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# The interaction of internal waves with an unsteady non-uniform current

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On the assumptions of incompressibility, and negligible thermal conduction, salinity diffusion and viscosity, simple expressions are derived for the conservation equations of mass, momentum and energy when internal waves encounter an unsteady non-uniform current. These expressions of conservation equations are valid for all kinds of internal waves without regard to their different characteristics. From the dynamical conservation equations, we find that a stress-like term, the 'excess momentum flux tensor', plays an important role in the interaction between internal waves and an unsteady non-uniform current. Furthermore, it is deduced from the energy balance equation that, in the encounter of interfacial waves with a steady non-uniform current in a two-liquid system, the waves are amplified in an adverse current but suppressed in an advancing current as a result of interaction of the waves with the current. This conclusion may explain the large amplitudes sometimes observed in internal waves near the confluence of currents and near fronts at the thermocline, the region in the ocean where the density gradient is a maximum.

## 1. Introduction

The interaction between internal waves and varying currents is frequently encountered in the atmosphere and oceans, particularly in tidewaters near coastlines and in estuaries. For instance, large-amplitude internal waves are sometimes detected near the confluence of currents and near fronts at the thermocline (Lafond 1962, pp. 731–57) presumably as a result of this type of interaction.

Although some general wave studies in inhomogeneous moving media have been recently published by Bretherton & Garrett (1968), no work has been done specifically on the interaction between internal waves and varying currents. However, the effect of a horizontal shearing flow on surface waves was successfully investigated by Longuet-Higgins & Stewart (1961) so that the old misconception was dispelled that no dynamical coupling between the surface waves and varying currents can take place. A more direct derivation of some of their results on the amplitude variation of surface waves propagating on a non-uniform stream was given later by Whitham (1962) by a different approach. Their results have been neatly given in the book by Phillips (1966). In this present paper the

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equations for the conservation of mass, momentum and energy for internal waves propagating on an unsteady non-uniform current are derived to the second order in the wave slope by an integral method.

As in the case of surface waves, we might anticipate that there is an interchange of momentum and energy between internal waves and an unsteady non-uniform current in which the velocity varies, i.e. $U_\alpha = U_\alpha(x_\alpha, t)$, where the subscript $\alpha = 1, 2$ denotes horizontal directions. Though, dynamically, internal waves are rather different from surface waves, the basic mechanism of interaction might be expected to have features in common with surface waves in their encounter with a varying current. In this paper the equations for the conservation of mass, momentum and energy in internal wave-trains are derived, and, based on them, the interaction between internal waves and an unsteady non-uniform current is studied. One particular example is studied in detail, that of an interfacial gravity wave propagating in a steady non-uniform current in a two-liquid system. Under these circumstances the interaction phenomenon is found analogous to that between surface waves and a steady non-uniform current in deep water.

## 2. Governing equations and boundary conditions

We will suppose that in the internal wave motion the variations in density are small compared with the average density $\bar{\rho}$, that the vertical length scale of the motion is small compared with the scale height, the time scale of the motion is small compared with 12 h, the Mach number of the flow is very small and the Reynolds number of the motion is large. Consequently the sea water can be regarded incompressible and inviscid, and in the momentum equation the Coriolis force is ignored.

With rectangular Cartesian co-ordinates the condition of incompressibility, the equation of continuity and the Navier–Stokes equation for inviscid fluid are, neglecting thermal conduction and salinity mass transfer,

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0, \tag{2}$$

and

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} + \frac{\partial P}{\partial x_i} + \rho g m_i = 0, \tag{3}$$

where $\mathbf{m}$ is a unit vector vertically upwards and $g$ the acceleration of gravity.

Since internal waves produce only very small vertical displacement at the free surface (see Phillips 1966), the vertical component $w$ of the velocity must vanish at $z = 0$:

$$w_0 = 0, \tag{4}$$

and the kinematic condition on the bottom $z = -d$ is

$$w_{-d} = 0, \tag{5}$$

where $w_0$ and $w_{-d}$ are the vertical velocities due to internal waves at the surface and at the bottom, respectively.

