# MIDTERM 1 REVIEW (SPRING 2013)

Conducted by HKN

## 1 Complex Numbers

1. Let $z_1 = 2e^{i\frac{21\pi}{9}}$, $z_2 = 1 - \sqrt{3}i$, $z_3 = e^{i\pi}$, $z_4 = i^i$, $z_5 = i^{1/\pi}$, $z_6 = \log_i e$. Evaluate the following:
   (a) $z_1 + z_2 + z_3$.
   (b) $\frac{z_1 z_2}{z_3 z_4}$.
   (c) $z_5^2$.
   (d) $z_5^{z_6}$.
2. Simplify and plot $(-1 + i\sqrt{3})^8$ on the complex plane.
3. Prove De Moivre's Theorem: For any integer $n$, $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$.

## 2 System Properties

1. For each system given below, determine if it is linear, time-invariant, or both.
   (a) $y(n) = x(n-2) + x(2-n)$.
   (b) $y(t) = |x(t)|$.
   (c) $y(t) = x(|t|)$.
   (d) $y(t) = \cos\left(\frac{\pi}{3}\right)$.
   (e) $y(t) = \sum_{k=-\infty}^{\infty} x(t - kT)$, $T \in \mathbb{Z}$.
   (f) $y(n) = \sum_{k=-\infty}^{n} x(k)$.

## 3 Periodicity

1. Determine if the following signals are periodic. If a signal is periodic, find its fundamental period:
   (a) $x(n) = (-1)^n$.
   (b) $x(n) = \cos\left(\frac{\pi}{2}n\right)$.
   (c) $x(t) = \cos\left(\frac{2\pi}{3}t + \frac{\pi}{2}\right)$.
   (d) $x(n) = \sin\left(\frac{2\pi}{5}n\right) + e^{i\frac{3\pi}{5}n}$.
   (e) $x(t) = \sin(t)$.
   (f) $x(n) = \sin(n)$.

## 4 Fourier Series

1. (*Signals and Systems* by Oppenheim and Willsky, exercise 3.1) A continuous-time periodic signal $x(t)$ is real-valued and has a fundamental period $p = 8$ and a fundamental frequency $\omega_0$. It has the Fourier series expansion $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$. The nonzero Fourier coefficients for $x(t)$ are given by $X_1 = X_{-1} = 2, X_3 = X_{-3}^* = 4j$. Determine $\omega_0$ and express $x(t)$ in the form
$$x(t) = \sum_{k=0}^{\infty} A_k \cos(k\omega_0 t + \phi_k).$$

2. (Oppenheim and Willsky, exercise 3.3) For the continuous-time periodic signal $x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$, determine the fundamental frequency $\omega_0$ and the Fourier series coefficients $X_k$ such that $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$.

3. Let $x(n)$ be a $p$-periodic discrete-time signal with the Fourier series expansion
$$x(n) = A_0 + \sum_{k=1}^{K} \left[\alpha_k \cos(k\omega_0 n) + \beta_k \sin(k\omega_0 n)\right],$$
where $K$ is $p/2$ if $p$ is even, or $(p-1)/2$ if $p$ is odd, and $\omega_0 = 2\pi/p$. If $x(n)$ is real and even ($x(n) = x(-n)$), show that the coefficients $\beta_k$ must be zero.

4. (Time-shifting) Let $x(t)$ be a $p$-periodic continuous-time signal with the Fourier series expansion $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$. Define another signal $\hat{x}(t) = x(t-T)$, for some integer $T$. It can be proven that $\hat{x}(t)$ is also $p$-periodic (try it!), with the Fourier series expansion $\hat{x}(t) = \sum_{k=-\infty}^{\infty} \hat{X}_k e^{ik\omega_0 t}$. Relate $X_k$ and $\hat{X}_k$.

5. (Frequency-shifting) Let $x(t)$ be a $p$-periodic continuous-time signal with the Fourier series expansion $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$. Define another signal $\hat{x}(t) = x(t)e^{i\omega t}$, for some frequency $\omega$. It can be proven that $\hat{x}(t)$ is also $p$-periodic (try it!), with the Fourier series expansion $\hat{x}(t) = \sum_{k=-\infty}^{\infty} \hat{X}_k e^{ik\omega_0 t}$. Relate $X_k$ and $\hat{X}_k$.

## 5 Frequency Responses

1. Recall the system that represents the two-path wireless channel from class:
$$y(t) = a_1 x(t-\tau_1) + a_2 x(t-\tau_2).$$
   (a) What is the frequency response $H(\omega)$ of this system?
   (b) What is the output of the system that corresponds to each of the following input signals?
      i. $x(t) = 1$.
      ii. $x(t) = \cos(\pi t)$.
      iii. Any signal $x$ such that $x(t+1) = x(t)$.

2. Consider a continuous-time system $\mathsf{G_C}$ with the frequency response $G_C(\omega)$. The magnitude of the frequency response is given by
$$|G_C(\omega)| = \begin{cases} A, & -\pi/2 < \omega < \pi/2, \\ 0, & \text{otherwise.} \end{cases}$$
The phase of the frequency response is given by $\angle G_C(\omega) = -\frac{3}{2}\omega$. What is the response $y(t)$ of the system to the input
$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik[(\pi/3)t+(\pi/2)]},$$
for some complex scalars $X_k$?