Q1. Consider a system whose input-output relationship is given by $y(n) = x(n-2) + x(2-n)$. (Oppenheim 2nd ed. chapter 1, 1.27(a))

(a) Is the system linear?

A. Let $x_1(n)$ and $x_2(n)$ be two input signals, and let $y_1(n)$ and $y_2(n)$ be the corresponding output signals.
By definition then, $y_1(n) = x_1(n-2) + x_1(2-n)$ and $y_2(n) = x_2(n-2) + x_2(2-n)$.
We define a new signal $\hat{x}(n) = \alpha x_1(n) + \beta x_2(n)$.
The corresponding output signal is $\hat{y}(n)$. By definition, $\hat{y}(n) = \hat{x}(n-2) + \hat{x}(2-n)$.
If the system is linear, then $\hat{y}(n) = \alpha y_1(n) + \beta y_2(n)$. ← We have to check this!
Now,
$$\hat{y}(n) = \hat{x}(n-2) + \hat{x}(2-n) = [\alpha x_1(n-2) + \beta x_2(n-2)] + [\alpha x_1(2-n) + \beta x_2(2-n)]$$
$$= \alpha[x_1(n-2) + x_1(2-n)] + \beta[x_2(n-2) + x_2(2-n)]$$
$$= \alpha y_1(n) + \beta y_2(n),$$ as required ⇒ System is linear. ☒

(b) Is the system time-invariant?

A. Define a new signal $\hat{x}(n) = x(n-N)$ $(n \in \mathbb{Z})$
The corresponding output signal is $\hat{y}(n)$. By definition, $\hat{y}(n) = \hat{x}(n-2) + \hat{x}(2-n)$.
If the system is time-invariant, then $\hat{y}(n) = y(n-N)$. ← We have to check this!
Now, $\hat{y}(n) = \hat{x}(n-2) + \hat{x}(2-n) = x((n-2)-N) + x((2-n)-N) = x(n-N-2) + x(2-n-N)$.
But, $y(n-N) = x((n-N)-2) + x(2-(n-N)) = x(n-N-2) + x(2-n+N) \neq \hat{y}(n)$.
⇒ System is not time-invariant. ☒

(c) Is the system causal?

A. Notice that $y(0) = x(-2) + x(2)$, which depends on a value of the input in the future.
⇒ System is not causal. ☒

(d) Is the system memoryless?

A. Method #1: There is no function f such that $y(n) = f(x(n))$ ⇒ System is not memoryless. ☒

Method #2: If a system is memoryless, it must be causal, or $M \Rightarrow C$. (Why?).
This is logically equivalent to the contraposition $\neg C \Rightarrow \neg M$.
So, since the system is not causal
it must not be memoryless. ☒

Method #3: If a system is memoryless, it must be time-invariant, or $M \Rightarrow TI$ (Why?)
This is logically equivalent to the contraposition $\neg TI \Rightarrow \neg M$.
So, since the system is not TI,
it must not be memoryless. ☒

Note: Remember that $y(t) = \underset{\text{system}}{F}(x)(t)$ is always true, but $y(t) = f(x(t))$ is only true for memoryless systems.

(e) Is the System BIBO-stable?

A. Let $x(n)$ be an arbitrary input signal that is <u>bounded</u>: in other words, $|x(n)| \le B_x \ \forall n,\ B_x \in \mathbb{R}$.
We have to show that the corresponding output signal is also bounded: $|y(n)| \le B_y \ \forall n$, for some $B_y \in \mathbb{R}$.

$$|y(n)| = |x(n-2) + x(2-n)| \underset{\substack{\uparrow \\ \text{Triangle inequality}}}{\le} |x(n-2)| + |x(2-n)| \le B_x + B_x = 2B_x \underset{\substack{\uparrow \\ \text{"Defined"}}}{\triangleq} B_y, \text{ as required.}$$

$\Rightarrow$ System is BIBO-stable. ⊠

Q2. Consider a system whose input-output relationship is given by $y(t) = [x(t)]^{x(t)}$.

(a) Is the system memoryless?

