# Deep Learning

## 2.3 Bias-Variance trade-off

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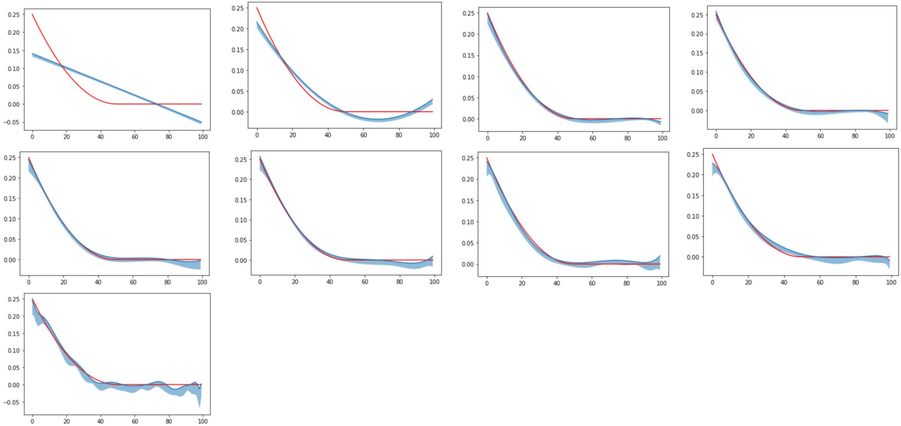
# Visualize overfitting



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2. Let $x$ be fixed, $y$ be the true value associated with it, $f*$ is what we leaned from dataset $\mathcal{D}$, and $Y = f*(x)$ is the predicted value
3. Let's consider that $\mathcal{D}$ is a random variable, then $f*$ and $Y$ get random

Consider

$$
\begin{aligned}
\mathbb{E}_{\mathcal{D}}((Y-y)^2) =& \mathbb{E}_{\mathcal{D}}(Y^2 - 2Yy + y^2) \\
=& \mathbb{E}_{\mathcal{D}}(Y^2) - 2\mathbb{E}_{\mathcal{D}}(Y)y + y^2 \\
=& \mathbb{E}_{\mathcal{D}}(Y^2) - \mathbb{E}_{\mathcal{D}}(Y)^2 + \mathbb{E}_{\mathcal{D}}(Y)^2 + 2\mathbb{E}_{\mathcal{D}}(Y)y + y^2 \\
=& \mathbb{E}_{\mathcal{D}}(Y^2) - \mathbb{E}_{\mathcal{D}}(Y)^2 + \mathbb{E}_{\mathcal{D}}(Y)^2 + 2\mathbb{E}_{\mathcal{D}}(Y)y + y^2 \\
=& (\mathbb{E}_{\mathcal{D}}(Y) - y)^2 + \mathbb{V}_{\mathcal{D}}(Y)
\end{aligned}
$$

# Bias-Variance Decomposition

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=& {\color{red}\mathbb{E}_{\mathcal{D}}(Y^2) - \mathbb{E}_{\mathcal{D}}(Y)^2} + {\color{blue}\mathbb{E}_{\mathcal{D}}(Y)^2 + 2\mathbb{E}_{\mathcal{D}}(Y)y + y^2} \\
=& {\color{blue}(\mathbb{E}_{\mathcal{D}}(Y) - y)^2} + {\color{red}\mathbb{V}_{\mathcal{D}}(Y)}
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2. Bias term quantifies how much the model fits the data on average

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1. This is known as Bias-Variance decomposition
2. Bias term quantifies how much the model fits the data on average
3. Variance term quantifies how much the model changes across datasets

# Bias-Variance Trade-off

1. Reducing the capacity makes $f^*$ fit the data less on average, which increases the bias term

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1. Reducing the capacity makes $f^*$ fit the data less on average, which increases the bias term
2. Increasing the capacity makes $f^*$ vary a lot with the training data, which increases the variance term