# Counterfactual Framework of Causal Inference

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February 28, 2022

# Outline

# Identification

- We have learned how to use samples to estimate and make statistical inference over some population quantity (e.g., $P(X)$ or $E(Y|X)$)
- What if we cannot observe some random variables?
- Statistical identification: use some observed random variables to infer properties about random variables that cannot be observed, or unobserved.
- To address identification problem, we need additional assumptions about our data

# Missing Data

- Missing data is one common identification problem
- E.g., in a survey, people answer "Don't know"
- Let us work with the simplest case: we are interested in only one random variable $Y$.
- And we draw a sample of $n$ points, $Y_1, \cdots, Y_n$ from population $Y$.
- Define $R_i$ be an indicator for whether or not we observe $Y_i$
- General solutions:
  1. Bounds: possible ranges of $Y$
  2. Deletion: discard missing ones
  3. Imputation: predict missing $Y$

# Missing data: bounds

- Assume we are interested in $E(Y)$
- How do we estimate $E(Y)$ in the presence of missing data?
- Suppose we see a data that looks like the below

| Unit | $Y_i$ | $R_i$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | ? | 0 |
| 3 | 1 | 1 |
| 4 | 0 | 1 |
| 5 | 1 | 1 |
| 6 | ? | 0 |

(1)

- And we know that $Y$ can take values between $[0, 1]$ ($Y$ can be continuous)
- What is the maximum possible value of $E(Y)$?

# Missing data: bounds

- The largest value of $Y$ is 1. We just fill in them, and calculate the largest possible value of $E(Y)$

| Unit | $Y_i$ | $R_i$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 1 | 0 |
| 3 | 1 | 1 |
| 4 | 0 | 1 |
| 5 | 1 | 1 |
| 6 | 1 | 0 |

(2)

- The largest possible $E(Y)$ is 5/6
- Likewise, we plug in the smallest value of $Y$
- The smallest possible value of $E(Y)$ is 3/6
- We obtained bounds for $E(Y|X)$: $[3/6, 5/6]$; this is known as Manski bounds.
- Note that bounds are not confidence intervals. WHY?

# Missing data: deletion

- Bounds can often be very wide, making them not that useful
- We can make stronger assumption to obtain more meaningful point estimation of $E(Y)$

## Definition (MCAR: Missing Complete at Random, Rubin, 1976)

$Y$ is missing completely at random if:

1. The missing $Y \perp\!\!\!\perp R$ (Response is independent of the missing $Y$ we are interested in).
2. $P(R = 1) > 0$ (non-zero response probability)

# Missing data: deletion

- MCAR assumption implies that

$$E(Y) = E(Y|R = 1) \tag{3}$$

- The right hand side is something we can estimate: the sample mean for those we can observe (apply plug-in principle)
- Practical implication: if MCAR holds, we can safely delete missing $Y$, and $E(Y|R = 1)$ is an unbiased estimates of $E(Y)$

# Missing data: imputation

- We can also impute missing values to estimate $E(Y)$

| Unit | $Y_i$ | $R_i$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | ? | 0 |
| 3 | 1 | 1 |
| 4 | 0 | 1 |
| 5 | 1 | 1 |
| 6 | ? | 0 |

- Instead of deleting missing rows, we can fill in values

# Imputation Method 1: Unconditional Mean Imputation

- Unconditional mean imputation fill in missing $Y$ by the sample mean of observed $Y$

| Unit | $Y_i$ | $R_i$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | $\hat{\mathrm{E}}[Y \mid R=1] = \frac{3}{4}$ | 0 |
| 3 | 1 | 1 |
| 4 | 0 | 1 |
| 5 | 1 | 1 |
| 6 | $\hat{\mathrm{E}}[Y \mid R=1] = \frac{3}{4}$ | 0 |

- After unconditional mean imputation, the sample mean of imputed $Y$ is an unbiased estimate of $Y$
  - Note: this is not the only way to make $E(Y) = E(Y \mid R=1)$
- Deletion and imputation all lead to unbiased estimate of $E(Y)$
- Their variance estimates are usually different!
  - $\hat{V}_{deletion}(Y) = 0.25$
  - $\hat{V}_{imputation}(Y) = 0.15$

