Review of Probability [1]

Jasmine(Yu) Hao

Faculty of Business and Economics
Hong Kong University

August 8, 2021

[1] This section is based on ?, Chapter 2.

Probabilities and Outcomes

- An **outcomes** is a specific result:
  - Coin toss: either H or T.
  - Roll of dice: $1, 2 \ldots, 6$.
- The **probability** of an **outcome** is the proportion of the time that the outcome occurs in the long run.
  - Fair coin toss: 50 % chance of heads.
- The **sample space** is the set of all possible outcomes.
  - In a coin flip the sample space is $S = \{H, T\}$.
  - If two coins are flipped the sample space is $S = \{HH, HT, TH, TT\}$.
- An **event** is a subset of the sample space.
  - Roll a die $A = \{1, 2\}$.

Probability Function

Definition 1 (Probability Function)

A function $\mathbb{P}$ which assigns a numerical value to events is called a probability function if it satisfies the following Axioms of Probability:

1. $\mathbb{P}(A) \geq 0$.
2. $\mathbb{P}(S) = 1$.
3. If $A_1, A_2, \ldots$ are disjoint then $\mathbb{P}| \cup_{j=1}^{N} A_j| = \sum_{j=1}^{N} A_j$.

Events

Theorem 2 (Properties of probability functions)

For two events, A and B,

1. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.
2. $\mathbb{P}(\emptyset) = 0$.
3. $\mathbb{P}(A) \leq 1$.
4. Monotone Probability Inequality*: If $A \subset B$, $\mathbb{P}(A) \leq \mathbb{P}(B)$.*
5. Inclusion-Exclusion Principle*:*
   $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.
6. Boole's Inequality*: $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$.*
7. Bonferroni's Inequality*: $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$.*

Exercise: show **Bonferroni's Inequality**.(Hint: use the above theorems.)

Conditional Probability

Definition 3 (Conditional Probability)

If $\mathbb{P}(B) > 0$, then the **conditional probability** of A given B is given by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cup B)}{\mathbb{P}(B)}.$$

$\mathbb{P}(B)$ is the **marginal probability** of event B.

$$\mathbb{P}(B) = \mathbb{P}(A \cup B) + \mathbb{P}(A^c \cup B).$$

Joint Events

Take two events H and C.

- let be H the event that an individual's monthly wage exceeds RMB 8000,
- let M be the event that the individual has a master's degree.

Table: Joint Distribution

	Master degree	Non-master degree	Any education
High wage	0.19	0.12	0.31
Low wage	0.17	0.52	0.69
Any wage	0.36	0.64	1.00

Conditional Probability - Example

The probability of earning a high wage conditional on high education is

$$
\begin{aligned}
&\mathbb{P}(\text{High wage}|\text{Master degree}) \\
=&\frac{\mathbb{P}(\text{High wage} \cup \text{Master degree})}{\mathbb{P}(\text{High wage} \cup \text{Master degree}) + \mathbb{P}(\text{Low wage} \cup \text{Master degree})} \\
=&\frac{0.16}{0.36} = 0.53.
\end{aligned}
$$

Similarly, the probability of earning a high wage conditional on non-master degree is
$\mathbb{P}(\text{High wage}|\text{Non-master degree}) = \frac{0.12}{0.64} = 0.19.$

Independence

Definition 4 (Independence)

The events A and B are independent if $\mathbb{P}(A \cup B) = \mathbb{P}(A)\mathbb{P}(B)$

Theorem 5 (Independence)

If A and B are independent with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$, then

$$\mathbb{P}(A) = \mathbb{P}(A|B), \mathbb{P}(B) = \mathbb{P}(B|A).$$

Some facts:

- When events are independent then joint probabilities can be calculated by multiplying individual probabilities.
- If A and B are disjoint then they cannot be independent.

Bayes Rule

Theorem 6 (Bayes Rule)

If $\mathbb{P}(A) > 0$ *and* $\mathbb{P}(B) > 0$ *then*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c)}.$$

Proof.

