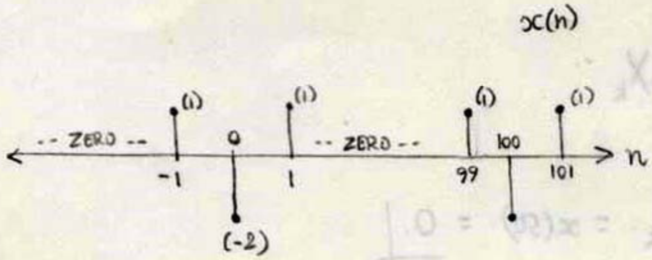
①



$p = 100$

$\omega_0 = \frac{2\pi}{100} = \frac{\pi}{50}$.

1. $x(n) = \sum_{k=\langle p\rangle} X_k e^{ik\omega_0 n}$ DFS Synthesis Equation.

Notice that we only need p DFS coefficients.

x has 100 DFS coefficients

$$X_k = \frac{1}{p} \sum_{n=\langle p\rangle} x(n) e^{-ik\omega_0 n} \quad \text{DFS Analysis Equation.}$$

$$= \frac{1}{100} \sum_{n=-1}^{98} x(n) e^{-ik\omega_0 n}$$

$$= \frac{1}{100} \left[x(-1) e^{-ik\omega_0(-1)} + x(0) e^{-ik\omega_0(0)} + x(1) e^{-ik\omega_0(1)} \right]$$

$$= \frac{1}{100} \left[e^{+ik\omega_0} - 2 + e^{-ik\omega_0} \right].$$

$$= \frac{1}{100} \left[2\cos(k\omega_0) - 2 \right].$$

2. $\sum_{k=\langle p\rangle} X_k = \sum_{k=\langle p\rangle} \frac{1}{100} \left[2\cos(k\omega_0) - 2 \right].$ We could use formulae like $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$.

But, there is an easier way!

$x(n) = \sum_{k=\langle p\rangle} X_k e^{ik\omega_0 n}$ ← We have no control over k, e, ω_0, but we can set n to be anything.

We want $\sum_{k=\langle p\rangle} X_k = \sum_{k=\langle p\rangle} X_k (1)^k = \sum_{k=\langle p\rangle} \left[e^{i\omega_0(0)} \right]^k$.

Set $n = 0$!

So, $x(0) = \sum_{k=\langle p\rangle} X_k e^{ik\omega_0(0)} = \sum_{k=\langle p\rangle} X_k$. But, $x(0) = -2$. So, $\sum_{k=\langle p\rangle} X_k = -2$.

<u>Cool aside:</u> $\sum_{k=\langle p\rangle} \frac{1}{100} \left[2\cos(k\omega_0) - 2 \right] = \frac{1}{100} \sum_{k=\langle p\rangle} 2\cos(k\omega_0) - \frac{1}{100} \sum_{k=\langle p\rangle} 2$

$$= \frac{1}{50} \sum_{k=\langle p\rangle} \cos(k\omega_0) - \frac{1}{100} \cdot 2 \cdot 100 = \frac{1}{50} \sum_{k=\langle p\rangle} \cos(k\omega_0) - 2$$

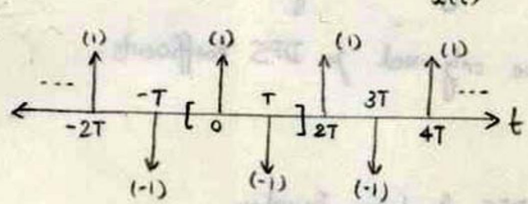
But, $\sum_{k=\langle p\rangle} \frac{1}{100} \left[2\cos(k\omega_0) - 2 \right] = -2$. So, $\frac{1}{50} \sum_{k=\langle p\rangle} \cos(k\omega_0) = 0 \Rightarrow \sum_{k=\langle p\rangle} \cos(k\omega_0) = 0.$

