Deep Learning

3.2 Multi-layer Perceptron (MLP)

Dr. Konda Reddy Mopuri
kmopuri@iittp.ac.in
Dept. of CSE, IIT Tirupati
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Recap: Linear classifier

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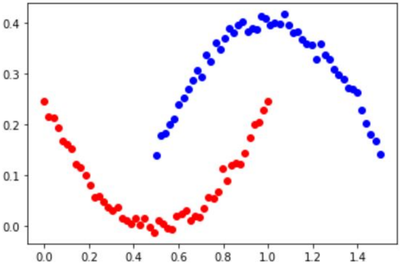
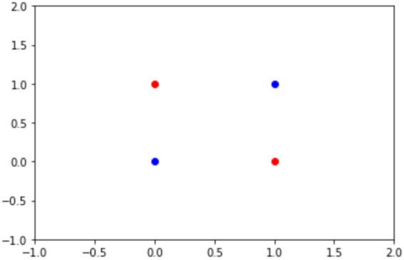
2. Seen a couple of simple examples: MP neuron and Perceptron

Linear Classifiers: Shortcomings

- Lower capacity: data has to be linearly separable
- Some times no hyper-plane can separate the data (e.g. XOR data)





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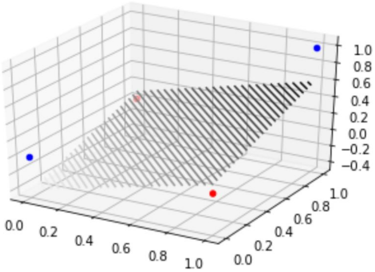
2. Consider the xor case
   $\phi(\mathbf{x}) = \phi(x_u, x_v) = (x_u, x_v, x_u x_v)$
3. Consider the perceptron in the new space $f(\mathbf{x}) = \sigma(\mathbf{w}^T \phi(\mathbf{x}) + b)$



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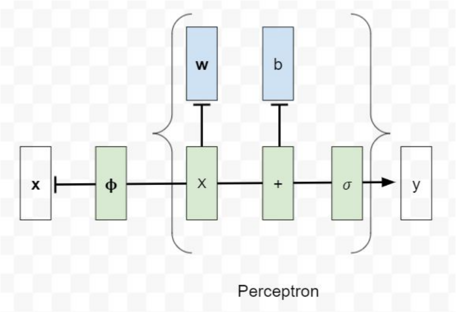

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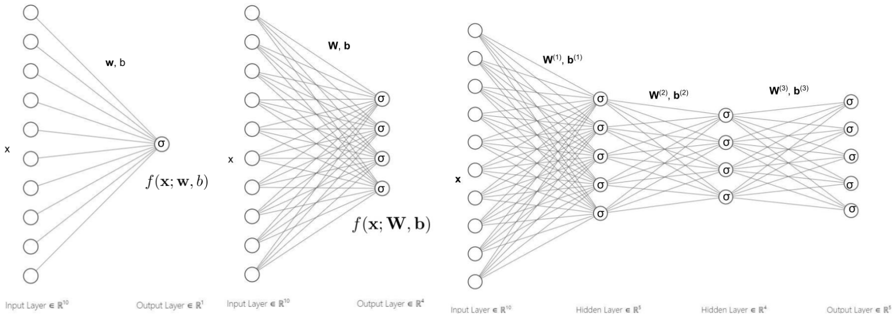
2. Also, remember the Bias-Variance decomposition: for reducing the bias error, we increased the model capacity
3. Feature design (or pre-processing) may also be another way to reduce the capacity without affecting (or improving) the bias

Extending Linear Classifier

1. Linear classifier $f(\mathbf{x}) = \sigma(\mathbf{w}^T\mathbf{x} + b)$ from $\mathcal{R}^D \rightarrow \mathcal{R}$ where $\mathbf{w}$ and $\mathbf{x} \in \mathcal{R}^D$
   can be extended to multi-dimension output
   $f(\mathbf{x}) = \sigma(\mathbf{W}\mathbf{x} + \mathbf{b})$ from $\mathcal{R}^D \rightarrow \mathcal{R}^C$ where
   $\mathbf{W} \in \mathcal{R}^{C \times D}$ and $\mathbf{b} \in \mathcal{R}^C$, and σ is applied element-wise

Single unit to a layer of Perceptrons



Formal Representation

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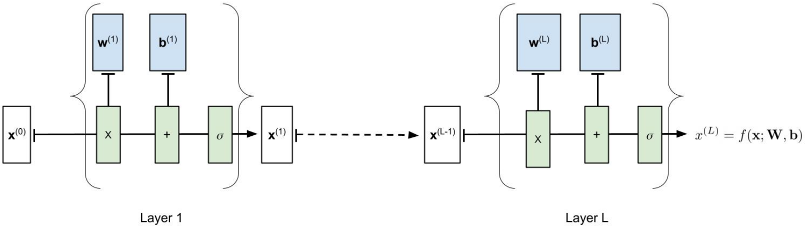
2. can be represented as:
   $\mathbf{x}^{(0)} = \mathbf{x}$,
   $\forall l = 1, \ldots, L, \quad \mathbf{x}^{(l)} = \sigma({\mathbf{W}^{(l)}}^T \mathbf{x}^{(l-1)} + \mathbf{b}^{(l)})$, and