# MCAR in multivariate case

- When we have multiple variables, we can extend MCAR assumptions: each variable is independent of response.
- And with MCAR assumptions, we can perform listwise deletion by removing any row that has missing entries.

| Unit | $Y_i$ | $R_i$ | $X_i$ |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 2 | ? | 0 | 0 |
| 3 | 1 | 1 | 0 |
| 4 | 0 | 1 | 0 |
| 5 | 1 | 1 | ? |
| 6 | ? | 0 | 1 |

(4)

- Or taking the imputation perspective, we can perform unconditional mean imputation for each variables

# MAR

- MCAR is often too strong in multivariate case
  - If there are many variables, we can delete a lot of observations
  - Often these variable are correlated with each other;
- One weaker assumption is MAR, also known as ignorability

**Definition (MAR: Missing at Random, Rubin, 1976)**

$Y$ is missing at random if:

1. $Y \perp\!\!\!\perp R | X$ (Response is independent of $Y$, given some other variables $X$).
2. $P(R = 1) > 0$ (non-zero response probability)

- That is, $Y$ is missing at random, once we condition on some control variables $X$.

# Post-stratification estimator of sample mean

- Under MAR, we can estimate the mean of $Y$ using post-stratification estimator

$$E(Y) = \sum_x E(Y|R = 1, X = x)p(X = x) \tag{5}$$

- In other words, we estimate $E(Y)$ as the weighted mean of the conditional expectation of $Y$ given $X$ in observed data, with weights $P(X = x)$
- Both terms on the right hand side can be estimated from samples (plug-in sample analog)
- Note: post-stratification estimator does not impute; directly estimate $E(Y)$

# MAR vs MCAR

- Under MCAR: $\hat{\mathrm{E}}\left[\mathrm{Y}_i\right] = 3/4$

| Unit | $Y_i$ | $R_i$ | $X_i$ |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 2 | ? | 0 | 0 |
| 3 | 1 | 1 | 0 |
| 4 | 0 | 1 | 0 |
| 5 | 1 | 1 | 1 |
| 6 | ? | 0 | 1 |

- Under MAR, with stratification estimator, $\hat{\mathrm{E}}\left[\mathrm{Y}_i\right] = 7/9$

$$\hat{\mathrm{E}}\left[\mathrm{Y}\right] = \hat{\mathrm{E}}\left[Y|R=1, X=0\right]\widehat{\mathrm{P}}\left[X=0\right] + \hat{\mathrm{E}}\left[Y|R=1, X=1\right]\widehat{\mathrm{P}}\left[X=1\right]$$

$$= \frac{2}{3}\cdot\frac{4}{6} + 1\cdot\frac{2}{6} = \frac{7}{9}$$

- MCAR and MAR will yield different estimates of $E(Y)$
- Each estimate is unbiased estimate only if the corresponding assumption is true

# Imputation method 2: Conditional Mean Imputation

- With MAR, we can also impute $Y$ using conditional mean imputation: use the conditional mean of $Y$ as our guesses of the missing $Y$
- $Y_i = \hat{E}(Y|R = 1, X = X_i)$

| Unit | $Y_i$ | $R_i$ | $X_i$ |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 2 | $\hat{\mathrm{E}}[Y_i \| X_i = 0] = \frac{2}{3}$ | 0 | 0 |
| 3 | 1 | 1 | 0 |
| 4 | 0 | 1 | 0 |
| 5 | 1 | 1 | 1 |
| 6 | $\hat{\mathrm{E}}[Y_i \| X_i = 1] = 1$ | 0 | 1 |

(6)

- Then we can calculate sample mean over imputed $Y$
- Under conditional mean imputation, $\hat{E}(Y)$ is again 7/9
- The below two gives the same estimate of $E(Y)$:
  - conditional mean imputation of $Y$, and then take sample mean of imputed $Y$
  - post-stratification estimator

