

Review of Probability ¹

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¹This section is based on Stock and Watson (2020), Chapter 2.

Probabilities and Outcomes

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- ▷ An **outcomes** is a specific result:
 - ◇ Coin toss: either H or T .
 - ◇ Roll of dice: $1, 2, \dots, 6$.
- ▷ The **probability** of an **outcome** is the proportion of the time that the outcome occurs in the long run.
 - ◇ Fair coin toss: 50 % chance of heads.
- ▷ The **sample space** is the set of all possible outcomes.
 - ◇ In a coin flip the sample space is $S = \{H, T\}$.
 - ◇ If two coins are flipped the sample space is $S = \{HH, HT, TH, TT\}$.
- ▷ An **event** is a subset of the sample space.
 - ◇ Roll a die $A = \{1, 2\}$.

Definition 1 (Probability Function)

A function \mathbb{P} which assigns a numerical value to events is called a probability function if it satisfies the following Axioms of Probability:

1. $\mathbb{P}(A) \geq 0$.
2. $\mathbb{P}(S) = 1$.
3. If A_1, A_2, \dots are disjoint then $\mathbb{P}(\cup_{j=1}^N A_j) = \sum_{j=1}^N \mathbb{P}(A_j)$.

Definition 2 (Conditional Probability)

If $\mathbb{P}(B) > 0$, then the **conditional probability** of A given B is given by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

$\mathbb{P}(B)$ is the **marginal probability** of event B .

$$\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B).$$

Take two events H and M .

- ▷ let H be the event that an individual's monthly wage exceeds HKD 20000,
- ▷ let M be the event that the individual has a master's degree.

Table: Joint Distribution

	Master degree	Non-master degree	Any education
High wage	0.19	0.12	0.31
Low wage	0.17	0.52	0.69
Any wage	0.36	0.64	1.00

Conditional Probability - Example

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The probability of earning a high wage conditional on high education is

$$\begin{aligned} & \mathbb{P}(\text{High wage} | \text{Master degree}) \\ &= \frac{\mathbb{P}(\text{High wage} \cap \text{Master degree})}{\mathbb{P}(\text{High wage} \cap \text{Master degree}) + \mathbb{P}(\text{Low wage} \cap \text{Master degree})} \\ &= \frac{0.19}{0.36} = 0.53. \end{aligned}$$

Similarly, the probability of earning a high wage conditional on non-master degree is

$$\mathbb{P}(\text{High wage} | \text{Non-master degree}) = \frac{0.12}{0.64} = 0.19.$$

- ▶ The events A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

Some facts:

- ▶ When events are independent then joint probabilities can be calculated by multiplying individual probabilities.
- ▶ If A and B are disjoint then they cannot be independent.

Theorem 3 (Bayes Rule)

If $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$ then

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c)}.$$

Consider the PCR test for Covid-19:

- ▷ If actual infected: 0.9 positive, 0.1 negative.
- ▷ If not infected: 0.05 positive, 0.95 negative.
- ▷ 1 % population is infected.

What is the chance of being infected given the test is positive?

Let A denote being infected, let B denote test result positive.

- ▷ $P(B|A) = 0.9$, $P(B|A^c) = 0.05$, $P(A) = 0.01$.
- ▷ $P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{0.05 \cdot 0.9}{0.05 \cdot 0.9 + 0.95 \cdot 0.01} \approx 0.8256$.

What is the chance of not infected given the test is negative?

- ▷ $P(A^c|B^c) = \frac{P(B^c|A^c)P(A^c)}{P(B^c|A^c)P(A^c) + P(B^c|A)P(A)} = \frac{0.95 \cdot 0.99}{0.95 \cdot 0.99 + 0.05 \cdot 0.01} \approx 0.9947$.

Definition 4 (Random variable)

A random variable is a real-valued outcome; a function from the sample space S to the real line \mathbb{R} .

For example, X is a mapping from the coin flip sample space to the real line, with T mapped to 0 and H mapped to 1.

$$X = \begin{cases} 1 & \text{if } H \\ 0 & \text{if } T. \end{cases}$$

Properties of random variables.

- ▷ The **expected value** is the long-run average of the random variable.
- ▷ The **standard deviation** measures the spread of a probability distribution.

