Generic Structural Stability in 2×2 Systems of Conservation Laws

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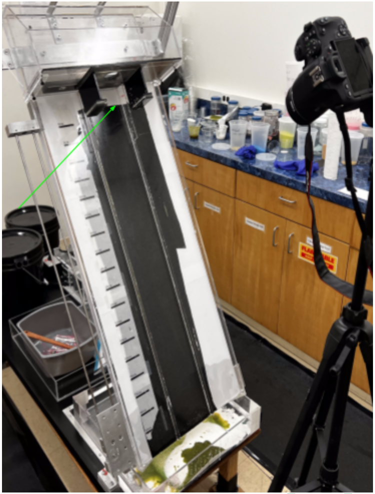
Motivation

Gravity-Driven Particle-Laden Flow

Experimental Setup:

- Inclined Slope
- Negatively buoyant monodisperse particles (i.e glass beads) mixed with oil.
- Mixture of particles and oil added with a gate before the start of the experiment.
- Release the gate to start.



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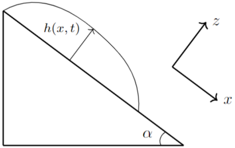

Assumptions: Fast Equilibrium + Lubrication Assumption.

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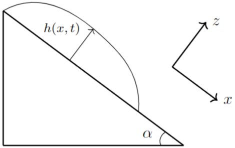

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- Conservation of suspension volume:
$$\partial_t h + \partial_x F(h, h_0) = 0.$$
- Conservation of the number of particles:
$$\partial_t (h\phi_0) + \partial_x G(h, h\phi_0) = 0.$$
- Functional form of flux functions:
$$F(h, h\phi_0) = h^3 f\left(\frac{h\phi_0}{h}\right) = h^3 f(\phi_0),$$
$$G(h, h\phi_0) = h^3 g\left(\frac{h\phi_0}{h}\right) = h^3 g(\phi_0).$$

Issue: f and g are computationally expensive to evaluate.

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Given:

$$\begin{cases} u_t + (F_0(u,v))_x = 0, \\ v_t + (G_0(u,v))_x = 0, \end{cases}$$

with given initial data $(u,v)(0,x)$.

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yield solutions are sufficiently close in the following sense:

- L^1 stability of $L^1 \cap BV$ solution with respect to flux functions, and
- Structural Stability of Riemann Problems; ie solutions with initial data of the form

$$(u,v)(0,x) = \begin{cases} (u_l, v_l) \text{ for } x < 0, \\ (u_r, v_r) \text{ for } x > 0. \end{cases}$$

Main Result

Consider:

$$\begin{cases} u_t + (F_0(u,v))_x = 0, \\ v_t + (G_0(u,v))_x = 0, \end{cases}$$

with Riemann initial data

$$(u,v)(0,x) = \begin{cases} (u_l, v_l) \text{ for } x < 0, \\ (u_r, v_r) \text{ for } x > 0, \end{cases}$$

for $(t,x) \in [0,\infty) \times \mathbb{R}$, $U \subset \mathbb{R}^2$ open, and $F_0, G_0 \in C^2(U)$.

Main Result

Assumptions: (F_0, G_0) forms a

(i) **Strictly hyperbolic** system in U:

Jacobian matrix $J(u,v;F_0,G_0) = \begin{pmatrix} (F_0)_u & (F_0)_v \\ (G_0)_u & (G_0)_v \end{pmatrix}$ possess two distinct real eigenvalues for each $(u,v) \in U$.

(ii) **Genuinely non-linear** system in U: For $k \in \{1,2\}$

$$\nabla \underbrace{\lambda_k}_{\text{k-Eigenvalue}} (u,v;F_0,G_0) \cdot \underbrace{\mathbf{r}_k}_{\text{k-Right Eigenvector}} (u,v;F_0,G_0) \neq 0.$$

Convention: $\lambda_1 < \lambda_2$.

(iii) **Uni-directional system** in U: Either

- $(F_0)_v(u,v) \neq 0$ for all $(u,v) \in U$ or
- $(G_0)_u(u,v) \neq 0$ for all $(u,v) \in U$.

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Perturbations:

$$\begin{cases} \tilde{F} = F_0 + F_\delta, \\ \tilde{G} = G_0 + G_\delta. \end{cases}$$

Main Result

Theorem (Generic Approximation Theorem & Structural Stability.)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For almost every $(u_l, v_l) \neq (u_r, v_r) \in U$, consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset $K \subset U$ containing (u_l, v_l) and (u_r, v_r) in its interior,

1. The unperturbed system satisfies the transversality property on K,
2. There exists $\varepsilon_1, \varepsilon_2 > 0$ such that for any perturbations $(F_\delta, G_\delta) \in C^2(K)^2$ with $\|(F_\delta, G_\delta)\|_{C^2(K)^2} < \varepsilon_1$, the corresponding perturbed 2×2 system admits a unique double-wave entropy solution with an intermediate state $(\tilde{u}^*, \tilde{v}^*) \in \mathrm{int}(K)$ satisfying $\|(\tilde{u}^*, \tilde{v}^*) - (u^*, v^*)\|_2 < \varepsilon_2$.
3. The perturbed system satisfies the transversality property on the same compact set K.

Main Result

Roughly speaking, this translates to:

For a system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption, for almost every (generic) Riemann initial data, unique solutions and their structures (shock/rarefactions) are preserved upon a sufficiently small C^2 perturbation to the flux functions.

Furthermore, the "amplitudes" of shock and rarefaction upon perturbation are only perturbed by a small amount.

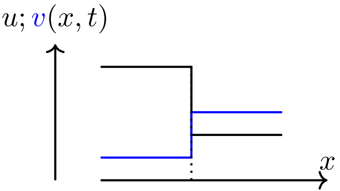
Crash Course: 2×2 System

General System: $\begin{cases} u_t + (F(u, v))_x = 0, \\ v_t + (G(u, v))_x = 0. \end{cases}$

Riemann initial data:

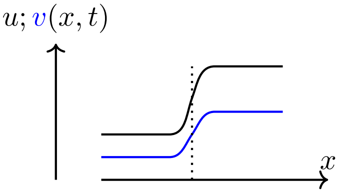
$$(u, v)(0, x) = \begin{cases} (u_l, v_l) \text{ for } x < 0, \\ (u_r, v_r) \text{ for } x > 0, \end{cases}$$

Shock:



and Lax Entropy Conditions.

Rarefaction:



State Space (u, v) Analysis :

Given a state (u_l, v_l),

- (Shocks) Hugoniot loci: (Rankine-Hugoniot) All (u, v) satisfying $\begin{pmatrix} F(u,v) - F(u_l, v_l) \\ G(u,v) - G(u_l, v_l) \end{pmatrix} = s \begin{pmatrix} u - u_l \\ v - v_l \end{pmatrix}$ for some s.
  Equivalently,
  $$(F(u,v) - F(u_l, v_l))(v - v_l) - (G(u,v) - G(u_l, v_l))(u - u_l) = 0.$$
  Required to satisfy 1-wave Lax Entropy condition.
- 1-Rarefaction Curves: All (u, v) solving
  $$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\lambda}(u(\lambda), v(\lambda)) = \mathbf{r}_1(u(\lambda), v(\lambda)) \\ (u(\lambda(u_l, v_l)), v(\lambda(u_l, v_l))) = (u_l, v_l) \end{cases}$$
  Integral curves of the right eigenvector $\mathbf{r}_1$.
  Solve for increasing λ.