A. Define the function f as $f(v) = v^v$. Then, $y(t) = f(x(t)) \Rightarrow$ System is memoryless. ⊠

(b) Is the system causal?

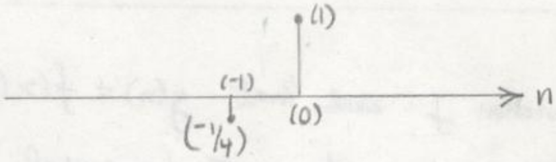
A. The system is memoryless $\Rightarrow$ System is causal. ⊠

Q3. Consider a (LTI) system F whose frequency response is given by $F(\omega) = 1 - \frac{1}{4} e^{i\omega}$.

(a) Determine and plot the impulse response of the system, $f(n)$.

A. We know that $F(\omega) = \sum_{n=-\infty}^{\infty} f(n) e^{-i\omega n} = \dots + f(-2)e^{i2\omega} + f(-1)e^{i\omega} + f(0) + f(1)e^{-i\omega} + f(2)e^{-i2\omega} + \dots$

But, $F(\omega) = 1 - \frac{1}{4}e^{i\omega}$.
Comparing both sides, we find that $f(0) = 1,\ f(-1) = -\frac{1}{4},\ f(n) = 0 \ \forall n \ne 0, -1$.



(b) Is the system causal?

A. Recall that, <u>for LTI systems</u>, a system is causal $\Longleftrightarrow f(n) = 0$ for $n < 0$.
But, $f(-1) \ne 0 \Rightarrow$ System is <u>not</u> causal. ⊠

(c) Is the system memoryless?

A. The system is not causal $\Rightarrow$ The system is not memoryless. ⊠

(d) Is the system BIBO-stable?

A. Recall that <u>for LTI systems</u>, a system is BIBO-stable $\Longleftrightarrow \sum_{n=-\infty}^{\infty} |f(n)| < \infty$.
Here, $\sum_{n=-\infty}^{\infty} |f(n)| = |f(0)| + |f(-1)| = 1 + 1/4 = 5/4 < \infty \Rightarrow$ The system is BIBO-stable. ⊠

Q4. Let the impulse responses of two systems G and H — $g(n)$ and $h(n)$, respectively — be related as $h(n) = g(n-1)$. Determine a relationship between their frequency responses $G(\omega)$ and $H(\omega)$.

A. We know,

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n} = \sum_{n=-\infty}^{\infty} g(n-1) e^{-i\omega n}$$ ← This almost looks like the expression for $G(\omega)$

Set $m = n-1$.

Then, $$H(\omega) = \sum_{m=-\infty}^{\infty} g(m) e^{-i\omega(m+1)} = e^{-i\omega} \sum_{m=-\infty}^{\infty} g(m) e^{-i\omega m} = e^{-i\omega} G(\omega).$$

↑ Mind the limits.

Q5. Consider a system H whose input-output relationship is given by

$$y(n) - \alpha y(n-1) = x(n) - \beta x(n-1), \quad -1 < \alpha < 0,\ 0 < \beta < 1.$$

(a) Determine an expression for the frequency response $H(\omega)$.

A. If $x(n) = e^{i\omega n}$, then $y(n) = H(\omega) e^{i\omega n}$.

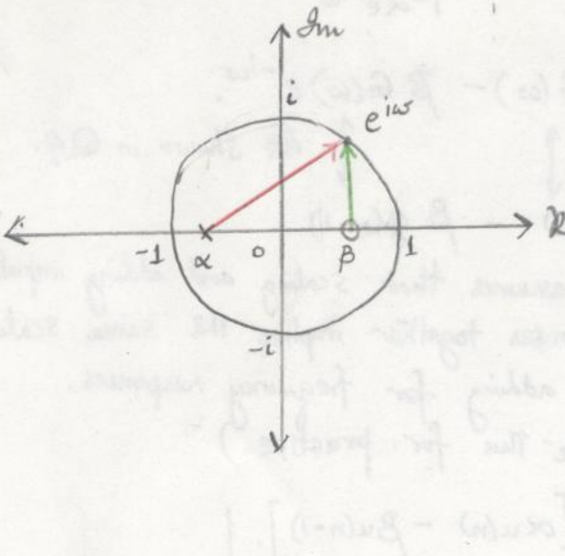
So, $$\underbrace{H(\omega) e^{i\omega n}}_{y(n)} - \alpha \underbrace{H(\omega) e^{i\omega(n-1)}}_{y(n-1)} = \underbrace{e^{i\omega n}}_{x(n)} - \beta \underbrace{e^{i\omega(n-1)}}_{x(n-1)}.$$

