Zero-Knowledge and Sigma Protocol

LIU Yi

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- **Example**: P : 26781 is not a prime since $26781 = 113 \times 237$.

- Given this factorization, other than that you are convinced that *Statement* is true, you gained some knowledge (the factorization).

Zero knowledge

- Informally, in a Zero Knowledge Proof, Alice will prove to Bob that a statement S is true. Bob will be completely convinced that S is true, but will not learn anything as a result of this process. That is, Bob will gain zero knowledge.

Zero knowledge

- S. Goldwasser, S. Micali, C. Rackoff, STOC' 85

The Knowledge Complexity of Interactive Proof-Systems

(Extended Abstract)

Shafi Goldwasser, MIT
Silvio Micali, MIT
Charles Rackoff, University of Toronto

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(Extended Abstract)

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Shafi, with Micali (and later Rackoff) [6], had been thinking for a while about expanding the traditional notion of "proof" to an interactive process in which a "prover" can convince a probabilistic "verifier" of the correctness of a mathematical proposition with overwhelming probability if and only if the proposition is correct. They called this interactive process an "interactive proof" (a name suggested by Mike Sipser). They wondered if one could prove some non-trivial statement (for example, membership of a string in a hard language) without giving away any knowledge whatsoever about why it was true. They defined that the verifier receives no knowledge from the prover if the verifier could simulate on his own the probability distribution that he obtains in interacting with the prover.The idea that "no knowledge" means simulatability was a very important contribution. They also gave the first example of these "zero knowledge interactive proofs" using quadratic residuosity. This paper won the first ACM SIGACT Gödel Prize. This zero-knowledge work led to a huge research program in the community that continues to this day, including results showing that (subject to an assumption such as the existence of one-way functions) a group of distrusting parties can compute a function of all their inputs without learning any knowledge about other people's inputs beyond that which follows from the value of the function.

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https://amturing.acm.org/award_winners/goldwasser_8627889.cfm

Zero knowledge

- Together with a paper by László Babai and Shlomo Moran, this landmark paper invented interactive proof systems, for which all five authors won the first Gödel Prize in 1993.

Recipients [edit]

Year	Name(s)	Notes	Publication year
1993	László Babai, Shafi Goldwasser, Silvio Micali, Shlomo Moran, and Charles Rackoff	for the development of interactive proof systems	1988,[paper 1] 1989[paper 2]
1994	Johan Håstad	for an exponential lower bound on the size of constant-depth Boolean circuits (for the parity function).	1989[paper 3]
1995	Neil Immerman and Róbert Szelepcsényi	for the Immerman–Szelepcsényi theorem regarding nondeterministic space complexity	1988,[paper 4] 1988[paper 5]
1996	Mark Jerrum and Alistair Sinclair	for work on Markov chains and the approximation of the permanent of a matrix	1989,[paper 6] 1989[paper 7]
1997	Joseph Halpern and Yoram Moses	for defining a formal notion of "knowledge" in distributed environments	1990[paper 8]
1998	Seinosuke Toda	for Toda's theorem which showed a connection between counting solutions (PP) and alternation of quantifiers (PH)	1991[paper 9]

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Proofs that Yield Nothing But their Validity and a Methodology of Cryptographic Protocol Design

(Extended Abstract)

Oded Goldreich
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Jerusalem, Israel

Applications of ZKPs

- **Authentication systems**
- **Blockchains**
- **…**

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The first two of these are properties of more general interactive proof systems. The third is what makes the proof zero-knowledge.

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- Informally, a zero-knowledge proof of knowledge is a special case when the statement consists only of the fact that the prover possesses the secret information.

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- Equivalently: The protocol securely computes the functionality $f_{zk}((x,w),x) = (-,R(x,w))$

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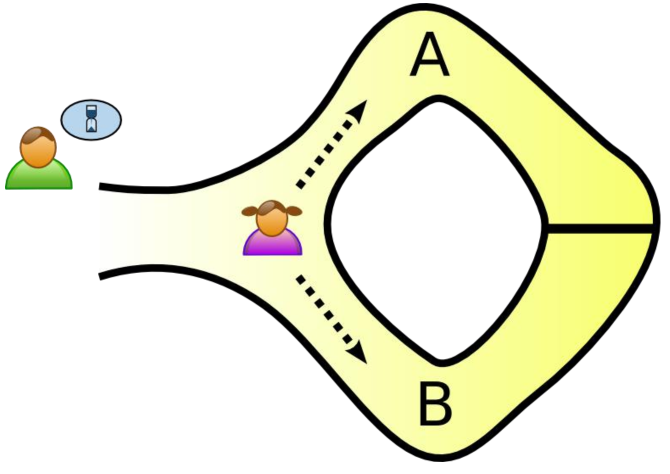

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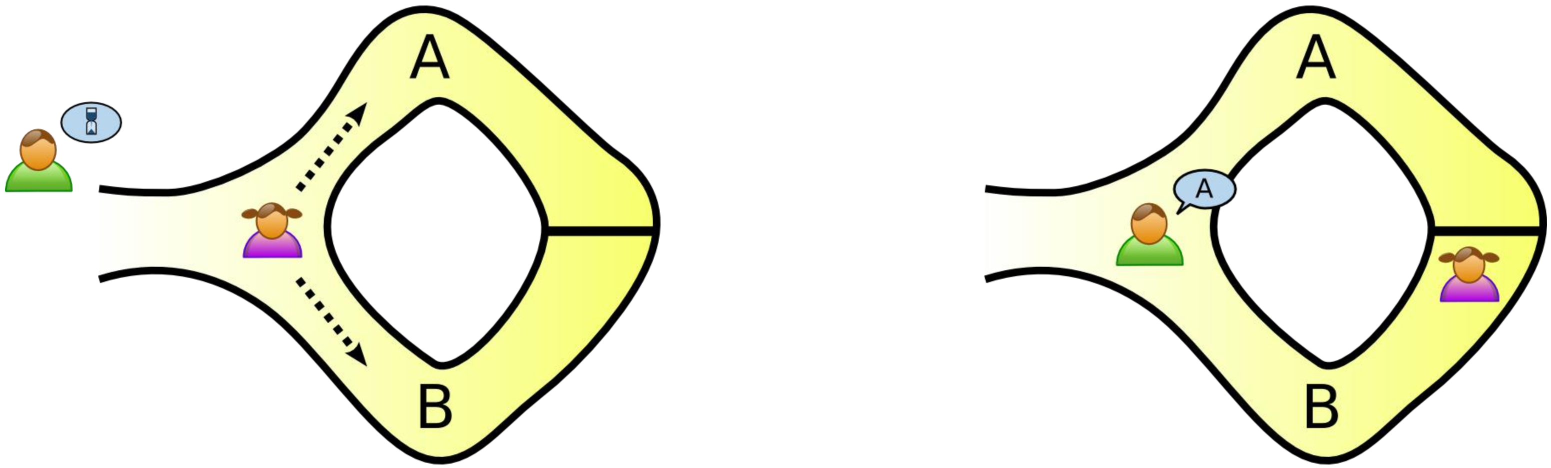

