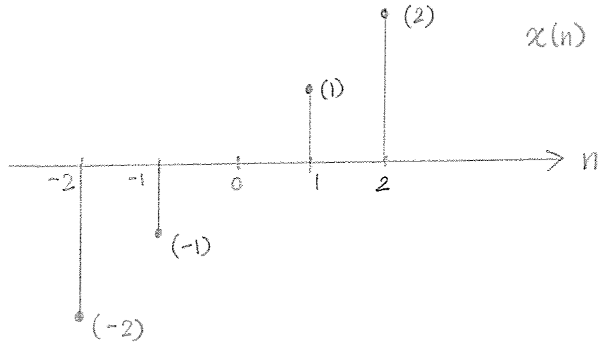
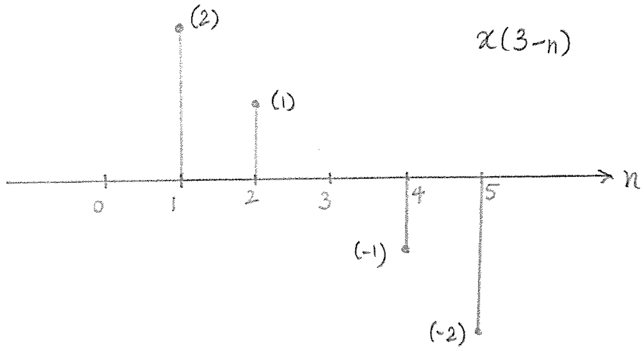
SIGNALS

1.
$$x(n) = \sum_{k=-2}^{2} k\,\delta(n-k) = -2\delta(n+2) + (-\delta(n+1)) + 0\delta(n) + \delta(n-1) + 2\delta(n-2).$$
$$= -2\delta(n+2) - \delta(n+1) + \delta(n-1) + 2\delta(n-2).$$

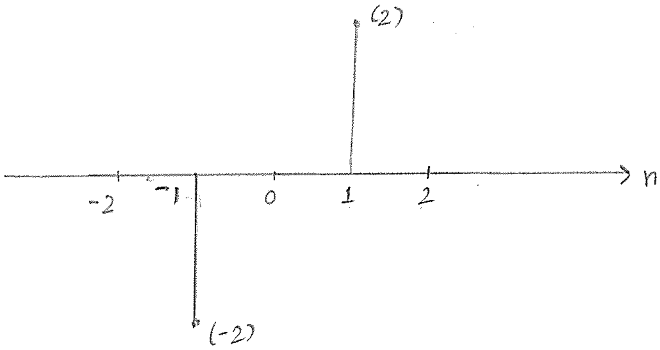
(a) $x(n)$



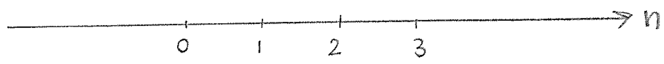
(b) $x(3-n)$



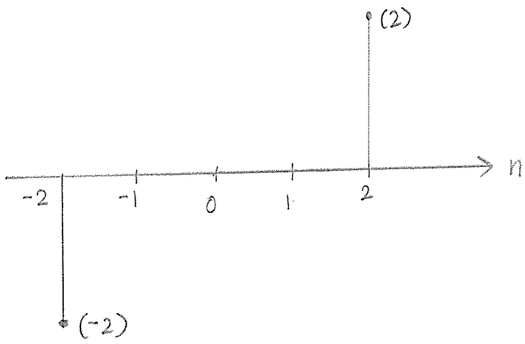
(c) $x(2n)$



(d) $x(n-2)\,\delta(n-2)$



(e) $\frac{1}{2}x(n) + \frac{1}{2}(-1)^n x(n) = \begin{cases} x(n), & n \text{ is even.} \\ 0, & n \text{ is odd.} \end{cases}$



DIRAC DELTA

Review: The Kronecker delta function and Dirac delta function have very similar properties.