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- ▷ The set \mathcal{X} is discrete if it has a **finite or countably infinite number of elements**.
- ▷ If there is a discrete set \mathcal{X} such that $\mathbb{P}(X \in \mathcal{X}) = 1$ then X is a discrete random variable.
- ▷ The **probability mass function** of a random variable is $\pi(x) = \mathbb{P}(X = x)$, the probability that X equals the value x .
- ▷ The **probability distribution** of a discrete random variable is the list of all possible values of the variable and the probability that each value will occur.

For a discrete variable X with the support of \mathcal{X} , the expectation is computed as

$$\mathbb{E}(X) = \sum_{x \in \mathcal{X}} \pi(x)x.$$

- ▷ $X = 1$ with the probability of p and $X = 0$ with probability $1 - p$. The expected value is $\mathbb{E}(X) = 1 * p + 0 * (1 - p) = p$.

The expectation of the function of X , $g(X)$ is computed as

$$\mathbb{E}(g(X)) = \sum_{x \in \mathcal{X}} \pi(x)g(x).$$

Cumulative distribution function

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The **cumulative probability distribution (CDF)** is the probability that the random variable is less than or equal to a particular value,

$$F(x) = \mathbb{P}(X \leq x),$$

where the probability event is $X \leq x$.

Properties of a CDF If $F(x)$ is a distribution function, then

1. $F(x)$ is non-decreasing.
2. $\lim_{x \rightarrow -\infty} F(x) = 0$.
3. $\lim_{x \rightarrow \infty} F(x) = 1$.
4. $F(x)$ is right-continuous, $\lim_{x \downarrow x_0} F(x) = F(x_0)$.

Example-Probability mass function

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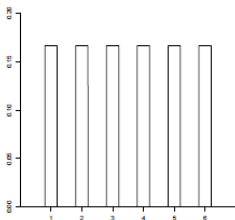
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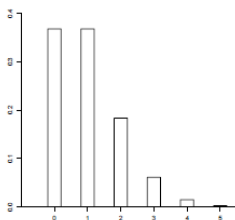
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Some examples for discrete variables.

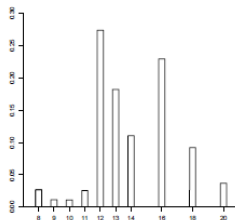
- ▶ For a fair dice toss, the support is $\mathcal{X} = \{1, 2, \dots, 6\}$ with the probability mass function is $\pi(x) = \frac{1}{6}$ for $x \in \mathcal{X}$.
- ▶ An example of infinite countable random variable is the Poisson distribution, the probability mass function is $\pi(x) = \frac{e^{-1}}{x!}, x = 0, 1, \dots$



(a) Die Toss



(b) Poisson



(c) Education

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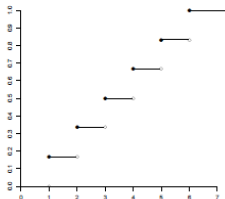
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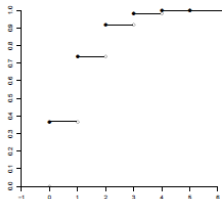
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Some examples for discrete variables.

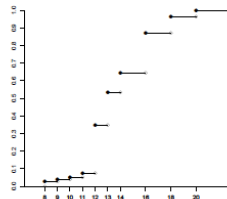
- ▷ For a fair dice toss, the support is $\mathcal{X} = \{1, 2, \dots, 6\}$ with the probability mass function is $\pi(x) = \frac{1}{6}$ for $x \in \mathcal{X}$.
- ▷ An example of infinite countable random variable is the Poisson distribution, the probability mass function is $\pi(x) = \frac{e^{-1}}{x!}$, $x = 0, 1, \dots$



(a) Die Toss



(b) Poisson



(c) Education

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- ▷ The **probability density function**(*p.d.f*) area under the probability density function between any two points is the probability that the random variable falls between those two points.
 - ◇ the probability for a continuous variable to take any value is 0.
 - ◇ definition is different from discrete random variables.
- ▷ When $F(x)$ is differentiable, the density function is $f(x) = \frac{dF(x)}{dx}$.

Theorem 5 (Properties of density function)

A function $f(x)$ is a density function if and only if

- ▷ $f(x) \geq 0 \forall x$.
- ▷ $\int_0^{\infty} f(x)dx = 1$.

