Structural Stability in 2×2 Systems of Conservation Laws

Advancement to Candidacy Talk

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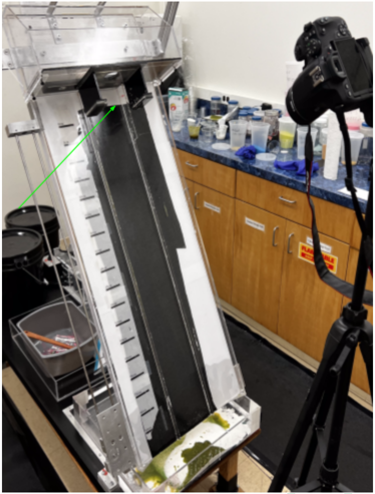
Generic Structural Stability: Approximating Flux Functions for 2×2 Systems of Hyperbolic Conservation Laws in Particle-Laden Flows and the p-system.

Motivation

Gravity-Driven Particle-Laden Flow

Experimental Setup:

- Inclined Slope
- Negatively buoyant monodisperse particles (i.e glass beads) mixed with oil.
- Mixture of particles and oil added with a gate before the start of the experiment.
- Release the gate to start.



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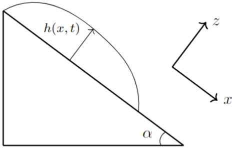

Assumptions: Fast Equilibrium + Lubrication Assumption.

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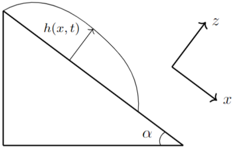

Assumptions: Fast Equilibrium + Lubrication Assumption.

- Conservation of suspension volume:

$$\partial_t h + \partial_x F(h, h_0) = 0.$$

- Conservation of the number of particles:

$$\partial_t (h\phi_0) + \partial_x G(h, h\phi_0) = 0.$$

- Functional form of flux functions:

$$F(h, h\phi_0) = h^3 f\left(\frac{h\phi_0}{h}\right) = h^3 f(\phi_0),$$

$$G(h, h\phi_0) = h^3 g\left(\frac{h\phi_0}{h}\right) = h^3 g(\phi_0).$$

Issue: f and g are computationally expensive to evaluate.

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Problem Statement

Given:

$$\begin{cases} u_t + (F_0(u,v))_x = 0, \\ v_t + (G_0(u,v))_x = 0, \end{cases}$$

with given initial data
$(u,v)(0,x)$.

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Can we approximate (F,G) with $(\tilde{F},\tilde{G})$ such that

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- L^1 stability of $L^1 \cap BV$ solution with respect to flux functions, and
- Structural Stability of Riemann Problems; ie solutions with initial data of the form

$$(u,v)(0,x) = \begin{cases} (u_l,v_l) \text{ for } x < 0, \\ (u_r,v_r) \text{ for } x > 0. \end{cases}$$

Main Result

Consider:

$$\begin{cases} u_t + (F_0(u,v))_x = 0, \\ v_t + (G_0(u,v))_x = 0, \end{cases}$$

with Riemann initial data

$$(u,v)(0,x) = \begin{cases} (u_l, v_l) \text{ for } x < 0, \\ (u_r, v_r) \text{ for } x > 0, \end{cases}$$

for $(t,x) \in [0,\infty) \times \mathbb{R}$, $U \subset \mathbb{R}^2$ open, and $F_0, G_0 \in C^2(U)$.

Main Result

Assumptions: (F_0, G_0) forms a

(i) **Strictly hyperbolic** system in U:

Jacobian matrix $J(u, v; F_0, G_0) = \begin{pmatrix} (F_0)_u & (F_0)_v \\ (G_0)_u & (G_0)_v \end{pmatrix}$ possess two distinct real eigenvalues for each $(u, v) \in U$.

(ii) **Genuinely non-linear** system in U: For $k \in \{1, 2\}$

$$\nabla \underbrace{\lambda_k}_{\text{k-Eigenvalue}} (u, v; F_0, G_0) \cdot \underbrace{\mathbf{r}_k}_{\text{k-Right Eigenvector}} (u, v; F_0, G_0) \neq 0.$$

Convention: $\lambda_1 < \lambda_2$.

(iii) **Uni-directional system** in U: Either

- $(F_0)_v(u, v) \neq 0$ for all $(u, v) \in U$ or
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Perturbations:

$$\begin{cases} \tilde{F} = F_0 + F_\delta, \\ \tilde{G} = G_0 + G_\delta. \end{cases}$$

Main Result

Theorem (Generic Approximation Theorem & Structural Stability.)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For almost every $(u_l, v_l) \neq (u_r, v_r) \in U$, consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset $K \subset U$ containing (u_l, v_l) and (u_r, v_r) in its interior,

1. The unperturbed system satisfies the transversality property on K,
2. There exists $\varepsilon_1, \varepsilon_2 > 0$ such that for any perturbations $(F_\delta, G_\delta) \in C^2(K)^2$ with $\|(F_\delta, G_\delta)\|_{C^2(K)^2} < \varepsilon_1$, the corresponding perturbed 2×2 system admits a unique double-wave entropy solution with an intermediate state $(\tilde{u}^*, \tilde{v}^*) \in \mathrm{int}(K)$ satisfying $\|(\tilde{u}^*, \tilde{v}^*) - (u^*, v^*)\|_2 < \varepsilon_2$.
3. The perturbed system satisfies the transversality property on the same compact set K.

Main Result

Roughly speaking, this translates to:

For a system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption, for almost every (generic) Riemann initial data, unique solutions and their structures (shock/rarefactions) are preserved upon a sufficiently small C^2 perturbation to the flux functions.

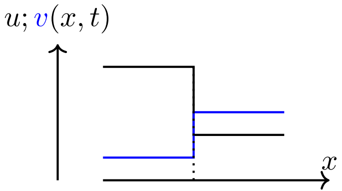
Furthermore, the "amplitudes" of shock and rarefaction upon perturbation are only perturbed by a small amount.

General System: $\begin{cases} u_t + (F(u,v))_x = 0, \\ v_t + (G(u,v))_x = 0. \end{cases}$

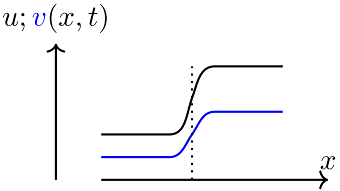
Riemann initial data:

$$(u,v)(0,x) = \begin{cases} (u_l, v_l) \text{ for } x < 0, \\ (u_r, v_r) \text{ for } x > 0, \end{cases}$$

Shock:



Rarefaction:



and Lax Entropy Conditions.

Crash Course: 2×2 System

State Space (u, v) Analysis :

Given a state (u_l, v_l),

- (Shocks) Hugoniot loci: (Rankine-Hugoniot) All (u, v) satisfying $\begin{pmatrix} F(u,v) - F(u_l, v_l) \\ G(u,v) - G(u_l, v_l) \end{pmatrix} = s \begin{pmatrix} u - u_l \\ v - v_l \end{pmatrix}$ for some s.

  Equivalently,

  $$(F(u,v) - F(u_l, v_l))(v - v_l) - (G(u,v) - G(u_l, v_l))(u - u_l) = 0.$$

  Required to satisfy 1-wave Lax Entropy condition.