State Space (u, v) Analysis :

Given a state (u_r, v_r),

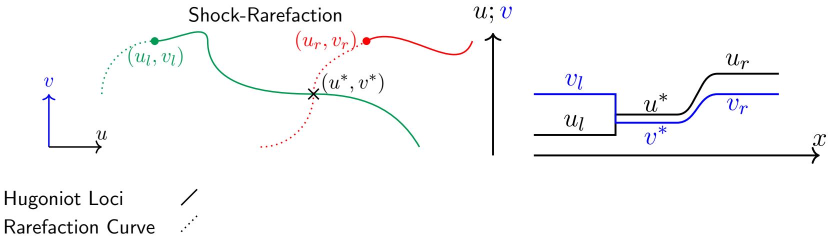
- (Shocks) Hugoniot loci: (Rankine-Hugoniot) All (u, v) satisfying $\begin{pmatrix} F(u,v) - F(u_r, v_r) \\ G(u,v) - G(u_r, v_r) \end{pmatrix} = s \begin{pmatrix} u - u_r \\ v - v_r \end{pmatrix}$ for some s.
  Equivalently,
  $(F(u,v) - F(u_r,v_r))(v - v_r) - (G(u,v) - G(u_r,v_r))(u - u_r) = 0.$
  Required to satisfy 2-wave Lax Entropy condition.
- 2-Rarefaction Curves: All (u, v) solving

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\lambda}(u(\lambda), v(\lambda)) = \mathbf{r}_2(u(\lambda), v(\lambda)) \\ (u(\lambda(u_l, v_l)), v(\lambda(u_l, v_l))) = (u_l, v_l) \end{cases}$$

  Integral curves of the right eigenvector $\mathbf{r}_2$.
  Solve for decreasing λ.

Constructing Composite Solutions:

$$\underbrace{(u_l, v_l) \xrightarrow[\text{1−shock}]{\text{Hugoniot Locus}} (u^*, v^*) \xrightarrow[\text{2−rarefaction}]{\text{Rarefaction Curve}} (u_r, v_r)}_{\text{Shock-Rarefaction Solution}}$$



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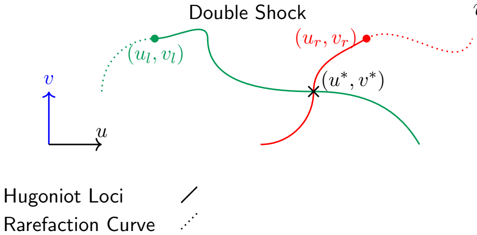
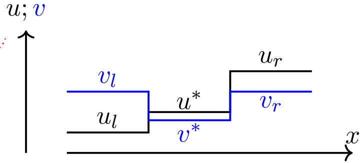
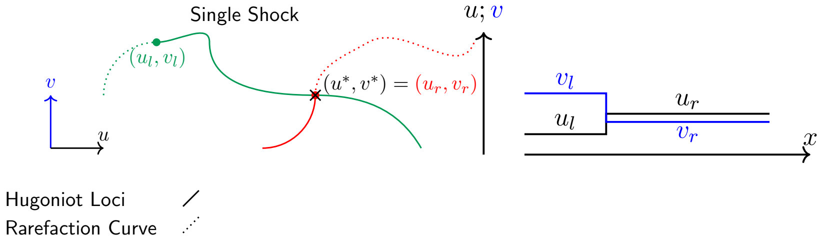




2×2 System

Unstable Case I: Single Wave Solution.

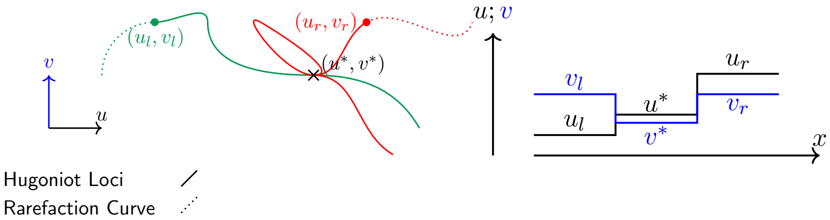
$$\underbrace{(u_l, v_l) \xrightarrow[\text{1−shock}]{\text{Hugoniot Locus}} (u^*, v^*) = (u_r, v_r)}_{\text{Single Shock}}$$



2×2 System

Unstable Case II: Self-Intersecting Hugoniot Loci.

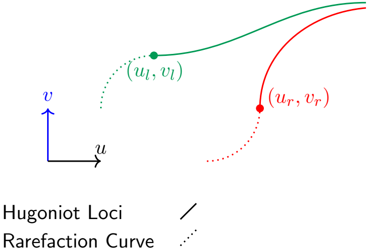
$$\underbrace{(u_l, v_l) \xrightarrow[\text{1−shock}]{\text{Hugoniot Locus}} (u^*, v^*) \xrightarrow[\text{2−shock}]{\text{Self-Intersecting Hugoniot Locus}} (u_r, v_r)}_{\text{Double Shock}}$$

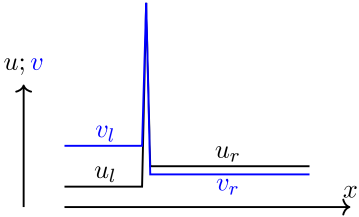


2×2 System

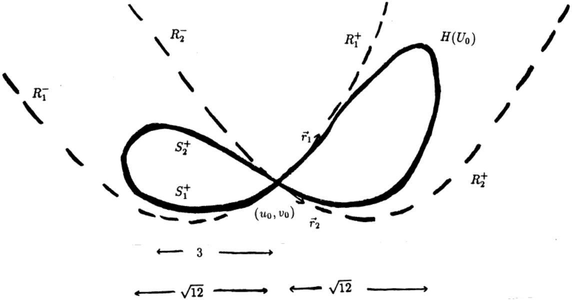
Unstable Case III: Singular (δ) Shock - Intersection at ∞.
To be interpreted in the sense of distributions.
(Wang and Bertozzi, 2014.)

$$\underbrace{(u_l, v_l) \xrightarrow[\text{1}-\text{shock}]{\text{Hugoniot Locus}} \infty \xleftarrow[\text{2}-\text{shock}]{\text{Hugoniot Locus}} (u_r, v_r)}_{\text{Singular Shock}}$$





Case IV: Singular Shock - Self-intersecting at given states. (Keyfitz and Kranzer, 1990.)



Hugoniot Loci is **not** a manifold (locally '$\times$', not Euclidean).

Regular Manifold Assumption

Recall: Hugoniot loci connects all (u, v) from a given state (u_g, v_g)

$$(F(u,v) - F(u_g, v_g))(v - v_g) - (G(u,v) - G(u_g, v_g))(u - u_g) = 0.$$

Define the **Hugoniot Objective Function**:

$$H_{(u_g,v_g)} = (F(u,v) - F(u_g, v_g))(v - v_g) - (G(u,v) - G(u_g, v_g))(u - u_g).$$

Hugoniot locus is the zero level set of $H_{(u_g,v_g)}$.

