Theoretical Background

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1 Introduction

The Gauss Siedel method is an iterative method that can be used for linear systems of equations

$$Ax = b$$

.It can be applied to any matrix with non-zero elements on it diagonals. However convergence is only guaranteed if the matrix is either diagonally dominant or symmetric and positive semi-definite. The Gauss–Seidel method is an iterative technique for solving a square system of n linear equations with unknown x.

2 The Gauss Siedel Mathod

Let's define:

$$A\mathbf{x} = \mathbf{b}$$

matrix equation for a system of linear equations where A the m x n matrix, **x** is a column vector with n entries, and **b** is a column vector with m entries. Then the Gauss-Siedel method is defined by the iteration:

$$L_* \mathbf{x}^{(k+1)} = \mathbf{b} - U\mathbf{x}^{(k)}$$

where $\mathbf{x}^{(k)}$ is the kth approximation or iteration of x , $x^{(k+1)}$ is the next or $k+1$ iteration of **x** , and the matrix A is decomposed into a *lower triangular component* L_*, and a *strictly upper triangular component* **U**:That is

$$A = L_* + U$$

.

where :

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n4} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

And:

$$L_* = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \mathbf{U} = \begin{bmatrix} 0 & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

The system of linear equation can rewritten as: The system of linear equations may be rewritten as:

$$L_*\mathbf{x} = \mathbf{b} - U\mathbf{x}$$

The Gauss–Seidel method now solves the left hand side of this expression for x, using previous value for x on the right hand side. Analytically, this may be written as:

$$\mathbf{x}^{(k+1)} = L_*^{-1}(\mathbf{b} - U\mathbf{x}^{(k)})$$

However, by taking advantage of the triangular form of L_* , the elements of $x(k+1)$ can be computed sequentially using forward substitution:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right), \quad i = 1, 2, \dots, n.$$

3 Conclusion

The element-wise formula for the Gauss–Seidel method is extremely similar to that of the Jacobi method.

The computation of $xi^{(}k+1)$ uses only the elements of $\mathbf{x}^{k+1} thathavealreadybeencomputed, andonlytheeleme$
1.[**1**]

References

[1] Gaussian-seidel method, wikipedia.