Use the Properties of probability functions. $\square$

Random Variables

Definition 7 (Random variable)

A random variable is a real-valued outcome; a function from the sample space S to the real line $\mathbb{R}$.

For example, X is a mapping from the coin flip sample space to the real line, with T mapped to 0 and H mapped to 1.

$$X = \begin{cases} 1 & \text{if H} \\ 0 & \text{if T.} \end{cases}$$

Properties of random variables.

- The **expected value** is the long-run average of the random variable.
- The **standard deviation** measures the spread of a probability distribution.

Discrete Random Variables

The set $\mathcal{X}$ is discrete if it has a finite or countably infinite number of elements.

Definition 8 (Discrete random variable)

If there is a discrete set $\mathcal{X}$ such that $\mathbb{P}(X \in \mathcal{X}) = 1$ then X is a discrete random variable.

The smallest set $\mathcal{X}$ with this property is the support of X.

Definition 9 (Probability mass function)

The probability mass function of a random variable is $\pi(x) = \mathbb{P}(X = x)$, the probability that X equals the value x.

- The **probability distribution** of a discrete random variable is the list of all possible values of the variable and the probability that each value will occur.

Expectation

For a dsicrete variable X with the support of $\mathcal{X}$, the expctation is computed as

$$\mathbb{E}(X) = \sum_{x \in \mathcal{X}} \pi(x) x.$$

- $X = 1$ with the probability of p and $X = 0$ with probability $1 - p$. The expected value is $\mathbb{E}(X) = 1 * p + 0 * (1 - p) = p$.

The expectation of the function of X, $g(X)$ is computed as

$$\mathbb{E}(g(X)) = \sum_{x \in \mathcal{X}} \pi(x) g(x).$$

St. Petersburg Paradox

Note that the expectation of a distribution does not neccesarily exist.

- A single player in which a fair coin is tossed at each stage.
- The initial stake begins at \$ 2 and is doubled every time heads appears.
- The first time a tail appears, the game ends and the player wins whatever is in the pot.
- How much would a rational agent pay to get in the bet.

X has the support of $\mathcal{X} = \{2^k : k = 1, \ldots\}$. The probability distribution is defined by $\pi(2^k) = 2^{-k}$. Then the expectation of X is

$$\mathbb{E}(x) = \sum_{k=1}^{\infty} 2^k \pi(2^k) = 1 + 1 \ldots = \infty.$$

Cumulative distribution function

The **cumulative probability distribution(*CDF*)** is the probability that the random variable is less than or equal to a particular value,

$$F(x) = \mathbb{P}(X \leq x),$$

where the probability event is $X \leq x$.

Theorem 10 (Properties of a CDF)

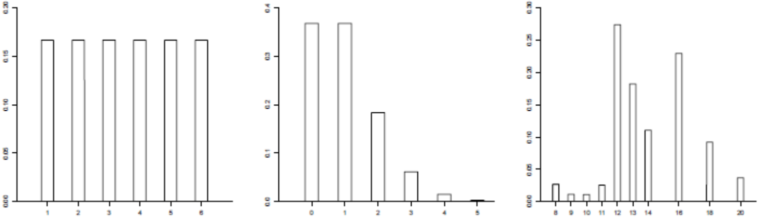
If $F(x)$ is a distribution function, then

1. *$F(x)$ is non-decreasing.*
2. $\lim_{x \to -\infty} F(x) = 0$.
3. $\lim_{x \to \infty} F(x) = 1$.
4. *$F(x)$ is right-continuous,* $\lim_{x \downarrow x_0} F(x) = F(x_0)$.

Example-Probability mass function

Some examples for discrete variables.

- For a fair dice toss, the support is $\mathcal{X} = \{1, 2\ldots, 6\}$ with the probability mass function is $\pi(x) = \frac{1}{6}$ for $x \in \mathcal{X}$.
- An example of infinite countable random variable is the Poisson distribution, the probability mass function is $\pi(x) = \frac{e^{-1}}{x!}, x = 0, 1, \ldots$.