# Conditional Mean Imputation using linear regression

- If we further assume all assumptions of linear regression are correct: $E(Y|R=1, X=x)$ is linear in $X$
- Then conditional mean imputation just uses predicted values of linear regression as imputed values

| Unit | $Y_i$ | $R_i$ | $X_{[1]i}$ | $X_{[2]i}$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 3 |
| 2 | ? | 0 | 0 | 7 |
| 3 | 1 | 1 | 0 | 9 |
| 4 | 0 | 1 | 0 | 5 |
| 5 | 1 | 1 | 1 | 4 |
| 6 | ? | 0 | 1 | 3 |

(7)

# Conditional Mean Imputation using regression

$$
\begin{array}{cccccc}
\text{Unit} & Y_i & R_i & X_{[1]i} & X_{[2]i} \\
\hline
1 & 1 & 1 & 0 & 3 \\
2 & \hat{\beta}_0 + \hat{\beta}_1 \cdot 0 + \hat{\beta}_2 \cdot 7 & 0 & 0 & 7 \\
3 & 1 & 1 & 0 & 9 \\
4 & 0 & 1 & 0 & 5 \\
5 & 1 & 1 & 1 & 4 \\
6 & \hat{\beta}_0 + \hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 3 & 0 & 1 & 3
\end{array} \tag{8}
$$

# Conditional Mean Imputation using other methods

- The interpretation advantage of linear regression is not relevant now; we do not care about interpreting $\beta$; we want our predictions of $E(Y|R=1, X=x)$ to be more precise
- So you can use GLM to predict $E(Y|R=1, X=x)$
    - GLM
- Or other more complex machine learning algorithms. It's a prediction problem!
- These options are all provided in `R` package `mice`

# Imputation Method 3: hot-deck imputation

- Hot-deck imputation uses nearest-neighbor matching
- For unit $i$ with missing $Y_i$, and non-missing $X_i$
  - Find the $X_j$ that has the smallest distance to/is closet $X_i$
  - Use the $Y_j$ associated with $j$ as the imputed $Y$ value for $i$

| Unit | $Y_i$ | $R_i$ | $X_i$ |
|---|---|---|---|
| 1 | 1 | 1 | 4 |
| 2 | ? | 0 | 8 |
| 3 | 1 | 1 | 1 |
| 4 | 0 | 1 | 12 |
| 5 | 1 | 1 | 20 |
| 6 | ? | 0 | 3 |

- Example: unit 6's $X$ is closest to unit 1's $X$. So we impute $Y_6$ as $Y_1 = 1$

# Hot-deck imputation using propensity scores

- When we have multivariate $X$, it is not easy to calculate their distances
- Instead, it is popular to estimate propensity score of response

$$P(R = 1|X) \tag{9}$$

- Propensity score of response provides an single-number summary of multivariate $X$
- Hot-deck imputation based on nearest propensity score, not based on original distances between $X$
  - In other words, you want to match units whose response propensity are similar
- Estimation of propensity scores
  - Logistic regression is the default choice
  - But apparently other machine learning methods are acceptable

# Hot-deck example

| Unit | $Y_i$ | $R_i$ | $X_{[1]i}$ | $X_{[2]i}$ | $P(R_i = 1 \mid X_i)$ |
|---|---|---|---|---|---|
| 1 | 2 | 1 | 0 | 3 | 0.33 |
| 2 | ? | 0 | 0 | 7 | 0.14 |
| 3 | 3 | 1 | 0 | 9 | 0.73 |
| 4 | 10 | 1 | 0 | 5 | 0.35 |
| 5 | 12 | 1 | 1 | 4 | 0.78 |
| 6 | ? | 0 | 1 | 3 | 0.70 |

(10)

- Unit 6's propensity score of response is closest to unit 3's propensity score. Thus $Y_6$ is imputed as $Y_3 = 3$

# Deletion vs Imputation

- In practice, assume we want to run a regression based on $Y$ and 10 predictors $X$
- Solution 1: Listwise deletion
  - Both R and Stata uses this strategy by default
  - Pros: simple; unbiased if MAR is true
  - Cons: large standard errors (since you will drop many cases)
- Solution 2: mean imputation (unconditional or conditional)
  - Pros:
    - give you more cases to work with
    - also unbiased if MCAR/MAR is true
  - Cons: small standard errors. Why?
    - Artificially fix the missing $Y$ to its mean.
- Solution 3: hot-deck imputation
  - Pros: preserve the support of original data
    - Bear similarity to propensity score matching
  - Cons: how to estimate propensity scores?