$$\Rightarrow H(\omega) \cdot e^{i\omega n} [1 - \alpha e^{-i\omega}] = e^{i\omega n} \cdot [1 - \beta e^{-i\omega}].$$

$$\Rightarrow H(\omega) = \frac{1 - \beta e^{-i\omega}}{1 - \alpha e^{-i\omega}} \qquad (e^{i\omega n} \neq 0\ \forall n).$$

(b) Plot the magnitude of the frequency response $|H(\omega)|$ and the phase of the frequency response $\measuredangle H(\omega)$.

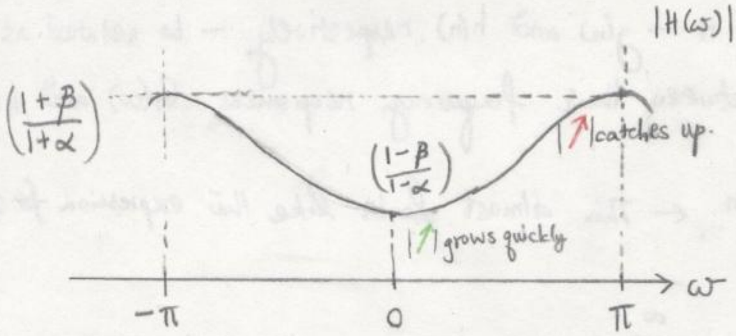
A. The frequency response for DT-LTI systems (<u>not</u> CT-LTI systems) is 2π-periodic, so we only need to plot it for any interval of size 2π. We choose $-\pi \le \omega < \pi$.

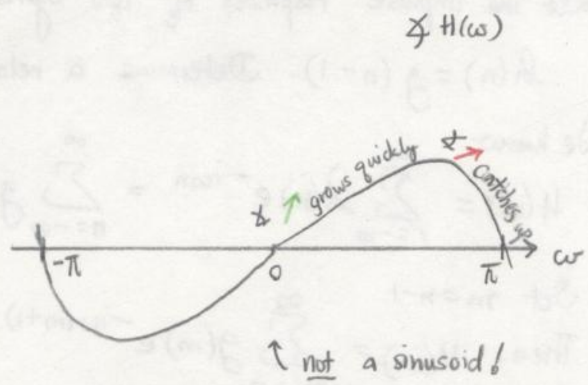


$$|H(\omega)| = \frac{|1 - \beta e^{-i\omega}|}{|1 - \alpha e^{-i\omega}|} = \frac{|e^{-i\omega}(e^{i\omega} - \beta)|}{|e^{-i\omega}(e^{i\omega} - \alpha)|} = \frac{|e^{i\omega} - \beta|}{|e^{i\omega} - \alpha|} = \frac{|\nearrow|}{|\nearrow|}$$

This ratio is minimized when $\omega = 0$: $|\nearrow|$ is its smallest, $|\nearrow|$ is its largest. When $\omega = \pm\pi$, $|\nearrow|$ is its largest, $|\nearrow|$ is its smallest, and the ratio is maximized.

Similarly, $\measuredangle H(\omega) = \measuredangle \nearrow - \measuredangle \nearrow$.





(c) Find the impulse response $h(n)$ of the system, assuming that it is causal.

