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# Eigen solutions of internal waves over subcritical topography

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**Abstract**

Diapycnal mixing plays an important role in the ocean circulation. Internal waves are a kind of bridge relating the diapycnal mixing to external sources of mechanical energy. Difficulty in obtaining eigen solutions of internal waves over curved topography is a limitation for further theoretical study on the generation problem and scattering process. In this study, a kind of transform method is put forward to derive the eigen solutions of internal waves over subcritical topography in two-dimensional and linear framework. The transform converts the curved topography in physical space to flat bottom in transform space while the governing equation of internal waves is still hyperbolic if proper transform function is selected. Thus, one can obtain eigen solutions of internal waves in the transform space. Several examples of transform functions, which convert the linear slope, the convex slope, and the concave slope to flat bottom, and the corresponding eigen solutions are illustrated. A method, using a polynomial to approximate the transform function and least squares method to estimate the undetermined coefficients in the polynomial, is introduced to calculate the approximate expression of the transform function for the given subcritical topography.

**Key words:** internal waves, transform method, eigen solutions, subcritical curved topography

## 1 Introduction

The ocean circulation is controlled by diapycnal mixing which consists of mechanical mixing and convective mixing (Munk and Wunsch, 1998; Huang, 1999). The intensity of mechanical mixing depends on the amount of mechanical energy offered by external processes, among which wind stress and tidal generation force are major sources (Munk and Wunsch, 1998). Internal waves are a kind of bridge relating the diapycnal mixing to external sources of mechanical energy, first, internal waves are driven by wind stress and barotropic tide so that large amount of mechanical energy are transported to abyssal ocean with propagation of internal waves; second, the mechanical energy can directly be cascaded to diapycnal mixing through instability and breaking of internal waves (Wunsch and Ferrari, 2004). Due to the importance on the diapycnal mixing, internal waves become an active topic of physical oceanography (Fang and Du, 2005).

Ray theory approach was usually used to investigate the generation and scattering of internal waves. Baines (1973; 1982) applied the ray theory to generation of internal tides over subcritical topography, which were reduced to solve a coupled pair of Fredholm integral equations of the second kind. By using the method similar to Baines, Craig (1987) considered the generation of internal tides over a continental slope. Scattering of internal waves over curved topographies were investigated by Müller and Liu(2000a; b), who derived a formal solution using a mapping function based on ray tracing. The ray theory method is applicable for studying internal waves however it is complicated and difficult to carry out (Balmforth et al., 2002). Therefore, there are still studies trying to obtain the analytical solutions of internal generation over subcritical topography, such as Bell(1975a; b), Llewellyn Smith and Young(2002), Khatiwala (2003), who investigated generation of internal tides over subcritical topography. However, only weak topography

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is taken into account in their work, where the weak topography means that the topographic slope is much smaller than the slope of internal tidal rays and the topographic height is much smaller than the water depth and the vertical wave length of internal tide. Balmforth et al. (2002) extended the generation problem to subcritical but large topography, but only an asymptotic solution was offered.

Substantial advances have been obtained in the generation, propagation, scattering of internal waves. However, it is still a difficult problem to obtain the eigen solutions of internal waves over curved topography. In this study, a transform method will be introduced in Section.2, by using which the eigen solutions of internal waves can be easily obtained. Some examples of transforms and the corresponding eigen solutions are also given in Section 2. A method to estimate the transform function is introduced in Section 3. Conclusions are given in Section 4.

## 2 Transform method and eigen solutions of internal waves

### 2.1 *Governing equations of internal waves*

Two-dimensional Cartesian coordinate system is set up as follows, $\tilde{x}$ is a horizontal axis along the propagation direction of internal waves, and $\tilde{z}$, a vertical axis positive up. Idealizing the ocean as an inviscid, stratified fluid in which the rotation of the earth is ignored, we have the linear governing equations of internal waves with Boussinesq approximation (Xu, 1999)

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = -\frac{1}{\bar{\tilde{\rho}}_0}\frac{\partial \tilde{p}}{\partial \tilde{x}} \tag{1}$$

$$\frac{\partial \tilde{w}}{\partial \tilde{t}} = -\frac{1}{\bar{\tilde{\rho}}_0}\frac{\partial \tilde{p}}{\partial \tilde{z}} - \frac{\tilde{\rho}'}{\tilde{\rho}_0}\tilde{g} \tag{2}$$