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Regular Manifold Assumption

The Jacobian map $(dH_{(u_g,v_g)})_{(u,v)} : \mathbb{R}^2 \to \mathbb{R}$ given by $\begin{pmatrix} D_u H_{(u_g,v_g)}(u,v) & D_v H_{(u_g,v_g)}(u,v) \end{pmatrix}$ is surjective for each $(u,v) \neq (u_g,v_g)$ on the Hugoniot locus.

- Always not satisfied at $(u,v) = (u_g,v_g)$.
- By the **Regular Value Theorem**, the Hugoniot locus restricted on $U \setminus \{(u_g,v_g)\}$ is a C^1 manifold.

Transversality

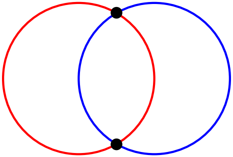
Let $\mathcal{M}$ and $\mathcal{N}$ be submanifolds of $\mathbb{R}^n$.

Definition: Transverse Intersection

We say that $\mathcal{M}$ and $\mathcal{N}$ **intersects transversely** if for every $x \in \mathcal{M} \cap \mathcal{N}$,

$$T_x\mathcal{M} + T_x\mathcal{N} = \mathbb{R}^n.$$

Notation: $\mathcal{M} \pitchfork \mathcal{N}$.



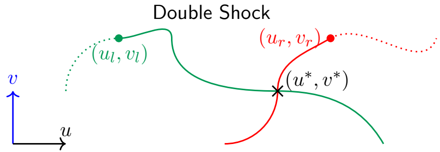
Transversality Property

Let K be a compact subset of U containing the given left and right states $(u_l, v_l) \neq (u_r, v_r)$.

Definition: Transversality Property

We say that the 2×2 system with Riemann initial data given by (u_l, v_l) and (u_r, v_r) as left and right states satisfies the **transversality property on** K if for the "correct" curves $\mathcal{W}_l$ (from (u_l, v_l)) and $\mathcal{W}_r$ (from (u_r, v_r)) intersecting at $(u^*, v^*) \neq (u_l, v_l)$ or (u_r, v_r), we have

$$\mathcal{W}_l \pitchfork \mathcal{W}_r.$$



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Step I: Implicit Function Theorem on Banach Spaces

Theorem A. (Structural Stability)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For given states $(u_l, v_l) \neq (u_r, v_r) \in U$, consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset $K \subset U$ containing (u_l, v_l) and (u_r, v_r) in its interior,

If The unperturbed system satisfies the transversality property on K,

Then There exists $\varepsilon_1, \varepsilon_2 > 0$ such that for any perturbations $(F_\delta, G_\delta) \in C^2(K)^2$ with $\|(F_\delta, G_\delta)\|_{C^2(K)^2} < \varepsilon_1$, the corresponding perturbed 2×2 system admits a unique double-wave entropy solution with an intermediate state $(\tilde{u}^*, \tilde{v}^*) \in \mathrm{int}(K)$ satisfying $\|(\tilde{u}^*, \tilde{v}^*) - (u^*, v^*)\|_2 < \varepsilon_2$.

Moreover, The perturbed system satisfies the transversality property on the same compact set K.

Step I: Implicit Function Theorem on Banach Spaces

Proof Sketch (Persistence of Existence):

- **Hugoniot Objective Function** $H(u,v;u_g,v_g,F,G)$ given by

$$\begin{aligned}&H(u,v;u_g,v_g,F,G)\\=&(F(u,v)-F(u_g,v_g))(v-v_g)-(G(u,v)-G(u_g,v_g))(u-u_g).\end{aligned}$$

- Hugoniot locus: All (u,v) such that $H(u,v;u_g,v_g,F,G)=0$.

Step I: Implicit Function Theorem on Banach Spaces

Rarefaction Curves:

- Rarefaction ODEs:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\lambda}(u(\lambda), v(\lambda)) = \mathbf{r}_k(u(\lambda), v(\lambda)) \\ (u(\lambda(u_g, v_g)), v(\lambda(u_g, v_g))) = (u_g, v_g) \end{cases}$$

- Use uni-direction assumption (iii) to normalize the 2nd component of the right eigenvector to be 1.
- Obtain a single ODE "$\frac{\mathrm{d}u}{\mathrm{d}v} = \frac{\mathrm{d}u/\mathrm{d}\lambda}{\mathrm{d}v/\mathrm{d}\lambda}$":

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}v} u_k(v; F, G) = \Xi(u_k(v; F, G), F, G) \\ u_k(v_g) = u_g. \end{cases}$$

Step I: Implicit Function Theorem on Banach Spaces

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- **Rarefaction Objective Function**:

$$R_k(u, v; u_g, v_g, F, G) = u - u_k(v; u_g, v_g, F, G).$$

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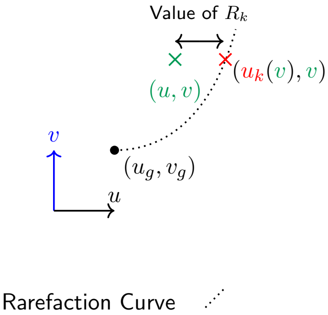
Interpretation: Signed Distance of u-coordinate to rarefaction curve integrated up to v.



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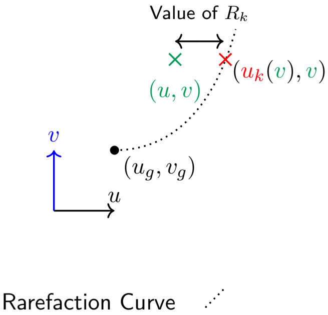
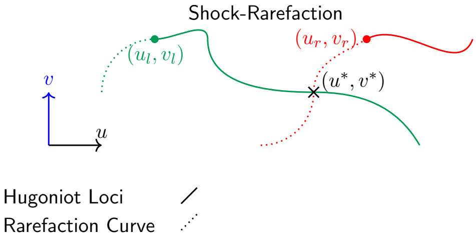


$k-$Rarefaction curve $=$ Zero-level set of R_k.