- 1-Rarefaction Curves: All (u, v) solving

  $$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\lambda}(u(\lambda), v(\lambda)) = \mathbf{r}_1(u(\lambda), v(\lambda)) \\ (u(\lambda(u_l, v_l)), v(\lambda(u_l, v_l))) = (u_l, v_l) \end{cases}$$

  Integral curves of the right eigenvector $\mathbf{r}_1$.

  Solve for increasing λ.

Crash Course: 2×2 System

State Space (u, v) Analysis :

Given a state (u_r, v_r),

- (Shocks) Hugoniot loci: (Rankine-Hugoniot) All (u, v) satisfying $\begin{pmatrix} F(u,v) - F(u_r, v_r) \\ G(u,v) - G(u_r, v_r) \end{pmatrix} = s \begin{pmatrix} u - u_r \\ v - v_r \end{pmatrix}$ for some s.

  Equivalently,

  $(F(u,v) - F(u_r,v_r))(v - v_r) - (G(u,v) - G(u_r,v_r))(u - u_r) = 0.$

  Required to satisfy 2-wave Lax Entropy condition.

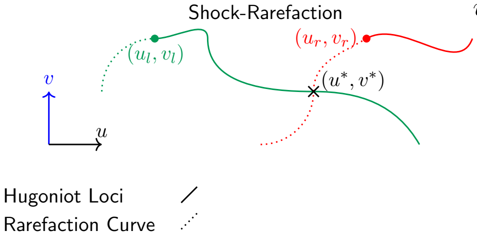
- 2-Rarefaction Curves: All (u, v) solving

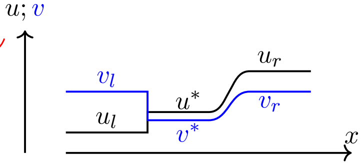
$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\lambda}(u(\lambda), v(\lambda)) = \mathbf{r}_2(u(\lambda), v(\lambda)) \\ (u(\lambda(u_l, v_l)), v(\lambda(u_l, v_l))) = (u_l, v_l) \end{cases}$$

  Integral curves of the right eigenvector $\mathbf{r}_2$.
  Solve for decreasing λ.

Constructing Composite Solutions:

$$\underbrace{(u_l, v_l) \xrightarrow[\text{1−shock}]{\text{Hugoniot Locus}} (u^*, v^*) \xrightarrow[\text{2−rarefaction}]{\text{Rarefaction Curve}} (u_r, v_r)}_{\text{Shock-Rarefaction Solution}}$$





Crash Course: 2×2 System

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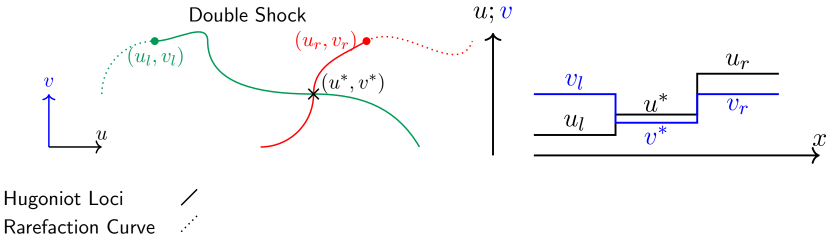
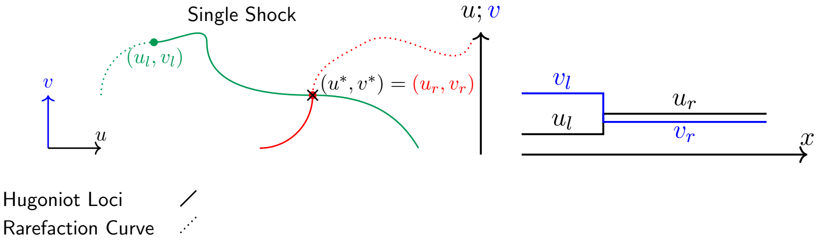


Unstable Case I: Single Wave Solution.

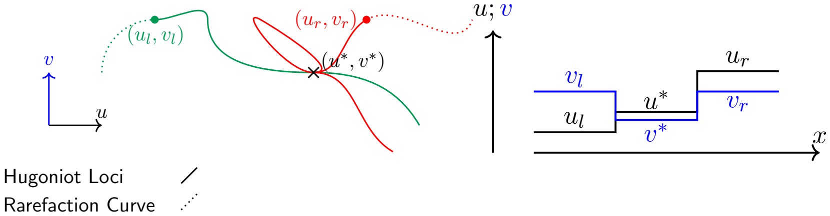
$$\underbrace{(u_l, v_l) \xrightarrow[\text{1-shock}]{\text{Hugoniot Locus}} (u^*, v^*) = (u_r, v_r)}_{\text{Single Shock}}$$



2×2 System

Unstable Case II: Self-Intersecting Hugoniot Loci.

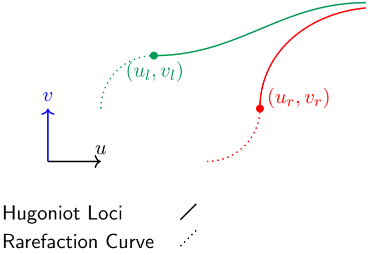
$$\underbrace{(u_l, v_l) \xrightarrow[\text{1-shock}]{\text{Hugoniot Locus}} (u^*, v^*) \xrightarrow[\text{2-shock}]{\text{Self-Intersecting Hugoniot Locus}} (u_r, v_r)}_{\text{Double Shock}}$$

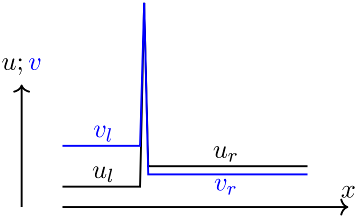


2×2 System

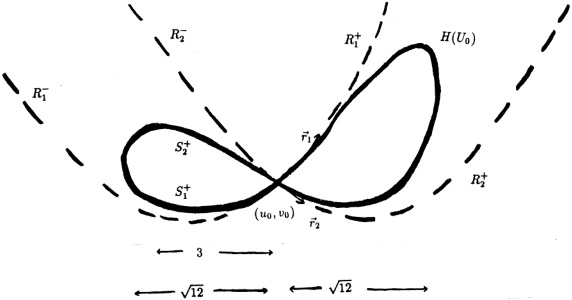
Unstable Case III: Singular (δ) Shock - Intersection at ∞.
To be interpreted in the sense of distributions.
(Wang and Bertozzi, 2014.)

$$\underbrace{(u_l, v_l) \xrightarrow[1-\text{shock}]{\text{Hugoniot Locus}} \infty \xleftarrow[2-\text{shock}]{\text{Hugoniot Locus}} (u_r, v_r)}_{\text{Singular Shock}}$$





Case IV: Singular Shock - Self-intersecting at given states.
(Keyfitz and Kranzer, 1990.)



Hugoniot Loci is **not** a manifold (locally '$\times$', not Euclidean).

