SOSC 5340: Logistic Regression and MLE

Han Zhang

Feb 16, 2021

Outline

How you wish I can be improved?

- *It is a little bit slow as most of us have already had some quantitative backgrounds. I hope we can move to causal inference as soon as possible.*

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  - Maybe you have talent to do methodology research

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- All are expected to read the article and ask questions

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- Hansen, *Econometrics*, 2020. Chapter 4, 5, 23. Free at the author's website `https://www.ssc.wisc.edu/~bhansen/econometrics/`

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  - Voting: $1 =$ vote; $0 =$ abstain

How do we model binary outcome?

- We already know that conditional expectation $E(Y|X)$ is the best predictor

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- $P(Y = 1|X)$ is the conditional probability of $\mathrm{Y} = 1$ given X
- What is different here: conditional probability must be between 0 and 1 by definition!

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- Alternatively: we can apply a function F onto $X\beta$ to ensure

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Logistic regression

- Two useful functions:

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  - We use the inverse-logit function as F

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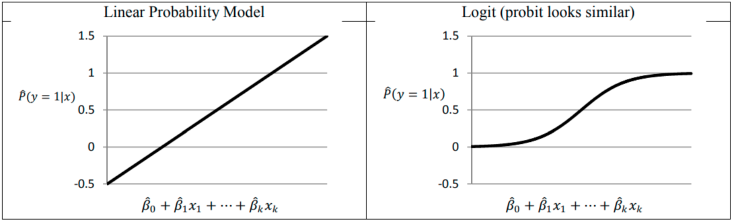
Logistic Regression vs Linear Probability Model



- inverse-logit function "squashs" $X\beta$ to $[0, 1]$

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- We can also "squash" $X\beta$ using standard normal CDF (normal cumulative density function)

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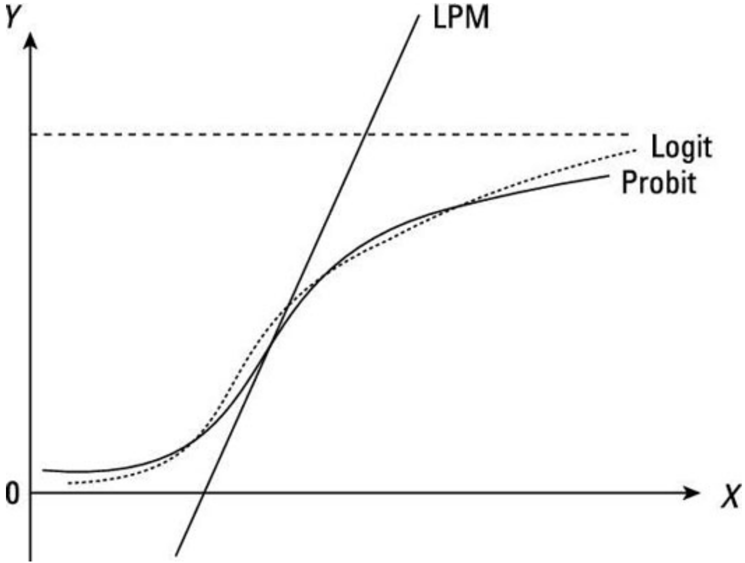
Probit vs Logit vs Linear Probability



More on linear probability model

- Binary data (and more general, most categorical data) always exhibit heteroscedasticity

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- So always use robust standard error if you decide to use OLS regression to model binary outcomes (linear probability model).

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Assumptions of Logistic/Probit regressions

- Assumption 1 and 2: shared by logit/probit regressions

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  - economists may say that Y^* is the underlying preference, and Y is revealed preference

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Assumptions 3 and 5 together lead to

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Estimation of parameters in OLS regressions: review

- There are two ways to estimate β in linear regression

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- Both solutions result in the same β estimate for OLS regression

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- For a single data point, the probability we observe Y_i is exactly given by $logit^{-1}(X_i\beta)$ or $1 - logit^{-1}(X_i\beta)$ (depending on observed Y_i)

Maximum Likelihood Estimation

- Because we have i.i.d. samples, we can multiple these empirical probabilities together, as the probability that we observe the entire sample.

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- Put it differently, our estimate of β should maximize the likelihood function.

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$$\hat{\beta}_{MLE} = \arg\max_{\beta} \log L$$

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- Last, we can use normal approximated intervals for confidence interval (below is an example for 95% confidence interval)

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MLE in practice: logistic regression

- Step 1: write single point probability distribution; this case it is easy:

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- Step 2: for all n points:

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  - How? There may be infinite choices of β
  - There are many mature optimization algorithms that help you find β quicker

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- in R, use `optim` package

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$$\beta_{new} = \beta_{old} + \eta \cdot \frac{\partial \log L}{\partial \beta} \tag{9}$$

$\frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{n} \left[Y_i - logit^{-1}(X\beta) \right] X_i$

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- You need to choose an starting β; try several random guess

Odds and Log Odds

- Let us move on to interpreting regression coefficients

$$X\beta = logit(E(Y|X)) = log\left[\frac{P(Y=1|X)}{1-P(Y=1|X)}\right] = log\left[\frac{P(Y=1|X)}{P(Y=0|X)}\right]$$

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  - Problem: it is very intuitive to think about what β increase in log-odds means

Logistic Regression Interpretations: Approach 1

- Example, we are interested in the effect of income and gender on whether a person vote or not. For gender, 1 is female and 0 is female. Income is in thousand dollars

$$P(Y = 1|X) = logit^{-1}(-1.92 + 0.032 * \text{income} + 0.67 * \text{gender})$$

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  - Do not write this in formal paper!

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- We can always calculate the marginal effect: how conditional probability changes for one unit increase in X: $\frac{\partial P(Y=1|X)}{\partial X}$

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- This is especially useful if you have interaction terms

Predicted probability (example)

See RMarkdown codes and files.

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    - (but I personally find them hard to grasp; and I am sure I am not the only one)

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Next week

- More on generalized linear model