Example - Continuous Variables

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- ▷ Uniform distribution. The CDF is

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases} \quad . \text{ The PDF is}$$

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases} .$$

- ▷ Exponential distribution. The CDF is

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - \exp(-x) & \text{if } x \geq 0 \end{cases} \quad . \text{ The PDF is}$$

$$f(x) = \exp(-x), x \geq 0.$$

If X is a continuous random variable with the density function $f(x)$, its expectation is defined as

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} xf(x)dx$$

when the integral is convergent.

The expectation of the function of X , $g(X)$ is computed as

$$\mathbb{E}(g(X)) = \sum_{x \in \mathcal{X}} \int_{-\infty}^{\infty} g(x)f(x)dx.$$

Some examples:

- ▷ $f(x) = 1$ if $0 \leq x \leq 1$, $\mathbb{E}(X) = \int_0^1 xf(x) = 0.5$.
- ▷ $f(x) = \exp(-x)$ if $x \geq 0$,
 $\mathbb{E}(X) = \int_0^{\infty} x \exp(-x)dx = 1$ (integration by part).

Mean, variance and Higher Moment

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Suppose X is a random variable (either discrete or continuous).

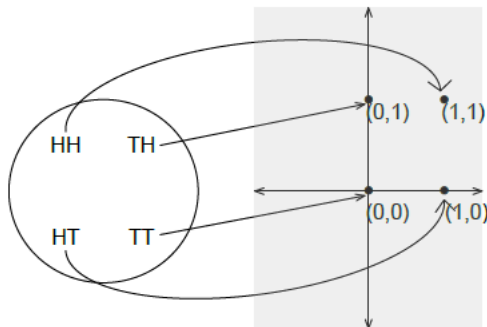
- ▷ The **mean** of X is $\mu = \mathbb{E}(X)$.
- ▷ The **variance** of X is $\sigma^2 = \text{var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$.
 - ◇ The **standard deviation** of X is the positive root of the variance, $\sigma = \sqrt{\sigma^2}$.
- ▷ The m -th **moment** of X is $\mu'_m = \mathbb{E}(X^m)$ and the m -th **central moment** of X is $\mu_m = \mathbb{E}((X - \mathbb{E}(X))^m)$.
 - ◇ The **skewness** of X is defined as $\text{skewness} = \frac{\mathbb{E}((X - \mathbb{E}(X))^3)}{\sigma^3}$. If the distribution is symmetric, the skewness is 0.
 - ◇ The **kurtosis** of X is defined as $\text{skewness} = \frac{\mathbb{E}((X - \mathbb{E}(X))^4)}{\sigma^4}$.

Bivariate random variables

A pair of **bivariate random variables** is a pair of numerical outcomes; a function from the sample space to \mathbb{R}^2 .

A pair of bivariate random variables are typically represented by a pair of uppercase Latin characters such as (X, Y) . Specific values will be written by a pair of lower case characters, e.g. (x, y) .

Figure: Tossing two coins



Joint distribution functions

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The **joint distribution function (Joint CDF)** of (X, Y) is defined as $F(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(\{X \leq x\} \cap \{Y \leq y\})$.

- ▷ A pair of random variables is **discrete** if there is a discrete set $(P) \in \mathbb{R}^2$ such that $\mathbb{P}((X, Y) \in \mathcal{P}) = 1$.
 - ◇ The set \mathcal{P} is the support of (X, Y) and consists of a set of points in \mathbb{R}^2 .
 - ◇ The **joint probability mass function** is defined as $p(x, y) = \mathbb{P}(X = x, Y = y)$.
- ▷ The pair (X, Y) has a continuous distribution if the joint distribution function $F(x, y)$ is **continuous** in (x, y) .
 - ◇ When $F(x, y)$ is continuous and differentiable its **joint density (joint PDF)** $f(x, y)$ equals $f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}$.