KRONECKER DELTA (DT)	DIRAC DELTA (CT)
① $\sum_{n=-\infty}^{\infty} \delta(n) = 1$	① $\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$
② $\underbrace{\underbrace{x(n)}_{\text{FUNCTION}}\;\underbrace{\delta(n-N)}_{\text{FUNCTION}}}_{\text{FUNCTION}} = \underbrace{\underbrace{x(N)}_{\text{CONSTANT}}\;\underbrace{\delta(n-N)}_{\text{FUNCTION}}}_{\text{FUNCTION}}$	② $\underbrace{\underbrace{x(t)}_{\text{FUNCTION}}\;\underbrace{\delta(t-T)}_{\text{FUNCTION}}}_{\text{FUNCTION}} = \underbrace{\underbrace{x(T)}_{\text{CONSTANT}}\;\underbrace{\delta(t-T)}_{\text{FUNCTION}}}_{\text{FUNCTION}}$ "SAMPLING PROPERTY": "Samples" the function at one specific point in time.
③ $\sum_{n=-\infty}^{\infty} x(n)\,\delta(n-N) = \sum_{n=-\infty}^{\infty} x(N)\,\delta(n-N) = x(N).$	③ $\int_{-\infty}^{\infty} x(t)\,\delta(t-T)\,dt = \int_{-\infty}^{\infty} x(T)\,\delta(t-T)\,dt = x(T)$ "SIFTING PROPERTY": "Sifts out" the value of the function at a specific point in time.

1

(a) $$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\delta(\tau)f(t-\tau)\,d\tau\right]\delta(t)\,dt$$

Only "active" when $\tau = 0$.

$$= \int_{-\infty}^{\infty} f(t-0)\ \delta(t)\,dt$$

Only "active" when $t = 0$.

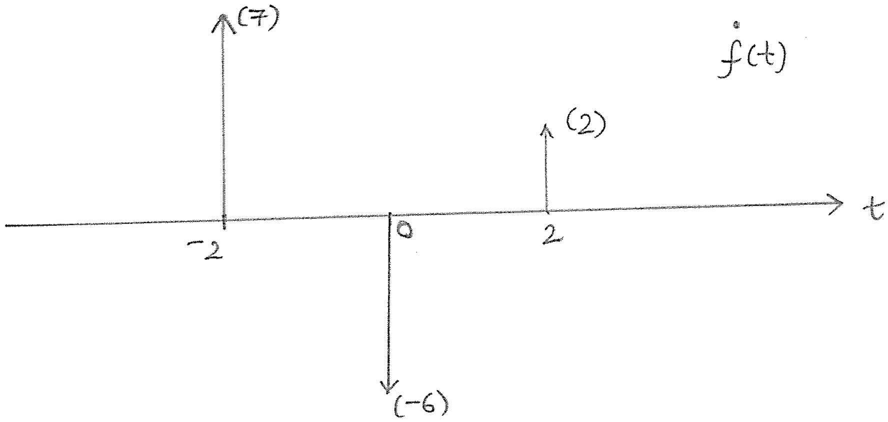
$$= \boxed{f(0)}$$

(b) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(\tau)\,\delta(t)\,d\tau\,dt.$$

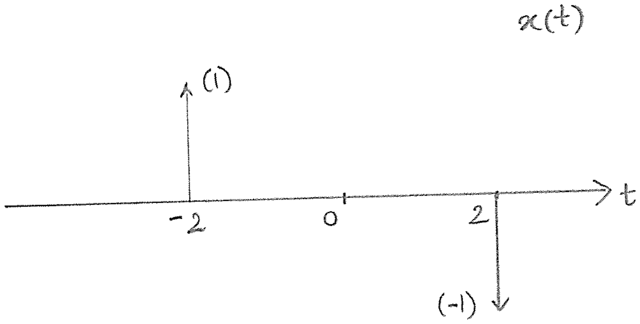
$$= \int_{-\infty}^{\infty}\delta(t)\left[\int_{-\infty}^{\infty}\delta(\tau)\,d\tau\right]dt.$$

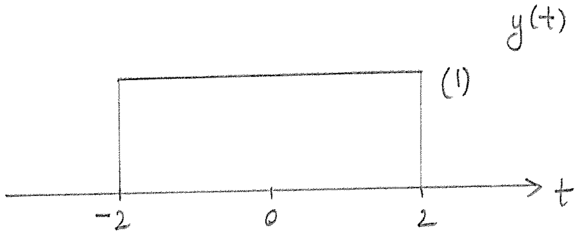
$$= \int_{-\infty}^{\infty}\delta(t)\cdot 1\,dt. \quad = \boxed{1.}$$

2



3



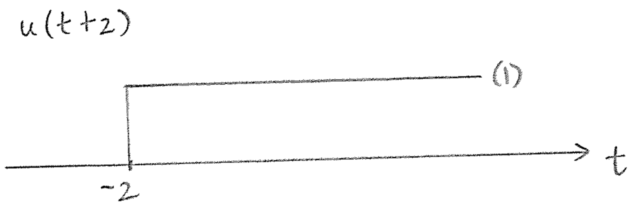


Alternatively, remember that

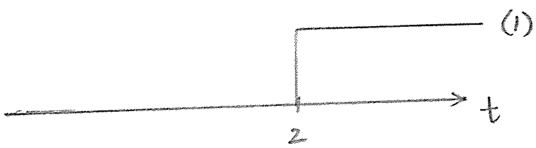
$\dfrac{du(t)}{dt} = \delta(t)$, or equivalently $\quad u(t) = \displaystyle\int_{-\infty}^{t}\delta(\tau)\,d\tau.$