# Stochastic Imputation

- Problem of mean imputation: small standard error issues when we use sample mean for imputation
- A workaround—stochastic imputation—add some random noise to the sample mean
  - Say, we still use regression to impute $Y$, but add some random noises to your predicted $Y$
  - If we are working with complex machine learning models, there may be some inherent stochastic component (results are not the same every time)
- Problem: stochastic imputation also have some uncertainty, based on what noises you use
  - These random noises are not added to your final analysis, thus still producing small standard error estimates

# Multiple Imputation

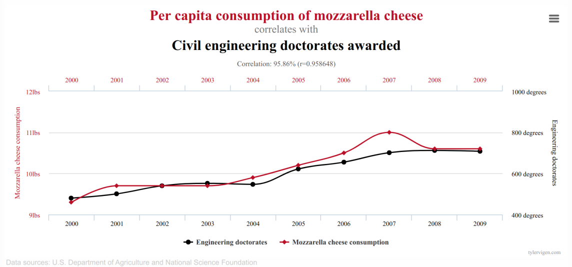
- Rubin, 1977, Multiple Imputation
  - Stochastic imputation for $m$ times; ending up with $m$ imputed datasets.
  - Analysis: Run your model (regression $Y$ on 10 $X$) $m$ times
  - Pooling: parameter estimates for $m$ different models can be used for estimation and inference:
    - The final parameter estimates of $\beta$ is the mean of $\beta$ across $m$ models
    - The standard error of final $\beta$ is more complex in math
    - basically it's the (within model standard error of $\beta$) + (between model standard error of $\beta$)
    - Or use bootstrap if $m$ is large enough

# Missing data by Chained Equations

- In practice, more than one $X$ can have missing values
- Assume we have 5 $X$; we use $1, 2, 3, 4$ to impute the 5th variable, and then use $1, 2, 3, 5$ to impute the 4th variable, and so on and so forth
  - Imputed values are allowed to use in the next step

# Prediction vs Causation

- Correlation $\neq$ causation
- We can use $X$ to predict $Y$, and use $Y$ to predict $X$
- $Y = g(X) \iff X = g^{-1}(Y)$
- This does not capture the intuitive idea that $X$ causes $Y$




https://www.tylervigen.com/spurious-correlations

# Counterfactual

- Does college education lead to higher wages?
- Observed (Factorial): on average, college graduates indeed earn more than people with only high school education
- Critique:
  - people who can go to college have higher ability
  - even if they did not go to education, they could still earn more
  - Therefore, correlation does not mean causation
- Counterfactual thinking:
  - Guess the (counterfactual) earning of college graduates if they did not go to college
  - If the counterfactual earning equals to the factual earning, then college education does not matter; there is no causal effect
  - Alternatively, if the factual earning is higher than counterfactual earning, then college education indeed lead to increase in wages

# Neyman-Rubin Causal Model: potential outcomes

- Neyman-Rubin Causal Model formally write down the counterfactual idea
- We have a binary treatment $D$; $D = 1$ if treated and 0 otherwise
- For a person $i$ in the population, her outcome $Y_i$ is assumed to be:

**Definition (Neyman-Rubin model)**

$$\begin{aligned} Y_i &= \begin{cases} Y_i^0 : D_i = 0 \\ Y_i^1 : D_i = 1 \end{cases} \\ &= Y_i^0 + D_i(Y_i^1 - Y_i^0) \end{aligned}$$

- $Y_i$ is observed outcome
- $Y_i^0$ is the potential outcome if $i$ is not treated
- $Y_i^1$ is the potential outcome if $i$ is treated