Regular Manifold Assumption

Recall: Hugoniot loci connects all (u, v) from a given state (u_g, v_g)

$$(F(u,v) - F(u_g, v_g))(v - v_g) - (G(u,v) - G(u_g, v_g))(u - u_g) = 0.$$

Define the **Hugoniot Objective Function**:

$$H_{(u_g,v_g)} = (F(u,v) - F(u_g, v_g))(v - v_g) - (G(u,v) - G(u_g, v_g))(u - u_g).$$

Hugoniot locus is the zero level set of $H_{(u_g,v_g)}$.

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Regular Manifold Assumption

The Jacobian map $(dH_{(u_g,v_g)})_{(u,v)} : \mathbb{R}^2 \to \mathbb{R}$ given by $\begin{pmatrix} D_u H_{(u_g,v_g)}(u,v) & D_v H_{(u_g,v_g)}(u,v) \end{pmatrix}$ is surjective for each $(u,v) \neq (u_g, v_g)$ on the Hugoniot locus.

- Always not satisfied at $(u,v) = (u_g, v_g)$.
- By the **Regular Value Theorem**, the Hugoniot locus restricted on $U \setminus \{(u_g, v_g)\}$ is a C^1 manifold.

Transversality

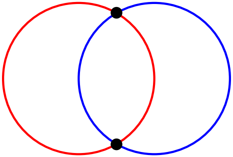
Let $\mathcal{M}$ and $\mathcal{N}$ be submanifolds of $\mathbb{R}^n$.

Definition: Transverse Intersection

We say that $\mathcal{M}$ and $\mathcal{N}$ **intersects transversely** if for every $x \in \mathcal{M} \cap \mathcal{N}$,

$$T_x\mathcal{M} + T_x\mathcal{N} = \mathbb{R}^n.$$

Notation: $\mathcal{M} \pitchfork \mathcal{N}$.



Transversality Property

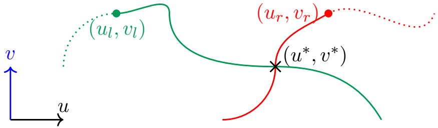
Let K be a compact subset of U containing the given left and right states $(u_l, v_l) \neq (u_r, v_r)$.

Definition: Transversality Property

We say that the 2×2 system with Riemann initial data given by (u_l, v_l) and (u_r, v_r) as left and right states satisfies the **transversality property on** K if for the "correct" curves $\mathcal{W}_l$ (from (u_l, v_l)) and $\mathcal{W}_r$ (from (u_r, v_r)) intersecting at $(u^*, v^*) \neq (u_l, v_l)$ or (u_r, v_r), we have

$$\mathcal{W}_l \pitchfork \mathcal{W}_r.$$

Double Shock



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Step I: Implicit Function Theorem on Banach Spaces

Theorem A. (Structural Stability)

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Moreover, The perturbed system satisfies the transversality property on the same compact set K.

Step I: Implicit Function Theorem on Banach Spaces

Proof Sketch (Persistence of Existence):

- **Hugoniot Objective Function** $H(u, v; u_g, v_g, F, G)$ given by

$$H(u, v; u_g, v_g, F, G)$$
$$= (F(u, v) - F(u_g, v_g))(v - v_g) - (G(u, v) - G(u_g, v_g))(u - u_g).$$

- Hugoniot locus: All (u, v) such that $H(u, v; u_g, v_g, F, G) = 0$.

Step I: Implicit Function Theorem on Banach Spaces

Rarefaction Curves:

- Rarefaction ODEs:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\lambda}(u(\lambda), v(\lambda)) = \mathbf{r}_k(u(\lambda), v(\lambda)) \\ (u(\lambda(u_g, v_g)), v(\lambda(u_g, v_g))) = (u_g, v_g) \end{cases}$$

- Use uni-direction assumption (iii) to normalize the 2nd component of the right eigenvector to be 1.
- Obtain a single ODE "$\frac{\mathrm{d}u}{\mathrm{d}v} = \frac{\mathrm{d}u/\mathrm{d}\lambda}{\mathrm{d}v/\mathrm{d}\lambda}$":

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}v} u_k(v; F, G) = \Xi(u_k(v; F, G), F, G) \\ u_k(v_g) = u_g. \end{cases}$$

Step I: Implicit Function Theorem on Banach Spaces

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- **Rarefaction Objective Function**:

$$R_k(u, v; u_g, v_g, F, G) = u - u_k(v; u_g, v_g, F, G).$$

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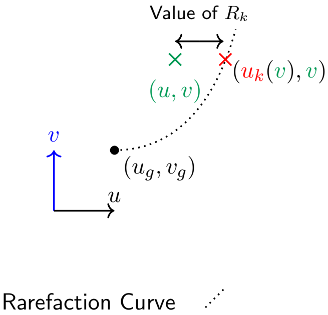
Interpretation: Signed Distance of u-coordinate to rarefaction curve integrated up to v.



Step I: Implicit Function Theorem on Banach Spaces

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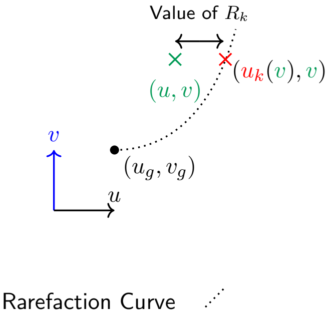
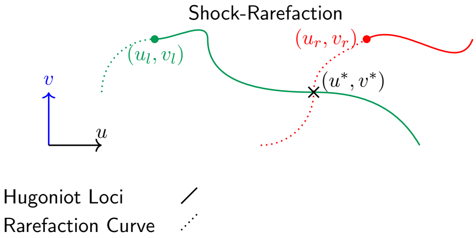


$k-$Rarefaction curve $=$ Zero-level set of R_k.

Step I: Implicit Function Theorem on Banach Spaces



Example: Unique intermediate state (u^*, v^*) and unperturbed fluxes (F_0, G_0) satisfy

$$\begin{cases} H(u^*, v^*, F_0, G_0; u_l, v_l) = 0, \\ R_2(u^*, v^*, F_0, G_0; u_r, v_r) = 0. \end{cases}$$

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Apply **Implicit Function Theorem on Banach Spaces** to

$$\mathbf{hr}(u, v, F, G) := \begin{cases} H(\mathbf{u},\mathbf{v},\mathbf{F},\mathbf{G}; u_l, v_l) = 0, \\ R_2(\mathbf{u},\mathbf{v},\mathbf{F},\mathbf{G}; u_r, v_r) = 0. \end{cases}$$

with $(u, v) \in K$ and $(F, G) \in C^2(K)^2$ to obtain a map $M : C^2(K)^2 \to K$ such that

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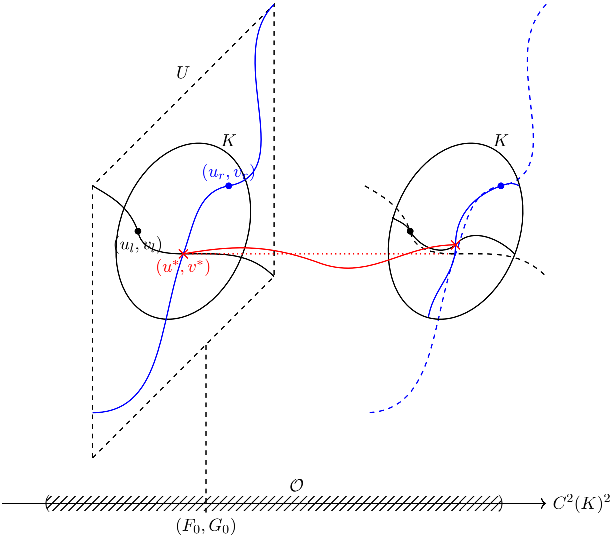
$$\begin{cases} H(M(F,G), F, G; u_l, v_l) = 0, \\ R_2(M(F,G), F, G; u_r, v_r) = 0. \end{cases}$$

with $M(F_0, G_0) = (u^*, v^*)$ in a $C^2(K)^2$ neighborhood of (F_0, G_0).