So, $\quad x(t) = \delta(t+2) - \delta(t-2)$

$$\Rightarrow y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{\infty}\left[\delta(\tau+2) - \delta(\tau-2)\right]d\tau = u(t+2) - u(t-2).$$



MINUS



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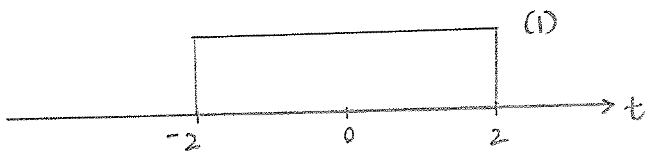

Review: Finding roots of a complex number.

Let $z = Re^{i\theta}$. It has n distinct n^{th} roots.

Also, $z = Re^{i\theta} = Re^{i(\theta + 2\pi k)}$, $k \in \mathbb{Z}$.

Then, $z^{1/n} = R^{1/n} e^{i\frac{(\theta + 2\pi k)}{n}}$
$= R^{1/n} e^{i(\theta/n + 2\pi k/n)}$

To get the n roots, we set $k = 0, 1, 2, \ldots, n-1$. When $k=n$, we get the same root as when $k=0$.
The roots are located on a circle of radius $R^{1/n}$, separated by an angle of $2\pi/n$.

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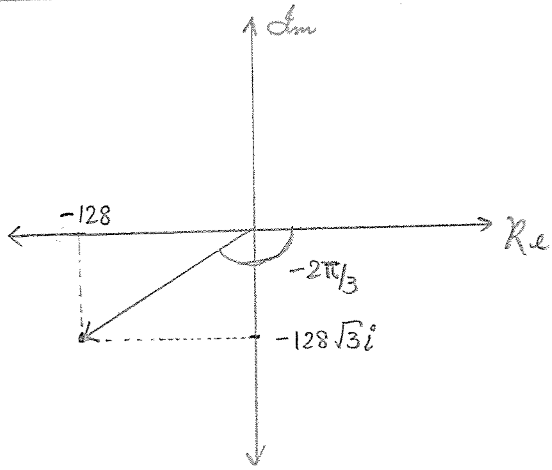
① $(-1 + i\sqrt{3})^8$

$= \left[\sqrt{(-1)^2 + (\sqrt{3})^2}\; e^{i\tan^{-1}(-\sqrt{3})}\right]^8$

$= \left[2e^{i2\pi/3}\right]^8 = 256\, e^{i16\pi/3}$

$= 256\, e^{-i2\pi/3} = 256\left[\cos(-2\pi/3) + i\sin(-2\pi/3)\right]$

$= 256\left[-1/2 - i\sqrt{3}/2\right] = \boxed{-128 - i\,128\sqrt{3}}$



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② $z^5 - z^3 + z = 0$

$\Rightarrow z(z^4 - z^2 + 1) = 0. \Rightarrow z = 0$ OR $z^4 - z^2 + 1 = 0$.

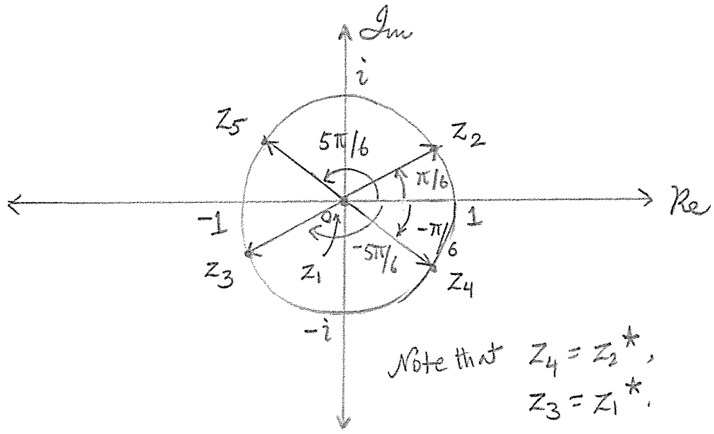
$\boxed{z_1 = 0}$ is one root.