# Individual Level Treatment Effect

## Definition (Unit-Level Treatment Effect)

For $i$ in the population, the causal effect of treatment for unit $i$ is : $\rho_i = Y_i^1 - Y_i^0$

- $Y_i = Y_i^0 + D_i(Y_i^1 - Y_i^0)$
- So the Neyman Rubin model suggests that the observed $Y$ for a treated unit $i$ are the two sums:
  - counterfactual outcome if $i$ were not treated
  - individual level treatment effect

# Counterfactual Outcome as Missing Data Problem

| Unit | $Y_i$ | $D_i$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 1 | 0 |
| 3 | 1 | 1 |
| 4 | 0 | 1 |
| 5 | 1 | 1 |
| 6 | 0 | 0 |

(11)

| Unit | $Y_i^0$ | $Y_i^1$ | $D_i$ |
|---|---|---|---|
| 1 | ? | 1 | 1 |
| 2 | 1 | ? | 0 |
| 3 | ? | 1 | 1 |
| 4 | ? | 0 | 1 |
| 5 | ? | 1 | 1 |
| 6 | 0 | ? | 0 |

(12)

# Fundamental Problem of Causal Inference

## Definition (Fundamental Problem of Causal Inference, Holland, 1986)

At any given time, we only observe one of the potential outcomes for unit $i$ — either $Y_i^1$ or $Y_i^0$ —but not both. Thus unit-level treatment effect $\rho_i$ is not identified.

- Similar to missing data problems, we have to make assumptions (here, assumptions about potential outcomes) to allow identification.
  - In particular, identification of the average of treatment effect $\rho_i$ because identifying the effect for every unit can be extremely hard

# ATE and ATT

Definition (Average Treatment Effect (ATE))

$ATE = E(\rho) = E(Y^1 - Y^0) = E(Y^1) - E(Y^0)$

- ATE is the mean of unit-level treatment effect
- ATE is the difference between the mean of two potential outcomes

Definition (Average Treatment Effect on the Teated (ATT))

$ATT = E(\rho|D = 1) = E(Y^1 - Y^0|D = 1) = E(Y^1|D = 1) - E(Y^0|D = 1)$

- ATT is the mean of unit-level treatment effect for treated units

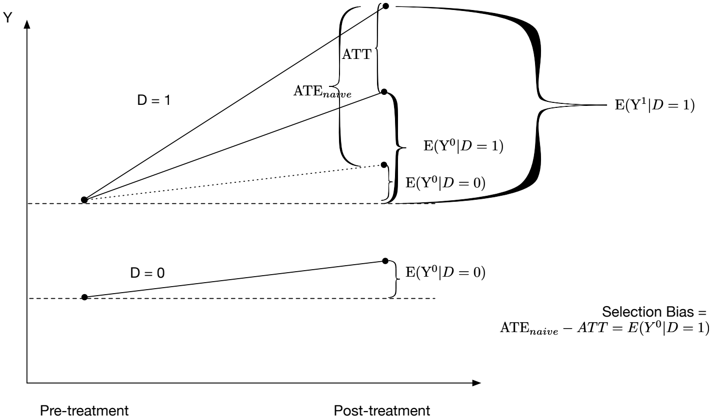
# Naive estimate of ATE

- Naive estimate of ATE is just the difference in means of observed data

$$ATE_{naive} = E\left[\mathrm{Y}|\mathrm{D} = 1\right] - E\left[\mathrm{Y}|\mathrm{D} = 0\right] \tag{13}$$

- For instance, $D = 1$ for college education and $D = 0$ for less than college education
- $ATE_{naive}$ is the mean earning of college education - mean earning of non-college educated
- In general, $ATE \neq ATT \neq ATE_{naive}$; what is their connection?