## 3. The conservation equations of mass, momentum and energy

In the derivation of the conservation equations, the total velocity $u_i(\mathbf{x}, t)$ can be decomposed into the ensemble mean (or current) horizontal velocity

$$U_\alpha = U_\alpha(x_\alpha, t)$$

and the velocity due to internal waves $u_i'(\mathbf{x}, t)$. Similarly, we can represent the pressure $p = \bar{p}(\mathbf{x}, t) + p'(\mathbf{x}, t)$ and the total density $\rho = \bar{\rho}(\mathbf{x}, t) + \rho'(\mathbf{x}, t)$. The ensemble mean velocity $U_\alpha = U_\alpha(x_\alpha, t)$, and $U_\alpha$ varies continuously with a length scale large compared with a typical wavelength. The mean density $\bar{\rho}$ is assumed to be a continuous function of $z$ and is slowly varying in $x_\alpha$ so that $\partial\bar{\rho}/\partial z \gg \partial\bar{\rho}/\partial x_\alpha$. In the following calculations, ensemble averages are taken and denoted by brackets $\langle\ \rangle$.

*Mass transport*

An equation for the transport of mass can be easily set up from (2). Substitution of $u_i = U_\alpha + u_i'$ and $\rho = \bar{\rho} + \rho'$ into (2) makes it

$$\frac{\partial\bar{\rho}}{\partial t} + \frac{\partial\rho'}{\partial t} + U_\alpha\frac{\partial\bar{\rho}}{\partial x_\alpha} + U_\alpha\frac{\partial\rho'}{\partial x_\alpha} + u_i'\frac{\partial\bar{\rho}}{\partial x_i} + u_i'\frac{\partial\rho'}{\partial x_i} = 0. \tag{6}$$

Taking ensemble average of (6), we get

$$\frac{\partial\bar{\rho}}{\partial t} + U_\alpha\frac{\partial\bar{\rho}}{\partial x_\alpha} + \left\langle u_i'\frac{\partial\rho'}{\partial x_i}\right\rangle = 0. \tag{7}$$

Subtracting (7) from (6), we are left with

$$\frac{\partial\rho'}{\partial t} + U_\alpha\frac{\partial\rho'}{\partial x_\alpha} + u_i'\frac{\partial\bar{\rho}}{\partial x_i} + u_i'\frac{\partial\rho'}{\partial x_i} - \left\langle u_i'\frac{\partial\rho'}{\partial x_i}\right\rangle = 0, \tag{8}$$

which is, of course, the continuity equation for $\rho'$.

In order to get the equation for mass transport, we integrate (7) over the whole depth of the fluid to have

$$\int_{-d}^{0}\frac{\partial\bar{\rho}}{\partial t}dz + \int_{-d}^{0}\frac{\partial(\bar{\rho}U_\alpha)}{\partial x_\alpha}dz + \int_{-d}^{0}\left\langle\frac{\partial(\rho' u_i')}{\partial x_i}\right\rangle dz = 0.\dagger \tag{9}$$

Using the formula that if $D$ is a differential operator, provided $f$ and $Df$ are continuous in the interval of integration,

$$\int_{-d}^{0} Df\,dz = D\int_{-d}^{0} f\,dz, \tag{10}$$

and using the boundary conditions (4) and (5), equation (9) becomes

$$\frac{\partial}{\partial t}\int_{-d}^{0}\bar{\rho}\,dz + \frac{\partial}{\partial x_\alpha}\left\{\int_{-d}^{0}\bar{\rho}U_\alpha\,dz + \int_{-d}^{0}\langle\rho' u_\alpha'\rangle\,dz\right\} = 0. \tag{11}$$

If we let

$$\bar{M}_\alpha = \int_{-d}^{0}\bar{\rho}U_\alpha\,dz, \tag{12}$$

and

$$M_\alpha = \int_{-d}^{0}\langle\rho' u_\alpha'\rangle\,dz, \tag{13}$$

† If all functions are continuous in the interval of integration, the operations of integrating and averaging are interchangeable.

representing the mean mass transports of the mean flow and the wave motion respectively, (11) is written as

$$\frac{\partial}{\partial t}\int_{-d}^{0}\bar{\rho}\,dz+\frac{\partial}{\partial x_\alpha}\{\bar{M}_\alpha+M_\alpha\}=0, \tag{14}$$

which states that the time rate of change of mass per unit area is equal to the negative divergence of the total mass flux $\bar{M}_\alpha+M_\alpha$.