Step I: Implicit Function Theorem on Banach Spaces



Example: Unique intermediate state (u^*, v^*) and unperturbed fluxes (F_0, G_0) satisfy

$$\begin{cases} H(u^*, v^*, F_0, G_0; u_l, v_l) = 0, \\ R_2(u^*, v^*, F_0, G_0; u_r, v_r) = 0. \end{cases}$$

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Apply **Implicit Function Theorem on Banach Spaces** to

$$\mathbf{hr}(u, v, F, G) := \begin{cases} H(\mathbf{u,v,F,G}; u_l, v_l) = 0, \\ R_2(\mathbf{u,v,F,G}; u_r, v_r) = 0. \end{cases}$$

with $(u, v) \in K$ and $(F, G) \in C^2(K)^2$ to obtain a map $M : C^2(K)^2 \to K$ such that

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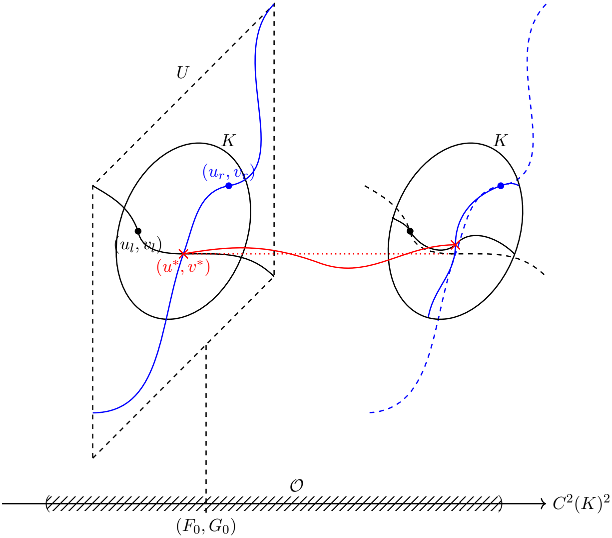
$$\begin{cases} H(M(F, G), F, G; u_l, v_l) = 0, \\ R_2(M(F, G), F, G; u_r, v_r) = 0. \end{cases}$$

with $M(F_0, G_0) = (u^*, v^*)$ in a $C^2(K)^2$ neighborhood of (F_0, G_0).

Step I: Implicit Function Theorem on Banach Spaces



Transition to Step II

$$\begin{cases} H(M(F,G),F,G;u_l,v_l)=0, \\ R_2(M(F,G),F,G;u_r,v_r)=0. \end{cases}$$

Transition to Step II

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To apply **Implicit Function Theorem on Banach Spaces**, we need to check the invertibility of a Jacobian map (matrix) $D_{(u,v)}\mathbf{hr}(u^*,v^*,F_0,G_0) : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$\begin{pmatrix} D_u H(u^*,v^*,F_0,G_0;u_l,v_l) & D_v H(u^*,v^*,F_0,G_0;u_l,v_l) \\ D_u R_2(u^*,v^*,F_0,G_0;u_r,v_r) & D_v R_2(u^*,v^*,F_0,G_0;u_r,v_r) \end{pmatrix}.$$

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$$\begin{pmatrix} D_uH(u^*,v^*,F_0,G_0;u_l,v_l) & D_vH(u^*,v^*,F_0,G_0;u_l,v_l) \\ D_uR_2(u^*,v^*,F_0,G_0;u_r,v_r) & D_vR_2(u^*,v^*,F_0,G_0;u_r,v_r) \end{pmatrix}.$$

If $\mathcal{W}_l =$ Hugoniot locus from (u_l,v_l) and
$\mathcal{W}_r =$ Rarefaction curve from (u_r,v_r), this is equivalent to

$$\mathcal{W}_l \pitchfork \mathcal{W}_r = \mathbb{R}^2.$$

Transition to Step II

Recall:

Theorem A. (Structural Stability)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For given states $(u_l, v_l) \neq (u_r, v_r) \in U$, consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset $K \subset U$ containing (u_l, v_l) and (u_r, v_r) in its interior,

If The unperturbed system satisfies the transversality property on K,

Then There exists $\varepsilon_1, \varepsilon_2 > 0$ such that for any perturbations $(F_\delta, G_\delta) \in C^2(K)^2$ with $\|(F_\delta, G_\delta)\|_{C^2(K)^2} < \varepsilon_1$, the corresponding perturbed 2×2 system admits a unique double-wave entropy solution with an intermediate state $(\tilde{u}^*, \tilde{v}^*) \in \mathrm{int}(K)$ satisfying $\|(\tilde{u}^*, \tilde{v}^*) - (u^*, v^*)\|_2 < \varepsilon_2$.

Moreover, The perturbed system satisfies the transversality property on the same compact set K.

Transition to Step II

Theorem B. (Transversality is Generic.)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For almost every $(u_l, v_l) \neq (u_r, v_r) \in U$, consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset $K \subset U$ containing (u_l, v_l) and (u_r, v_r) in its interior, we

Get: The unperturbed system satisfies the transversality property on K.

Transition to Step II

Theorem A + Theorem B = Main Theorem.

Theorem. (Generic Approximation Theorem & Structural Stability.)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For almost every $(u_l, v_l) \neq (u_r, v_r) \in U$, consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset $K \subset U$ containing (u_l, v_l) and (u_r, v_r) in its interior,

1. The unperturbed system satisfies the transversality property on K,
2. There exists $\varepsilon_1, \varepsilon_2 > 0$ such that for any perturbations $(F_\delta, G_\delta) \in C^2(K)^2$ with $\|(F_\delta, G_\delta)\|_{C^2(K)^2} < \varepsilon_1$, the corresponding perturbed 2×2 system admits a unique double-wave entropy solution with an intermediate state $(\tilde{u}^*, \tilde{v}^*) \in \mathrm{int}(K)$ satisfying $\|(\tilde{u}^*, \tilde{v}^*) - (u^*, v^*)\|_2 < \varepsilon_2$.
3. The perturbed system satisfies the transversality property on the same compact set K.

Step II: Parametric Transversality Theorems

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Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For almost every $(u_l, v_l) \neq (u_r, v_r) \in U$, consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset $K \subset U$ containing (u_l, v_l) and (u_r, v_r) in its interior, we

Get: The unperturbed system satisfies the transversality property on K.

Step II: Parametric Transversality Theorems

Let $\mathcal{X}$ and $\mathcal{Y}$ be C^r manifolds, and $\mathcal{Z}$ be a C^r submanifold of $\mathcal{Y}$ for $r \geq 1$.

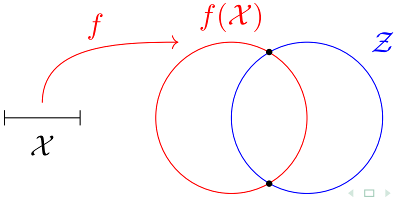
Definition: Transversality of a Map

Let $f : \mathcal{X} \to \mathcal{Y}$ be a C^r map. We say that f is **transverse** to $\mathcal{Z}$ if for every $a \in f^{-1}(\mathcal{Z})$, we have

$$df(T_a\mathcal{X}) + T_{f(a)}\mathcal{Z} = T_{f(a)}\mathcal{Y}.$$

Notation: $f \pitchfork \mathcal{Z}$.

Intuition: "$f(\mathcal{X}) \pitchfork \mathcal{Z}$".



Step II: Parametric Transversality Theorems

Typical genericity arguments utilize:

Thom's Parametric Transversality Theorem

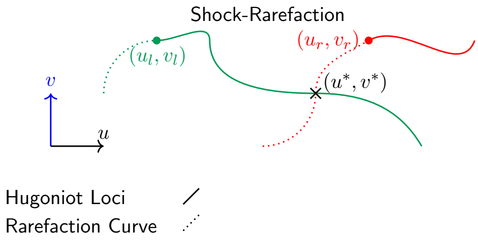
Let $\mathcal{X}, \mathcal{P}$, and $\mathcal{Y}$ be C^r manifolds and $\mathcal{Z}$ be a C^r submanifolds of $\mathcal{N}$. Consider

- The map $F : \mathcal{X} \times \mathcal{P} \to \mathcal{Y}$, and
- The associated parametric maps $F_p : \mathcal{X} \to \mathcal{Y}$ for each $p \in \mathcal{P}$.