Similarly, $\sum_{k=\langle p\rangle} (-1)^k X_k = \sum_{k=\langle p\rangle} (e^{i\pi})^k X_k$

$$= \sum_{k=\langle p\rangle} \left(e^{i\pi/50 \cdot 50}\right)^k X_k$$

$$= \sum_{k=\langle p\rangle} \left(e^{i\omega_0 50}\right)^k X_k = x(50) = \underline{0.}$$

② Sketch the plot of the signal x:



We see (and can easily prove) that the period $p = 2T$ and so $\omega_0 = \frac{2\pi}{2T} = \frac{\pi}{T}$.

$$X_k = \frac{1}{2T}\int_{\langle 2T\rangle} x(t)\, e^{-ik\omega_0 t}\, dt$$

$$= \frac{1}{2T}\int_{-T/2}^{3T/2} x(t)\, e^{-ik\omega_0 t}\, dt.$$

$$= \frac{1}{2T}\int_{-T/2}^{3T/2} [\delta(t) - \delta(t-T)]\, e^{-ik\omega_0 t}\, dt.$$

$$= \frac{1}{2T}\left[\int_{-T/2}^{3T/2} \delta(t)\, e^{-ik\omega_0 t}\, dt\right] - \frac{1}{2T}\int_{-T/2}^{3T/2} \delta(t-T)\, e^{-ik\omega_0 t}\, dt.$$

Recall the sifting property of the Dirac delta: $\int_I \delta(t-T) f(t)\, dt = f(T)$, where I is an interval that contains T.

Then, $X_k = \frac{1}{2T}\left[e^{-ik\omega_0(0)} - e^{-ik\omega_0 T}\right].$

$\omega_0 = \frac{\pi}{T} \Rightarrow \omega_0 T = \pi.$

So, $X_k = \frac{1}{2T}[1 - e^{-ik\pi}] = \frac{1}{2T}[1-(-1)^k] = \begin{cases} \frac{1}{2T}[1-1] = 0, & k \text{ is even.} \\ \frac{1}{2T}[1-(-1)] = \frac{1}{T}, & k \text{ is odd.} \end{cases}$

③

1. We know

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-i\omega n}, \quad \hat{X}(\omega) = \sum_{n=-\infty}^{\infty} \hat{x}(n)\, e^{-i\omega n}.$$

But, $\hat{x}(n) = x(n-N)$.

$$\Rightarrow \hat{X}(\omega) = \sum_{n=-\infty}^{\infty} \underbrace{x(n-N)}_{\text{Change variables!}}\, e^{-i\omega n}.$$

Let $m = n-N$. When $n=-\infty$, $m=-\infty$; when $n=\infty$, $m=\infty$.

$$\text{So, } \hat{X}(\omega) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-i\omega(m+N)}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, e^{-i\omega m} \cdot e^{-i\omega N}$$

Does not depend on m.

$$= e^{-i\omega N} \underbrace{\sum_{m=-\infty}^{\infty} x(m)\, e^{-i\omega m}}_{X(\omega)}$$

$$\Rightarrow \hat{X}(\omega) = X(\omega)\, e^{-i\omega N}$$

2. We know

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt, \quad \hat{X}(\omega) = \int_{-\infty}^{\infty} \hat{x}(t)\, e^{-i\omega t}\, dt.$$

But, $\hat{x}(t) = \frac{d}{dt} x(t)$.

$$\Rightarrow \hat{X}(\omega) = \int_{-\infty}^{\infty} \overbrace{\frac{d}{dt}(x(t))}^{dv}\, \overbrace{e^{-i\omega t}}^{u}\, dt, \quad \text{Can integrate by parts, but...}$$

$$= \underbrace{e^{-i\omega t}}_{u} \underbrace{\int_{-\infty}^{\infty} \frac{d}{dt}(x(t))\, dt}_{v} + \int_{-\infty}^{\infty} \Big[\underbrace{\int_{-\infty}^{\infty} \frac{d}{dt}(x(t))\, dt}_{v} \cdot \underbrace{-i\omega\, e^{-i\omega t}}_{du} \Big]\, dt.$$

$$= e^{-i\omega t}\left[x(\infty) - x(-\infty)\right] + \dots$$

This gets far too complicated! ‼

Try the other equation!