*Momentum transport*

To derive the equation for the balance of total momentum, we rewrite (3) as

$$\frac{\partial(\rho u_i)}{\partial t}+\frac{\partial(\rho u_i u_j)}{\partial x_j}+\frac{\partial P}{\partial x_i}+\rho g m_i=0, \tag{15}$$

by means of the continuity equation (2). Then substitute $u_\alpha = U_\alpha + u'_\alpha$ and $\rho=\bar{\rho}+\rho'$ into the horizontal component of the momentum equation which is the restricted equation (3) with the index $i$ replaced by $\alpha=(1,2)$, then integrate throughout the depth and average as before to obtain

$$\begin{aligned}\int_{-d}^{0}\frac{\partial}{\partial t}\{\bar{\rho}U_\alpha+\langle\rho' u'_\alpha\rangle\}\,dz+\int_{-d}^{0}U_\beta\frac{\partial(\bar{\rho}U_\alpha)}{\partial x_\beta}\,dz+\int_{-d}^{0}U_\beta\frac{\partial\langle\rho' u'_\alpha\rangle}{\partial x_\beta}\,dz\\ +\int_{-d}^{0}\left\langle u'_j\frac{\partial(\rho' U_\alpha)}{\partial x_j}\right\rangle dz+\int_{-d}^{0}\left\langle u'_j\frac{\partial(\bar{\rho}u'_\alpha)}{\partial x_j}\right\rangle dz\\ +\int_{-d}^{0}\left\langle u'_j\frac{\partial(\rho' u'_\alpha)}{\partial x_j}\right\rangle dz+\int_{-d}^{0}\frac{\partial\bar{P}}{\partial x_\alpha}\,dz=0.\end{aligned} \tag{16}$$

Using the formula (10) and the boundary conditions (4) and (5), equation (16) becomes

$$\begin{aligned}\frac{\partial}{\partial t}(\bar{M}_\alpha+M_\alpha)+U_\beta\frac{\partial}{\partial x_\beta}(\bar{M}_\alpha+M_\alpha)+\frac{\partial}{\partial x_\beta}(U_\alpha M_\beta)\\ +\frac{\partial}{\partial x_\beta}\int_{-d}^{0}\bar{\rho}\langle u'_\alpha u'_\beta\rangle\,dz+\frac{\partial}{\partial x_\alpha}\int_{-d}^{0}\bar{P}\,dz=0,\end{aligned} \tag{17}$$

correct to the second order.

If we define

$$S_{\alpha\beta}=\int_{-d}^{0}\{\bar{\rho}\langle u'_\alpha u'_\beta\rangle+\delta_{\alpha\beta}\bar{P}\}\,dz \tag{18}$$

as the 'excess momentum flux' tensor where the Kronecker delta $\delta_{\alpha\beta}=1$ if $\alpha=\beta$, 0 if $\alpha\neq\beta$, then (17) becomes

$$\frac{\partial}{\partial t}(\bar{M}_\alpha+M_\alpha)+U_\beta\frac{\partial}{\partial x_\beta}(\bar{M}_\alpha+M_\alpha)+\frac{\partial}{\partial x_\beta}(S_{\alpha\beta}+U_\alpha M_\beta)=0, \tag{19}$$

where $\bar{M}_\alpha$ is interpreted as the mean momentum of the mean flow per unit area, while $M_\alpha$ is the mean momentum of the wave motion per unit area. The term $U_\alpha M_\beta$ can be regarded as 'convective momentum flux'.†

The balance of total momentum per unit area of the motion is expressed by (19). The terms in the equation represent respectively the time rate of change of total momentum, the flux of total momentum, the gradients of the excess momentum flux and convective momentum flux.

† This term is suggested by Professor Phillips.

The terms $U_\beta \partial M_\alpha/\partial x_\beta$ and $\partial(U_\alpha M_\beta)/\partial x_\beta$ indicate clearly an interchange of momentum between the mean flow and the wave motion.