Let $x = z^2$. Then, $z^4 - z^2 + 1 = x^2 - x + 1 = 0$.

The roots of this new polynomial are $x = \dfrac{+1 \pm i\sqrt{3}}{2} = 1/2 \pm i\sqrt{3}/2 = e^{i\pi/3}$ OR $e^{-i\pi/3}$.

When $x = e^{i\pi/3}$, $z^2 = e^{i\pi/3}$.
The two roots are given by $z_2 = (e^{i\pi/3})^{1/2}$ and $z_3 = \left[e^{i(\pi/3 + 2\pi)}\right]^{1/2}$, or
$\boxed{z_2 = e^{i\pi/6}}$ and $\boxed{z_3 = e^{-i5\pi/6}}$

When $x = e^{-i\pi/3}$, $z^2 = e^{-i\pi/3}$.
The two roots are given by $z_4 = (e^{-i\pi/3})^{1/2}$ and $z_5 = \left[e^{i(-\pi/3 + 2\pi)}\right]^{1/2}$, or
$\boxed{z_4 = e^{-i\pi/6}}$ and $\boxed{z_5 = e^{i5\pi/6}}$



Note that $z_4 = z_2^*$,
$z_3 = z_5^*$.

Sanity check: If a complex number z is a root of a polynomial, then its conjugate (z^*) is also a root of the polynomial. (Can you prove this?)

From the diagram, we see that when we add all of the roots, the real and imaginary parts cancel.

So, $\left(\sum_{k=1}^{5} z_k\right)^{e^{e^{20}}} = 0^{e^{e^{20}}} = \boxed{0}$.

③ $F(\omega) = \cos(\omega) e^{i\omega}$.

For this function, we make the plots over an interval of length 2π, such as $(-\pi, \pi]$ or $[0, 2\pi)$, since the graph will repeat beyond the interval. Why?

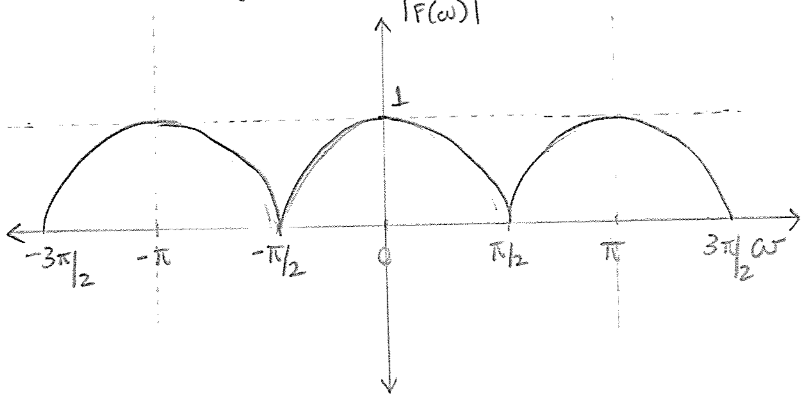
④

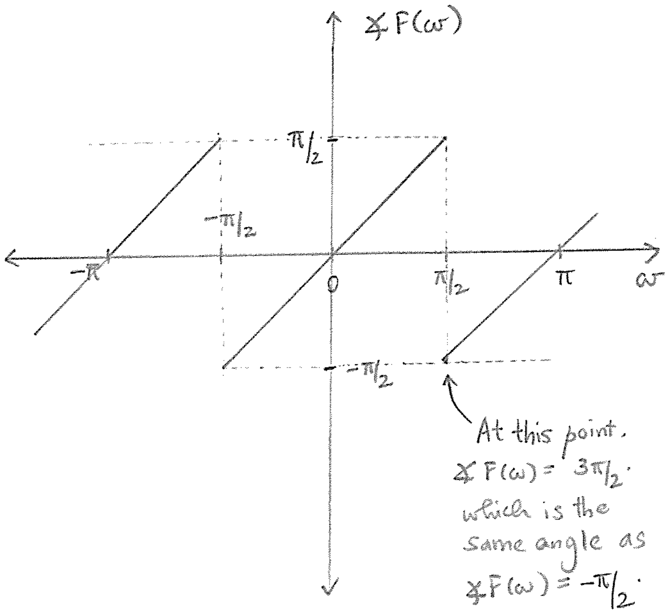
$|F(\omega)| = |\cos(\omega)||e^{i\omega}| = |\cos(\omega)| \cdot 1 = |\cos(\omega)|$.