A. Non-slick way: Since the system is causal, $h(n)=0$ for $n<0$.

Now, $h(0)+\alpha h(-1) = \delta(0) - \beta\delta(-1)$

$\Rightarrow h(0) - \alpha\cdot(0) = 1 - \beta\cdot(0)$

$\Rightarrow h(0) = 1.$

Also, $h(1) - \alpha h(0) = \delta(1) - \beta\delta(0).$

$\Rightarrow h(1) - \alpha\cdot 1 = 0 - \beta\cdot 1$

$\Rightarrow h(1) = \alpha - \beta.$

Also, $h(2) - \alpha h(1) = \delta(2) - \beta\delta(1).$

$\Rightarrow h(2) = \alpha h(1) = \alpha(\alpha-\beta).$

Similarly, $h(3) = \alpha h(2) = \alpha^2(\alpha-\beta).$

$h(4) = \alpha h(3) = \alpha^3(\alpha-\beta).$

Notice that the β term only "kicks in" after $n=1$.

So, a suggested closed form is $h(n) = \alpha^{n-1}(\alpha u(n) - \beta u(n-1)).$

Check: $h(0) = \alpha^{-1}(\alpha u(0) - \beta u(-1)) = 1,$

$h(1) = \alpha^0(\alpha u(1) - \beta u(0)) = \alpha - \beta,$

$h(2) = \alpha^1(\alpha u(0) - \beta u(1)) = \alpha(\alpha-\beta)$, and so on.

Slick way: We know that $g(n) = \alpha^n u(n) \longleftrightarrow G(\omega) = \dfrac{1}{1-\alpha e^{-i\omega}}.$

Notice that $H(\omega) = \dfrac{1}{1-\alpha e^{-i\omega}} - \dfrac{\beta e^{-i\omega}}{1-\alpha e^{-i\omega}} = G(\omega) - \beta G(\omega)e^{-i\omega}.$

As shown in Q4.

So, $h(n) = g(n) - \beta g(n-1)$

(This assumes that scaling and adding impulse responses together implies the same scaling and adding for frequency responses. Prove this for practice.)

Thus, $h(n) = \alpha^n u(n) - \beta[\alpha^{n-1}u(n-1)] = \alpha^{n-1}[\alpha u(n) - \beta u(n-1)].$

(d) Find the quantity $\sum_{n=-\infty}^{\infty} h(n)$.

A.

Non-slick way: $\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \alpha^{n-1}[\alpha u(n) - \beta u(n-1)] = \sum_{n=-\infty}^{\infty}[\alpha^{n-1}\cdot\alpha u(n)] - \sum_{n=-\infty}^{\infty}[\alpha^{n-1}\beta u(n-1)]$.

$$= \sum_{n=0}^{\infty} \alpha^{n-1}\alpha - \sum_{n=1}^{\infty}\alpha^{n-1}\beta = \alpha\sum_{n=0}^{\infty}\alpha^{n-1} - \beta\sum_{n=1}^{\infty}\alpha^{n-1}$$

$$= \frac{\alpha}{\alpha}\sum_{n=0}^{\infty}\alpha^n - \beta\left[\sum_{m=0}^{\infty}\alpha^m\right]$$

$$= \frac{1}{1-\alpha}\cdot\frac{\alpha}{\alpha} - \beta\cdot\frac{1}{1-\alpha} = \frac{1-\beta}{1-\alpha}$$

Slick way: $H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n}$

Do not want this term

Set $\omega = 0$! $H(0) = \sum_{n=-\infty}^{\infty} h(n) e^{-i0n} = \sum_{n=-\infty}^{\infty} h(n)$.

But, $H(0) = \dfrac{1-\beta e^{-i0}}{1-\alpha e^{-i0}} = \dfrac{1-\beta}{1-\alpha}$

(e) Find the quantity $\sum_{n=-\infty}^{\infty}(-1)^n h(n)$.

A. Set $\omega = \pm\pi$. Then, $H(\omega)\Big|_{\omega=\pm\pi} = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n}\Big|_{\omega=\pm\pi} = \sum_{n=-\infty}^{\infty} h(n) e^{-i(\pm\pi)n} = \sum_{n=-\infty}^{\infty} h(n)(-1)^n$.

But, $H(\pm\pi) = \dfrac{1-\beta e^{-i(\pm\pi)}}{1-\alpha e^{-i(\pm\pi)}} = \dfrac{1-\beta(-1)}{1-\alpha(-1)} = \dfrac{1+\beta}{1+\alpha}$

Q6. Feedback question (Q1 from Fall 2007 Midterm 3).