*Energy transport*

Dealing with energy transport, we shall consider it for the mean flow and the wave motion separately. In order to set up the energy balance equation for the mean flow, we multiply the horizontal momentum equation by $U_\alpha$, then integrate and average the resulting equation as before to get

$$\int_{-d}^{0} U_\alpha \frac{\partial(\bar\rho U_\alpha)}{\partial t} dz + \int_{-d}^{0} U_\alpha \frac{\partial\langle \rho' u_\alpha'\rangle}{\partial t} dz + \int_{-d}^{0} U_\alpha \frac{\partial(\bar\rho U_\alpha U_\beta)}{\partial x_\beta} dz + \int_{-d}^{0} U_\alpha \frac{\partial\langle \rho' u_\alpha'\rangle}{\partial x_\beta} U_\beta dz + \int_{-d}^{0} U_\alpha \frac{\partial U_\alpha \langle \rho' u_j'\rangle}{\partial x_j} dz + \int_{-d}^{0} U_\alpha \frac{\partial \bar\rho\langle u_\alpha' u_j'\rangle}{\partial x_j} dz + \int_{-d}^{0} U_\alpha \frac{\partial \bar P}{\partial x_\alpha} dz = 0, \tag{20}$$

correct to the second order.

With the formula (10) and the boundary conditions (4) and (5), equation (20) takes the form

$$\frac{\partial \mathscr{E}}{\partial t} + \frac{\partial}{\partial x_\alpha}(U_\alpha \mathscr{E}) + U_\alpha \left\{\frac{\partial M_\alpha}{\partial t} + \frac{\partial}{\partial x_\beta}(U_\alpha M_\beta) + U_\beta \frac{\partial M_\alpha}{\partial x_\beta}\right\} + U_\alpha \frac{\partial}{\partial x_\beta}\left\{\int_{-d}^{0} (\bar\rho\langle u_\alpha' u_\beta'\rangle + \delta_{\alpha\beta}\bar P)\, dz\right\} = 0, \tag{21}$$

where
$$\mathscr{E} = \frac{1}{2}\int_{-d}^{0} \bar\rho U_\alpha^2\, dz$$

is the energy of the mean flow per unit area.

With the definition of (19), equation (21) becomes

$$\frac{\partial \mathscr{E}}{\partial t} + \frac{\partial}{\partial x_\alpha}(U_\alpha \mathscr{E}) + U_\alpha \left\{\frac{\partial M_\alpha}{\partial t} + U_\beta \frac{\partial M_\alpha}{\partial x_\beta} + \frac{\partial}{\partial x_\beta}(S_{\alpha\beta} + U_\alpha M_\beta)\right\} = 0, \tag{22}$$

where $S_{\alpha\beta}$ is the excess momentum flux just defined. Equation (22) is the energy budget equation for the mean flow, which states that the time rate of change of the kinetic energy of mean flow plus the energy flux of the mean flow is balanced by the rate of work done by the mean flow against the negative time rate of change and convective rate of change of wave momentum, against the negative gradient of the excess momentum flux and the convective momentum flux.

To formulate the mean energy transport equation for the wave motion, we multiply (16) by $u_i'$, the velocity due to wave motion, then integrate and average as before to obtain

$$\int_{-d}^{0} \frac{\partial}{\partial t}\langle \tfrac{1}{2}\bar\rho (u_i')^2\rangle dz + \int_{-d}^{0} \frac{\partial U_\alpha}{\partial t}\langle u_\alpha' \rho'\rangle dz + U_\alpha \int_{-d}^{0} \left\langle u_\alpha' \frac{\partial \rho'}{\partial t}\right\rangle dz + \int_{-d}^{0} \left\langle u_\alpha' \frac{\partial(\rho' U_\alpha U_\beta)}{\partial x_\beta}\right\rangle dz + \int_{-d}^{0} \left\langle u_i' \frac{\partial(\bar\rho u_i' U_\beta)}{\partial x_\beta}\right\rangle dz + \int_{-d}^{0} \left\langle u_\alpha' \frac{\partial(\bar\rho U_\alpha u_j')}{\partial x_j}\right\rangle dz + \int_{-d}^{0} \left\langle \frac{\partial(u_i' P')}{\partial x_i}\right\rangle dz + g\int_{-d}^{0} \langle \rho' w\rangle dz = 0, \tag{23}$$

correct to the second order.