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The **expected value** of real-valued $g(X, Y)$ is

$$\mathbb{E}(g(X, Y)) = \sum_{(x,y) \in \mathbb{R}^2, \pi(x,y) > 0} g(x, y) \pi(x, y),$$

and

$$\mathbb{E}(g(X, Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy.$$

Marginal distribution

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The **marginal distribution(marginal CDF)** of X is

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \leq x, Y \leq \infty) = \lim_{y \rightarrow \infty} F(x, y).$$

▷ In the continuous case,

$$F_X(x) = \lim_{y \rightarrow \infty} \int_{-\infty}^y \int_{-\infty}^x f(u, v) du dv = \int_{-\infty}^{\infty} \int_{-\infty}^x f(u, v) du dv.$$

The **marginal densities(marginal PDF)** of X is the derivative of the marginal CDF of X ,

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \int_{-\infty}^{\infty} \int_{-\infty}^x f(u, v) du dv = \int_{-\infty}^{\infty} f(x, y) dy.$$

Conditional distributions I

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The conditional cumulative distributions:

- ▶ The **conditional distribution function** of Y given $X = x$ is

$$F_{Y|X}(y|x) = \mathbb{P}(Y \leq y | X = x)$$

for any x such that $\mathbb{P}(X = x) > 0$, If X has a discrete distribution.

- ▶ For continuous X, Y , the **conditional distribution** of Y given $X = x$ is

$$F_{Y|X}(y|x) = \lim_{\epsilon \downarrow 0} \mathbb{P}(Y \leq y | x - \epsilon \leq X \leq x + \epsilon).$$

The conditional density:

- ▶ For continuous variable (X, Y) , the conditional density function (conditional PDF) is defined by $f_{Y|X}(y|x) = \frac{d}{dy} F_{Y|X}(y|x)$.

- ▶ Recall that two events A and B are independent if the probability that they both occur equals the product of their probabilities, thus $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
- ▶ Consider the events $A = \{X \leq x\}$ and $B = \{Y \leq y\}$.
- ▶ The probability that they both occur is $\mathbb{P}(A \cap B) = \mathbb{P}(X \leq x, Y \leq y) = F(x, y)$.
- ▶ If $F(x, y) = F_X(x)F_Y(y)$ then $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
- ▶ The random variables X and Y are **statistically independent** if for all x, y , $F(x, y) = F_X(x)F_Y(y)$.
- ▶ If X, Y have differentiable density function, X, Y are statistically independent if $f(x, y) = f_X(x)f_Y(y)$.
- ▶ The discrete random variables X and Y are **statistically independent** if for all x, y , $\pi(x, y) = \pi_X(x)\pi_Y(y)$.

- ▷ If X, Y are independent and continuously distributed, then

$$f_{Y|X}(y|x) = f_Y(y),$$

$$f_{X|Y}(x|y) = f_X(x).$$

- ▷ If X and Y are independent then for any functions, $g : \mathbb{R} \rightarrow \mathbb{R}$ and $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $\mathbb{E}|g(X)| < \infty$ and $\mathbb{E}|h(Y)| < \infty$, then

$$\mathbb{E}(g(X)h(Y)) = \mathbb{E}_X(g(X))\mathbb{E}_Y(h(Y)).$$

Theorem 6

(Bayes Theorem for Continuous Variables)

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)} = \frac{f_{X|Y}(x|y)f_Y(y)}{\int_{-\infty}^{\infty} f(x, y)dy}.$$

Covariance and correlation I

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- ▷ If X and Y have finite variances, the **covariance** between X and Y is

$$\text{cov}(X, Y) = \mathbb{E}(X - \mathbb{E}(X))(Y - \mathbb{E}(Y)) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

- ▷ The **correlation** between X and Y is

$$\text{corr}(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}}.$$

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- ▷ If X and Y are independent with finite variances, then X and Y are uncorrelated.
 - ◇ The reverse is not true. For example, suppose that $X \sim U[-1, 1]$. Since it is symmetrically distributed about 0 we see that $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^3] = 0$. Set $Y = X^2$. Then $\text{cov}(X, Y) = \mathbb{E}[X^3] - \mathbb{E}[X^2]\mathbb{E}[X] = 0$. Thus X and Y are uncorrelated yet are fully dependent!
- ▷ If X and Y have finite variances,

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y).$$
- ▷ If X and Y are independent, $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$.

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Just as the expectation is the central tendency of a distribution, the conditional expectation is the central tendency of a conditional distribution.

The **conditional expectation(conditional mean)** of Y given $X = x$ is the expected value of the conditional distribution $F_{Y|X}(y|x)$ and is written as $\mathbb{E}(Y|X = x)$.