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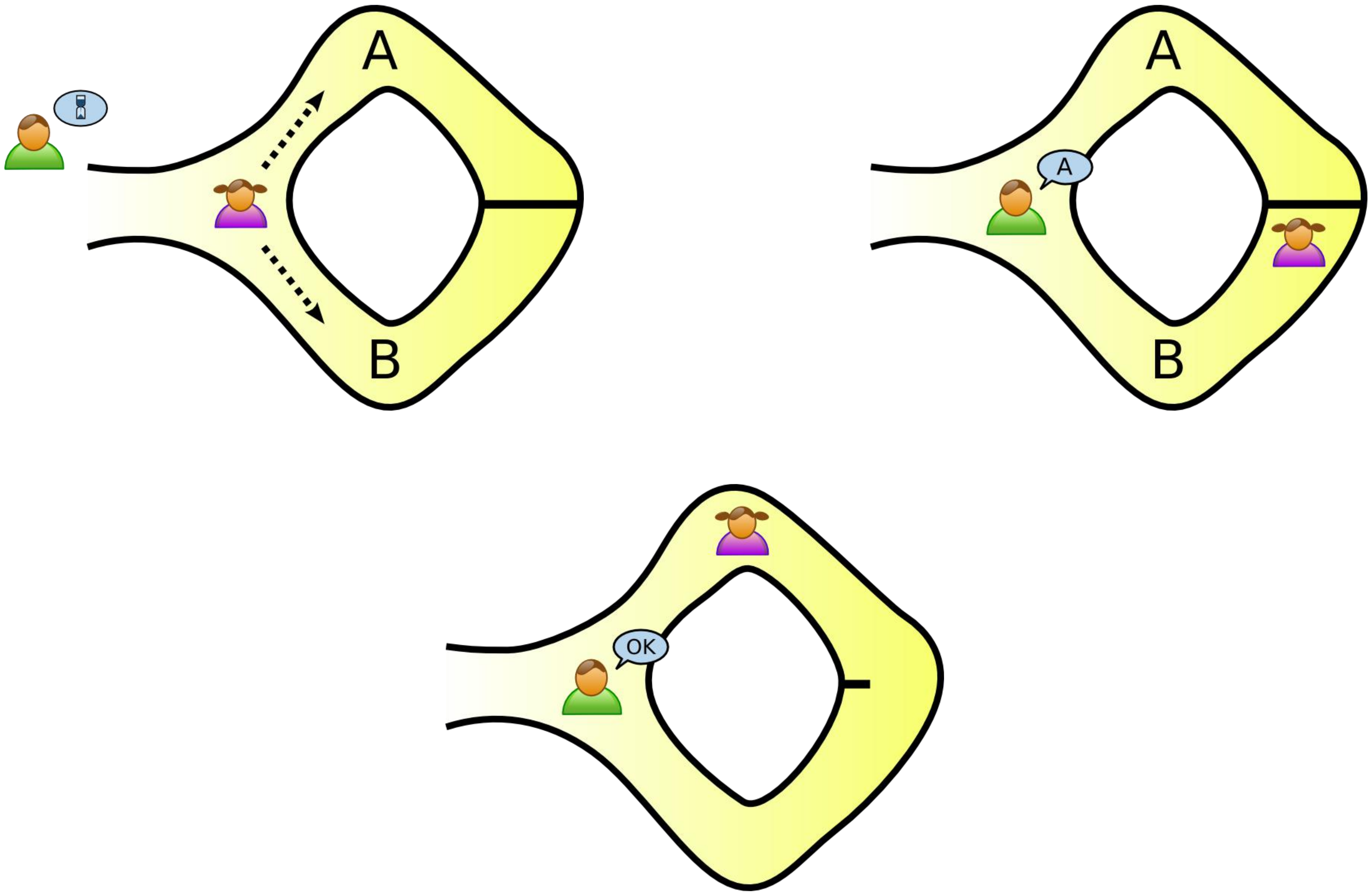

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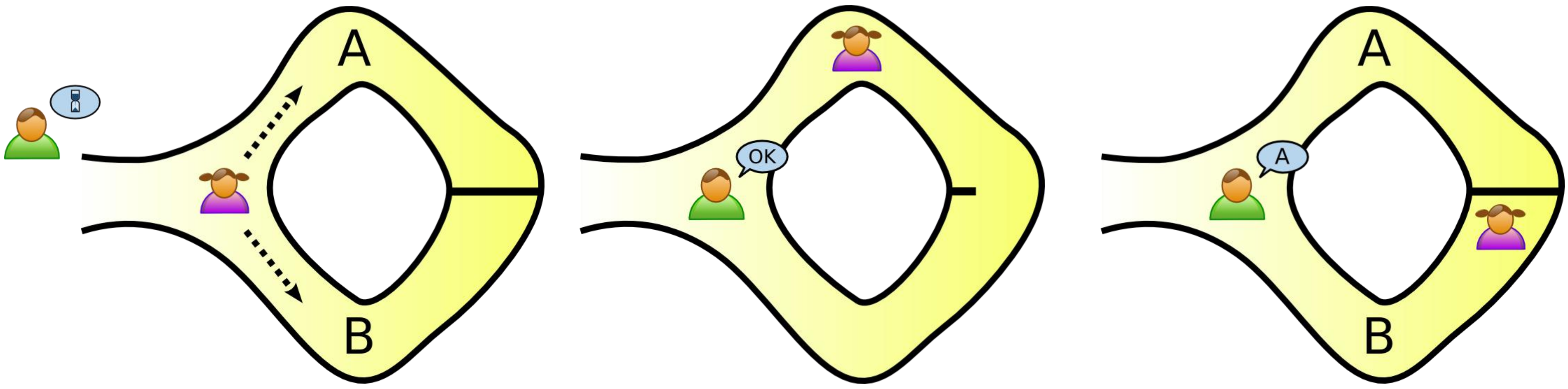

- Third-party observers

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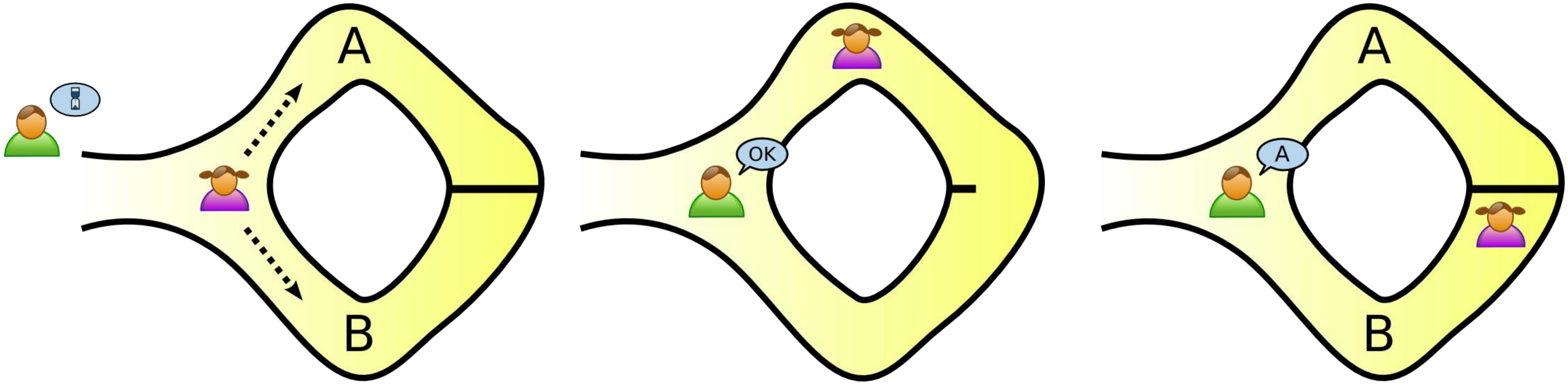