Suppose that

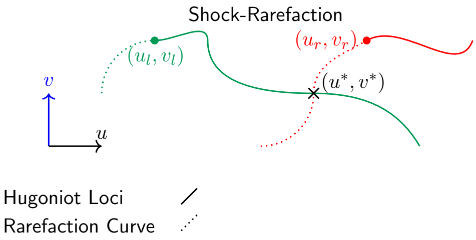
1. $r > \max\{0, \dim \mathcal{Y} + \dim \mathcal{Z} - \dim \mathcal{X}\}$,
2. The map $(x, p) \mapsto F_p(x)$ is C^r, and
3. $F \pitchfork \mathcal{Z}$.

Then, for almost every $p \in \mathcal{P}$, $F_p \pitchfork \mathcal{Z}$.



Strategy 1: $\mathbf{hr}(\cdot; u_l, v_l, u_r, v_r) : U \to \mathbb{R}^2$ with

$$\mathbf{hr}(u, v; u_l, v_l, u_r, v_r) := \begin{cases} H(u, v; u_l, v_l), \\ R_2(u, v; u_r, v_r). \end{cases}$$



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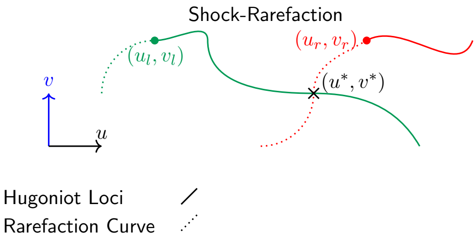
Recall: $\mathbf{hr}(u^*, v^*; u_l, v_l, u_r, v_r) = \mathbf{0}$.



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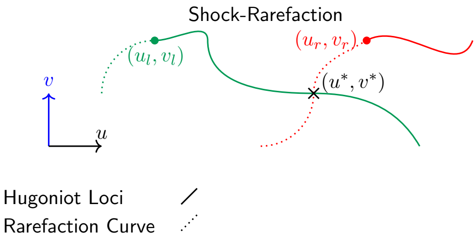
Recall: $\mathbf{hr}(u^*, v^*; u_l, v_l, u_r, v_r) = \mathbf{0}$.
Hope: $\mathbf{hr}(\cdot; u_l, v_l, u_r, v_r) \pitchfork \{(0,0)\}$ for almost every $(u_l, v_l) \neq (u_r, v_r)$.

Step II: Parametric Transversality Theorems



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Hope: $\mathbf{hr}(\cdot; u_l, v_l, u_r, v_r) \pitchfork \{(0,0)\}$ for almost every $(u_l, v_l) \neq (u_r, v_r)$.
Conclude: At each intersection point $\implies$ transverse intersection.

Define $\Delta_{U^2} = \{(u_l, v_l, u_r, v_r) : (u_l, v_l) \neq (u_r, v_r)\}$.

To apply **Thom's Parametric Transversality Theorem**, we need to check that

$$\mathbf{hr} : \underbrace{U}_{\text{Allowed Intersection Points}} \times \underbrace{(U^2 \setminus \Delta_{U^2})}_{\text{Parameters: Left and Right States}} \to \mathbb{R}^2$$

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Then, $\mathbf{hr}(\cdot; u_l, v_l, u_r, v_r) \pitchfork \{(0,0)\}$ for almost every $(u_l, v_l, u_r, v_r) \in U^2 \setminus \Delta_{U^2}$.

Step II: Parametric Transversality Theorems

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Hugoniot Loci from (u_g, v_g) are manifolds on $U \setminus (u_g, v_g)$. (i.e Keyfitz-Kranzer system.)

Step II: Parametric Transversality Theorems

Strategy 2: Puncture the domain U at (u_l, v_l) and (u_r, v_r) for **each** given left and right states.

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Set $U_{(u_l,v_l,u_r,v_r)} := U \setminus \{(u_l, v_l), (u_r, v_r)\}$ and define

$$ULR = \bigcup_{(u_l,v_l,u_r,v_r)\in U^2\setminus\Delta_{U^2}} \underbrace{U_{(u_l,v_l,u_r,v_r)}}_{\text{Intersection Points}} \times \underbrace{\{(u_l, v_l, u_r, v_r)\}}_{\text{Parameters}}.$$

Step II: Parametric Transversality Theorems

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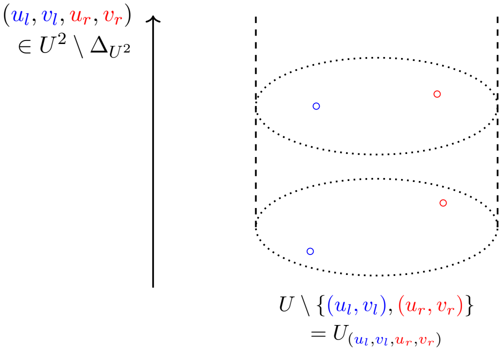
Now, check $\mathbf{hr} : ULR \to \mathbb{R}^2$ satisfies

$$\mathbf{hr} \pitchfork \{(0, 0)\}.$$



$$ULR = \bigcup_{(u_l,v_l,u_r,v_r) \in U^2 \setminus \Delta_{U^2}} U_{(u_l,v_l,u_r,v_r)} \times \{(u_l, v_l, u_r, v_r)\}.$$

ULR is a 6-dimensional submanifold of $\mathbb{R}^6$.

Foliated Parametric Transversality Theorem

Let $\mathcal{P}$ and $\mathcal{Y}$ be C^r manifolds, and $\mathcal{Z}$ be a C^r submanifold of $\mathcal{Y}$. Suppose that for each $p \in \mathcal{P}$, we consider a collection of C^r manifolds given by $\{\mathcal{X}_p\}_{p\in\mathcal{P}}$ each with the same dimension $\dim \mathcal{X}$, and the following foliated set:

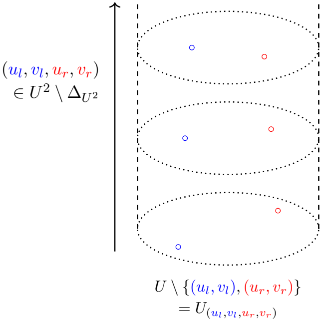
$$\mathcal{X}\mathcal{P} := \bigcup_{p\in\mathcal{P}} \mathcal{X}_p \times \{p\}. \tag{1}$$

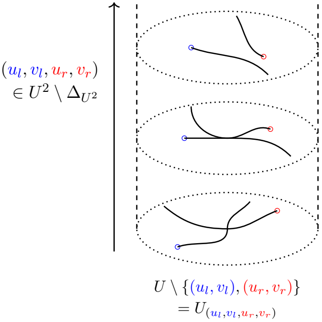
Consider the maps $F : \mathcal{X}\mathcal{P} \to \mathcal{Y}$ and the associated map $F_p : \mathcal{X}_p \to \mathcal{Y}$ for each parameter $p \in P$. Suppose that

1. $r > \max\{0, \dim \mathcal{Y} + \dim \mathcal{Z} - \dim \mathcal{X}\}$,
2. $\mathcal{X}\mathcal{P}$ is a C^r manifold with dimension $\dim \mathcal{X}\mathcal{P} = \dim \mathcal{X} + \dim \mathcal{P}$,
3. $T_{(x,p)}\mathcal{X}\mathcal{P} = T_x\mathcal{X}_p \times T_p\mathcal{P}$ for each $(x, p) \in \mathcal{X}\mathcal{P}$,
4. The map $(x, p) \mapsto F_p(x)$ is C^r, and
5. $F \pitchfork \mathcal{Z}$.