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0 \tag{3}$$

$$\frac{\partial \tilde{\rho}'}{\partial \tilde{t}} + \tilde{w}\frac{\partial \tilde{\rho}_0}{\partial \tilde{z}} = 0 \tag{4}$$

where $(\tilde{u}, \tilde{w})$ are the velocity components of internal waves, $\tilde{p}$ the pressure perturbation, $\tilde{\rho}_0 = \tilde{\rho}_0(z)$ the density in static equilibrium, $\tilde{\rho}'$ the density perturbation from this state due to the wave motion, $\bar{\tilde{\rho}}_0$ the mean value of $\tilde{\rho}_0$, $\tilde{g}$ the acceleration due to gravity.

From Eqs (1)–(4), one can obtain the dimensional equation

$$\frac{\partial^4 \tilde{\varphi}}{\partial \tilde{x}^2 \partial \tilde{t}^2} + \frac{\partial^4 \tilde{\varphi}}{\partial \tilde{z}^2 \partial \tilde{t}^2} + \tilde{N}^2 \frac{\partial^2 \tilde{\varphi}}{\partial \tilde{x}^2} = 0 \tag{5}$$

with $\tilde{u} = -\dfrac{\partial \tilde{\varphi}}{\partial \tilde{z}}, \tilde{w} = \dfrac{\partial \tilde{\varphi}}{\partial \tilde{x}}$, where $\tilde{\varphi}$, the stream function and $\tilde{N} = (-\dfrac{\tilde{g}}{\bar{\tilde{\rho}}_0}\dfrac{\partial \tilde{\rho}_0}{\partial \tilde{z}})^{\frac{1}{2}}$, the Brunt-Väisälä frequency. The boundary conditions are

$$\tilde{\varphi} = 0, \text{at} \quad \tilde{z} = 0 \quad \text{and} \quad \tilde{z} = \tilde{h}(\tilde{x}) \tag{6}$$

where $\tilde{z} = \tilde{h}(\tilde{x})$ is the bottom boundary and $\tilde{z} = 0$ denotes the oceanic surface. Eqs (1)–(4) can describe the internal waves which frequency $\tilde{\omega}$ is much larger than the Coriolis parameter $\tilde{f}$.

Nondimensionalizing the stream function, distance, time as

$$\tilde{\varphi} = \tilde{\psi}\varphi, \tilde{x} = \tilde{H}x, \tilde{z} = \tilde{H}z, \tilde{t} = \frac{1}{\tilde{\omega}}t \tag{7}$$

one can obtain the nondimensional equation:

$$\frac{\partial^4 \varphi}{\partial x^2 \partial t^2} + \frac{\partial^4 \varphi}{\partial z^2 \partial t^2} + N^2 \frac{\partial^2 \varphi}{\partial x^2} = 0 \tag{8}$$

where $\tilde{\psi}$ is the amplitude of $\tilde{\varphi}$, $\tilde{H}$, the maximum water depth of the study area, and $\tilde{\omega}$, the frequency of internal waves, $N = \tilde{N}/\tilde{\omega}$.

For monochromatic wave with dimensional frequency $\tilde{\omega}$, define $\varphi = \phi(x, z)\exp(\mathrm{i}t)$, then we have

$$\phi_{zz} - \frac{1}{c^2}\phi_{xx} = 0 \tag{9}$$

where

$$c^2 = \frac{1}{N^2 - 1} \tag{10}$$

The boundary conditions become

$$\phi = 0, \quad \text{at} \quad z = 0 \quad \text{and} \quad z = h(x) \tag{11}$$

We will assume $N = const$ and hence $c = const$ in the following.

If the bottom is flat, i.e., $h(x) = -1$, it is easy to get the eigen solutions of Eq.,

$$\phi_n = \sin(k_n z)[A_n \cos(ck_n x) + B_n \sin(ck_n x)] \tag{12}$$

where $k_n = n\pi$, $n = 1, 2, 3 \cdots$. However the flat bottom is only a special case, the actual bottom topography is usually complex and irregular. Generally speaking, it is difficult to find the eigen solutions of Eq. (9) when $h(x)$ is dependent of $x$, which may be a reason why the ray-tracing method is usually used to study the propagation of internal waves up a curved topography.