Step I: Implicit Function Theorem on Banach Spaces



Transition to Step II

$$\begin{cases} H(M(F,G),F,G;u_l,v_l)=0, \\ R_2(M(F,G),F,G;u_r,v_r)=0. \end{cases}$$

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To apply **Implicit Function Theorem on Banach Spaces**, we need to check the invertibility of a Jacobian map (matrix) $D_{(u,v)}\mathbf{hr}(u^*,v^*,F_0,G_0):\mathbb{R}^2\to\mathbb{R}^2$, given by

$$\begin{pmatrix} D_uH(u^*,v^*,F_0,G_0;u_l,v_l) & D_vH(u^*,v^*,F_0,G_0;u_l,v_l) \\ D_uR_2(u^*,v^*,F_0,G_0;u_r,v_r) & D_vR_2(u^*,v^*,F_0,G_0;u_r,v_r) \end{pmatrix}.$$

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If $\mathcal{W}_l=$ Hugoniot locus from (u_l,v_l) and $\mathcal{W}_r=$ Rarefaction curve from (u_r,v_r), this is equivalent to

$$\mathcal{W}_l \pitchfork \mathcal{W}_r=\mathbb{R}^2.$$

Transition to Step II

Recall:

Theorem A. (Structural Stability)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For given states $(u_l, v_l) \neq (u_r, v_r) \in U$, consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset $K \subset U$ containing (u_l, v_l) and (u_r, v_r) in its interior,

If The unperturbed system satisfies the transversality property on K,

Then There exists $\varepsilon_1, \varepsilon_2 > 0$ such that for any perturbations $(F_\delta, G_\delta) \in C^2(K)^2$ with $\|(F_\delta, G_\delta)\|_{C^2(K)^2} < \varepsilon_1$, the corresponding perturbed 2×2 system admits a unique double-wave entropy solution with an intermediate state $(\tilde{u}^*, \tilde{v}^*) \in \mathrm{int}(K)$ satisfying $\|(\tilde{u}^*, \tilde{v}^*) - (u^*, v^*)\|_2 < \varepsilon_2$.

Moreover, The perturbed system satisfies the transversality property on the same compact set K.

Transition to Step II

Theorem B. (Transversality is Generic.)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For almost every $(u_l, v_l) \neq (u_r, v_r) \in U$, consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset $K \subset U$ containing (u_l, v_l) and (u_r, v_r) in its interior, we

Get: The unperturbed system satisfies the transversality property on K.

Transition to Step II

Theorem A + Theorem B = Main Theorem.

Theorem. (Generic Approximation Theorem & Structural Stability.)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For almost every $(u_l, v_l) \neq (u_r, v_r) \in U$, consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset $K \subset U$ containing (u_l, v_l) and (u_r, v_r) in its interior,

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Step II: Parametric Transversality Theorems

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Get: The unperturbed system satisfies the transversality property on K.

Step II: Parametric Transversality Theorems

Let $\mathcal{X}$ and $\mathcal{Y}$ be C^r manifolds, and $\mathcal{Z}$ be a C^r submanifold of $\mathcal{Y}$ for $r \geq 1$.

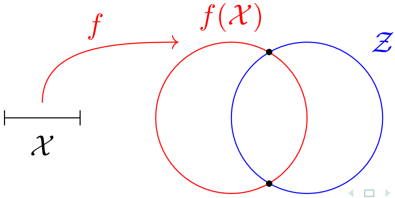
Definition: Transversality of a Map

Let $f : \mathcal{X} \to \mathcal{Y}$ be a C^r map. We say that f is **transverse** to $\mathcal{Z}$ if for every $a \in f^{-1}(\mathcal{Z})$, we have

$$df(T_a\mathcal{X}) + T_{f(a)}\mathcal{Z} = T_{f(a)}\mathcal{Y}.$$

Notation: $f \pitchfork \mathcal{Z}$.

Intuition: "$f(\mathcal{X}) \pitchfork \mathcal{Z}$".



Step II: Parametric Transversality Theorems

Typical genericity arguments utilize:

Thom's Parametric Transversality Theorem

Let $\mathcal{X}, \mathcal{P}$, and $\mathcal{Y}$ be C^r manifolds and $\mathcal{Z}$ be a C^r submanifolds of $\mathcal{N}$. Consider

- The map $F : \mathcal{X} \times \mathcal{P} \to \mathcal{Y}$, and
- The associated parametric maps $F_p : \mathcal{X} \to \mathcal{Y}$ for each $p \in \mathcal{P}$.

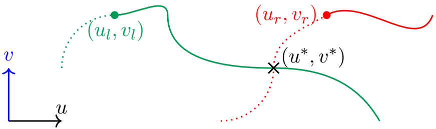
Suppose that

1. $r > \max\{0, \dim \mathcal{Y} + \dim \mathcal{Z} - \dim \mathcal{X}\}$,
2. The map $(x, p) \mapsto F_p(x)$ is C^r, and
3. $F \pitchfork \mathcal{Z}$.

Then, for almost every $p \in \mathcal{P}$, $F_p \pitchfork \mathcal{Z}$.

Step II: Parametric Transversality Theorems

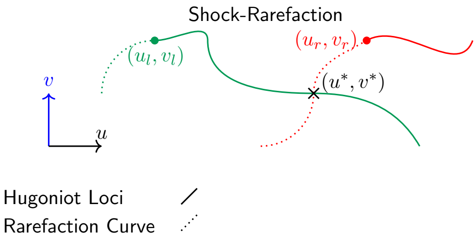
Shock-Rarefaction



Strategy 1: $\mathbf{hr}(\cdot; u_l, v_l, u_r, v_r) : U \to \mathbb{R}^2$ with

$$\mathbf{hr}(u, v; u_l, v_l, u_r, v_r) := \begin{cases} H(u, v; u_l, v_l), \\ R_2(u, v; u_r, v_r). \end{cases}$$

Step II: Parametric Transversality Theorems



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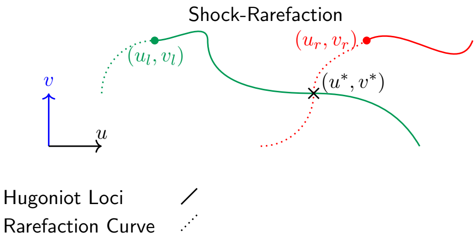
Recall: $\mathbf{hr}(u^*, v^*; u_l, v_l, u_r, v_r) = \mathbf{0}$.