# Selection Bias




Selection Bias =
$\mathrm{ATE}_{naive} - ATT = E(Y^0|D=1) - E(Y^0|D=0)$

# Selection bias

- $ATE_{naive}$ neither estimates $ATE$ nor $ATT$
- $ATE_{naive}$ differs from $ATT$ by selection bias

$$\text{selection bias} = ATE_{naive} - ATT = E(Y^0|D=1) - E(Y^0|D=0)$$

- Intuitively, this is the counterfactual earning of college educated if they did not go to college, minus the factual earning of non-college educated
  - This selection bias could be caused by ability, for example
- If selection bias is 0, $ATE_{naive} = ATT$

# Random Assignment

- In randomized controlled experiments, we randomly assign subjects into treatment and control groups; we have random assignment

Definition ((Completely) Random Assignment)

- $Y_i^0, Y_i^1 \perp\!\!\!\perp D_i$ (Potential outcome is independent of treatment assignment)
- $P(D = 1) > 0$ (non-zero treatment probability)

- Cautions:
  - Observed outcome $Y$ is not independent of treatment assignment.

# Random Assignment Solves the Identification Problem

- Under random assignment of $D$, we have:

$$ATE_{naive} = E\left[\mathrm{Y}|\mathrm{D}=1\right] - E\left[\mathrm{Y}|\mathrm{D}=0\right] = ATE \qquad (14)$$

- Proof (the first line to second line is due to independence between $D$ and $Y^0, Y^1$)

$$\begin{aligned} E\left[\mathrm{Y}|\mathrm{D}=1\right] - E\left[\mathrm{Y}|\mathrm{D}=0\right] &= E\left[\mathrm{Y}^1|\mathrm{D}=1\right] - E\left[\mathrm{Y}^0|\mathrm{D}=0\right] \\ &= E\left[\mathrm{Y}^1|\mathrm{D}=1\right] - E\left[\mathrm{Y}^0|\mathrm{D}=1\right] \\ &= E\left[\mathrm{Y}^1 - \mathrm{Y}^0|\mathrm{D}=1\right] \\ &= E\left[\mathrm{Y}^1 - \mathrm{Y}^0\right] \\ &= E\left[\mathrm{Y}^1\right] - E\left[\mathrm{Y}^0\right] \end{aligned} \qquad (15)$$

# Non-parametric estimator: difference-in-means

- With random assignment, estimating $ATE$ is very simple: $ATE_{naive}$, which is just the difference in mean outcome of the treatment and the control group
- This is a non-parametric estimator
- Another important observation: $ATT = ATE$ for randomized experiments

# Experiment as Imputation

| Unit | $Y_i^0$ | $Y_i^1$ | $D_i$ |
|---|---|---|---|
| 1 | ? | 1 | 1 |
| 2 | 1 | ? | 0 |
| 3 | ? | 1 | 1 |
| 4 | ? | 0 | 1 |
| 5 | ? | 1 | 1 |
| 6 | 0 | ? | 0 |

(16)

- Random assignment implies that we can impute the missing values using observed sample mean; similar to the MCAR assumption in missing data
  - But here, random assignment is a fact, not an assumption

| Unit | $Y^0$ | $Y^1$ | $D$ |
|---|---|---|---|
| 1 | $\hat{\mathrm{E}}[Y \mid D=0]=\frac{1}{2}$ | 1 | 1 |
| 2 | 1 | $\hat{\mathrm{E}}[Y \mid D=1]=\frac{3}{4}$ | 0 |
| 3 | $\hat{\mathrm{E}}[Y \mid D=0]=\frac{1}{2}$ | 1 | 1 |
| 4 | $\hat{\mathrm{E}}[Y \mid D=0]=\frac{1}{2}$ | 0 | 1 |
| 5 | $\hat{\mathrm{E}}[Y \mid D=0]=\frac{1}{2}$ | 1 | 1 |
| 6 | 0 | $\hat{\mathrm{E}}[Y \mid D=1]=\frac{3}{4}$ | 0 |

(17)