### 2.2 *Transform method*

Introduce the following transform

$$\begin{cases} \zeta = \Psi(cx - z) - \Psi(cx + z) \\ \eta = \Psi(cx - z) + \Psi(cx + z) \end{cases} \tag{13}$$

is introduced to Eq. (9), due to

$$\begin{aligned}
\frac{\partial\zeta}{\partial z} &= -\dot{\psi}(cx-z) - \dot{\psi}(cx+z),\\
\frac{\partial\zeta}{\partial x} &= c\dot{\psi}(cx-z) - c\dot{\psi}(cx+z)\\
\frac{\partial\eta}{\partial z} &= -\dot{\psi}(cx-z) + \dot{\psi}(cx+z),\\
\frac{\partial\eta}{\partial x} &= c\dot{\psi}(cx-z) + c\dot{\psi}(cx+z)\\
\frac{\partial^2\zeta}{\partial z^2} &= \ddot{\Psi}(cx-z) - \ddot{\Psi}(cx+z),\\
\frac{\partial^2\zeta}{\partial x^2} &= c^2\ddot{\Psi}(cx-z) - c^2\ddot{\Psi}(cx+z)\\
\frac{\partial^2\eta}{\partial z^2} &= \ddot{\Psi}(cx-z) + \ddot{\Psi}(cx+z),\\
\frac{\partial^2\eta}{\partial x^2} &= c^2\ddot{\Psi}(cx-z) + c^2\ddot{\Psi}(cx+z)
\end{aligned}\tag{14}$$

and

$$\begin{aligned}
\frac{\partial\phi}{\partial z} &= \frac{\partial\phi}{\partial\zeta}\frac{\partial\zeta}{\partial z} + \frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial z}\\
\frac{\partial\phi}{\partial x} &= \frac{\partial\phi}{\partial\zeta}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial x}\\
\frac{\partial^2\phi}{\partial z^2} &= \frac{\partial^2\phi}{\partial\zeta^2}(\frac{\partial\zeta}{\partial z})^2 + \frac{\partial^2\phi}{\partial\eta^2}(\frac{\partial\eta}{\partial z})^2 +\\
&2\frac{\partial^2\phi}{\partial\zeta\partial\eta}\frac{\partial\zeta}{\partial z}\frac{\partial\eta}{\partial z} + \frac{\partial\phi}{\partial\zeta}\frac{\partial^2\zeta}{\partial z^2} + \frac{\partial\phi}{\partial\eta}\frac{\partial^2\eta}{\partial z^2}\\
\frac{\partial^2\phi}{\partial x^2} &= \frac{\partial^2\phi}{\partial\zeta^2}(\frac{\partial\zeta}{\partial x})^2 + \frac{\partial^2\phi}{\partial\eta^2}(\frac{\partial\eta}{\partial x})^2 +\\
&2\frac{\partial^2\phi}{\partial\zeta\partial\eta}\frac{\partial\zeta}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial\zeta}\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial\phi}{\partial\eta}\frac{\partial^2\eta}{\partial x^2}
\end{aligned}\tag{15}$$

we have

$$\begin{aligned}
&\frac{\partial^2\phi}{\partial\zeta^2}[4\dot{\Psi}(cx-z)\dot{\Psi}(cx+z)] +\\
&\frac{\partial^2\phi}{\partial\eta^2}[-4\dot{\Psi}(cx-z)\dot{\Psi}(cx+z)] = 0
\end{aligned}\tag{16}$$

where $\dot{\Psi}(\alpha) = \frac{\partial\Psi(\alpha)}{\partial\alpha}$ and $\ddot{\Psi}(\alpha) = \frac{\partial^2\Psi(\alpha)}{\partial\alpha^2}$. Since $4\dot{\Psi}(cx-z)\dot{\Psi}(cx+z) \neq 0$ for subcritical topography, Eq. (16) can be reduced to

$$\frac{\partial^2\phi}{\partial\zeta^2} - \frac{\partial^2\phi}{\partial\eta^2} = 0 \tag{17}$$