- ▶ For discrete random variables, it is defined as

$$\mathbb{E}(Y|X = x) = \frac{\sum_y y\pi(x, y)}{\pi_X(x)}.$$

- ▶ For continuous random variables, it is defined as

$$\mathbb{E}[Y|X = x] = \frac{\int_y yf(x, y)}{f_X(x)}.$$

Law of Iterated Expectations I

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- ▷ Consider $m(X) = \mathbb{E}[Y|X]$ a transformation of X .
- ▷ We can take expectation with respect to $m(X)$

Theorem 7 (Law of Iterated Expectations(LIE))

If $\mathbb{E}[Y] < \infty$, then $\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[Y]$.

Definition 8 (Standard Normal Dist.)

A random variable Z has the **standard normal distribution**, write $Z \sim N(0, 1)$, if it has the density

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), x \in \mathbb{R}.$$

Note the standard normal density is typically written as $\phi(x)$. The CDF does not have closed form but is written as $\Phi(x)$.

If $X \sim N(\mu, \sigma^2)$ and $\sigma > 0$ then X has the density

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right), x \in \mathbb{R}.$$

Multivariate Normal I

Let Z_1, \dots, Z_m be i.i.d $N(0, 1)$. The joint density is the product of the marginal densities:

$$\begin{aligned} f(x_1, \dots, x_m) &= f(x_1) \dots f(x_m) \\ &= \frac{1}{(2\pi)^{m/2}} \exp \left(-\frac{1}{2} \sum_{i=1}^m x_i^2 \right). \end{aligned}$$

Multivariate Normal II

Let $\mathbf{Z} = [Z_1, \dots, Z_m]^\top$ be an m -component random vector following standard normal distribution $\mathbf{Z} \sim N(0, \mathbf{I}_m)$ and $\mathbf{X} = \mu + \mathbf{B}\mathbf{Z}$ for $q \times m$ matrix \mathbf{B} , then \mathbf{X} has the **multivariate normal distribution**, written as $\mathbf{X} \sim N(\mu, \Sigma)$, where $\Sigma = \mathbf{B}\mathbf{B}^\top$.

The PDF of \mathbf{X} is given by

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{q/2}(\det \Sigma)^{1/2}} \exp \left(-\frac{(\mathbf{x} - \mu)^\top \Sigma^{-1}(\mathbf{x} - \mu)}{2} \right).$$

Properties of multivariate normal distributions.

1. Any linear combination of X_1, \dots, X_m is normally distributed.
2. The marginal distribution of each random variable is normal.
3. If the covariance of X_1 and X_2 is 0, then X_1 and X_2 are independent. The reverse is true.
4. If X_1 and X_2 are normally distributed with the joint density of $f(x_1, x_2)$, then the marginal distribution of X_1 given X_2 is a linear function of X_2 : $\mathbb{E}[X_1|X_2 = x_2] = a + bx_2$.

χ -squared, t and F distribution

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The **chi-squared distribution** is the distribution of the sum of m squared independent standard normal distributed variables.

- ▷ Let $\mathbf{Z} \sim N(0, \mathbf{I}_m)$ be multivariate standard normal, then $\mathbf{Z}^\top \mathbf{Z} \sim \chi_m^2$.
- ▷ If $\mathbf{X} \sim N(0, \Sigma)$ with Σ positive definite, then $\mathbf{X}^\top \Sigma^{-1} \mathbf{X}$.
- ▷ Let $Q_m \sim \chi_m^2$ and $Q_r \sim \chi_r^2$ be independent. Then $\frac{Q_m/m}{Q_r/r} \sim F_{m,r}$.
- ▷ Let $Z \sim N(0, 1)$ and $Q_m \sim \chi_m^2$ be independent, then $\frac{Z}{\sqrt{Q_m/r}}$ follows the t -distribution with m degree of freedom, t_m .

Almost all the statistical and econometric procedures used in this text involve averages or weighted averages of a sample of data.

- ▶ Characterizing the distributions of sample averages.
- ▶ The act of random sampling is a random variable, its distribution is **sampling distribution**.