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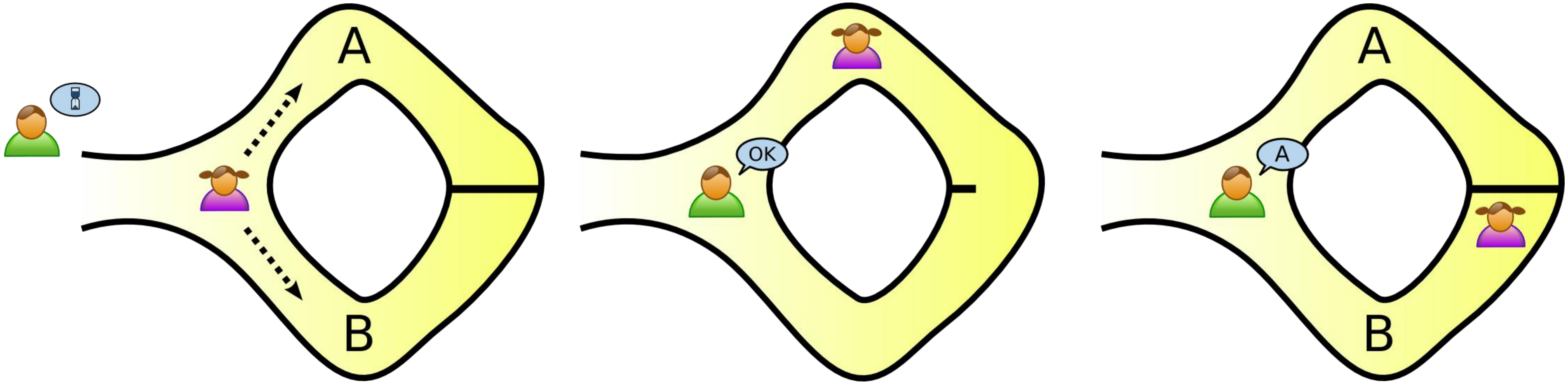

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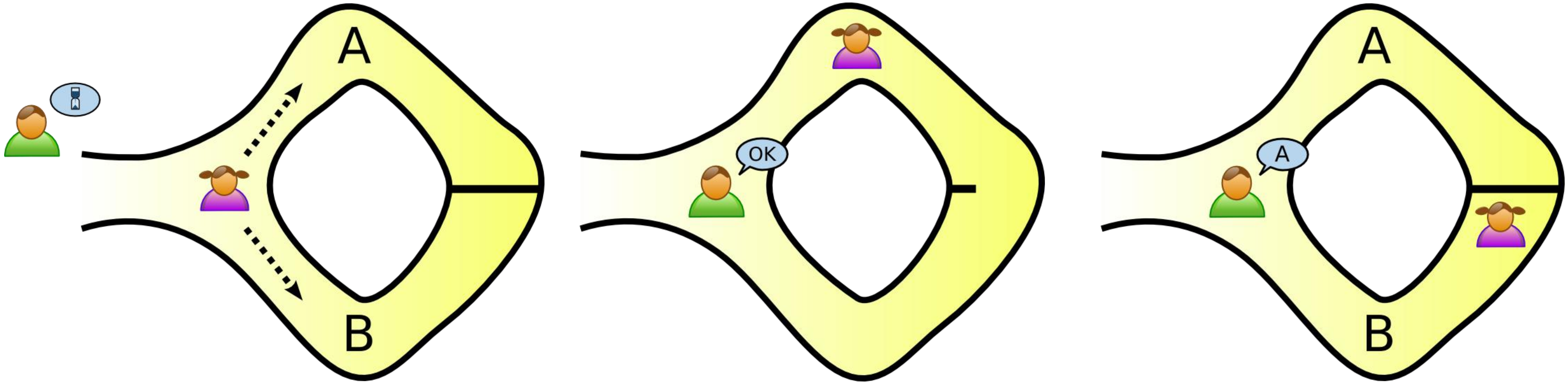

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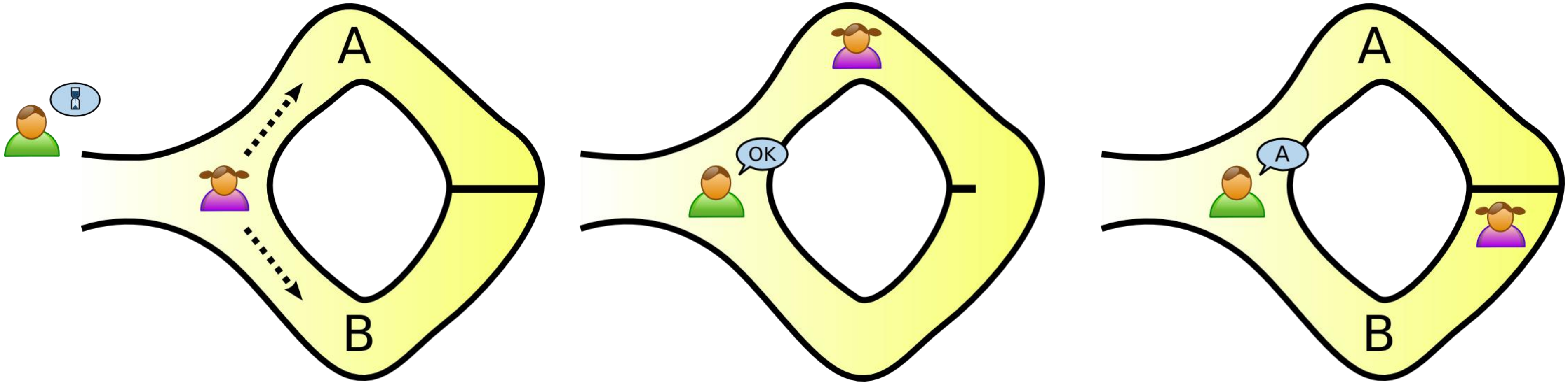

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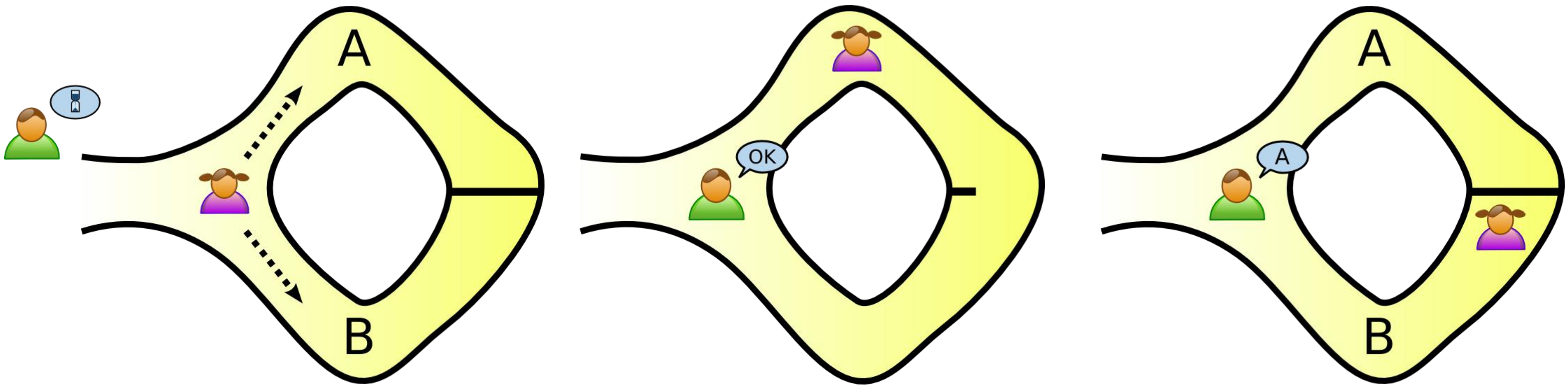

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- In a single trial
  - ok but could be observed by a third party, or recorded

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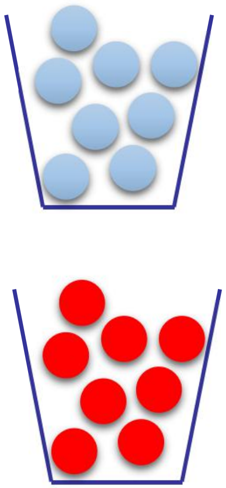
- You (the prover) want to convince me (the verifier) that you are not color-blind (two colors are completely identical)



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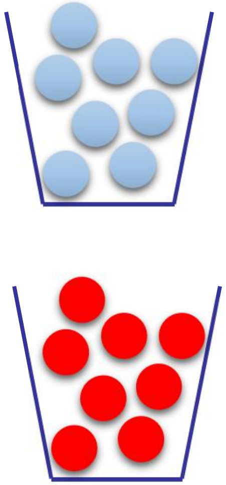