We know

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}\,d\omega.$$

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}(\omega)\,e^{i\omega t}\,d\omega$$

Now, $\dot{x}(t) = \dfrac{d}{dt}\left[\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega\right].$

$$= \frac{1}{2\pi}\left[\underbrace{\frac{d}{dt}\int_{-\infty}^{\infty}}_{\text{Switch}} X(\omega)\,e^{i\omega t}\,d\omega\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\frac{d}{dt}\left[X(\omega)e^{i\omega t}d\omega\right]\right].$$

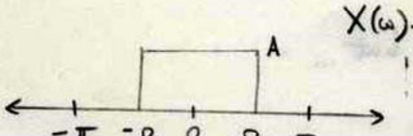
$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} X(\omega)\,d\omega\;\frac{d}{dt}\left(e^{i\omega t}\right)\right].$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} i\omega X(\omega)\,d\omega\right].$$

But, $\hat{x}(t) = \dot{x}(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\hat{X}(\omega)\,d\omega.$

Comparing, we see that $\hat{X}(\omega) = i\omega X(\omega).$

3.



$$x(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)\,e^{i\omega n}\,d\omega.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} A e^{i\omega n}\,d\omega = \frac{A}{2\pi}\int_{-\Omega}^{\Omega} e^{i\omega n}\,d\omega.$$

$$= \frac{A}{2\pi}\cdot\left[\frac{e^{i\omega n}}{in}\right]_{-\Omega}^{\Omega}$$

$$= \frac{A}{\pi n}\,\frac{\left[e^{i(\Omega)n} - e^{-i(\Omega)n}\right]}{2i}.$$

$$= A\cdot\frac{\sin(\Omega n)}{\pi n}$$

4. We know that

$$x(n) = A \cdot \frac{\sin(\Omega n)}{\pi n} \xleftrightarrow{\mathcal{F}} X(\omega) = \begin{cases} A, & |\omega| < \Omega. \\ 0, & \text{otherwise.} \end{cases}$$

So, $x_1(n) = \dfrac{\sin(\frac{\pi}{3} n)}{\pi n} \xleftrightarrow{\mathcal{F}} X_1(\omega) = \begin{cases} 1, & |\omega| < \pi/3. \\ 0, & \text{otherwise.} \end{cases}$

Now, $x_2(n) = \dfrac{\sin(\frac{\pi}{4}(n-2))}{\pi (n-2)}$ looks like a time-shifted version of $\dfrac{\sin(\frac{\pi}{4} n)}{\pi n}$.

Let $x_3(n) = \dfrac{\sin(\frac{\pi}{4} n)}{\pi n} \xleftrightarrow{\mathcal{F}} X_3(\omega) = \begin{cases} 1, & |\omega| < \pi/4. \\ 0, & \text{otherwise} \end{cases}$

Now, $x_2(n) = x_3(n-2)$. So, $X_2(\omega) = X_3(\omega) e^{-i2\omega}$. (From part 1)

$$= \begin{cases} e^{-i2\omega}, & |\omega| < \pi/4. \\ 0, & \text{otherwise} \end{cases}$$

Again, $(x_1 \star x_2) \xleftrightarrow{\mathcal{F}} X_1 \cdot X_2$.

Notice that $X_1 \cdot X_2 = X_2$! And, $X_2 \xleftrightarrow{\mathcal{F}^{-1}} x_2$.

So, $(x_1 \star x_2) \xleftrightarrow{\mathcal{F}} X_1 \cdot X_2 = X_2 \xleftrightarrow{\mathcal{F}^{-1}} x_2$.

$(x_1 \star x_2)(n) = \dfrac{\sin(\frac{\pi}{4}(n-2))}{\pi n}$ ... Woah..