The transform obviously converts $z = 0$ to $\zeta = 0$. If it also converts $z = h(x)$ to $\zeta = \zeta_0 = const$, i.e.,

$$\Psi[cx - h(x)] - \Psi[cx + h(x)] = \zeta_0 \tag{18}$$

the eigen solutions of internal waves can be written in the transform space as

$$\phi_n = \sin(q_n\zeta)[A_n\cos(q_n\eta) + B_n\sin(q_n\eta)] \tag{19}$$

where $q_n = n\pi/\zeta_0, \quad n = 1, 2, 3\cdots$.

The above analysis shows that the transform converts the upper boundary $z = 0$ in the physical space to $\zeta = 0$ in the transform space while the governing equation still remains linear and hyperbolic. The difficulty is now focused on how to get the transform function $\Psi(\alpha)$ which satisfies Eq.(18).

### 2.3 *Eigen solutions of internal waves over some idealized topography*

There are two approaches to derive the transform function $\Psi(\alpha)$ from Eq. (18). One is to directly solve Eq. (18) to get the analytical expression of $\Psi(\alpha)$ for the given $h(x)$. However, we have not found a universal method to solve the equation up to date. A method, using a polynomial to approximate the transform function and least squares method to estimate the undetermined coefficients, to obtain the asymptotic expression will be described in section. 3. Another method is to deduce the shape of topography $h(x)$ as $\Psi(\alpha)$ is provided. Examinations show that this method can help us understand the transform method and the patterns of internal waves over some idealized topographies. Several useful examples are presented in the following.

(1) $\Psi(\alpha) = \ln\alpha$

Eq. (18) becomes

$$\ln(cx - h) - \ln(cx + h) = \zeta_0 \tag{20}$$

from which we can get

$$h(x) = \frac{1 - \exp(\zeta_0)}{1 + \exp(\zeta_0)}cx = rx \tag{21}$$

where $r = \frac{1 - \exp(\zeta_0)}{1 + \exp(\zeta_0)}c$. It denotes that the transform corresponding to $\Psi(\alpha) = \ln\alpha$ can transform the subcritical linear slope in the physical space to flat bottom in the transform space. Eigen solutions of internal waves, expressed as (19) with $q_n = 2\pi/\zeta_0$, over the linear slope was shown in Fig. 1a. In fact, this transform should be attributed to Wunsch (1968) who employed the polar coordinates to derive the eigen solutions of internal waves over linear slope

$$\begin{aligned}
\phi_n &= \sin[q_n\ln(\frac{cx - z}{cx + z})]\cos[q_n\ln(c^2x^2 - z^2)]\\
\phi_n &= \sin[q_n\ln(\frac{cx - z}{cx + z})]\sin[q_n\ln(c^2x^2 - z^2)]
\end{aligned}\tag{22}$$

where $q_n = n\pi/\ln(\dfrac{c-r}{c+r})$. Using the eigen solutions, Dai et al. (2005) discussed the scattering process of internal waves propagating over a subcritical linear slope.

(2) $\Psi(\alpha) = \alpha^2$

Eq. (18) becomes

$$(cx-h)^2 - (cx+h)^2 = \zeta_0 \tag{23}$$

from which we can get

$$h(x) = -\frac{\zeta_0}{4cx} \tag{24}$$

which means the transform with $\Psi(\alpha) = \alpha^2$ can convert the hyperbolic slope to flat bottom. Fig. 1b shows the eigen solutions of internal waves over hyperbolic slope. The hyperbolic slope can offer good approximation for the continental slope and shelf. The eigen solutions of internal waves were used to study the propagation of internal waves from the abyssal ocean to the continental slope and shelf (Dai et al., 2008).

(3) $\Psi(\alpha) = \sqrt{\alpha}$

Eq. (18) becomes

$$\sqrt{cx-h} - \sqrt{cx+h} = \zeta_0 \tag{25}$$

from which we can get

$$h(x) = -\zeta_0\sqrt{cx - \frac{\zeta_0^2}{4}} \tag{26}$$

The topography expressed in (26) shows the characteristic of concave slope. The corresponding eigen solutions of internal waves are shown in Fig. 1c.