If $\eta$ represents the vertical displacement of fluid elements in the motion, then

$$w = \frac{\partial \eta}{\partial t} + u_i \frac{\partial \eta}{\partial x_i},$$

and since $\rho' = \eta \partial \bar{\rho}/\partial z$† correct to the first order, the last term on the left-hand side of (23) can be written

$$g\int_{-d}^{0} \langle \rho' w \rangle \, dz = -g\int_{-d}^{0} \left\langle \frac{\partial}{\partial t}\left(\frac{\eta^2}{2}\frac{\partial \bar{\rho}}{\partial z}\right) + \frac{u_j}{2}\frac{\partial}{\partial x_j}\left(\eta^2 \frac{\partial \bar{\rho}}{\partial z}\right)\right\rangle dz$$

to the second order

$$= -\frac{g}{2}\frac{\partial}{\partial t}\int_{-d}^{0}\left(\frac{\partial \bar{\rho}}{\partial z}\right)\langle \eta^2 \rangle \, dz - \frac{g}{2} U_\alpha \frac{\partial}{\partial x_\alpha}\int_{-d}^{0}\left(\frac{\partial \bar{\rho}}{\partial z}\right)\langle \eta^2 \rangle \, dz$$

$$= \frac{\partial V}{\partial t} + U_\alpha \frac{\partial V}{\partial x_\alpha} \tag{24}$$

to the second order, where

$$V = -\frac{g}{2}\int_{-d}^{0}\left(\frac{\partial \bar{\rho}}{\partial z}\right)\langle \eta^2 \rangle \, dz‡$$

is the mean potential energy of internal waves per unit area.

With (4), (5), (10) and (24), equation (23) provides the mean energy transport equation for the wave motion in the form

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_\alpha}(U_\alpha E + D_\alpha) + M_\alpha \frac{\partial U_\alpha}{\partial t} + S_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta} + U_\alpha N_\alpha = 0, \tag{25}$$

where

$$E = \frac{1}{2}\int_{-d}^{0} \bar{\rho}\langle u_i'^2 \rangle \, dz + V$$

is the mean total energy of internal waves per unit area, and $S_{\alpha\beta}$ the excess momentum flux. The new quantities $D_\alpha$ and $N_\alpha$ are defined as the following:

$$D_\alpha = \int_{-d}^{0} \langle u_\alpha' P' \rangle \, dz, \tag{26}$$

and

$$N_\alpha = \int_{-d}^{0} \left\langle u_\alpha' \frac{\partial \rho'}{\partial t} + u_\beta' \frac{\partial(\rho' U_\beta)}{\partial x_\alpha} + u_\alpha' u_j \frac{\partial \bar{\rho}}{\partial x_j}\right\rangle dz. \tag{27}$$

In the case of a single wave-train, expression (26) is simply $D_\alpha = (C_g)_\alpha E$, where $\mathbf{C}_g$ is the group velocity of the wave-train. With equation (8), it can be easily shown that

$$U_\alpha N_\alpha = U_\alpha M_\beta \frac{\partial U_\beta}{\partial x_\alpha}, § \tag{28}$$

correct to the second order.

If we define

$$\tilde{E} = E + U_\alpha M_\alpha, \tag{29}$$

as the convected energy of the internal wave-trains propagating in the mean current, then the mean energy transport equation for the wave motion becomes

$$\frac{\partial \tilde{E}}{\partial t} + \frac{\partial}{\partial x_\alpha}(U_\alpha \tilde{E} + D_\alpha) + S_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta} - U_\alpha\left(\frac{\partial M_\alpha}{\partial t} + U_\beta \frac{\partial M_\alpha}{\partial x_\beta}\right) = 0. \tag{30}$$

† This expression is obtained from equation (8) under the restriction $\partial\bar{\rho}/\partial x_\alpha \ll \partial\bar{\rho}/\partial z$.

‡ The proof of this expression is in appendix A.

§ This relation is shown in appendix B.

The terms in equation (30) are respectively the time rate of change of mean convected energy of internal waves, the mean energy flux of internal waves convected by the mean current and transmitted by internal wave-trains as well, the rate of working by the excess momentum flux against the mean rate of shear and the work done by the mean flow against the negative time and convective rates of change of wave momentum.