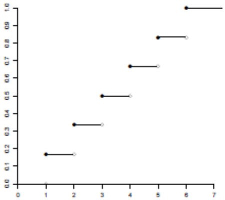


(a) Die Toss (b) Poisson (c) Education

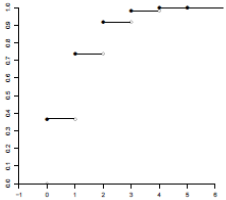
Example-Probability function

Some examples for discrete variables.

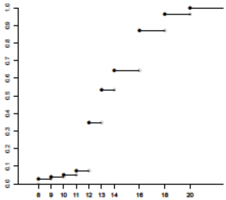
- For a fair dice toss, the support is $\mathcal{X} = \{1, 2 \ldots, 6\}$ with the probability mass function is $\pi(x) = \frac{1}{6}$ for $x \in \mathcal{X}$.
- An example of infinite countable random variable is the Poisson distribution, the probability mass function is $\pi(x) = \frac{e^{-1}}{x!}, x = 0, 1, \ldots$.



(a) Die Toss



(b) Poisson



(c) Education

Continuous Random Variables

- The **probability density function($p.d.f$)** area under the probability density function between any two points is the probability that the random variable falls between those two points.
  - the probability for a continous variable to take any value is 0.
  - definition is different from discrete random variables.
- When $F(x)$ is differentiable, the density function is $f(x) = \frac{dF(x)}{dx}$.

Theorem 11 (Properties of density function)

A function $f(x)$ is a density function if and only if

- $f(x) \geq 0 \forall x$.
- $\int_0^\infty f(x)dx = 1$.

Example - Continuous Variables

- Uniform distribution. The CDF is
  $$F(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x & \text{if} \quad 0 \leq x \leq 1 \\ 1 & \text{if} \quad x > 1 \end{cases}.$$ The PDF is
  $$f(x) = \begin{cases} 1 & \text{if} \qquad 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}.$$
- Exponential distribution. The CDF is
  $$F(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 - \exp(-x) & \text{if} x \geq 0 \end{cases}.$$ The PDF is
  $f(x) = \exp(-x), x \geq 0.$

Expectation

If X is a continuous random variable with the density function $f(x)$, its expectation is defined as

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} xf(x)dx$$

when the integral is convergent.
The expectation of the function of X, $g(X)$ is computed as

$$\mathbb{E}(g(X)) = \sum_{x \in \mathcal{X}} \int_{-\infty}^{\infty} g(x)f(x)dx.$$

Some examples:

- $f(x) = 1$ if $0 \leq x \leq 1$, $\mathbb{E}(X) = \int_0^1 xf(x) = 0.5$.
- $f(x) = \exp(-x)$ if $x \geq 0$,
  $\mathbb{E}(X) = \int_0^{\infty} x \exp(-x)dx = 1$(integration by part).

Mean, variance and Higher Moment

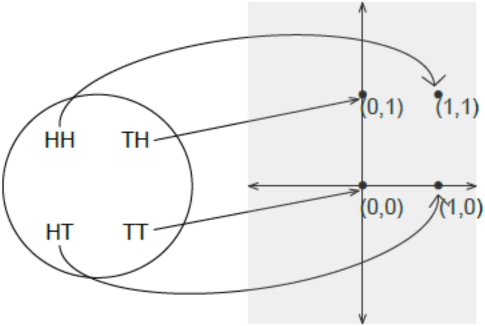
Suppose X is a random variable(either discrete or continous).

- The **mean** of X is $\mu = \mathbb{E}(X)$.
- The **variance** of X is $\sigma^2 = var(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$.
    - The **standard deviation** of X is the positive root of the variance, $\sigma = \sqrt{\sigma^2}$.
- The $m-th$ **moment** of X is $\mu'_m = \mathbb{E}(X^m)$ and the $m-th$ **central moment** of X is $\mu_m = \mathbb{E}((X - \mathbb{E}(X))^m)$.
    - The **skewness** of X is defined as $skewness = \frac{\mathbb{E}((X-\mathbb{E}(X))^3)}{\sigma^3}$. If the distribution is symmetric, the skewness is 0.
    - The **kurtosis** of X is defined as $skewness = \frac{\mathbb{E}((X-\mathbb{E}(X))^4)}{\sigma^4}$.