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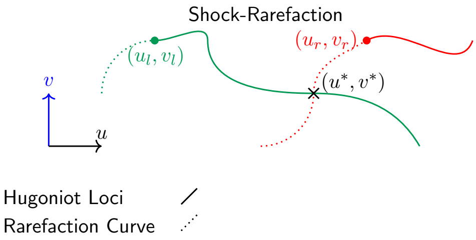
Hope: $\mathbf{hr}(\cdot; u_l, v_l, u_r, v_r) \pitchfork \{(0,0)\}$ for almost every $(u_l, v_l) \neq (u_r, v_r)$.

Step II: Parametric Transversality Theorems



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Conclude: At each intersection point $\implies$ transverse intersection.

Define $\Delta_{U^2} = \{(u_l, v_l, u_r, v_r) : (u_l, v_l) \neq (u_r, v_r)\}$.

To apply **Thom's Parametric Transversality Theorem**, we need to check that

$$\mathbf{hr} : \underbrace{U}_{\text{Allowed Intersection Points}} \times \underbrace{(U^2 \setminus \Delta_{U^2})}_{\text{Parameters: Left and Right States}} \to \mathbb{R}^2$$

satisfies $\mathbf{hr} \pitchfork \{(0,0)\}$.

Step II: Parametric Transversality Theorems

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Then, $\mathbf{hr}(\cdot; u_l, v_l, u_r, v_r) \pitchfork \{(0,0)\}$ for almost every $(u_l, v_l, u_r, v_r) \in U^2 \setminus \Delta_{U^2}$.

Step II: Parametric Transversality Theorems

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Hugoniot Loci from (u_g, v_g) are manifolds on $U \setminus (u_g, v_g)$. (i.e Keyfitz-Kranzer system.)

Step II: Parametric Transversality Theorems

Strategy 2: Puncture the domain U at (u_l, v_l) and (u_r, v_r) for **each** given left and right states.

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Set $U_{(u_l,v_l,u_r,v_r)} := U \setminus \{(u_l, v_l), (u_r, v_r)\}$ and define

$$ULR = \bigcup_{(u_l,v_l,u_r,v_r)\in U^2\setminus\Delta_{U^2}} \underbrace{U_{(u_l,v_l,u_r,v_r)}}_{\text{Intersection Points}} \times \underbrace{\{(u_l, v_l, u_r, v_r)\}}_{\text{Parameters}}.$$

Step II: Parametric Transversality Theorems

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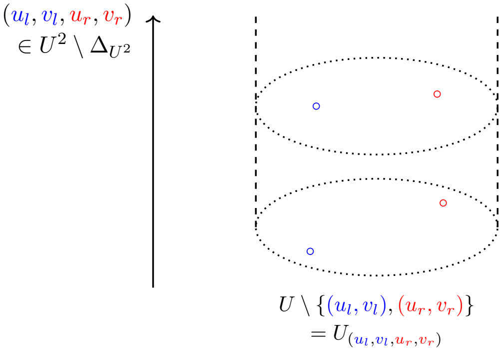
Now, check $\mathbf{hr} : ULR \to \mathbb{R}^2$ satisfies

$$\mathbf{hr} \pitchfork \{(0,0)\}.$$

Step II: Parametric Transversality Theorems



$$ULR = \bigcup_{(u_l,v_l,u_r,v_r)\in U^2\setminus\Delta_{U^2}} U_{(u_l,v_l,u_r,v_r)} \times \{(u_l, v_l, u_r, v_r)\}.$$

ULR is a 6-dimensional submanifold of $\mathbb{R}^6$.

Step II: Parametric Transversality Theorems

Foliated Parametric Transversality Theorem

Let $\mathcal{P}$ and $\mathcal{Y}$ be C^r manifolds, and $\mathcal{Z}$ be a C^r submanifold of $\mathcal{Y}$. Suppose that for each $p \in \mathcal{P}$, we consider a collection of C^r manifolds given by $\{\mathcal{X}_p\}_{p\in\mathcal{P}}$ each with the same dimension $\dim \mathcal{X}$, and the following foliated set:

$$\mathcal{X}\mathcal{P} := \bigcup_{p\in\mathcal{P}} \mathcal{X}_p \times \{p\}. \tag{1}$$

Consider the maps $F : \mathcal{X}\mathcal{P} \to \mathcal{Y}$ and the associated map $F_p : \mathcal{X}_p \to \mathcal{Y}$ for each parameter $p \in P$. Suppose that

1. $r > \max\{0, \dim \mathcal{Y} + \dim \mathcal{Z} - \dim \mathcal{X}\}$,
2. $\mathcal{X}\mathcal{P}$ is a C^r manifold with dimension $\dim \mathcal{X}\mathcal{P} = \dim \mathcal{X} + \dim \mathcal{P}$,
3. $T_{(x,p)}\mathcal{X}\mathcal{P} = T_x\mathcal{X}_p \times T_p\mathcal{P}$ for each $(x, p) \in \mathcal{X}\mathcal{P}$,
4. The map $(x, p) \mapsto F_p(x)$ is C^r, and
5. $F \pitchfork \mathcal{Z}$.

Then, for almost every $p \in \mathcal{P}$, $F_p \pitchfork \mathcal{Z}$.

Main Result

Theorem (Generic Approximation Theorem & Structural Stability.)

Consider the system satisfying assumptions (i), (ii), (iii), and the regular manifold assumption. For almost every $(u_l, v_l) \neq (u_r, v_r) \in U$, consider the system with Riemann initial data such that there is a unique double-wave entropy solution. Then, for any compact subset $K \subset U$ containing (u_l, v_l) and (u_r, v_r) in its interior,

1. The unperturbed system satisfies the transversality property on K,
2. There exists $\varepsilon_1, \varepsilon_2 > 0$ such that for any perturbations $(F_\delta, G_\delta) \in C^2(K)^2$ with $\|(F_\delta, G_\delta)\|_{C^2(K)^2} < \varepsilon_1$, the corresponding perturbed 2×2 system admits a unique double-wave entropy solution with an intermediate state $(\tilde{u}^*, \tilde{v}^*) \in \mathrm{int}(K)$ satisfying $\|(\tilde{u}^*, \tilde{v}^*) - (u^*, v^*)\|_2 < \varepsilon_2$.
3. The perturbed system satisfies the transversality property on the same compact set K.

$$\mathbf{hr}(\underbrace{u, v}_{\text{Double-Wave Solutions}}; \underbrace{u_l, v_l, u_r, u_r}_{\text{Generically}}, \underbrace{F, G}_{\text{Perturbation} \rightarrow \text{Stable}})$$

Existing Literature

L^1 Stability:

Existing Literature

L^1 Stability:

- (Holden and Holden, 1992.) L^1 stability for scalar conservation laws:

$$\|u_f(t,\cdot) - u_g(t,\cdot)\|_{L^1} \lesssim t\mathsf{Lip}(f-g).$$

Done using the front-tracking algorithm.

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- (Bianchini and Colombo, 2002.) L^1 stability for systems:

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix}(t,\cdot) - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}(t,\cdot) \right\|_{L^1} \lesssim C_{(F_0,G_0)} C_{(\tilde{F},\tilde{G})} \hat{d}((F_0,G_0),(\tilde{F},\tilde{G})).$$

Done using (semi-)standard PDE techniques on Riemann semigroup.