An equation for the balance of the mean total energy of the fluid system can be readily obtained by summing (22) and (30). It is

$$\frac{\partial(\mathscr{E}+\bar{E})}{\partial t}+\frac{\partial}{\partial x_\alpha}\{U_\alpha(\mathscr{E}+\bar{E})+D_\alpha+U_\beta S_{\alpha\beta}\}+U_\alpha\left\{\frac{\partial}{\partial x_\beta}(U_\alpha M_\beta)\right\}=0. \tag{31}$$

This can be derived, of course, from (15) multiplied by total velocity $u_i$ undergoing the same operation of integrating and averaging used before. Note that $(\partial/\partial x_\alpha)(U_\beta S_{\alpha\beta})$ is a divergence term. In contrast with this, the interaction term in (30), for example, $S_{\alpha\beta}\partial U_\alpha/\partial x_\beta$, is not of the form of a simple divergence; it represents a source of energy of the wave motion rather than a spatial redistribution of energy.

These dynamical conservation equations, derived for internal waves propagating in an unsteady horizontally non-uniform current, demonstrate clearly that the excess momentum flux tensor $S_{\alpha\beta}$ plays an important role in the interaction between internal waves and an unsteady non-uniform current. Through the excess momentum flux tensor (which is closely analogous to the integrated Reynolds stress in a turbulent shear flow) internal waves and current exchange both momentum and energy. The precise form of $S_{\alpha\beta}$ depends on the mean pressure and the detailed nature of the internal waves, but in the case of a single wave-train, when referred to co-ordinate axes perpendicular and parallel to the local wave front, it can be shown simply that the non-diagonal components vanish.

## 4. The interaction between internal gravity waves and a steady non-uniform current

As a result of the interaction between internal waves and an unsteady non-uniform current, the amplitude of the waves will change as they propagate through a region of varying current. To study the amplitude variation of internal waves, we take, as a specific example, the incidence of an interfacial gravity wave-train moving in the $x$-direction on a steady variable current $U_\alpha(x_\alpha)$ in a two-liquid system.

For an interfacial gravity wave propagating in a two-liquid system of densities $\rho_1$ and $\rho_2>\rho_1$, the frequency relation (Lamb 1945) relative to a frame moving with $\mathbf{U}$ is, ignoring the interfacial surface tension,

$$\sigma^2=\left(\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\right)gk, \tag{32}$$

where $\sigma$ is the frequency and $k$ the wave-number. In a steady wave-train advancing in the system,

$$kU+\sigma=\text{constant}=\sigma_0, \tag{33}$$

where
$$\sigma_0 = \left\{\left(\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\right) gk_0\right\}^{\frac{1}{2}}$$
is the wave frequency at a location where $U = 0$ and relative to a fixed observer at all positions. With frequency relation (32), equation (33) becomes
$$\frac{C}{C_0} = 1+\frac{U}{C}, \tag{34}$$
where $C = \sigma/k$ is the phase velocity relative to $U$. Equation (34) admits a solution
$$\frac{C}{C_0} = \frac{1}{2}+\frac{1}{2}\left(1+\frac{4U}{C_0}\right)^{\frac{1}{2}}, \tag{35}$$
where the positive sign is taken since $C = C_0$ when $U = 0$. It is noted that the square root becomes imaginary when $-U$ exceeds $\frac{1}{4}C_0$; at this critical point, $C = \frac{1}{2}C_0$ and $U/C = \frac{1}{2}$.

When the convergence of the current is balanced by lateral spreading, the continuity equation yields
$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y} = 0. \tag{36}$$
With the definition of (18) and continuity equation (36), the wave energy equation (30) reduces to
$$\frac{\partial}{\partial x}\left\{E\left(\frac{C}{2}+U\right)\right\}-E\frac{\partial V}{\partial y}+T\frac{\partial U}{\partial x}+MU\frac{\partial U}{\partial x} = 0. \tag{37}$$
If the variation in density over the depth $d$ is substantial, the dependences of the mean flow may require $U$ to be a function of $z$ as well as of $x$ and $y$. In the Boussinesq approximation, however, in which variations in inertial density are neglected, the last term in (37) vanishes; the equation then reduces to the one given by Longuet-Higgins & Stewart (1961).