Bivariate random variables

A pair of **bivariate random variables** is a pair of numerical outcomes; a function from the sample space to $\mathbb{R}^2$.
A pair of bivariate random variables are typically represented by a pair of uppercase Latin characters such as (X, Y). Specific values will be written by a pair of lower case characters, e.g. (x, y).

Figure: Tossing two coins



Joint distribution functions

The **joint distribution function(Joint CDF)** of (X, Y) is defined as $F(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(\{X \leq x\} \cap \{Y \leq y\})$.

- A pair of random variables is **discrete** if there is a discrete set $(P) \in \mathbb{R}^2$ such that $\mathbb{P}((X, Y) \in \mathscr{P}) = 1$.
  - The set $\mathcal{P}$ is the support of (X, Y) and consists of a set of points in $\mathbb{R}^2$.
  - The **joint probability mass function** is defined as $p(x,y) = \mathbb{P}(X = x, Y = y)$.
- The pair (X, Y) has a continuous distribution if the joint distribution function $F(x,y)$ is **continuous** in (x,y).
  - When $F(x,y)$ is continuous and differentiable its **joint density (joint PDF)** $f(x,y)$ equals $f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$.

Bivariate expectation

The **expected value** of real-valued $g(X, Y)$ is

$$\mathbb{E}(g(X, Y)) = \sum_{(x,y)\in\mathbb{R}^2, \pi(x,y)>0} g(x, y)\pi(x, y),$$

for discrete variables and

$$\mathbb{E}(g(X, Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)f(x, y)dxdy.$$

Marginal distribution

The **marginal distribution(marginal CDF)** of X is

$$F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(X \le x, Y \le \infty) = \lim_{y\to\infty} F(x, y).$$

- In the continuous case,

$$F_X(x) = \lim_{y\to\infty} \int_{-\infty}^{y} \int_{-\infty}^{x} f(u, v)dudv = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(u, v)dudv.$$

- The **marginal densities(marginal PDF)** of X is the derivative of the marginal CDF of X,

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{d}{dx} \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(u, v)dudv = \int_{-\infty}^{\infty} f(x, y)dy.$$

- Similarly, the marginal PDF of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx.$$

Conditional distributions I

The conditional cumulative distributions:

- The **conditional distribution function** of Y given $X = x$ is

$$F_{Y|X}(y|x) = \mathbb{P}(Y \leq y|X = x)$$

  for any x such that $\mathbb{P}(X = x) > 0$, If X has a discrete distribution.

- For continous X, Y, the **conditional distribution** of Y given $X = x$ is

$$F_{Y|X}(y|x) = \lim_{\epsilon \downarrow 0} \mathbb{P}(Y \leq y|x - \epsilon \leq X \leq x + \epsilon).$$

  If $F(x, y)$ is differentiable w.r.t x and $f_X(x) > 0$,

$$F_{Y|X}(y|x) = \frac{\frac{\partial}{\partial x} F(x, y)}{f_X(x)}.$$

The conditional density:

- For continous variable (X, Y), the conditional density function (conditional PDF) is defined by $f_{Y|X}(y|x) = \frac{d}{dy} f_{Y|X}(y|x)$.

Independence I

- Recall that two events A and B are independent if the probability that they both occur equals the product of their probabilities, thus $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
- Consider the events $A = \{X \leq x\}$ and $B = \{Y \leq y\}$.
- The probability that they both occur is $\mathbb{P}(A \cap B) = \mathbb{P}(X \leq x, Y \leq y) = F(x, y)$.
- If $F(x, y) = F_X(x)F_Y(y)$ then $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
- The random variables X and Y are **statistically independent** if for all x, y, $F(x, y) = F_X(x)F_Y(y)$.
- The discrete random variables X and Y are **statistically independent** if for all x, y, $\pi(x, y) = \pi_X(x)\pi_Y(y)$.
- If X, Y have differentiable density function, X, Y are statistically independent if $f(x, y) = f_X(x)f_Y(y)$.

