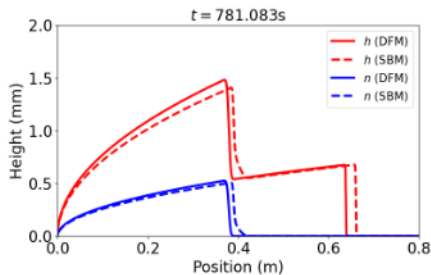
(d)

# Application II-3: Comparing Models

$\alpha = 50^\circ$ , PDE Simulations.



(b)



(c)



- **Main Result:** Under the usual assumptions and some additional mild assumptions, unique double-wave entropy solutions are preserved upon a sufficiently good approximation of flux functions.
- Understanding how each of the assumptions fails allows us to predict properties that are not expected to be preserved upon perturbation.

## Generalizing the result to $n \times n$ systems.

- (Wong and Bertozzi, 2016.)  
 $n = 3$ : Bidensity/Bisize Particle Laden Flow  
(Additional Parameter  $\rightarrow$  Additional Conservation Law.)
- General  $n \times n$  using “more differential topology”.

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## Other Variants - Regularity:

- Lower Regularity required for flux functions and their perturbations.
- Smooth except at finite points (corresponding to phase transitions).
- Perturbations to initial data (left and right states).

## Allowing Linear Degenerate Waves:

- Example:  $n = 3$ , Compressible Euler Equations for gas dynamics.
- Expectation: Preserved under the class of “shocks, rarefactions, and contact discontinuities” for a class of perturbations.

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## Violating Genuine Non-linearity:

- (Liu, 1973.)  
Alternative to Lax's Entropy Condition  $\rightarrow$  Liu's Entropy Condition.
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## Numerical Schemes Motivated by Transversality.



Thank you for your attention!



## Particle-Laden Flow:



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