The complete equation (37) can also be integrated, as an interfacial gravity wave, with
$$T = V = \tfrac{1}{2}E,$$
and
$$M = \frac{E}{C};$$
then, through (36), equation (37) becomes
$$\frac{\partial}{\partial x}\left\{E\left(\frac{C}{2}+U\right)\right\}-\frac{E}{2}\frac{\partial U}{\partial x}+\frac{E}{C}U\frac{\partial U}{\partial x} = 0. \tag{38†}$$
The above system of equations (34), (36) and (38) describes the amplitude variation of the interfacial gravity wave moving on the steady non-uniform current in a two-liquid system.

In order to integrate (38), we first establish from (34) a differential relation:
$$\frac{\partial U}{\partial x} = \left(1+\frac{2U}{C}\right)\frac{\partial C}{\partial x}. \tag{39}$$

† The equation (38) to the Boussinesq approximation can be obtained from Bretherton & Garrett's result. This is shown in appendix C on referees' suggestion.

Multiplying (38) by $C^{-3}\, e^{2C/C_0}$ and using relations (34) and (35), we can write the resulting equation, first

$$C^{-2}\, e^{2C/C_0}\frac{\partial}{\partial x}\left\{\frac{E}{C}\left(U+\frac{C}{2}\right)\right\}+C^{-4}\, e^{2C/C_0}\, EU\frac{\partial U}{\partial x}=0,$$

then

$$\frac{\partial}{\partial x}\left\{\frac{E}{C^3}\, e^{2C/C_0}\left(U+\frac{C}{2}\right)\right\}=0, \tag{40}$$

and consequently

$$\frac{E}{E_0}=\frac{(C/C_0)^2\, e^{2(1-C/C_0)}}{(1+2U/C)}, \tag{41}$$

where $E_0$ is the wave energy when $U=0$. If we denote the amplitude of the interfacial gravity wave by $a$, equation (41) implies

$$\frac{a}{a_0}=\frac{\exp\left\{-\dfrac{U}{C}+\ln\dfrac{C}{C_0}\right\}}{(1+2U/C)^{\frac{1}{2}}}. \tag{42}$$

For $U/C \geqslant -\frac{1}{2}$, $\ln(C/C_0)$ can be expanded as

$$\begin{aligned}\ln(C/C_0)&=\ln(1+U/C)\\&=\frac{(U/C)}{(1+U/C)}+\frac{(U/C)^2}{2(1+U/C)^2}+\frac{(U/C)^3}{3(1+U/C)^3}+\ldots,\end{aligned}$$

which is definitely less than $U/C$. Therefore, relation (32) together with (34) and (35) indicates that in an adverse current the waves are amplified and in an advancing current the waves are suppressed, the analogous phenomena when surface waves propagate on a variable current in deep water. Because of the relatively slow phase speeds of internal waves, the effects will become evident for considerably smaller changes in the current speed. At the critical point when $U/C=-\frac{1}{2}$, $a/a_0$ will approach infinity; physically, this might imply the breakdown of the wave.

This paper is based upon a part of the author's dissertation, 'On internal gravity waves', submitted to the Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy.

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## Appendix A

The following is to show the expression of the mean potential energy of internal wave-trains propagating in a continuously stratified fluid in terms of the vertical displacement of the fluid particles. If we let $\eta$ denote the vertical displacement of the fluid particle, the mean potential energy of internal wave-trains propagating at level $z_0$ can be expressed by

$$\left\langle\int_{z_0}^{z_0+\eta}\rho g z\, dz\right\rangle$$

and, because $\rho = \rho_0 - (\partial\bar{\rho}/\partial z)\,z$ to the first order in the neighbourhood of $z_0$, the above expression is equal to $-\frac{1}{2}g(\partial\bar{\rho}/\partial z)\langle\eta^2\rangle$. Finally, the total mean potential energy per unit area of the internal waves $V$ is

$$V = -\tfrac{1}{2}g\int_{-d}^{0}\left(\frac{\partial\bar{\rho}}{\partial z}\right)\langle\eta^2\rangle\,dz.$$

## Appendix B

The relation
$$U_\alpha N_\alpha = U_\alpha M_\beta \frac{\partial U_\beta}{\partial x_\alpha}$$

is shown correct to the second order as follows:

By definition,
$$N_\alpha = \int_{-d}^{0}\left\langle u'_\alpha\left(\frac{\partial\rho'}{\partial t} + u'_j\frac{\partial\bar{\rho}}{\partial x_j}\right) + u'_\beta\frac{\partial(\rho' U_\beta)}{\partial x_\alpha}\right\rangle dz.$$

From equation (8), then
$$N_\alpha = \int_{-d}^{0}\left\langle u'_\alpha\left(-U_\beta\frac{\partial\rho'}{\partial x_\beta}\right) + u'_\beta\frac{\partial(\rho' U_\beta)}{\partial x_\alpha}\right\rangle dz,$$

correct to the second order.