(4) $\Psi(\alpha) = \sin(b\alpha)$

Eq. (18) becomes

$$\sin[b(cx-h)] - \sin[b(cx+h)] = \zeta_0 \tag{27}$$

from which we can get

$$h(x) = -\frac{1}{b}\mathrm{Arcsin}[-\frac{\zeta_0}{2\cos(bcx)}] \tag{28}$$

The topography expressed in (28) shows a characteristic of convex slope, similar to the hyperbolic slope, which can also be used to approximate the continental slope and shelf. The corresponding eigen solutions of internal waves are shown in Fig. 1d.

(5) $\Psi(\alpha) = \mathrm{tg}(b\alpha)$

Eq. (18) becomes

$$\mathrm{tg}[b(cx-h)] - \mathrm{tg}[b(cx+h)] = \zeta_0 \tag{29}$$

from which we can get

$$h(x) = \frac{\mathrm{Acrsin}[-\dfrac{\zeta_0\cos(2bcx)}{\sqrt{4+\zeta_0^2}}] - \theta}{2b} \tag{30}$$

where $\theta = \mathrm{Arctg}(\dfrac{\zeta_0}{2})$. The topography expressed in (30) and corresponding eigen solutions of internal waves are shown in Fig. 1e, from which we can see that this topography can be used to approximate the oceanic trench.

(6) $\Psi(\alpha) = \mathrm{Arctg}(b\alpha)$

Eq. (18) becomes

$$\mathrm{Arctg}[b(cx-h)] - \mathrm{Arctg}[b(cx+h)] = \zeta_0 \tag{31}$$

from which we can get

$$\begin{cases} x = [\mathrm{tg}(\dfrac{\eta-\zeta_0}{2}) + \mathrm{tg}(\dfrac{\eta+\zeta_0}{2})]/2bc \\ h = [\mathrm{tg}(\dfrac{\eta-\zeta_0}{2}) - \mathrm{tg}(\dfrac{\eta+\zeta_0}{2})]/2b \end{cases} \tag{32}$$

The topography expressed in (32) and corresponding eigen solutions of internal waves are shown in Fig 1f. This topography can approximate the oceanic ridge when the proper parameters in (32) are chosen.

In the several examples, the linear slope, the concave slope, the convex slope in the physical space can all be transformed to flat bottom if the transform function is selected properly. From Fig. 1, we can see that both the upper and bottom boundaries are exactly the stream lines of the internal waves, which implying that the transform method works well. Another characteristic shown in the figure is that the wave length of internal waves is larger over the deeper water while the wave length is smaller when the water depth is shallower. It is consistent to the general understandings about the evolution of internal waves.

## 3 Least squares method to estimate the transform function

The transform method is useful to obtain the eigen solutions of internal waves over the subcritical curved topography. In the above section, we offer the transform functions for several idealized curved topographies. However, how to obtain the transform function for any subcritical curved topography is still a key and difficult problem. Investigation shows that it is difficult to exactly obtain the analytical expression of the transform function, but the asymptotic expression can be obtained by using the least squares method.