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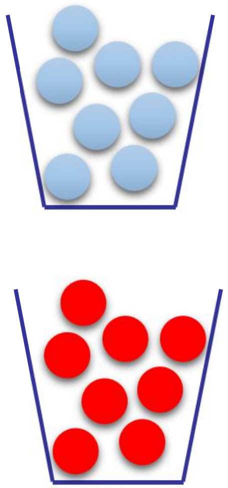

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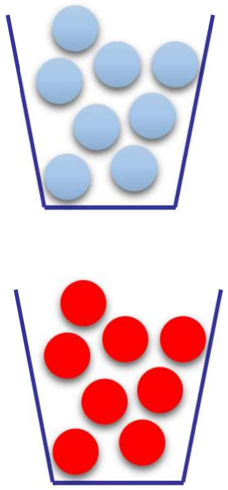

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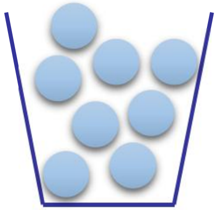

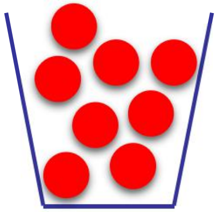

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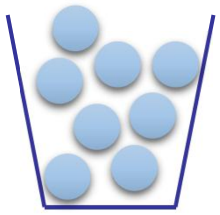

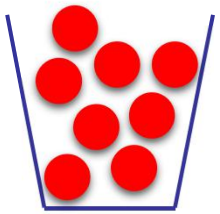

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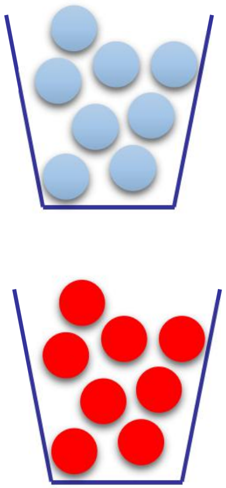

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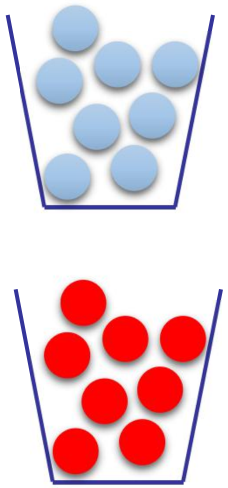

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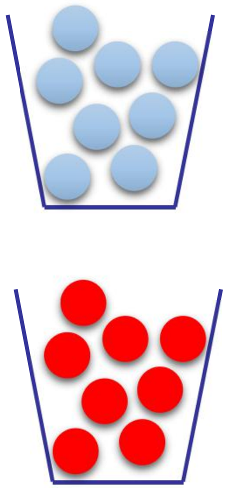

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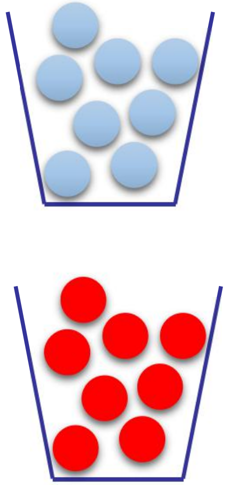

- Soundness
  - If the verifier accepts then the property (the prover is not color blind) holds with high probability
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  - If property (the prover is not color blind) holds then the verifier always accepts

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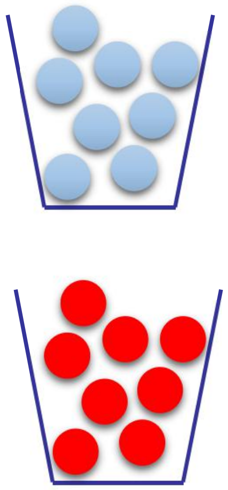

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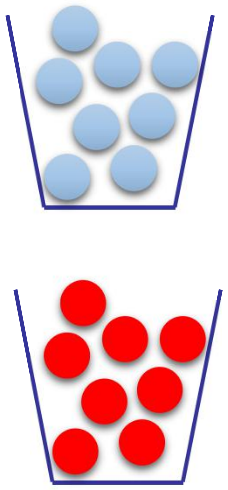

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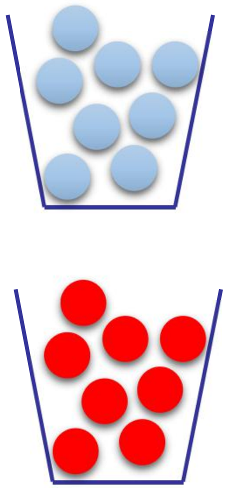

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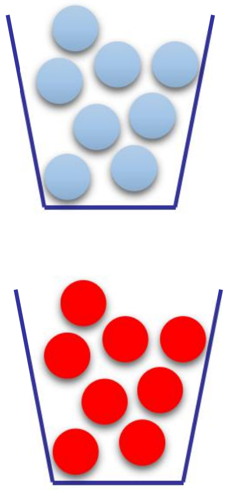

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  Suppose I was color blind, then I used you to gain some extra knowledge!

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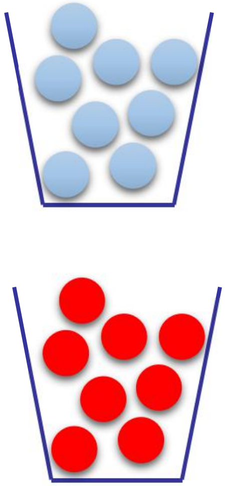

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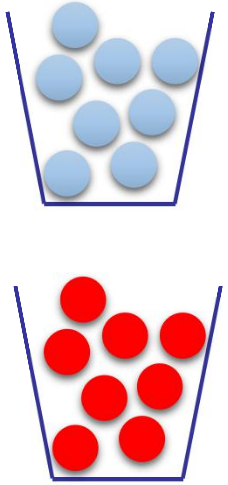

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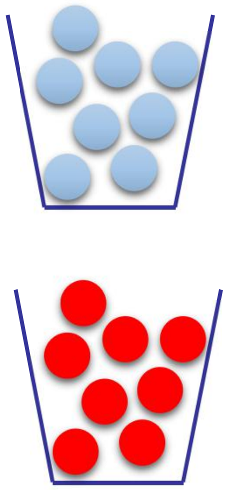

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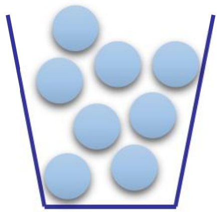

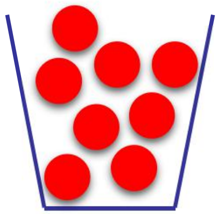

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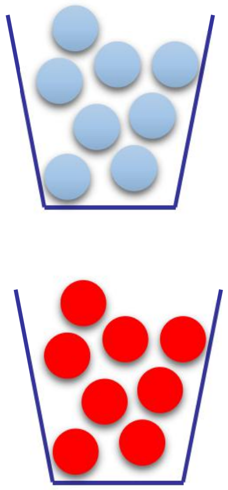

Zero-knowledge:

Suppose I was color blind but you are not.

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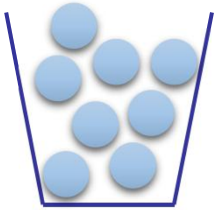

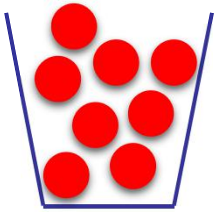

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Suppose I was color blind but you are not.

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Example: Non-color-blindness

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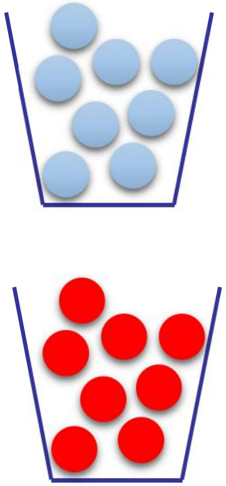

Zero-knowledge:

Suppose I was color blind but you are not.

- In the first protocol, I cannot predict your Answer ahead of time.
- In the second protocol, I know what you will say, so I do not gain knowledge when you say it.