Existing Literature

Structural Stability of Riemann Problem:

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- (Schecter, Marchesin, and Plohr, 1994.)
  Structurally Stable Riemann Solutions.
  - Conclusion depends on the given left and right states and transversality condition of intersecting curves that could not be checked *a priori*.
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  - Done using viscous regularization, traveling waves, and phase portrait analysis.
- (Azevedo et. al., 2010 and Eschenazi et. al., 2023.)
  Topological Approach for 2×2 systems.
  - Quadratic flux and perturbations; some work in progress.
  - Similar issue with transversality condition.
  - Done by employing desingularization methods (motivated by singularity theorem) specific to quadratic fluxes.

Existing Literature

Genericity for Conservation Laws:

Existing Literature

Genericity for Conservation Laws:

- (Schaeffer, 1973.)
  Schaeffer Regularity Theorem (for scalar conservation laws): For almost any $u(0, x) \in \mathcal{S}(\mathbb{R})$, the solution is piecewise smooth with a finite number of shock curves.
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  Generic Singularities for 2D Pressureless Flow.
  - $x \in \mathbb{R}^2$, only for smooth initial data and a specific problem.

Application I: p-system

(Wendroff, 1972 and Wendroff, 1972.)

$$\begin{cases} u_t + (p(v))_x = 0, \\ v_t - u_x = 0. \end{cases}$$

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Compressible Isentropic Flow in Lagrangian Coordinates:

- Lagrangian Coordinates x
- Velocity in Lagrangian Coordinates $u \in \mathbb{R}$
- Specific Volume $v > 0$
- Pressure $p(v) \in C^2((0, \infty))$

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Modelling Assumptions:

- Thermodynamics : $p'(v) < 0$ for $v > 0$.
- Experimental Evidence (Bethe, 1942): $p''(v) > 0$ for $v > 0$.

Application I: p-system

Jacobian Matrix:

$$J(u,v) = \begin{pmatrix} 0 & p'(v) \\ -1 & 0 \end{pmatrix}$$

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$$J(u,v) = \begin{pmatrix} 0 & p'(v) \\ -1 & 0 \end{pmatrix}$$

Assumptions:

(i) $p'(v) < 0$ for $v > 0$ implies **strictly hyperbolic** system in $(0,\infty) \times \mathbb{R}$.

(ii) $p''(v) > 0$ for $v > 0$ implies **genuinely non-linear** system in $(0,\infty) \times \mathbb{R}$.

(iii) $-1 \neq 0$ implies **uni-directional system** in $(0,\infty) \times \mathbb{R}$.

Manifold Assumption:

Application I: p-system

Manifold Assumption:

- Hugoniot Objective Function:

$$H(u, v; u_g, v_g) = (u - u_g)^2 + (p(v) - p(v_g)) \cdot (v - v_g).$$

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$$(dH_{(u_g,v_g)})_{(u,v)} = \begin{pmatrix} 2(u - u_g) & p'(v)(v - v_g) + (p(v) - p(v_g)) \end{pmatrix}.$$

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- Show that $(dH_{(u_g,v_g)})_{(u,v)} : \mathbb{R}^2 \to \mathbb{R}$ is surjective for any $(u,v) \neq (u_g,v_g)$ on the Hugoniot locus.

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Physical Interpretation:

For a sufficiently good C^2 approximation of the pressure relation (on a compact subset), unique double-wave entropy solutions are preserved.

Application II: Gravity-Driven Particle-Laden Flow

$$\begin{cases} h_t + \left(\underbrace{h^3 f\left(\frac{h\phi_0}{h}\right)}_{F(h,h\phi_0)} \right)_x = 0, \\ (h\phi_0)_t + \left(\underbrace{h^3 g\left(\frac{h\phi_0}{h}\right)}_{G(h,h\phi_0)} \right)_x = 0. \end{cases}$$

- $f(\phi_0), g(\phi_0)$, $\phi_0 \in [0, \phi_m]$.
  Physical Interpretation: $\phi_m =$ Maximum Packing Fraction.

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- $f(\phi_0), g(\phi_0)$, $\phi_0 \in [0, \phi_m]$.
  Physical Interpretation: ϕ_m = Maximum Packing Fraction.
- If the assumptions of the theorem hold, by picking a sufficiently good approximation of f and g, hence F and G, unique double-wave entropy solutions are preserved.

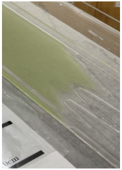
Application II: Gravity-Driven Particle-Laden Flow

Fix $\alpha = 30^\circ$. $\phi_0 \in [0, \phi_m]$,

$\phi_m = 0.61$: Maximum packing fraction.

$\phi_c \approx 0.459$: Phase transition from settled to ridged.

Settled: $\phi_0 < \phi_c$.



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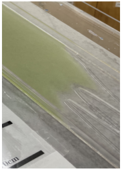

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Polynomial Approximations:

$$f(\phi_0) = \begin{cases} \sum_{j=1}^{10} \beta_{f,j}^{S} (\phi_c - \phi_0)^{j-1} & \text{for } \phi_0 < \phi_c, \\ \sum_{j=1}^{10} \beta_{f,j}^{R} (\phi_0 - \phi_c)^{j-1} & \text{for } \phi_0 > \phi_c, \end{cases}$$

and

$$g(\phi_0) = \begin{cases} \sum_{j=1}^{10} \beta_{g,j}^{S} (\phi_c - \phi_0)^{j-1} & \text{for } \phi_0 < \phi_c, \\ \sum_{j=1}^{10} \beta_{g,j}^{R} (\phi_0 - \phi_c)^{j-1} & \text{for } \phi_0 > \phi_c. \end{cases}$$

$$\boldsymbol{\beta}_f = \mathrm{argmin}_{\boldsymbol{\beta}_f \in \mathbb{R}^{20}} \|\mathbf{f} - \mathbf{X}\boldsymbol{\beta}_f\|_2^2 + \lambda \|\mathbf{f}' - \mathbf{X}'\boldsymbol{\beta}_f\|_2^2$$

subject to (I), (II), (III), and (IV) (similar for g).

Physical "Constraints":

$$\begin{aligned}
\text{(I)} &: f \text{ and } g \text{ are continuous at } \phi_c,\\
\text{(II)} &: f' \text{ and } g' \text{ are continuous at } \phi_c,\\
\text{(III)} &: f(0) = \frac{\mu_l}{3}, g(0) = 0,\\
\text{(IV)} &: f(\phi_m) = g(\phi_m) = f'(\phi_m) = g'(\phi_m) = 0.
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Compare:

- $\lambda = 0$ (not fitting for derivatives) and
- $\lambda = 0.1$ (fitting for derivatives).