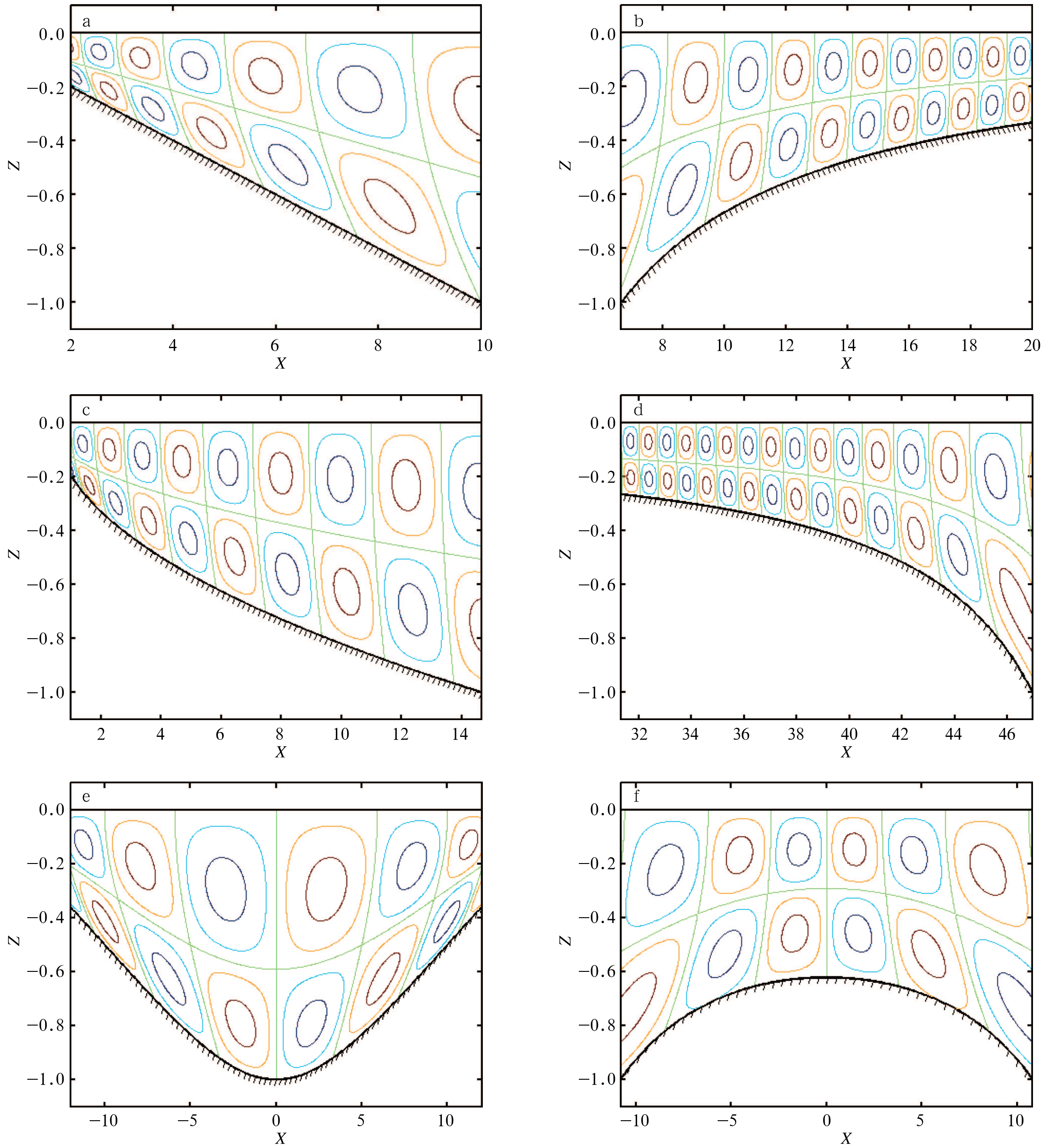


**Fig.1.** Eigen solutions of internal waves over some idealized subcritical topographies that correspond to the transform function: (a) $\Psi(\alpha) = \ln(\alpha)$; (b) $\Psi(\alpha) = (\alpha)^2$; (c) $\Psi(\alpha) = \sqrt{\alpha}$; (d) $\Psi(\alpha) = \sin(b\alpha)$; (e) $\Psi(\alpha) = \mathrm{tg}(b\alpha)$; (f) $\Psi(\alpha) = \mathrm{Arctg}(b\alpha)$.

As have shown in the above, the transform method requires that Eq. (18) must be satisfied. Using a $M$-th-order polynomial to approximate the transform function, i.e.,

$$\Psi(\alpha) = \sum_{m=1}^{M} a_m \alpha^m \tag{33}$$

Eq. (18) becomes

$$\sum_{m=1}^{M} a_m \{[cx - h(x)]^m - [cx + h(x)]^m\} = \zeta_0 \tag{34}$$

Discretizing $h(x)$, we have

$$\sum_{m=1}^{M} a_m \{[cx_i - h(x_i)]^m - [cx_i + h(x_i)]^m\} = \zeta_0 \tag{35}$$

where $a_m$ are the undetermined coefficients. The least squares method requires that the undetermined coefficients must obey

$$\sum_{m=1}^{M} a_m (\varphi_m, \varphi_j) = d_j \quad (j = 1, 2, 3 \cdots M) \tag{36}$$