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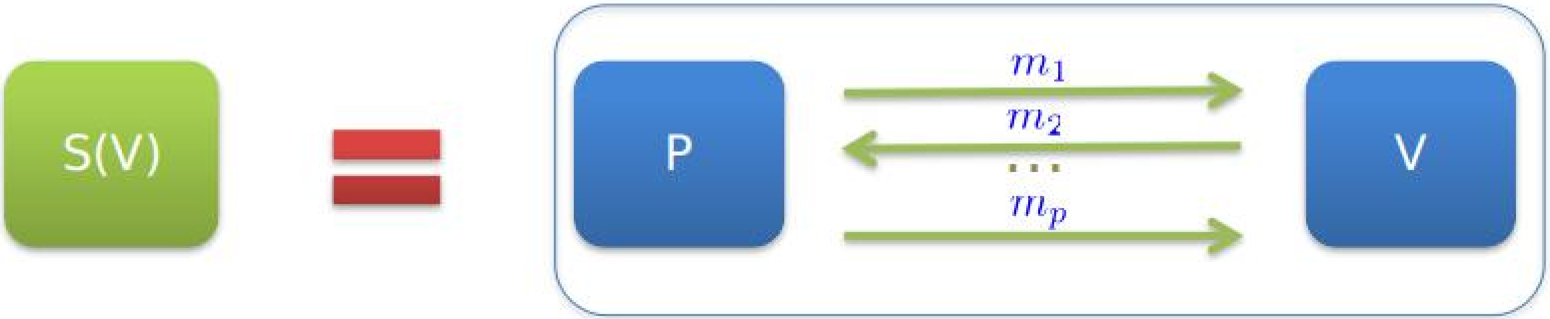

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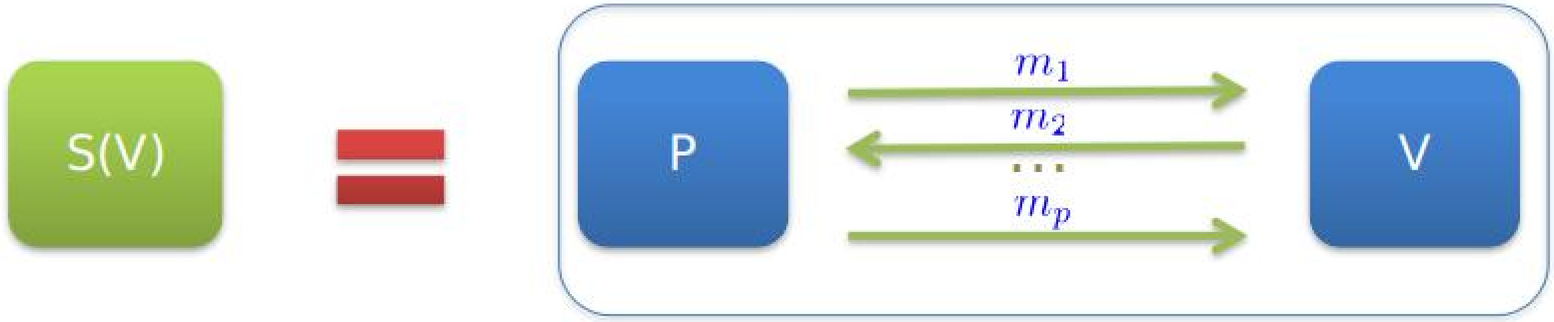

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- Zero knowledge is often avoided at significant cost

Sigma Protocol

- A way to obtain efficient zero knowledge

Sigma Protocol

- A way to obtain efficient zero knowledge
  - Many general tools
  - Many interesting languages can be proven with a sigma protocol

Schnorr DLOG

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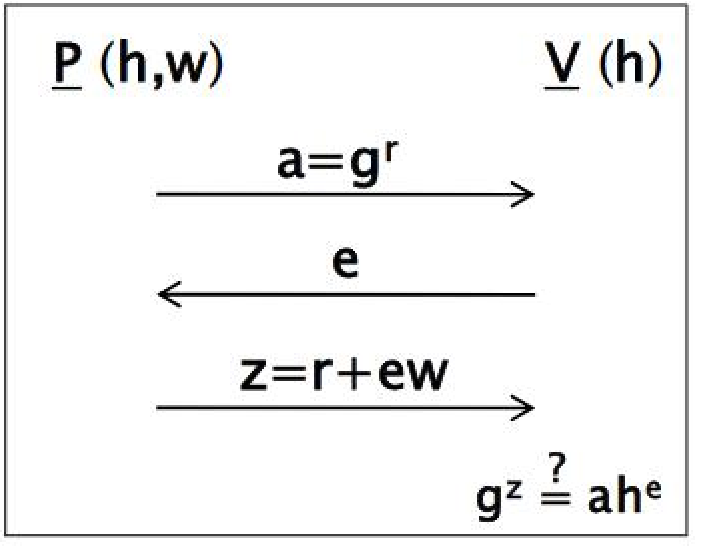
  - V sends P a random $e \in \{0,1\}^t$
  - P sends $z=r+ew \bmod q$ to V
  - V checks that $g^z = ah^e$

- **Completeness**
  - $g^z = g^{r+ew} = g^r(g^w)^e = ah^e$

Schnorr DLOG

- **Proof of knowledge**



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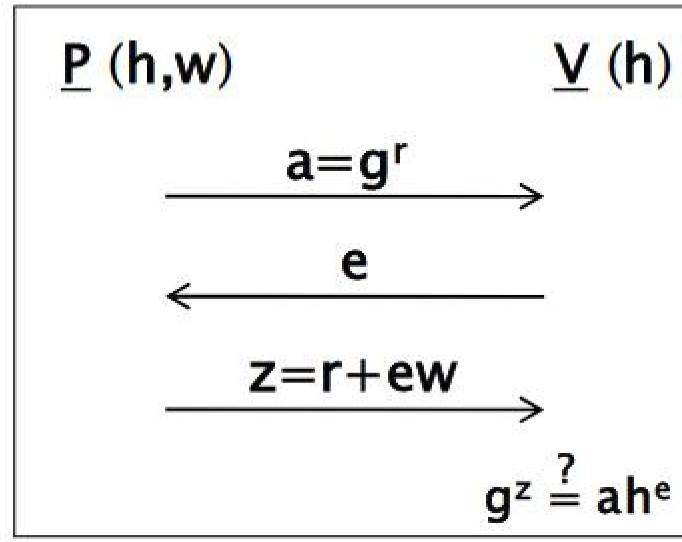

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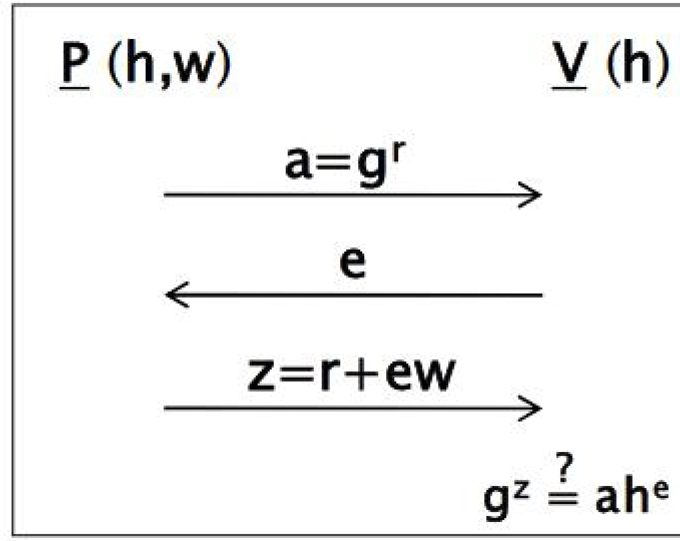