Thus
$$\begin{aligned} U_N N_\alpha &= \int_{-d}^{0}\left\langle -U_\alpha u'_\alpha U_\beta\frac{\partial\rho'}{\partial x_\beta} + U_\alpha u'_\beta\frac{\partial(\rho' U_\beta)}{\partial x_\alpha}\right\rangle dz \\ &= U_\alpha\frac{\partial U_\beta}{\partial x_\alpha}\int_{-d}^{0}\langle\rho' u'_\beta\rangle\,dz \\ &= U_\alpha\left(\frac{\partial U_\beta}{\partial x_\alpha}\right)M_\beta, \quad \text{by the definition of (13).}\end{aligned}$$

## Appendix C

The derivation of (38) to the Boussinesq approximation from equation (A) of Bretherton & Garrett (1968):

Their main result on the wave energy $E$ of wave-trains in inhomogeneous moving media is written as in their notation,

$$\frac{d}{dt}\left(\frac{E}{\omega'}\right) + (\nabla\,.\,\mathbf{c})\frac{E}{\omega'} = 0, \tag{A}$$

which is equivalent to

$$\frac{dE}{dt} + (\nabla\,.\,\mathbf{c})\,E - \frac{E}{\omega'}\frac{d\omega'}{dt} = 0, \tag{B}$$

where $d/dt = \partial/\partial t + (\mathbf{U}+\mathbf{c})\,.\,\nabla$, $\mathbf{c}$ is the group velocity and $\omega' = \sigma$ in this paper. For the special case in § 4 of this paper, equation (B) becomes

$$\left(U + \frac{c}{2}\right)\frac{\partial E}{\partial x} + \frac{E}{2}\frac{\partial c}{\partial x} - \frac{E}{kc}\left(U + \frac{c}{2}\right)\frac{\partial(kc)}{\partial x} = 0, \tag{C 1}$$

where scalar $c$ is the phase speed.

From (33), we have

$$\frac{\partial k}{\partial x}(U+c)+k\left(\frac{\partial U}{\partial x}+\frac{\partial c}{\partial x}\right) = 0 \tag{C 2}$$

and

$$\frac{\partial U}{\partial x} = \left(1+\frac{2U}{c}\right)\frac{\partial c}{\partial x}. \tag{39}$$

Using equations (C 2) and (39), the last term of (C 1) on left-hand side is written in terms of $\partial U/\partial x$ as follows:

$$\begin{aligned}
&\frac{E}{kc}\left(U+\frac{c}{2}\right)\frac{\partial(kc)}{\partial x} \\
&\quad = \frac{E}{kc}\left(U+\frac{c}{2}\right)k\frac{\partial c}{\partial x}+\frac{E}{kc}\left(U+\frac{c}{2}\right)c\left(-\frac{k}{U+c}\right)\left(\frac{\partial U}{\partial x}+\frac{\partial c}{\partial x}\right) \\
&\quad = -\frac{E}{2}\left(\frac{2U+c}{U+c}\right)\frac{\partial U}{\partial x}+\frac{E}{2}\left(\frac{U}{U+c}\right)\frac{\partial U}{\partial x} = -\frac{E}{2}\frac{\partial U}{\partial x}.
\end{aligned}$$

Therefore equation (C 1) becomes

$$\left(U+\frac{c}{2}\right)\frac{\partial E}{\partial x}+\frac{E}{2}\frac{\partial c}{\partial x}+\frac{E}{2}\frac{\partial U}{\partial x} = 0$$

or

$$\frac{\partial}{\partial x}\left\{E\left(U+\frac{c}{2}\right)\right\}-\frac{E}{2}\frac{\partial U}{\partial x} = 0.$$

This is equation (38) under the restriction of Boussinesq approximation.

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