S15 MA 265 – Exam 1 02/19/2015 12:00pm or 1:30pm NAME: ______________________

You have 75 minutes to finish the exam. You must show work to receive credit.

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1. (14 pts) Compute $\det(A)$, given

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution. Using cofactor expansion, we expand the determinant along the first row

$$\begin{aligned}
\det(A) &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + a_{14}A_{14} \\
&= 2(-1)^{1+1}\det\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} + 1(-1)^{1+2}\det\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} + 0 + 0 \\
&= 2(8 + 0 + 0 - 0 - 2 - 2) - 1(4 + 0 + 0 - 0 - 0 - 1) \\
&= 8 - 3 \\
&= 5
\end{aligned}$$

2. (14 pts) Determine the **reduced row echelon form** of

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 2 & -2 & -2 \\ -3 & 1 & -3 \end{bmatrix}$$

Solution. Performing elementary row operations on A, we have

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 1 & 5 \\ 2 & -2 & -2 \\ -3 & 1 & -3 \end{bmatrix} \\
\xrightarrow[3r_1 + r_2 \to r_2]{-2r_1 + r_2 \to r_2} & \begin{bmatrix} 1 & 1 & 5 \\ 0 & -4 & -12 \\ 0 & 4 & 12 \end{bmatrix} \\
\xrightarrow{-\frac{1}{4}r_2 \to r_2} & \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 4 & 12 \end{bmatrix} \\
\xrightarrow{-4r_2 + r_3 \to r_3} & \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(in row echelon form)} \\
\xrightarrow{-r_2 + r_1 \to r_1} & \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(in reduced row echelon form)}
\end{aligned}$$

3. (14 pts) Is the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ a linear combination of the following matrices?

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solution. We need to find constants a_1, a_2, a_3, and a_4 such that

$$a_1 A_1 + a_2 A_2 + a_3 A_3 + a_4 A_4 = A.$$

Plugging in the matrices we have

$$a_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

That is

$$\begin{bmatrix} a_1 & a_1 \\ a_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & a_2 \\ a_2 & 0 \end{bmatrix} + \begin{bmatrix} a_3 & a_3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_4 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + a_3 + a_4 & a_1 + a_2 + a_3 \\ a_1 + a_2 & a_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

It is equivalent to the linear system

$$\begin{cases} a_1 + a_2 + a_3 + a_4 & = 1, \\ a_1 + a_2 + a_3 & = 2, \\ a_1 + a_2 & = 3, \\ a_1 & = 4. \end{cases}$$

By back substitution, we find the solution of the system

$$a_1 = 4. \quad a_2 = -1, \quad a_3 = -1, \quad a_4 = -1.$$

Hence, the matrix A is a linear combination of A_1, A_2, A_3, and A_4.

4. (14 pts) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution. To find A^{-1}, we proceed as follows:

$$\begin{aligned}
[A|I_4] \quad &= \quad \left[\begin{array}{cccc|cccc} 1 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \\
&\xrightarrow{-ar_4 + r_3 \to r_3} \left[\begin{array}{cccc|cccc} 1 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \\
&\xrightarrow{-ar_3 + r_2 \to r_2} \left[\begin{array}{cccc|cccc} 1 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -a & a^2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \\
&\xrightarrow{-ar_2 + r_1 \to r_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -a & a^2 & -a^3 \\ 0 & 1 & 0 & 0 & 0 & 1 & -a & a^2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]
\end{aligned}$$

Hence,

$$A^{-1} = \begin{bmatrix} 1 & -a & a^2 & -a^3 \\ 0 & 1 & -a & a^2 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5. (14 pts) Solve the linear system $A\mathbf{x} = \mathbf{b}$ by **Cramer's rule** where

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 2 & 2 \\ 3 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Solution. By Cramer's rule, the solution of the linear system is given by

$$x_1 = \frac{det(A_1)}{det(A)}, \qquad x_2 = \frac{det(A_2)}{det(A)}, \qquad x_3 = \frac{det(A_3)}{det(A)},$$

where A_i, $i = 1, 2, 3$ are obtained from A by replacing i-th column by $\mathbf{b}$.
The determinant of A is

$$det(A) = \begin{vmatrix} 3 & 4 & 2 \\ 1 & 2 & 2 \\ 3 & 0 & 1 \end{vmatrix} = 6 + 24 + 0 - 12 - 4 - 0 = 14.$$