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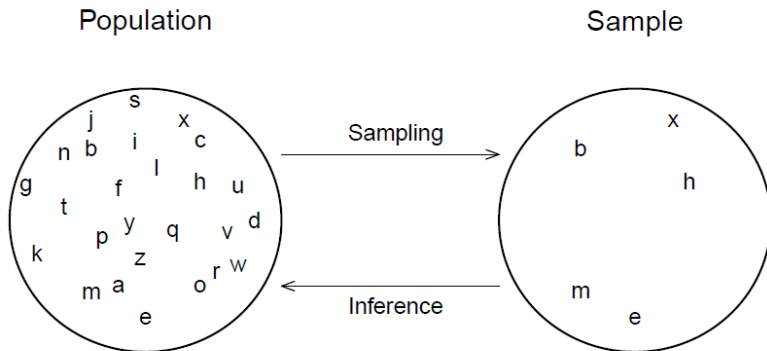
1. **Simple random sampling.** e.g. A commuting student record daily traffic time.
2. **i.i.d. draws.** e.g. Y_1, \dots, Y_n are randomly drawn from the same population.

Definition 9 (independent and identically distributed(i.i.d))

The collection of random vectors $\{X_1, \dots, X_n\}$ are **independent and identically distributed(i.i.d)** if they are mutually independent with identical marginal distributions.

- ▷ A collection of random vectors $\{X_1, \dots, X_n\}$ is a **random sample** from the population F if X_i are i.i.d with distribution F .
- ▷ The distribution F is called the **population distribution**. We refer to the distribution as the **data generating process(DGP)**.
- ▷ The **sample size** n is the number of individuals in the sample.

Figure: Sampling and Inference



There are two approaches to characterizing sampling distributions:

1. **the exact approach** derives: the **exact distribution** or **finite-sample distribution** of Y .
2. **the approximate approach** uses approximations to the sampling distribution that rely on the sample size being large, often called the asymptotic distribution.

Two useful tools when the sample size is large: **the law of large numbers** and the **central limit theorem**.

- ▷ The law of large numbers says that when the sample size is large, \bar{Y} will be close to μ_Y with very high probability.
- ▷ The central limit theorem says that when the sample size is large, the sampling distribution of the standardized sample average, $(\bar{Y} - \mu_Y)/\sigma_Y$, is approximately normal.

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- ▷ A sequence of random variables $Z_n \in \mathbb{R}$ **converges in probability** to c as $n \rightarrow \infty$, denoted by $Z_n \rightarrow_p c$ or $\text{plim}_{n \rightarrow \infty} Z_n = c$, if for all $\delta > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|Z_n - c| \leq \delta) = 1.$$

Theorem 10 (Weak Law of Large Numbers(WLLN))

If X_i are independent and identically distributed and $\mathbb{E}(X) < \infty$, then as $n \rightarrow \infty$,

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i \rightarrow_p \mathbb{E}(X).$$

An estimator $\hat{\theta}$ of a parameter θ is **consistent** if $\hat{\theta} \rightarrow_p \theta$ as $n \rightarrow \infty$.

Counter examples:

- ▷ $X_i = Z + U_i$, Z is common component and $\mathbb{E}[U] = 0$, $\bar{X}_i \rightarrow_p Z$ but not the sample mean.
- ▷ Suppose $\text{var}(X_i) = 1$ for $i \leq n/2$ and $\text{var}(X_i) = n$ for $i > n/2$. $\text{var}(\bar{X}_n) \rightarrow 1/2$, \bar{X}_n does not converge in probability.

Continuous Mapping Theorem

Theorem 11 (Continuous Mapping Theorem)

If $Z_n \rightarrow_p c$ as $n \rightarrow \infty$ and $h(\cdot)$ is continuous at c then $h(Z_n) \rightarrow_p h(c)$ as $n \rightarrow \infty$.

Theorem 12 (Central Limit Theorem(CLT))

If X_i are i.i.d. and $\mathbb{E}(X^2) < \infty$ then as $n \rightarrow \infty$

$$\sqrt{n}(\bar{X}_n - \mu) \rightarrow_d N(0, \sigma^2),$$

where $\mu = \mathbb{E}(X)$ and $\sigma^2 = \mathbb{E}[(X - \mu)^2]$.

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Theorem 13 (Slutsky's Theorem)

If $Z_n \rightarrow_d Z$ and $c_n \rightarrow_p c$ as $n \rightarrow \infty$, then

1. $Z_n + c_n \rightarrow_d Z + c$
2. $Z_n c_n \rightarrow_d Z c$
3. $\frac{Z_n}{c_n} \rightarrow_d \frac{Z}{c}$ if $c \neq 0$.

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