Independence II

Theorem 12

If X, Y are independent and continuously distributed, then

$$f_{Y|X}(y|x) = f(y),$$
$$f_{X|Y}(x|y) = f(x).$$

Theorem 13

(Bayes Theorem for Densities)

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)} = \frac{f_{X|Y}(x|y)f_Y(y)}{\int_{-\infty}^{\infty} f(x,y)dy}.$$

Independence III

Theorem 14

If X and Y are independent then for any functions, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}|g(X)| < \infty$ and $\mathbb{E}|h(Y)| < \infty$, then

$$\mathbb{E}(g(X)h(G)) = \mathbb{E}_X(g(X))\mathbb{E}_Y(h(Y)).$$

Covariance and correlation I

- If X and Y have finite variances, the **covariance** between X and Y is

$$cov(X,Y) = \mathbb{E}(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

- The **correlation** between X and Y is

$$corr(X,Y) = \frac{cov(X,Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}}.$$

Covariance and correlation II

- If X and Y are independent with finite variances, then X and Y are uncorrelated.
  - The reverse is not true. For example, suppose that $X \sim U[-1,1]$. Since it is symmetrically distributed about 0 we see that $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^3] = 0$. Set $Y = X^2$. Then $cov(X, Y) = \mathbb{E}[X^3] - \mathbb{E}[X^2]\mathbb{E}[X] = 0$. Thus X and Y are uncorrelated yet are fully dependent!
- If X and Y have finite variances, $\text{var}(X, Y) = \text{var}(X) + \text{var}(Y) + 2cov(X, Y)$.
- If X and Y are independent, $\text{var}(X, Y) = \text{var}(X) + \text{var}(Y)$.

Conditional Expectation

Just as the expectation is the central tendency of a distribution, the conditional expectation is the central tendency of a conditional distribution.

The **conditional expectation(conditional mean)** of Y given $X = x$ is the expected value of the conditional distribution $F_{Y|X}(y|x)$ and is written as $\mathbb{E}(Y|X = x)$.

- For discrete random variables, it is defined as

$$\mathbb{E}(Y|X = x) = \frac{\sum_y y\pi(x, y)}{\pi_X(x)}.$$

- For continuous random variables, it is defined as

$$\mathbb{E}[Y|X = x] = \frac{\int_y yf(x, y)}{f_X(x)}.$$

Law of Iterated Expectations I

- The function $\mathbb{E}[Y|X=x]$ is not random, it is a feature of the distribution function. But it is useful to treat the conditional expectation as a random variable.
- Consider $m(X) = \mathbb{E}[Y|X]$ a transformation of X.
- We can take expectation with respect to $m(X)$

Theorem 15 (Law of Iterated Expectations(LIE))

If $\mathbb{E}[Y] < \infty$, *then* $\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[Y]$.

Law of Iterated Expectations II

Proof.

For discrete random variables X, Y,

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[Y|X]] &= \sum_x \pi_X(x)\mathbb{E}[Y|X=x] \\
&= \sum_x \pi_X(x)\frac{\sum_y y\pi(x,y)}{\pi_X(x)} \\
&= \sum_x \sum_y y \\
&= \mathbb{E}[Y].
\end{aligned}$$

Law of Iterated Expectations III

Proof.(Cont.)

For continuous random variables X, Y,

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[Y|X]] &= \int_{-\infty}^{\infty} f_X(x)\mathbb{E}[Y|X=x]dx \\
&= \int_{-\infty}^{\infty} f_X(x)\frac{\int_{-\infty}^{\infty} yf(x,y)fy}{\pi_X(x)}dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \ dy \ dx \\
&= \mathbb{E}[Y].
\end{aligned}$$

$\square$

Conditional Variance

The **conditional variance** of Y given $X = x$ is the variance of the condition distribution $F_{Y|X}(y|x)$ and is written as $\mathrm{var}(Y|X = x)$ or $\sigma_Y^2(x)$. It equals

$$\mathrm{var}(Y|X = x) = \mathbb{E}[(Y - m(x))^2|X = x],$$

where $m(x) = \mathbb{E}[Y|X = x]$.
Note that $\mathrm{var}(Y) = \mathbb{E}[\mathrm{var}(Y|X)] + \mathrm{var}(\mathbb{E}[Y|X])$, where

- $\mathbb{E}[\mathrm{var}(Y|X)]$ is the **within group variance**.
- $\mathrm{var}(\mathbb{E}[Y|X])$ is the **across group variance**.