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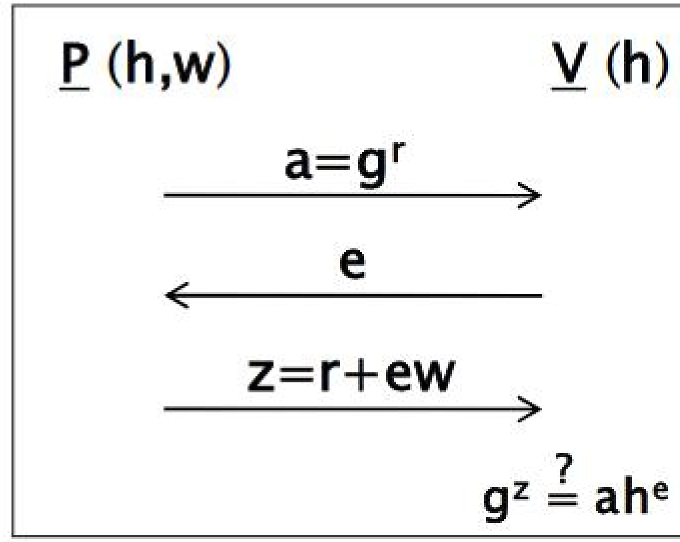

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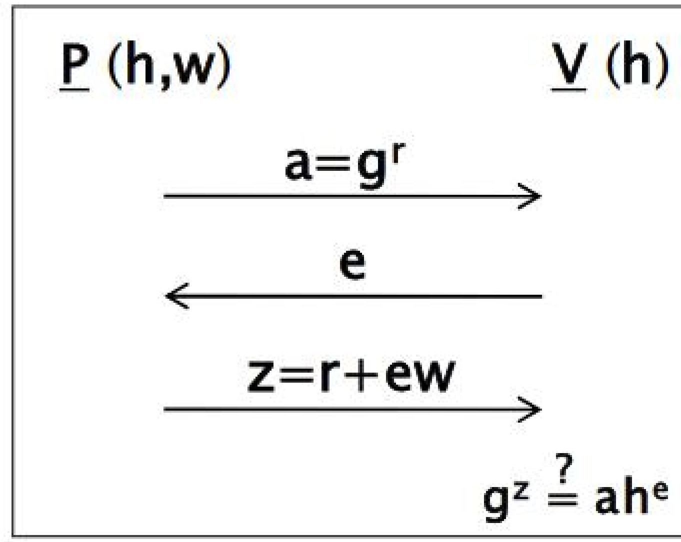

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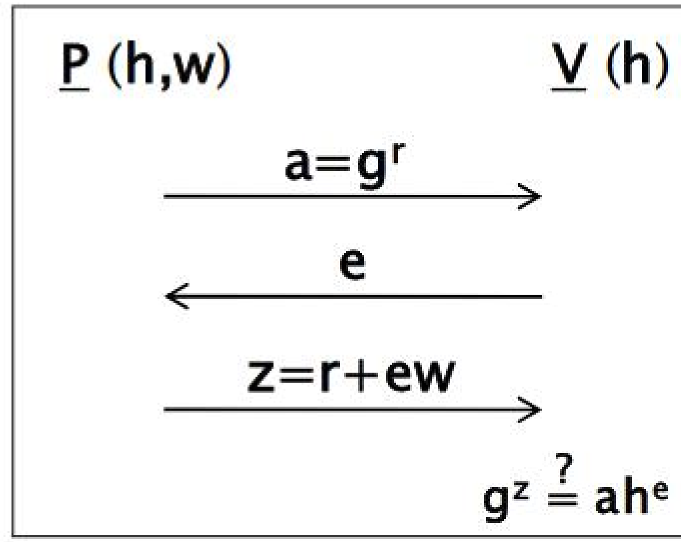

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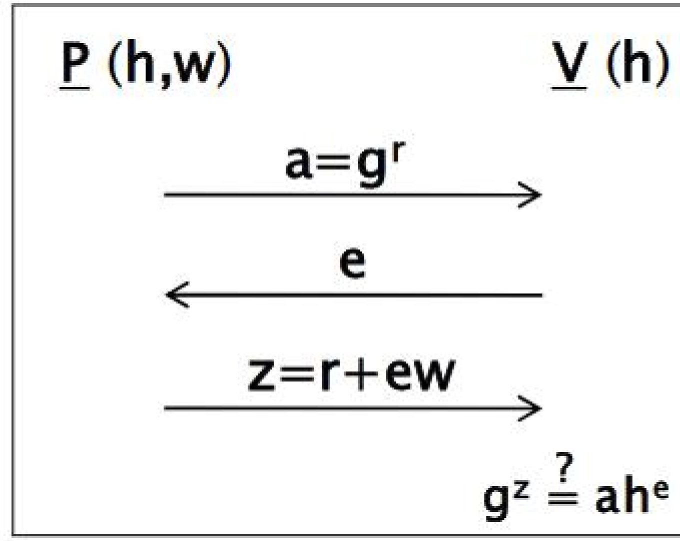
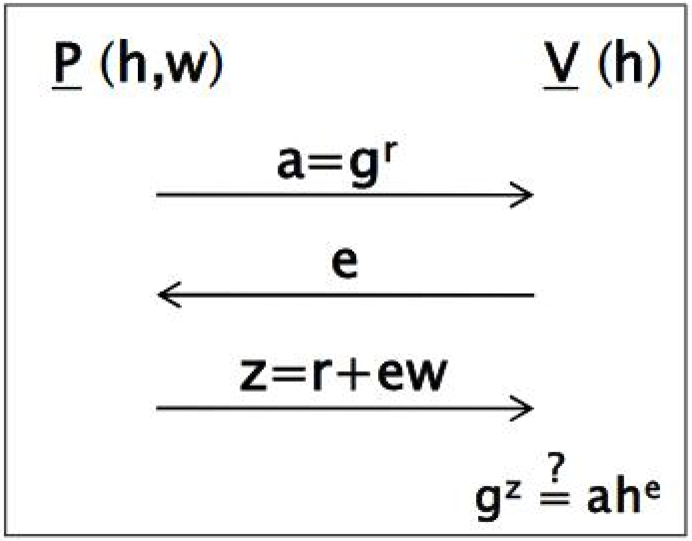


- **Conclusion:**
  - If P can answer with probability greater than $1/2^t$, then it must know the dlog

Schnorr DLOG

- **What about zero knowledge?**



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$$
\begin{array}{lcr}
\underline{P}\ (h,w) & & \underline{V}\ (h) \\
& \xrightarrow{\quad a=g^r \quad} & \\
& \xleftarrow{\quad e \quad} & \\
& \xrightarrow{\quad z=r+ew \quad} & \\
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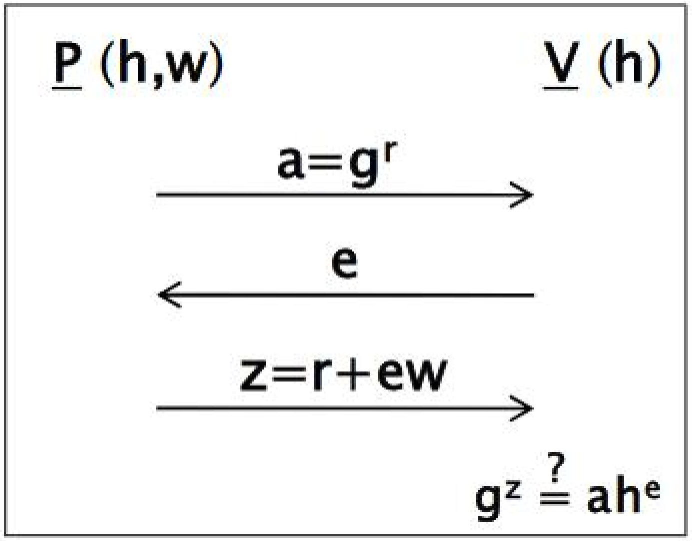