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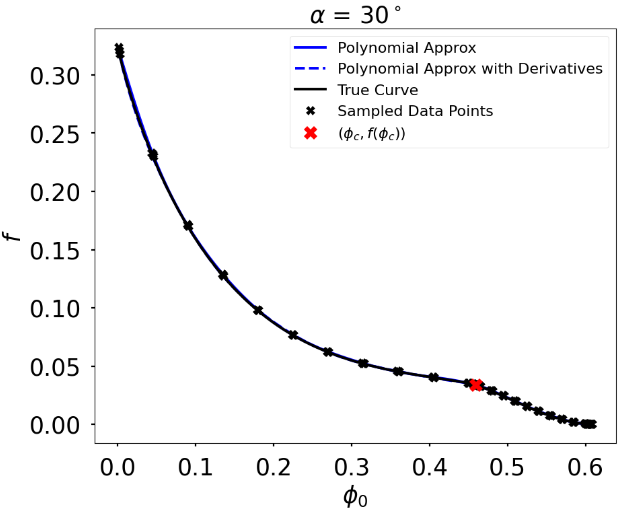
subject to (I), (II), (III), and (IV) (similar for g).

Sampled Data Points:

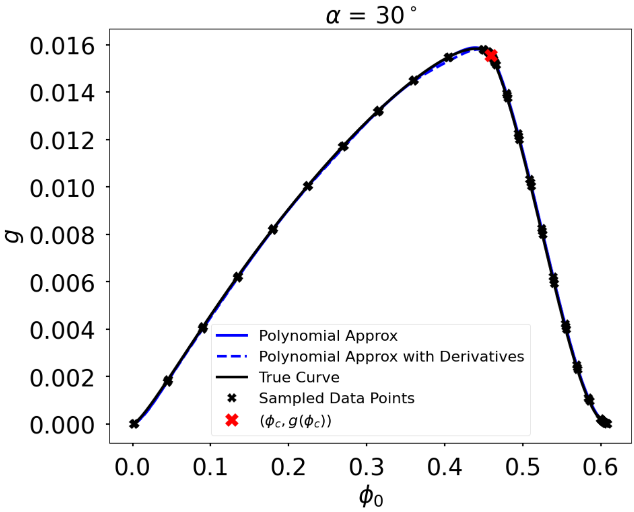
- A couple of points close to ϕ_m,
- A couple of points close to ϕ_c,
- A couple of sparse points,
- Points are in triplets to provide derivative information at the middle point.

Application II: Gravity-Driven Particle-Laden Flow

Quality of Approximation - Flux Function f:

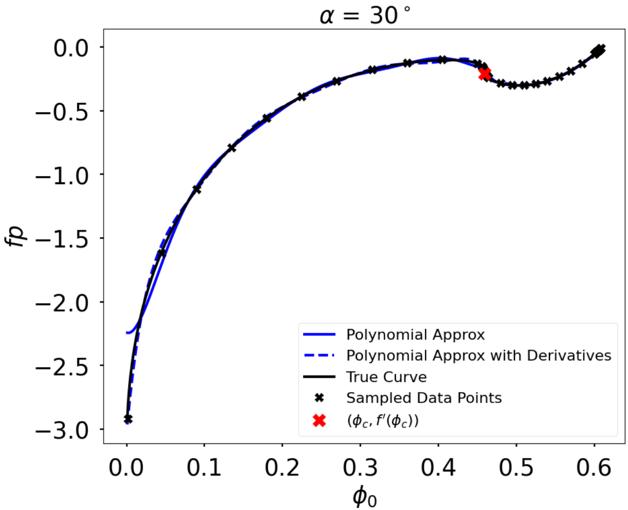


Quality of Approximation - Flux Function g:



Application II: Gravity-Driven Particle-Laden Flow

Quality of Approximation - Derivative of Flux Function f':



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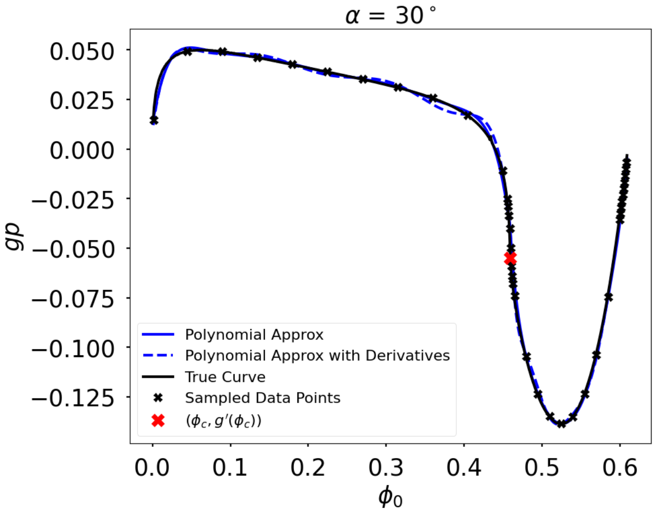
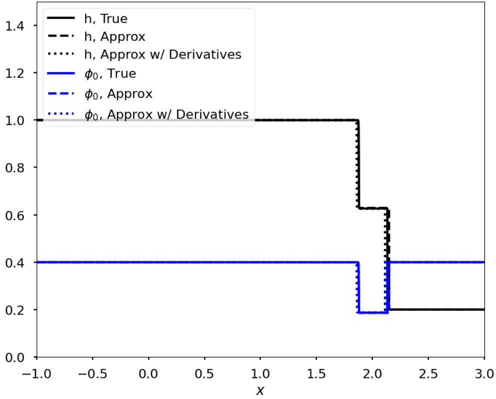


Application II: Gravity-Driven Particle-Laden Flow

Riemann Initial Data:

$$(h, \phi_0)(0, x) = \begin{cases} (1, 0.4) & \text{for } x > 0, \\ (0.2, 0.4) & \text{for } x < 0. \end{cases}$$

Solution for $(h, \phi_0) = (40, x)$:



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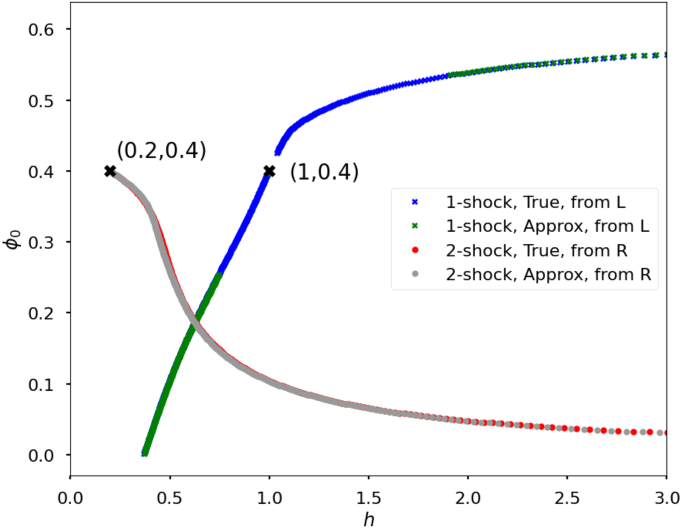
Phase Space Plots: $\lambda = 0$ (not fitting for derivatives)