$\measuredangle F(\omega) = \measuredangle[\cos(\omega) e^{i\omega}] = \measuredangle \cos(\omega) + \measuredangle e^{i\omega} = \measuredangle \cos(\omega) + \omega$.

Now, $\cos(\omega)$ is a positive real number for $-\pi/2 \le \omega \le \pi/2$, and a negative real number for $-\pi < \omega < -\pi/2$ and $\pi/2 < \omega \le \pi$.

So, $\measuredangle F(\omega) = \begin{cases} \omega + 0, & -\pi/2 \le \omega \le \pi/2 \\ \omega \pm \pi, & -\pi < \omega < -\pi/2,\ \pi/2 < \omega \le \pi. \end{cases}$

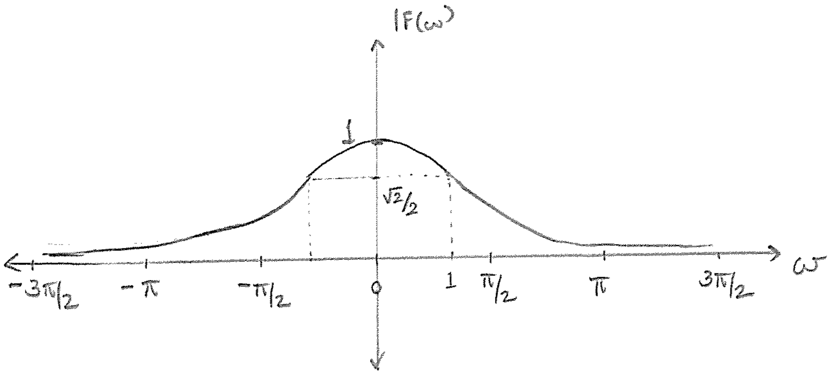




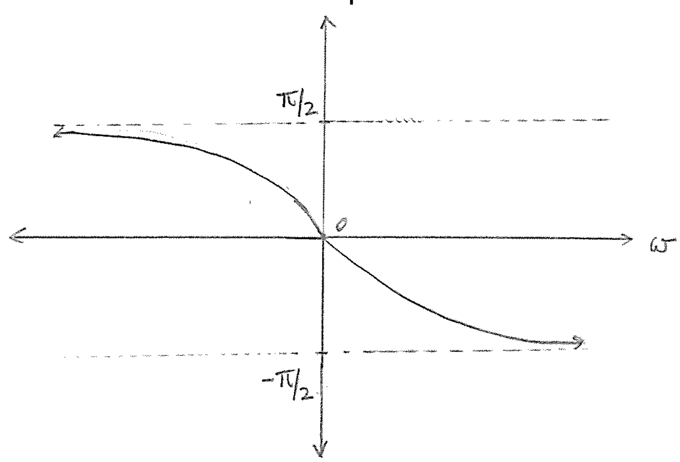
④ $G(\omega) = \dfrac{1}{1 + i\omega}$.

$|G(\omega)| = \left|\dfrac{1}{1+i\omega}\right| = \dfrac{1}{\sqrt{1+\omega^2}}$

$\measuredangle G(\omega) = \measuredangle\left(\dfrac{1}{1+i\omega}\right) = \measuredangle 1 - \measuredangle(1 + i\omega)$
$= 0 - \tan^{-1}(\omega) = -\tan^{-1}(\omega)$



Phase plot



5. $D_K = |z_K - z_0| = |e^{ik\theta} - 1|$, where $\theta = 2\pi/N$.

$= \left| e^{ik\theta/2}\left(e^{ik\theta/2} - e^{-ik\theta/2}\right)\right|$. ← Useful trick: "balance the exponents".

$= \left| e^{ik\theta/2} \cdot 2i\sin(k\theta/2)\right|$.

$= 2\,|\sin(k\theta/2)| \cdot |i| \cdot |e^{ik\theta/2}|$.

$= 2 \cdot |\sin(k\theta/2)|$.

So, $\displaystyle\prod_{k=1}^{N-1} D_K = \prod_{k=1}^{N-1}\left[2\,|\sin(k\theta/2)|\right] = 2^{N-1}\prod_{k=1}^{N-1}|\sin(k\theta/2)| = 2^{N-1}\prod_{k=1}^{N-1}|\sin(k\pi/N)|$.

Then, $\displaystyle\prod_{k=1}^{N-1} D_K = 2^{N-1}\prod_{k=1}^{N-1}|\sin(k\pi/N)| = N \Rightarrow \prod_{K=1}^{N-1}|\sin(k\pi/N)| = N/2^{N-1}$ ⊠

GOOD LUCK!