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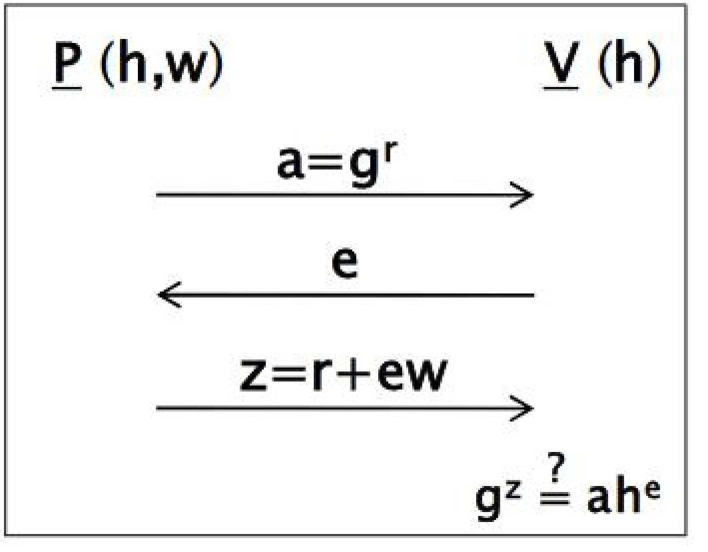
- **This is not very strong, but we will see that it yields efficient general ZK**



Sigma Protocol

Definitions

- **Sigma protocol template**
  - **Common input: P** and **V** both have **x**
  - **Private input: P** has **w** such that **(x,w)**$\in$**R**
  - **Protocol:**
    - **P** sends a message **a**
    - **V** sends a <u>random</u> **t**-bit string **e**
    - **P** sends a reply **z**
    - **V** accepts based solely on (**x,a,e,z**)

Sigma Protocol

- **Completeness: as usual**
- **<u>Special</u> soundness:**
  - There exists an algorithm **A** that given any **x** and pair of transcripts (**a**,**e**,**z**),(**a**,**e′**,**z′**) with **e**≠**e′** outputs **w** s.t. (**x**,**w**)∈**R**
- **<u>Special</u> honest-verifier ZK**
  - There exists an **M** that given **x** and **e** outputs (**a**,**e**,**z**) which is distributed exactly like a real execution where **V** sends **e**

Sigma Protocol DH Tuple

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- **Relation R of Diffie-Hellman tuples**
  - (**g,h,u,v**) $\in$**R** iff exists **w** s.t. **u=g^w^** and **v = h^w^**
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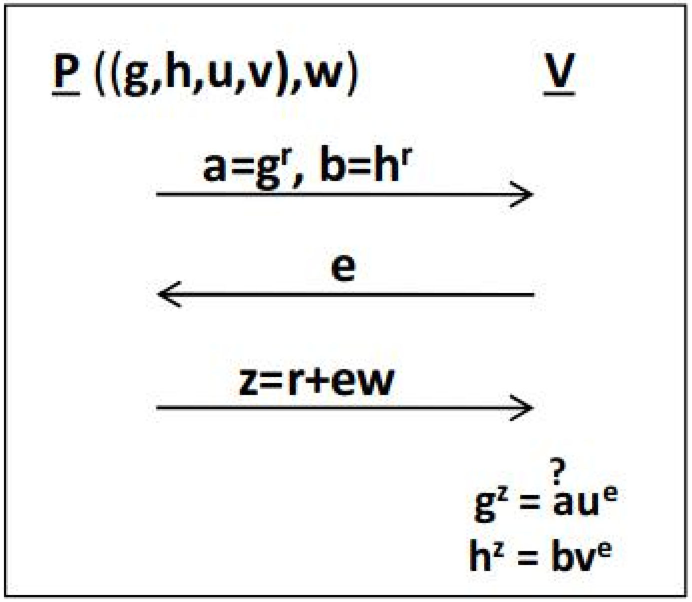

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    - $w = (z - z')(e - e')$

<u>P</u> ((g,h,u,v),w) <u>V</u>

$$\xrightarrow{a = g^r,\ b = h^r}$$

$$\xleftarrow{e}$$

$$\xrightarrow{z = r + ew}$$

$$g^z \stackrel{?}{=} au^e$$

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    - $\mathbf{w = (z\text{-}z')(e\text{-}e')}$
- **Special HVZK**
  - Given (**g,h,u,v**) and **e**, choose random **z** and compute
    - $\mathbf{a = g^z u^{-e}}$
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$$\underline{P}\ ((g,h,u,v),w) \qquad\qquad \underline{V}$$

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- **Properties of sigma protocols are invariant under parallel composition**
- **Any sigma protocol is a proof of knowledge with error 2^{-t}**
  – The difference between the probability that $\mathbf{P}^*$ convinces **V** and the probability that **K** obtains a witness is at most $\mathbf{2^{-t}}$

Tools for Sigma Protocols

- **Prove compound statements**
  - AND, OR, subset
  - Can be done efficiently
- **ZK from sigma protocols**
  - Can first make a compound sigma protocol and then compile it
- **ZKPOK from sigma protocols**

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- Sometimes it’s possible to do better than this
  - Statements can be batched

OR of Sigma Protocol

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- The solution
  - Using the simulator, if **e** is known ahead of time it is possible to cheat
  - We construct a protocol where the prover can cheat in one out of the two proofs

OR of Sigma Protocol

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  - **V** accepts if $\mathbf{e}_0 \oplus \mathbf{e}_1 = \mathbf{e}$ and $(\mathbf{a}_0,\mathbf{e}_0,\mathbf{z}_0),(\mathbf{a}_1,\mathbf{e}_1,\mathbf{z}_1)$ are both accepting

OR of Sigma Protocol

- **P sends two first messages (a_0,a_1)**
  - P has a witness for x_0 (and not for x_1)
  - P chooses a random e_1 and runs SIM to get (a_1,e_1,z_1)
  - P sends (a_0,a_1)
- **V sends a single challenge e**
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<u>Soundness</u>

- P doesn't know a witness for x_1, so can only answer for a single e_1
- This means that **e** defines a single challenge e_0, like in a regular proof

OR of Sigma Protocol

- **Special soundness**
  - Relative to first message ($\mathbf{a}_0$,$\mathbf{a}_1$), and two different $\mathbf{e}$,$\mathbf{e}'$, at least one of $\mathbf{e}_0 \neq \mathbf{e}'_0$ or $\mathbf{e}_1 \neq \mathbf{e}'_1$ (because $\mathbf{e}_0 \oplus \mathbf{e}_1 = \mathbf{e}$ and $\mathbf{e}'_0 \oplus \mathbf{e}'_1 = \mathbf{e}'$)
  - Thus, we will obtain two different continuations for at least one of the statements
- **Honest verifier ZK**
  - Can choose both $\mathbf{e}_0$,$\mathbf{e}_1$, so no problem

ZK from Sigma: Preliminaries

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  - **Hiding**: the receiver does not know anything about the commitment
- **Variants**
  - Perfect and computational binding
  - Perfect and computational hiding
  - Cannot have both perfect binding and hiding

ZK from Sigma Protocols

- **The basic idea**
  - Have **V** first commit to its challenge **e** using a **perfectly-hiding commitment**

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- **The protocol**
  - **P** sends the 1st message α of the commit protocol
  - **V** sends a commitment c=$\mathbf{Com}_{\alpha}$(**e;r**)
  - **P** sends a message **a**
  - **V** sends (**e,r**)
  - **P** checks that c=$\mathbf{Com}_{\alpha}$(**e;r**) and if yes sends a reply **z**
  - **V** accepts based on (**x,a,e,z**)

ZK from Sigma Protocols

- **Soundness:**
  - The perfectly hiding commitment reveals nothing about **e** and so soundness is preserved

ZK from Sigma Protocols

- **Soundness:**
  - The perfectly hiding commitment reveals nothing about **e** and so soundness is preserved
- **Zero knowledge**
  - In order to simulate:
    - Send **a**′ generated by the simulator, for a random **e**′
    - Receiver **V**'s decommitment to **e**
    - Run the simulator again with **e**, rewind **V** and send **a**
      - Repeat until **V** decommits to **e** again
    - Conclude by sending **z**
  - Analysis…

ZKPOK from Sigma Protocols

- Is the previous protocol a proof of knowledge?