Univariate Normal

Definition 16 (Standard Normal Dist.)

A random variable Z has the **standard normal distribution**, write $Z \sim N(0,1)$, if it has the density

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}), x \in \mathbb{R}.$$

Note the standard normal density is typically written as $\phi(x)$. The CDF does not have closed form but is written as $\Phi(x)$.
If $X \sim N(\mu, \sigma^2)$ and $\sigma > 0$ then X has the density

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-\mu)^2}{2\sigma^2}), x \in \mathbb{R}.$$

Multivariate Normal I

Let $Z_1, \dots, Z_m$ be i.i.d $N(0,1)$. The joint density is the product of the marginal densities:

$$\begin{aligned} f(x_1, \dots, x_m) &= f(x_1) \dots f(x_m) \\ &= \frac{1}{(2\pi)^{m/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{m} x_i^2\right). \end{aligned}$$

Multivariate Normal II

Let $\mathrm{Z} = [Z_1, \ldots, Z_m]^\top$ be an m-component random vector following standard normal distribution $\mathrm{Z} \sim N(0, \mathrm{I}_m)$ and $\mathrm{X} = \mu + \mathrm{BZ}$ for $q \times m$ matrix B, then X has the **multivariate normal distribution**, written as $\mathrm{X} \sim N(\mu, \Sigma)$, where $\Sigma = \mathrm{BB}^\top$.
The PDF of X is given by

$$f(\mathrm{x}) = \frac{1}{(2\pi)^{q/2}(\det\Sigma)^{1/2}} \exp\left(-\frac{(x-\mu)^\top\Sigma^{-1}(x-\mu)}{2}\right).$$

Multivariate Normal III

Properties of multivariate normal distributions.

1. Any linear combination of $X_1, \ldots, X_m$ is normally distributed.
2. The marginal distribution of each random variable is normal.
3. If the covariance of X_1 and X_2 is 0, then X_1 and X_2 are independent. The reverse is true.
4. If X_1 and X_2 are normally distributed with the joint density of $f(x_1, x_2)$, then the marginal distribution of X_1 given X_2 is a linear function of X_2: $\mathbb{E}[X_1|X_2 = x_2] = a + bx_2$.

χ-squared, t and F distribution

The **chi-squared distribution** is the distribution of the sum of m squared independent standard normal distributed variables.

- Let $\mathrm{Z} \sim N(0, \mathrm{I}_m)$ be multivariate standard normal, then $\mathrm{Z}^\top \mathrm{Z} \sim \chi^2_m$.
- If $mbX \sim N(0, \Sigma)$ with Σ positive definite, then $\mathrm{X}^\top \Sigma^{-1} \mathrm{X}$.
- Let $Q_m \sim chi^2_m$ and $Q_r \sim chi^2_r$ be independent. Then $\frac{Q_m/m}{Q_r/r} \sim F_{m,r}$.
- Let $Z \sim N(0,1)$ and $Q_m \sim chi^2_m$ be independent, then $\frac{Z}{\sqrt{Q_r/r}}$ follows the t-distribution with m degree of freedom, t_m.

Sampling I

Almost all the statistical and econometric procedures used in this text involve averages or weighted averages of a sample of data.

- Characterizing the distributions of sample averages therefore is an essential step toward understanding the performance of econometric procedures.
- The act of random sampling—that is, randomly drawing a sample from a larger population—has the effect of making the sample average itself a random variable.
- Because the sample average is a random variable, it has a probability distribution, which is called its **sampling distribution**.