The determinant of A_1 is

$$det(A_1) = \begin{vmatrix} 1 & 4 & 2 \\ -1 & 2 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 2 + 16 + 0 - 8 - (-4) - 0 = 14.$$

The determinant of A_2 is

$$det(A_2) = \begin{vmatrix} 3 & 1 & 2 \\ 1 & -1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = -3 + 6 + 4 - (-6) - 1 - 12 = 0.$$

The determinant of A_3 is

$$det(A_3) = \begin{vmatrix} 3 & 4 & 1 \\ 1 & 2 & -1 \\ 3 & 0 & 2 \end{vmatrix} = 12 + (-12) + 0 - 6 - 8 - 0 = -14.$$

Hence, the solution is

$$x_1 = 1, \quad x_2 = 0, \quad x_3 = -1$$

6. (16 pts) Let M_{22} be the vector space of all 2×2 matrices of the form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

The operations $\oplus$ and $\odot$ are standard matrix addition and scalar multiplication, respectively.

(a) (8 pts) Let W be a subset of M_{22} such that $a_{11} + a_{22} = 0$. Is W a subspace of M_{22}? Explain.

(b) (8 pts) Let V be a subset of M_{22} such that $a_{11} + a_{22} \neq 0$. Is V a subspace of M_{22}? Explain.

Solution. (a) Choose any two matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ in W, then

$$A \oplus B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

and

$$(a_{11} + b_{11}) + (a_{22} + b_{22}) = (a_{11} + a_{22}) + (b_{11} + b_{22}) = 0 + 0 = 0.$$

So $A \oplus B$ is also in W. Also, for any real number c, we have

$$c \odot A = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

and

$$ca_{11} + ca_{22} = c(a_{11} + a_{22}) = 0.$$

So $c \odot A$ is also in W. Hence, W is a subspace of M_{22}.

Solution. (b) The set V is not a subspace of M_{22}. To see this, we choose

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

Note that A and B are both in V, but

$$A \oplus B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is not in V because the sum of diagonal entries equals zero. Hence the set V is not closed under the operation $\oplus$.

7. (14 pts) Find an equation relating a, b, and c so that the linear system

$$\begin{cases} 2x + 2y + 3z = a \\ 3x - y + 5z = b \\ x - 3y + 2z = c \end{cases}$$

is consistent for any values of a, b, c that satisfy that equation.

Solution. The augmented matrix of this system is

$$[A \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 2 & 2 & 3 & a \\ 3 & -1 & 5 & b \\ 1 & -3 & 2 & c \end{array}\right].$$

We transform $[A \mid \mathbf{b}]$ to a matrix in a row echelon form as follows

$$[A \mid \mathbf{b}] \xrightarrow{r_1 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & c \\ 3 & -1 & 5 & b \\ 2 & 2 & 3 & a \end{array}\right] \xrightarrow{-3r_1 + r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & c \\ 0 & 8 & -1 & b - 3c \\ 2 & 2 & 3 & a \end{array}\right]$$

$$\xrightarrow{-2r_1 + r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & c \\ 0 & 8 & -1 & b - 3c \\ 0 & 8 & -1 & a - 2c \end{array}\right] \xrightarrow{-r_2 + r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & c \\ 0 & 8 & -1 & b - 3c \\ 0 & 0 & 0 & a - b + c \end{array}\right]$$

$$\xrightarrow{\frac{1}{8}r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & c \\ 0 & 1 & -\frac{1}{8} & \frac{b-3c}{8} \\ 0 & 0 & 0 & a - b + c \end{array}\right].$$

The system is consistent (i.e., it has at least a solution), if and only if the entry on the bottom right is zero. That is

$$a - b + c = 0.$$

Bonus True of False Problems: (1pt each)

1. If A is an $n \times n$ matrix, and k is a real number, then $\det(kA) = k\det(A)$. **T** or $\boxed{\textbf{F}}$
2. If a linear system has more unknowns than equations, then it has infinitely many solutions. **T** or $\boxed{\textbf{F}}$
3. If A and B are both nonsingular matrices, then $A + B$ is also nonsingular. **T** or $\boxed{\textbf{F}}$
4. If $AB = AC$, and A is nonsingular, then $B = C$. $\boxed{\textbf{T}}$ or **F**

Honor Pledge: I have neither given nor received aid on this exam. Signature: ______________________