ZKPOK from Sigma Protocols

- **Is the previous protocol a proof of knowledge?**
  - It seems not to be
  - The extractor for the Sigma protocol needs to obtain two transcripts with the same **a** and different **e**
  - The prover may choose its first message **a** differently for every commitment string, so if the extractor changes **e**, the prover changes **a**

ZKPOK from Sigma Protocols

$\underline{\text{P}}$ (h,w) $\qquad\qquad\qquad\qquad$ $\underline{\text{V}}$ (h)

$$\xrightarrow{\quad a=g^r \quad}$$

$$\xleftarrow{\quad e \quad}$$

$$\xrightarrow{\quad z=r+ew \quad}$$

$$g^z \stackrel{?}{=} ah^e$$

ZKPOK from Sigma Protocols

- **Solution: use a trapdoor (equivocal) commitment scheme**
  - Given a trapdoor, it is possible to open the commitment to any value

ZKPOK from Sigma Protocols

- **Solution: use a trapdoor (equivocal) commitment scheme**
  - Given a trapdoor, it is possible to open the commitment to any value
- **Pedersen has this property, and the previous protocol can be modified only slightly to get a proof of knowledge**

Pedersen Commitments

- **Highly efficient perfectly-hiding commitments**
  - **Parameters:** generator $\boldsymbol{g}$, order $\boldsymbol{q}$
  - **Commit protocol** (commit to $x \in \mathbb{Z}_q$):
    - Receiver chooses random $k \leftarrow \mathbb{Z}_q$ and sends $h = g^k$
    - Sender sends $c = g^r \cdot h^x$, for a random $r \leftarrow \mathbb{Z}_q$
  - **Perfect hiding:**
    - For every $x, y \in \mathbb{Z}_q$ there exist $r, s \in \mathbb{Z}_q$ such that $r + kx = s + ky \bmod q$
  - **Computational binding:**
    - If can find $(x, r), (y, s)$ such that $g^r \cdot h^x = g^s \cdot h^y$ then can compute $k = DLOG_g(h) = {}^{r-s}/_{y-x} \bmod q$

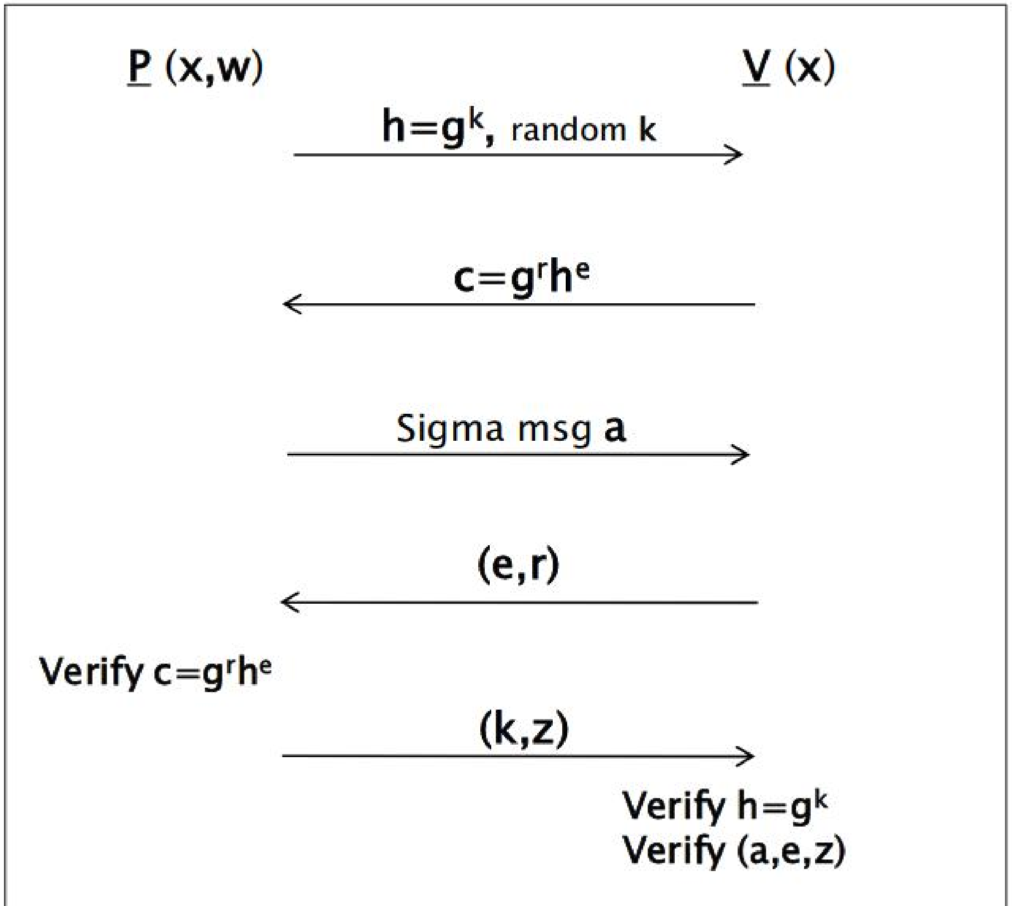
Pedersen has this property – given the discrete log k of h, can decommit to any value

- Commit to **x**: $c = g^r h^x$
- To decommit to **y**, find **s** such that $r+kx = s+ky$
- Compute $s = r+k(x-y) \bmod q$

ZKPOK from Sigma Protocols

- **The basic idea**
  - Have **V** first to its challenge **e** using a perfectly-hiding trapdoor (equivocal) commitment
- **The protocol**
  - **P** sends the 1st message α of the commit protocol
  - **V** sends a commitment c=$\mathbf{Com}_{\alpha}$(**e;r**)
  - **P** sends a message **a**
  - **V** sends (**e,r**)
  - **P** checks that c=$\mathbf{Com}_{\alpha}$(**e;r**) and if yes sends the **trapdoor** and **z**
  - **V** accepts if the **trapdoor** is correct and (**x,a,e,z**) is accepting

ZKPOK from Sigma Protocols



ZKPOK from Sigma Protocols

- **Why does this help?**
  - **Zero-knowledge** remains the same
  - **Extraction:** after verifying the proof once, the extractor obtains **k** and can rewind back to the decommitment of **c** and send any (**e′**,**r′**)

ZK and Sigma Protocols

- **We typically want zero knowledge, so why bother with sigma protocols?**
  - We have many useful general transformations
    - E.g., parallel composition, compound statements
    - The ZK and ZKPOK transformations can be applied on top of the above, so obtain transformed ZK
  - It is **much harder** to prove ZK than Sigma
    - ZK – distributions and simulation
    - Sigma: only HVZK and special oundness

Using Sigma Protocols and ZK

- **Prove that the El Gamal encryption (u,v) under public-key (g,h) is to the value m**
  - By encryption definition $\mathbf{u=g^r}$, $\mathbf{v=h^r \cdot m}$
  - ThUS (**g,h,u,v/m**) is a DH tuple
  - So, given (**g,h,u,v,m**), just prove that (**g,h,u,v/m**) is a DH tuple

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- **Database of ElGamal(K_i),E_{Ki}(T_i)**
  - Can release $\mathbf{T_i}$ without revealing anything about $\mathbf{T_j}$ for $\mathbf{j \neq I}$

Non-Interactive ZK (ROM)

- **The Fiat-Shamir paradigm**
  - To prove a statement **x**
  - Generate **a**, compute **e=H**(**a**,**x**), compute **z**
  - Send (**a**,**e**,**z**)
- **Properties:**
  - **Soundness:** follows from random oracle property
  - **Zero knowledge:** same
  - Can achieve simulation-soundness (non malleability) by including unique **sid** in **H**

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