Sampling II

1. **Simple random sampling**. A commuting student aspires to be a statistician and decides to record her commuting times on various days. She selects these days at random from the school year, and her daily commuting time has the cumulative distribution function. Because these days were selected at random, knowing the value of the commuting time on one of these randomly selected days provides no information about the commuting time on another of the days; that is, because the days were selected at random, the values of the commuting time on the different days are independently distributed random variables.
2. **i.i.d. draws**. Because $Y_1, \ldots, Y_n$ are randomly drawn from the same population, the marginal distribution of Y_i is the same for each $i = 1, \ldots, n$; this marginal distribution is the distribution of Y in the population being sampled. When Y_i has the same marginal distribution for $i = 1, \ldots, n$, then $Y_1, \ldots, Y_n$ are said to be identically distributed. Under simple random sampling, knowing the value of Y_1 provides no information about Y_2,

i.i.d I

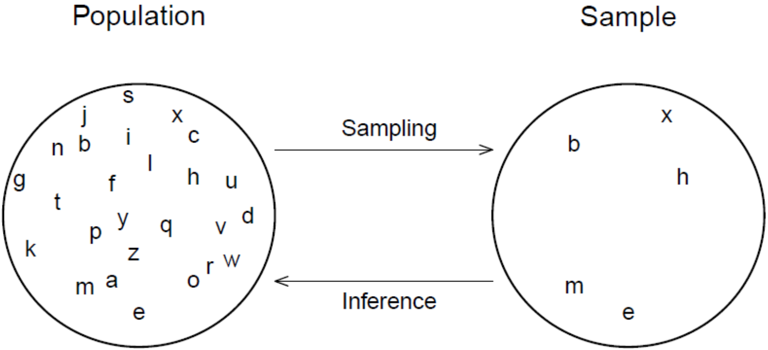
Definition 17 (independent and identically distributed(i.i.d))

The collection of random vectors $\{X_1, \ldots, X_n\}$ are **independent and identically distributed(i.i.d)** if they are mutually independent with identical marginal distributions.

- A collection of random vectors $\{X_1, \ldots, X_n\}$ is a **random sample** from the population F if X_i are i.i.d with distribution F.
- The distribution F is called the **population distribution**. We refer to the distribution as the **data generating process(DGP)**.
- The **sample size** n is the number of individuals in the sample.

i.i.d II

Figure: Sampling and Inference



Identification I

A critical and important issue in structural econometric modeling is **identification**, meaning that a parameter is uniquely determined by the distribution of the observed variables. [2]

- It is relatively straightforward in the context of the unconditional and conditional expectation, but it is worthwhile to introduce and explore the concept at this point for clarity.
- Let F denote a probability distribution, for example the distribution of the pair (Y, X). Let F be a collection of distributions. Let μ be a parameter of interest (for example, the mean $E[Y]$).

Definition 18

A parameter $\theta \in \mathbb{R}^k$ is identified on F if for all $F \in \mathcal{F}$ there is a unique value of θ.

Identification II

- Let (Y, X) have a joint distribution. If $E[Y] < \infty$ the conditional expectation $m(X) = \mathbb{E}[Y|X]$ is identified almost everywhere.

[2] For further reading, see **?** Sec. 4.26.

Sampling distributions play a central role in the development of statistical and econometric procedures, so it is important to know, in a mathematical sense, what the sampling distribution of Y is.

There are two approaches to characterizing sampling distributions: an "exact" approach and an "approximate" approach.

1. The exact approach entails deriving a formula for the sampling distribution that holds exactly for any value of n. The sampling distribution that exactly describes the distribution of Y for any n is called the **exact distribution** or **finite-sample distribution** of Y. For example, if Y is normally distributed and $Y_1, \ldots, Y_n$ are i.i.d., then the exact distribution of Y is normal with mean μ_Y and variance σ_Y^2/n. Unfortunately, if the distribution of Y is not normal, then in general the exact sampling distribution of Y is very complicated and depends on the distribution of Y.