MLP



x(0)
w(1)
b(1)
X
+
σ
x(1)
x(L-1)
w(L)
b(L)
X
+
σ
$x^{(L)} = f(\mathbf{x}; \mathbf{W}, \mathbf{b})$
Layer 1
Layer L

Nonlinear Activation

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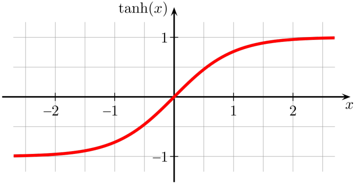
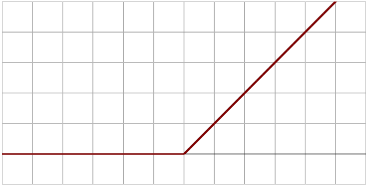
2. If it is an affine function, the full MLP becomes a complex affine transformation (composition of a series of affine mappings)

Nonlinear Activation

Familiar activation functions





Hyperbolic Tangent (Tanh) $x \rightarrow \frac{2}{1+e^{-2x}} - 1$ and Rectified Linear Unit (ReLU) $x \rightarrow \max(0, x)$ respectively

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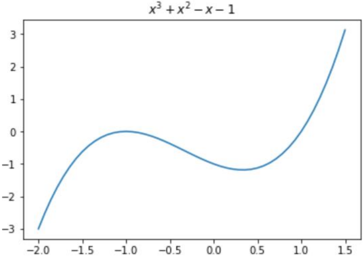
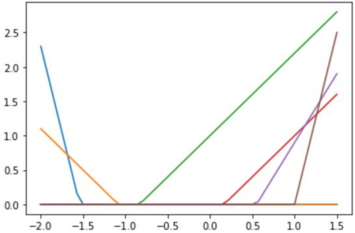
2. Let's approximate the following function using a bunch of ReLUs:

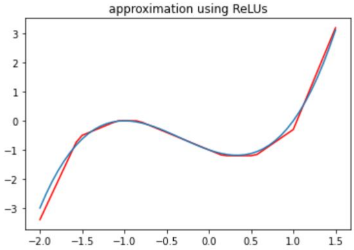
$n_1 = ReLU(-5x - 7.7), n_2 = ReLU(-1.2x - 1.3), n_3 = ReLU(1.2x + 1), n_4 = ReLU(1.2x - 0.2), n_5 = ReLU(2x - 1.1), n_6 = ReLU(5x - 5)$





Universal Approximation using ReLU functions

1. Appropriate combination of these ReLUs:
$-n_1 - n_2 - n_3 + n_4 + n_5 + n_6$



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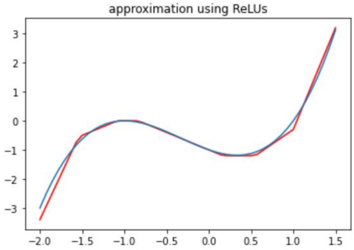
2. Note that this also holds in case of other activation functions with mild assumptions.



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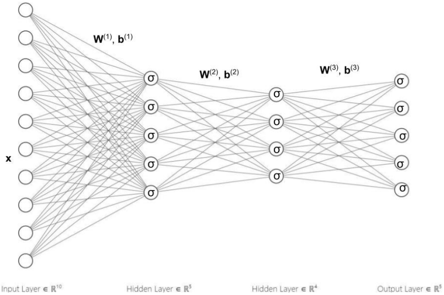
3. Better approximation requires larger hidden layer (C)
   - Theorem doesn't discuss their relation

MLP for regression

1. Output is a continuous variable in $\mathcal{R}^D$
   - Output layer has that many perceptrons (When $D = 1$, regresses a scalar value)
   - Generally employs a squared error loss



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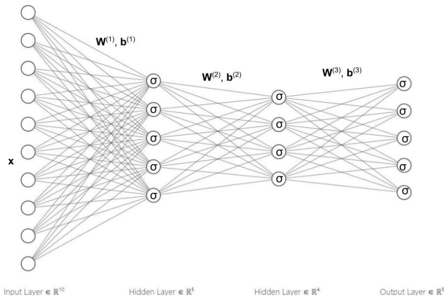

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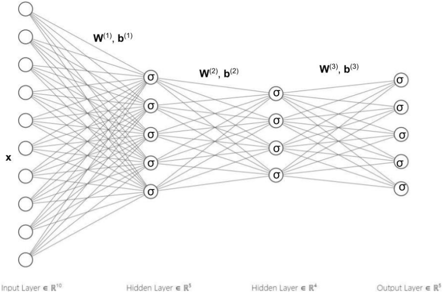

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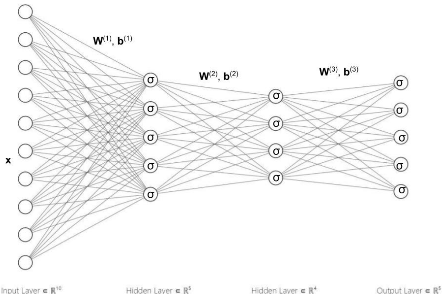

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