Then, for almost every $p \in \mathcal{P}$, $F_p \pitchfork \mathcal{Z}$.

Step II: Parametric Transversality Theorems





Main Result

Theorem (Generic Approximation Theorem & Structural Stability.)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For almost every $(u_l, v_l) \neq (u_r, v_r) \in U$, consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset $K \subset U$ containing (u_l, v_l) and (u_r, v_r) in its interior,

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3. The perturbed system satisfies the transversality property on the same compact set K.

$$\mathbf{hr}(\underbrace{u, v}_{\text{Double-Wave Solutions}}; \underbrace{u_l, v_l, u_r, u_r}_{\text{Generically}}, \underbrace{F, G}_{\text{Perturbation} \to \text{Stable}})$$

Existing Literature

L^1 Stability:

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- (Holden and Holden, 1992.) L^1 stability for scalar conservation laws:

$$\|u_f(t,\cdot) - u_g(t,\cdot)\|_{L^1} \lesssim t\mathsf{Lip}(f-g).$$

Done using the front-tracking algorithm.

Existing Literature

L^1 Stability:

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  Done using the front-tracking algorithm.
- (Bianchini and Colombo, 2002.) L^1 stability for systems:

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix}(t,\cdot) - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}(t,\cdot) \right\|_{L^1} \lesssim C_{(F_0,G_0)} C_{(\tilde{F},\tilde{G})} \hat{d}((F_0,G_0),(\tilde{F},\tilde{G})).$$

  Done using (semi-)standard PDE techniques on Riemann semigroup.

Existing Literature

Structural Stability of Riemann Problem:

- (Schecter, Marchesin, and Plohr, 1994.)
  Structurally Stable Riemann Solutions.
    - Conclusion depends on the given left and right states and transversality condition of intersecting curves that could not be checked *a priori*.
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  - Done using viscous regularization, traveling waves, and phase portrait analysis.
- (Azevedo et. al., 2010 and Eschenazi et. al., 2025.)
  Topological Approach for 2×2 systems.
  - Quadratic flux and perturbations; some work in progress.
  - Similar issue with transversality condition.
  - Done by employing desingularization methods (motivated by singularity theorem) specific to quadratic fluxes.

Existing Literature

Genericity for Conservation Laws:

- (Schaeffer, 1973.)
  Schaeffer Regularity Theorem (for scalar conservation laws): For almost any $u(0,x) \in \mathcal{S}(\mathbb{R})$, the solution is piecewise smooth with a finite number of shock curves.
  - Only for scalar conservation laws; strong assumptions on initial data.
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- (Bressan, Chen, and Huang, 2024.)
  Generic Singularities for 2D Pressureless Flow.
    - $x \in \mathbb{R}^2$, only for smooth initial data and a specific problem.

Application I: p-system

(Wendroff, 1972 and Wendroff, 1972.)

$$\begin{cases} u_t + (p(v))_x = 0, \\ v_t - u_x = 0. \end{cases}$$

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Compressible Isentropic Flow in Lagrangian Coordinates:

- Lagrangian Coordinates x
- Velocity in Lagrangian Coordinates $u \in \mathbb{R}$
- Specific Volume $v > 0$
- Pressure $p(v) \in C^2((0, \infty))$

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Modelling Assumptions:

- Thermodynamics : $p'(v) < 0$ for $v > 0$.
- Experimental Evidence (Bethe, 1942): $p''(v) > 0$ for $v > 0$.

Application I: p-system

Jacobian Matrix:

$$J(u,v) = \begin{pmatrix} 0 & p'(v) \\ -1 & 0 \end{pmatrix}$$

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Assumptions:

(i) $p'(v) < 0$ for $v > 0$ implies **strictly hyperbolic** system in $(0,\infty) \times \mathbb{R}$.

(ii) $p''(v) > 0$ for $v > 0$ implies **genuinely non-linear** system in $(0,\infty) \times \mathbb{R}$.

(iii) $-1 \neq 0$ implies **uni-directional system** in $(0,\infty) \times \mathbb{R}$.

Application I: p-system

Manifold Assumption:

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- Hugoniot Objective Function:

$$H(u, v; u_g, v_g) = (u - u_g)^2 + (p(v) - p(v_g)) \cdot (v - v_g).$$

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Physical Interpretation:

For a sufficiently good C^2 approximation of the pressure relation (on a compact subset), unique double-wave entropy solutions are preserved.

Application II: Gravity-Driven Particle-Laden Flow

$$\begin{cases} h_t + \left(\underbrace{h^3 f\left(\frac{h\phi_0}{h}\right)}_{F(h, h\phi_0)} \right)_x = 0, \\ (h\phi_0)_t + \left(\underbrace{h^3 g\left(\frac{h\phi_0}{h}\right)}_{G(h, h\phi_0)} \right)_x = 0. \end{cases}$$

- $f(\phi_0), g(\phi_0)$, $\phi_0 \in [0, \phi_m]$.
  Physical Interpretation: $\phi_m =$ Maximum Packing Fraction.

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- $f(\phi_0), g(\phi_0)$, $\phi_0 \in [0, \phi_m]$.
  Physical Interpretation: $\phi_m =$ Maximum Packing Fraction.
- If the assumptions of the theorem hold, by picking a sufficiently good approximation of f and g, hence F and G, unique double-wave entropy solutions are preserved.

Application II-1: Interpolating Flux Functions

Algorithm:

1. Place a grid on $\phi_0 = [0, \phi_m]$ with $\phi_m = 0.610$, say step size $\Delta\phi_0 = 0.001$.
2. Solve the nonlinear ODE for $\phi_0 = 0.001i$ for $i = 1, \cdots, 610$ to obtain $f(\phi_0)$ and $g(\phi_0)$.
3. Obtain $f(\phi_0)$ and $g(\phi_0)$ by interpolation.
4. Obtain $f'(\phi_0)$ and $g'(\phi_0)$ by interpolation too (if needed).

$$\text{Global Error for } f = \|f - f_{\text{int}}\|_{C^1(K)} \lesssim o(\Delta\phi_0)$$

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$$\xrightarrow{\Delta\phi_0 \to 0} 0$$

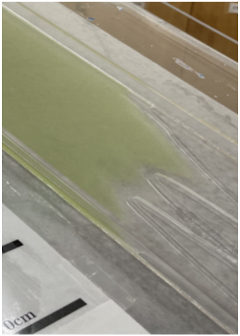
Interpretation:

The solutions exhibit structural stability for a sufficiently small grid size, with solutions converging to the original system as grid size goes to 0.