where

$$(\varphi_m, \varphi_j) = \sum_{i=1}^{I} \{[cx_i - h(x_i)]^m - [cx_i + h(x_i)]^m\} \{[cx_i - h(x_i)]^j - [cx_i + h(x_i)]^j\}$$

$$d_j = \sum_{i=1}^{I} \{[cx_i - h(x_i)]^j - [cx_i + h(x_i)]^j\}\zeta_0 \qquad (37)$$

and $I$ is the number of discretized points of $h(x)$. Solving the above equation, we can obtain the undetermined coefficients $a_m$, then the transform function expressed as (33) is obtained.

Figure 2 illustrates the transform function obtained from the least squares method for both the linear and hyperbolic slopes, where 20th-order polynomial is used, i.e., $M = 20$. For the linear slope, the transform function obtained by the least squares method is different from the analytical function, $\Psi(\alpha) = \ln(\alpha)$ while examination shows that the transform function obtained by the least squares meth-

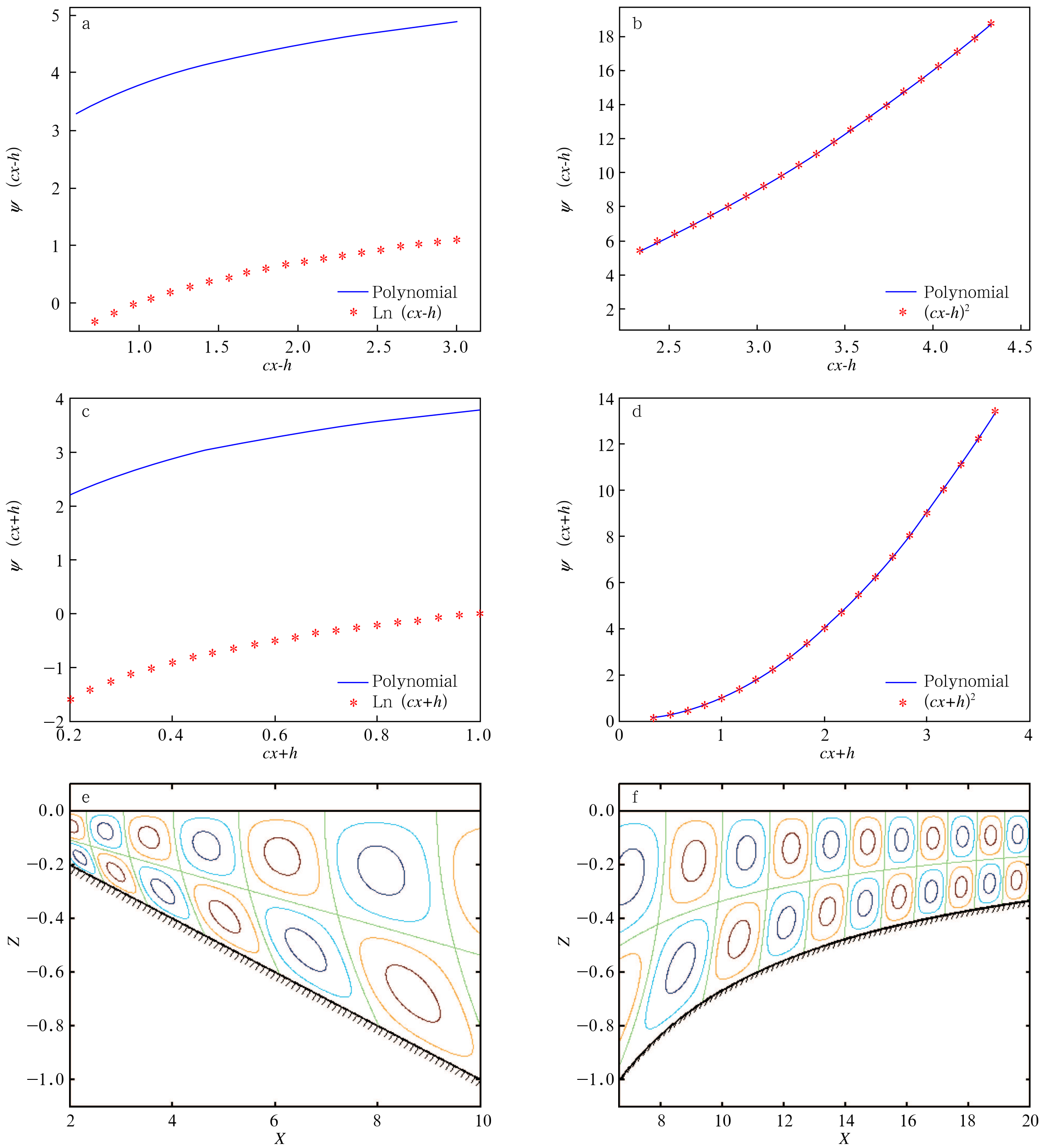


**Fig.2.** Transform functions $\Psi(cx-h)$ (upper panel) and $\Psi(cx+h)$ (middle panel), and the eigen solutions of internal waves (lower panel) obtained by the Least squares method for the linear and hyperbolic slopes. The left column is for the linear slope and the right for the hyperbolic slope.

od can be written as $\Psi(\alpha) = a\ln(\alpha) + b$ where both $a$ and $b$ are constant. For the transform function shown in Fig. 2a and Fig. 2c, $a = 1$ and $b = 3.79$. In fact, $\Psi(\alpha) = a\ln(\alpha) + b$ also satisfies Eq. (18) which is required by the transform method. For the hyperbolic slope, the transform function obtained by the least squares method is same to the analytical function, $\Psi(\alpha) = \alpha^2$, as shown in Figs 2b and Fig 2d. Figures 2e and Fig 2f also show the eigen solutions of internal waves over the linear and hyperbolic slopes by using the transform functions obtained by the least squares method. It is obvious that the eigen solutions of internal waves obtained by the least squares method are also similar to those shown Figs 1a and 1b.

The method, using a polynomial to approximate the transform function and the least squares method to estimate the undetermined coefficients, is applicable to estimate the asymptotic expression of the transform function. But there are still two issues need to be addressed. One problem is how to choose the order of polynomial so that the polynomial can offer good approximation. When the order is lower, the polynomial generally can not approximate the transform