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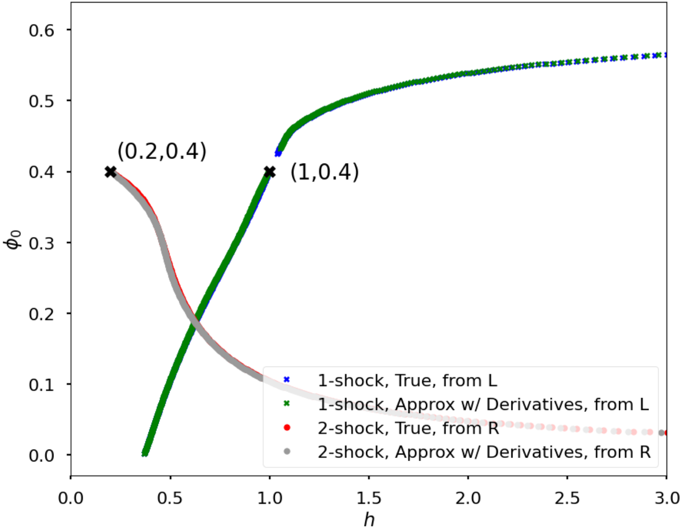
Phase Space Plots: $\lambda = 0.1$ (fitting for derivatives)



$(h_r, (\phi_0)_r) \to (0.5, 0.4)$

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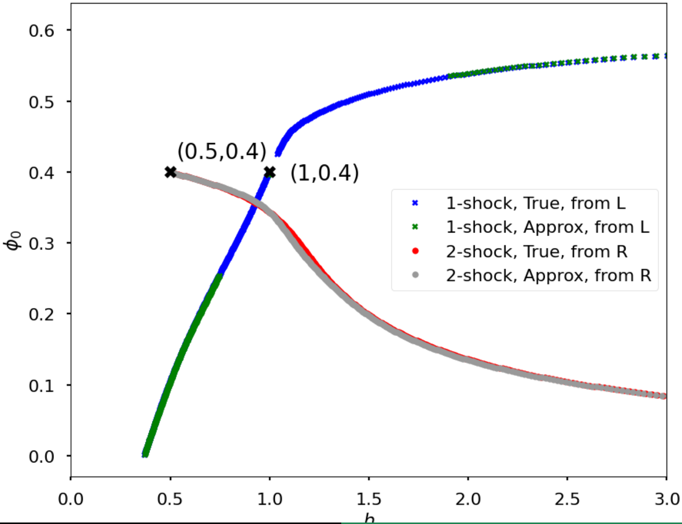

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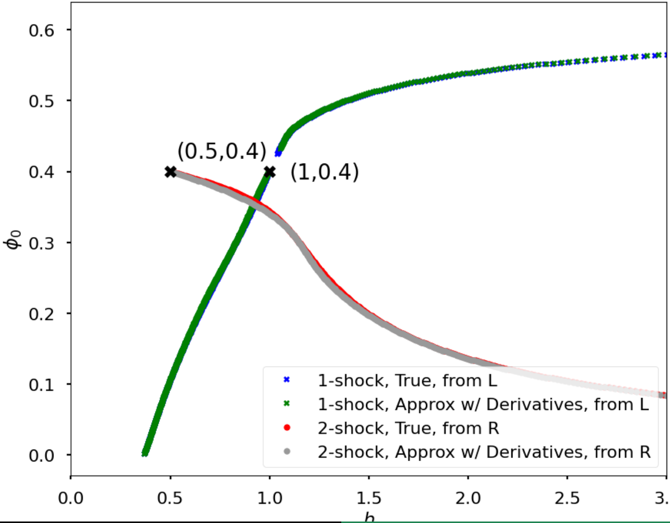

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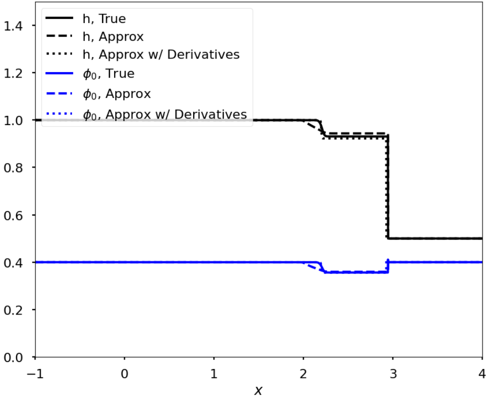

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Observation:

$$\text{Double Shock} \xrightarrow{\text{Perturbation}} \text{Rarefaction-Shock?}$$

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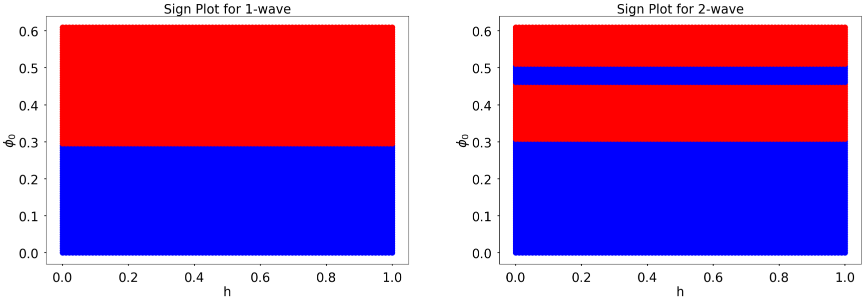
Observation:

$$\text{Double Shock} \xrightarrow{\text{Perturbation}} \text{Rarefaction-Shock?}$$

Possible Reason: Failure of Genuine Non-linearity.

Genuine Nonlinearity: $\nabla\lambda_k(u,v)\cdot r_k(u,v) \neq 0$ for k-wave.
Blue: $\nabla\lambda_k(u,v)\cdot r_k(u,v) > 0$.
Red: $\nabla\lambda_k(u,v)\cdot r_k(u,v) < 0$.

Application II: Gravity-Driven Particle-Laden Flow



Conclusion and Discussion

- **Main Result:** Under the usual assumptions and some additional mild assumptions, unique double-wave entropy solutions are preserved upon a sufficiently good approximation of flux functions.
- Understanding how each of the assumptions fails allows us to predict properties that are not expected to be preserved upon perturbation.

Future Work

Generalizing the result to $n \times n$ systems.

- (Wong and Bertozzi, 2016.)
  $n = 3$: Bidensity/Bisize Particle Laden Flow
  (Additional Parameter $\rightarrow$ Additional Conservation Law.)
- Figuring out an equivalent method to apply Implicit Function Theorem on Banach spaces (and corresponding assumptions).

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Other Variants - Regularity:

- Lower Regularity required for flux functions and their perturbations.
- Smooth except at finite points (corresponding to phase transitions).

Future Work

Allowing Linear Degenerate Waves:

- Example: $n = 3$, Compressible Euler Equations for gas dynamics.
- Expectation: Preserved under the class of "shocks, rarefactions, and contact discontinuities".

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Violating Genuine Non-linearity:

- (Liu, 1973.)
  Alternative to Lax's Entropy Condition $\rightarrow$ Liu's Entropy Condition.
- Generalizing the above arguments for a different entropy condition.

Future Work

Other Projects:

- Approximation techniques for fluxes (supported by our main theorem).
  Examples: Deep Learning, Asymptotics, etc.
- Hyperbolic Conservation Laws on a Network.

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  Work in Progress: Modeling Hyperbolic Conservation Laws on Traffic Flow Networks: Evacuation Route Analysis with Applications to Lahaina Wildfire (with Annie Lu, Isaac Ramos, and Alex Xue).

Thank you for your attention!

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