Application II-2: Gravity-Driven Particle-Laden Flow

Fix $\alpha = 25°$. $\phi_0 \in [0, \phi_m]$,
$\phi_m = 0.61$: Maximum packing fraction.
$\phi_c \approx 0.503$: Phase transition from settled to ridged.

Settled: $\phi_0 < \phi_c$.



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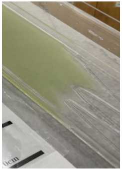

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Polynomial Approximations:

$$f(\phi_0) = \begin{cases} \sum_{j=1}^{10} \beta_{f,j}^S (\phi_c - \phi_0)^{j-1} & \text{for } \phi_0 < \phi_c, \\ \sum_{j=1}^{10} \beta_{f,j}^R (\phi_0 - \phi_c)^{j-1} & \text{for } \phi_0 > \phi_c, \end{cases}$$

and

$$g(\phi_0) = \begin{cases} \sum_{j=1}^{10} \beta_{g,j}^S (\phi_c - \phi_0)^{j-1} & \text{for } \phi_0 < \phi_c, \\ \sum_{j=1}^{10} \beta_{g,j}^R (\phi_0 - \phi_c)^{j-1} & \text{for } \phi_0 > \phi_c. \end{cases}$$

$$\boldsymbol{\beta}_f = \mathrm{argmin}_{\boldsymbol{\beta}_f \in \mathbb{R}^{20}} \|\mathbf{f} - \mathbf{X}\boldsymbol{\beta}_f\|_2^2 + \lambda \|\mathbf{f}' - \mathbf{X}'\boldsymbol{\beta}_f\|_2^2$$

subject to the assumptions above (similar for g).

Physical "Constraints":

$$\begin{aligned}
\text{(I)} &: f, f', f'', g, g', \text{and } g'' \text{ are continuous at } \phi_c, \\
\text{(II)} &: f(0) = \frac{\mu_l}{3}, g(0) = 0, \\
\text{(III)} &: \text{ Values of } f(\phi_c) \text{ and } g(\phi_c), \\
\text{(IV)} &: f(\phi_m) = g(\phi_m) = f'(\phi_m) = g'(\phi_m) = 0.
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Compare:

- $\lambda = 0$ (not fitting for derivatives) and
- $\lambda = 0.03$ (fitting for derivatives, obtained via leave-one-out cross validation).

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subject to the assumptions above (similar for g). Sampled Data Points:

- A couple of points close to ϕ_m,
- A couple of points close to ϕ_c,
- A couple of sparse points,
- Points are in triplets to provide derivative information at the middle point.

Optimization Algorithm:

- Quadratic program with linear equality constraints.
- Determine λ by using a leave-one-out cross validation algorithm.

Order of Phase Transition

C^1 vs C^2?

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Asymptotically,

$$f(\phi_0), g(\phi_0) \sim |\phi_0 - \phi_c|^\beta$$

with

- $\beta < 1$ if $\alpha > 70.309^\circ$
- $\beta \in (1, 2)$ if $\alpha \in (27.895^\circ, 70.309^\circ)$
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It can be "numerically verified" that $f(\phi_0)$ and $g(\phi_0)$ are

- C^2 across $\phi_0 = \phi_c$ for $\alpha = 17°$.
- C^1 only across $\phi_0 = \phi_c$ for $\alpha = 30°, 60°, 80°$.

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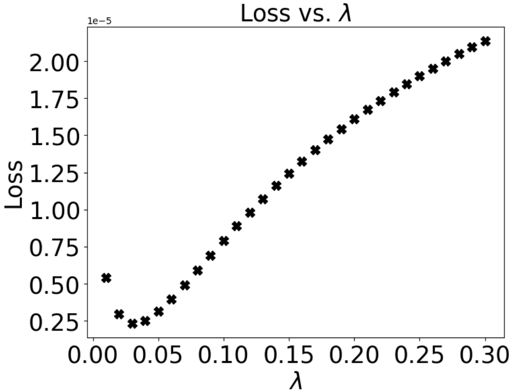
- C^1 only across $\phi_0 = \phi_c$ for $\alpha = 30°, 60°, 80°$.

Furthermore, most parts of the proof suggest that the above argument might work with the Sobolev Space $W^{2,\infty}(K)$ (i.e "derivatives are Lipschitz continuous").

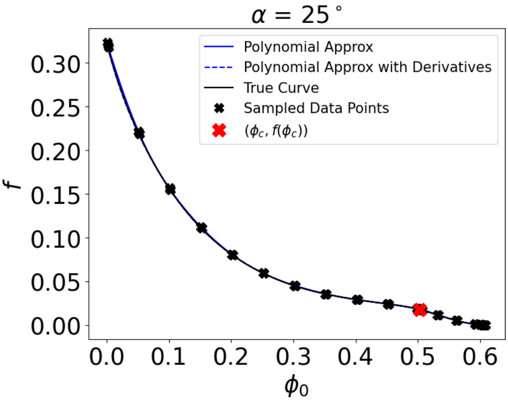
Application II-2: Gravity-Driven Particle-Laden Flow

Optimal λ from leave-one-out cross validation:

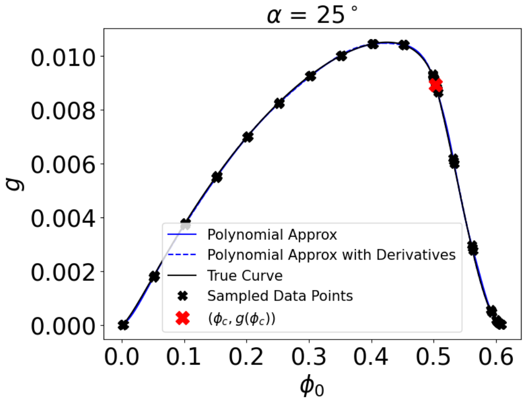


Application II-2: Gravity-Driven Particle-Laden Flow

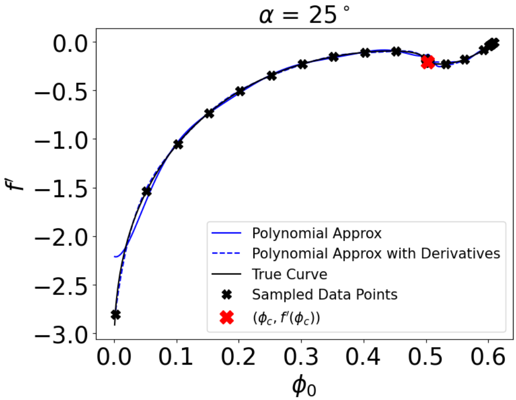
Quality of Approximation - Flux Function f:



Quality of Approximation - Flux Function g:



Quality of Approximation - Derivative of Flux Function f':