Q7(a) Find the convolution of the two signals $x_1(n) = \alpha^n u(n)$, $x_2(n) = \beta^n u(n)$ $(\alpha \neq \beta)$.

A. $(x_1 * x_2)(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) = \sum_{k=-\infty}^{\infty}[\alpha^k u(k) \cdot \beta^{n-k} u(n-k)]$

$$= \sum_{k=0}^{\infty} \alpha^k \beta^{n-k} u(n-k) \qquad (k \geq 0, \text{ since } u(k) = 1 \text{ for } k \geq 0)$$

$$= \sum_{k=0}^{n} \alpha^k \beta^{n-k} \text{ for } n \geq k \geq 0 \qquad (u(n-k) = 1 \text{ for } n \geq k).$$

Another way of writing this (without the annotation "for $n \geq k \geq 0$") is to say that

$$(x_1 * x_2)(n) = \left[\sum_{k=0}^{n} \alpha^k \beta^{n-k}\right] u(n) = \left[\sum_{k=0}^{n} (\alpha/\beta)^k\right]\beta^n u(n).$$

(b) Use your result from part (a) to prove that, when $\alpha \neq \beta$, for some $n \geq 0$,

$$\sum_{k=0}^{\infty} \alpha^k \beta^{n-k} = \sum_{m=0}^{\infty} \alpha^{n-k} \beta^k.$$

A. Convolution is commutative! So,

$$(x_1 \star x_2)(n) = \left[\sum_{k=0}^{\infty} \alpha^k \beta^{n-k}\right] u(n) = (x_2 \star x_1)(n)$$

$$= \sum_{k=-\infty}^{\infty} x_2(k)\, x_1(n-k).$$

$$= \sum_{k=-\infty}^{\infty} \beta^k u(k) \cdot \alpha^{n-k} u(n-k).$$

$$= \sum_{k=0}^{\infty} \beta^k \alpha^{n-k} u(n-k)$$

$$= \left[\sum_{k=0}^{n} \beta^k \alpha^{n-k}\right] u(n).$$

Comparing signals, we see that, for $n \geq 0$,

$$\sum_{k=0}^{\infty} \alpha^k \beta^{n-k} = \sum_{k=0}^{\infty} \beta^k \alpha^{n-k}, \text{ as required. } \boxtimes$$

Q8. Consider an LTI system H and an input signal $x(t) = 2e^{-3t} u(t-1)$. (Oppenheim 2nd ed., Chapter 2, 2.46).
If the output signal that corresponds to $x(t)$ is $y(t)$, and to $\frac{dx(t)}{dt}$ is $-3y(t) + e^{-2t} u(t)$, find the impulse response $h(t)$.

A. We know that $x(t) = 2e^{-3t} u(t-1) \rightarrow \boxed{H} \rightarrow y(t)$, and

$$\frac{dx(t)}{dt} \rightarrow \boxed{H} \rightarrow -3y(t) + e^{-2t} u(t).$$

$$\text{But, } \frac{dx(t)}{dt} = \frac{d}{dt}\left[2e^{-3t} u(t-1)\right] = 2(-3) e^{-3t} u(t-1) + 2\,\delta(t-1)\, e^{-3t}$$

$$= -6e^{-3t} u(t-1) + 2e^{-3}\, \delta(t-1).$$

SAMPLING PROPERTY

$$= -3x(t) + 2e^{-3}\, \delta(t-1).$$

Since the system is linear, we have

$$\left[\frac{dx(t)}{dt}\right] + 3x(t) \rightarrow \boxed{H} \rightarrow \left[-3y(t) + e^{-2t} u(t)\right] + 3y(t).$$

OR $$2e^{-3}\,\delta(t-1) \rightarrow \boxed{H} \rightarrow e^{-2t} u(t).$$

OR $$\delta(t-1) \rightarrow \boxed{H} \rightarrow \frac{e^3}{2}\, e^{-2t} u(t) \quad \text{(LINEARITY)}.$$

OR $$\delta(t) \rightarrow \frac{e^3}{2} \cdot e^{-2(t+1)} u(t+1) = \frac{e}{2} \cdot e^{-2t} u(t+1).$$