2. The approximate approach uses approximations to the sampling distribution that rely on the sample size being large. The large-sample approximation to the sampling distribution is often called the asymptotic distribution—"asymptotic" because the approximations become exact in the limit that $n \to \infty$. As we see in this section, these approximations can be very accurate even if the sample size is only $n = 30$ observations. Because sample sizes used in practice in econometrics typically number in the hundreds or thousands, these asymptotic distributions can be counted on to provide very good approximations to the exact sampling distribution.

This section presents the two key tools used to approximate sampling distributions when the sample size is large: **the law of large numbers** and the **central limit theorem**.

- The law of large numbers says that when the sample size is large, $\bar{Y}$ will be close to μ_Y with very high probability.
- The central limit theorem says that when the sample size is large, the sampling distribution of the standardized sample average, $(\bar{Y} - \mu_Y)/\sigma_Y$, is approximately normal.

Convergence

- A sequence of random variables $Z_n \in \mathbb{R}$ **converges in probability** to c as $n \to \infty$, denoted by $Z_n \to_p c$ or $plim_{n\to\infty} Z_n = c$, if for all $\delta > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|Z_n - c| \leq \delta) = 1.$$

- Let Z_n be a sequence of random variables or vectors with distribution $G_n(u) = \mathbb{P}(Z_n \leq u)$. We say that Z_n **converges in distribution** to Z as $n \to \infty$, denoted with $Z_n \to_d Z$, if for all u at which $G(u) = \mathbb{P}(Z \leq u)$ is continous, $G_n(u) \to G(u)$ as $n \to \infty$.

Law of Large Numbers I

Theorem 19 (Weak Law of Large Numbers(WLLN))

If X_i are independent and identically distributed and $\mathbb{E}(X) < \infty$, then as $n \to \infty$,

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \to_p \mathbb{E}(X).$$

An estimator $\hat{\theta}$ of a parameter θ is **consistent** if $\hat{\theta} \to_p \theta$ as $n \to \infty$.
Counter examples:

- $X_i = Z + U_i$, Z is common component and $\mathbb{E}[U] = 0$, $\bar{X}_i \to_p Z$ but not the sample mean.
- Suppose $\mathrm{var}(X_i) = 1$ for $i \le n/2$ and $\mathrm{var}(X_i) = n$ for $i > n/2$. $\mathrm{var}(\bar{X}_n) \to 1/2$, $\bar{X}_n$ does not converge in probability.

Continuous Mapping Theorem

Theorem 20 (Continuous Mapping Theorem)

If $Z_n \to_p c$ as $n \to \infty$ and $h(\cdot)$ is continuous at c then $h(Z_n) \to_p h(c)$ as $n \to \infty$.

Central Limit Theorem I

Theorem 21 (Central Limit Theorem(CLT))

If X_i are i.i.d. and $\mathbb{E}(X^2) < \infty$ then as $n \to \infty$

$$\sqrt{n}(\bar{X}_n \to \mu) \to_d N(0, \sigma^2),$$

where $\mu = \mathbb{E}(X)$ and $\sigma^2 = \mathbb{E}[(X - \mu)^2]$.

Theorem 22 (Slutsky's Theorem)

If $Z_n \to_d Z$ and $c_n \to_p c$ as $n \to \infty$, then

1. $Z_n + c_n \to_d Z + c$
2. $Z_n c_n \to_d Z_c$
3. $\frac{Z_n}{c_n} \to_d \frac{Z}{c}$ *if $c \neq 0$.*

Central Limit Theorem II

Theorem 23 (Delta Method)

If $\sqrt{n}(\hat{\theta} - \theta) \to_d \xi$ and $h(u)$ is continuously differentiable in a neighborhood of θ then as $n \to \infty$

$$\sqrt{n}(h(\hat{\theta}) - h(\theta)) \to_d \mathrm{H}'\xi$$

where $\mathrm{H}(u) = \frac{\partial}{\partial u} h(u)^\top$ and $\mathrm{H} = \mathrm{H}(\theta)$.

References I