Application II-2: Gravity-Driven Particle-Laden Flow

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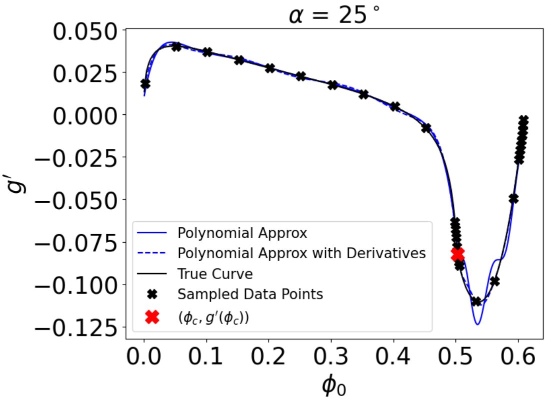
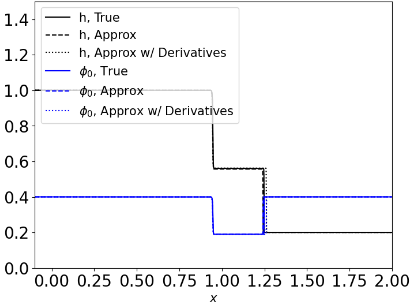


Application II-2: Gravity-Driven Particle-Laden Flow

Riemann Initial Data:

$$(h, \phi_0)(0, x) = \begin{cases} (1, 0.4) & \text{for } x > 0, \\ (0.2, 0.4) & \text{for } x < 0. \end{cases}$$

Solution for $(h, \phi_0)(30, x)$:



Riemann Initial Data:

$$(h, \phi_0)(0, x) = \begin{cases} (1, 0.485) & \text{for } x > 0, \\ (0.2, 0.485) & \text{for } x < 0. \end{cases}$$

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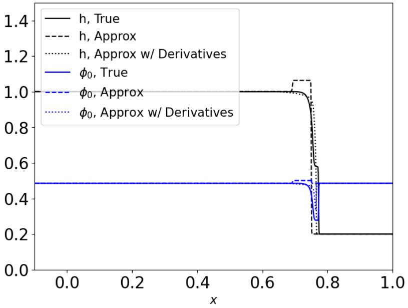
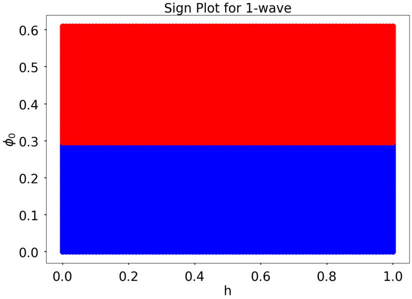
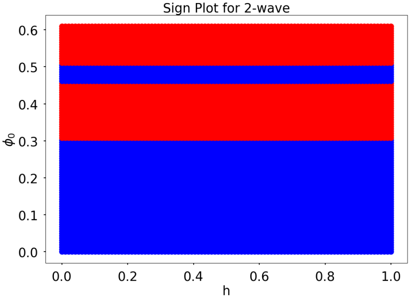


Violating Genuine Nonlinearity





Application II-3: Comparing Models

Lubrication Assumption gives the **same system of conservation laws** for both diffusive flux and shear-induced migration models:

$$\begin{cases} h_t + \left(h^3 f\left(\dfrac{h\phi_0}{h}\right)\right)_x = 0, \\ (h\phi_0)_t + \left(h^3 g\left(\dfrac{h\phi_0}{h}\right)\right)_x = 0. \end{cases}$$

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Different models yield different pairs of flux functions f and g.

Observation: If the flux functions from different models are sufficiently close, solutions to the Riemann problems are sufficiently close!

Application II-3: Comparing Models

- Reference: A comparative study of dynamic models for gravity-driven particle-laden flows. (Lee W.P. et. al, 2025.)
- Authors: 2023 REU students, S.C. Burnett, L. Ding, A. L. Bertozzi.
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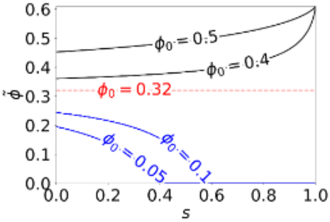
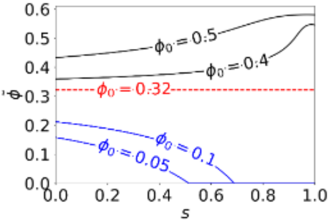
$\alpha = 50^\circ$, Equilibrium Profile - I.

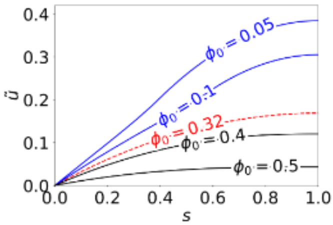


(a)

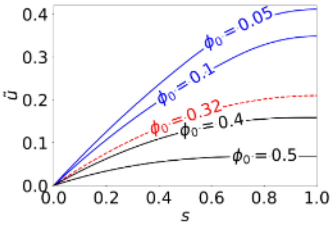


(b)

$\alpha = 50°$, Equilibrium Profile - II.

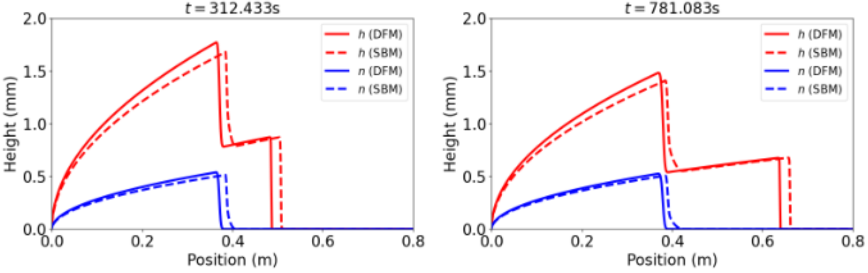


(c)



(d)

$\alpha = 50°$, PDE Simulations.



(b) (c)

Conclusion and Discussion

- **Main Result:** Under the usual assumptions and some additional mild assumptions, unique double-wave entropy solutions are preserved upon a sufficiently good approximation of flux functions.
- Understanding how each of the assumptions fails allows us to predict properties that are not expected to be preserved upon perturbation.

Future Work

Generalizing the result to $n \times n$ systems.

- (Wong and Bertozzi, 2016.)
  $n = 3$: Bidensity/Bisize Particle Laden Flow
  (Additional Parameter $\rightarrow$ Additional Conservation Law.)
- General $n \times n$ using "more differential topology".

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Other Variants - Regularity:

- Lower Regularity required for flux functions and their perturbations.
- Smooth except at finite points (corresponding to phase transitions).
- Perturbations to initial data (left and right states).

Future Work

Allowing Linear Degenerate Waves:

- Example: $n = 3$, Compressible Euler Equations for gas dynamics.
- Expectation: Preserved under the class of "shocks, rarefactions, and contact discontinuities" for a class of perturbations.

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  Alternative to Lax’s Entropy Condition $\rightarrow$ Liu’s Entropy Condition.
- Generalizing the above arguments for a different entropy condition.

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Numerical Schemes Motivated by Transversality.

Thank you for your attention!

References

Particle-Laden Flow:

N. Murisic, J. Ho, V. Hu, P. Latterman, T. Koch, K.Lin, M. Mata, and A. L. Bertozzi.
Particle-laden viscous thin-film flows on an incline: Exper- iments compared with a theory based on shear-induced migration and particle settling.
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N. Murisic, B. Pausader, D. Peschka, and A. L. Bertozzi.
Dynamics of particle settling and resuspension in viscous liquid films.
Journal of Fluid Mechanics 717 (2013), 203–31.

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