function well. On the contrary, a singular coefficient matrix will appear if the order of polynomial is very high. Another issue is that the interval of independent variable of the transform function must be limited in a fixed domain. For example, when considering the internal waves over the topography $h(x)$ with $x$ ranging from $x_h$ to $x_t$, the discretized $cx - h$ on the closed interval $[cx_h - h(x_h)\ cx_t - h(x_t)]$ and $cx + h$ on $[cx_h + h(x_h)\ cx_t - h(x_t)]$ are used to estimate the undetermined coefficients. The independent variable of obtained $\Psi(\alpha)$ must be limited on the $[\alpha_h\ \alpha_t]$ where $\alpha_h$ is the minimum of $cx_h - h(x_h)$ and $cx_h + h(x_h)$, $\alpha_t$ is the maximum of $cx_t - h(x_t)$ and $cx_t + h(x_t)$. Otherwise, an extremely large value of $\Psi(\alpha)$ may occur. The limitation of the interval of independent variable is another shortcoming for the method.

## 4 Conclusions

The governing equations of internal waves are reduced to linear hyperbolic equation in two dimensional and linear framework. A transform method is introduced in this study to solve the hyperbolic equation. The transform converts the upper boundary in the physical space to the flat surface in the transform space and the governing equation still remains the hyperbolic type. If the transform can also transform the curved topography to flat bottom, then the eigen solutions of internal waves can be obtained. However, it is a key and difficult problem how to derive the transform function which can convert the curved topography to flat bottom. Two methods are introduced in this study. One is to get the corresponding topography when the transform function is provided. Several examples are illustrated by using this method, from which the linear slope, the concave slope, and the convex slope can all be transformed to flat bottom in the transform space. The eigen solutions of those idealized topographies are also given. Another method to obtain the transform function is using a polynomial to approximate the transform function, discretizing the curved topography, then using the least squares method to estimate the undetermined coefficients in the polynomial. Examination shows that this method can lead to an asymptotic expression of transform function although there are still some issues need further investigations.

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