# Regression estimator of ATE

- We can rewrite $Y_i$ in the following way (MHE, 2.3.1)

$$\begin{aligned} \mathrm{Y}_i = E\left(\mathrm{Y}_i^0\right) &\quad + \left(\mathrm{Y}_i^1 - \mathrm{Y}_i^0\right) D_i \quad + \quad \mathrm{Y}_i^0 - E\left(\mathrm{Y}_i^0\right) \\ = \alpha &\quad + \rho_i \mathrm{D}_i + \quad\quad \eta_i \end{aligned} \tag{18}$$

- This equation looks like linear regression! But each individual has its own regression coefficient $\rho_i$, which is the individual-level treatment effect
- Constant treatment assumption: assume that $\rho_i$ is the same for every one, $\rho$, $ATE = E(\rho_i) = \rho$

$$\begin{aligned} E\left[\mathrm{Y}_i|\mathrm{D}_i = 1\right] = \alpha + \rho + E\left[\eta_i|\mathrm{D}_i = 1\right] \\ E\left[\mathrm{Y}_i|\mathrm{D}_i = 0\right] = \alpha + E\left[\eta_i|\mathrm{D}_i = 0\right] \end{aligned} \tag{19}$$

$$ATE_{naive} = E\left[\mathrm{Y}_i|\mathrm{D}_i = 1\right] - E\left[\mathrm{Y}_i|\mathrm{D}_i = 0\right]$$

$$= \underbrace{\rho}_{\text{ATE}} + \underbrace{E\left[\eta_i|\mathrm{D}_i = 1\right] - E\left[\eta_i|\mathrm{D}_i = 0\right]}_{\text{selection bias}} \tag{20}$$

# Regression estimator of ATE

- Selection bias is 0, since $Y^0 \perp\!\!\!\perp D$ under random assignment
- Therefore,

$$ATE_{naive} = E\left[\mathrm{Y}_i|\mathrm{D}_i = 1\right] - E\left[\mathrm{Y}_i|\mathrm{D}_i = 0\right] = \underbrace{\rho}_{\text{ATE}}$$

- Therefore, if you are running a random experiment,
  - Non-parametric estimator: the difference in mean outcome of treated and control units
  - Parametric estimator:
    - assume constant treatment effect
    - run a regression of observed outcome on treatment $D$, and use coefficient of $D$ as the estimate of $ATE$

# Regression as Imputation

- The regression estimator of ATE is implicitly making counterfactual imputation using linear regression:

| Unit | $Y_i^0$ | $Y_i^1$ | $D_i$ | $X_{[1]i}$ | $X_{[2]i}$ |
|---|---|---|---|---|---|
| 1 | ? | 2 | 1 | 1 | 7 |
| 2 | 5 | ? | 0 | 8 | 2 |
| 3 | ? | 3 | 1 | 9 | 3 |
| 4 | ? | 10 | 1 | 3 | 1 |
| 5 | ? | 2 | 1 | 5 | 2 |
| 6 | 0 | ? | 0 | 7 | 0 |

# Regression as Imputation

- Fit a regression $Y = \beta_0 + \beta_1 D_i + \beta_2 X_{[1]i} + \beta_3 X_{[2]i}$, and impute counterfactual outcome using the linear regression:

| Unit | $Y_i^0$ | $Y_i^1$ | $D_i$ | $X_{[1]i}$ | $X_{[2]i}$ |
|---|---|---|---|---|---|
| 1 | $\hat{\beta}_0 + \hat{\beta}_1 \cdot 0 + \hat{\beta}_2 \cdot 1 + \hat{\beta}_3 \cdot 7$ | 2 | 1 | 1 | 7 |
| 2 | 5 | $\hat{\beta}_0 + \hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 8 + \hat{\beta}_3 \cdot 2$ | 0 | 8 | 2 |
| 3 | $\hat{\beta}_0 + \hat{\beta}_1 \cdot 0 + \hat{\beta}_2 \cdot 9 + \hat{\beta}_3 \cdot 3$ | 3 | 1 | 9 | 3 |
| 4 | $\hat{\beta}_0 + \hat{\beta}_1 \cdot 0 + \hat{\beta}_2 \cdot 3 + \hat{\beta}_3 \cdot 1$ | 10 | 1 | 3 | 1 |
| 5 | $\hat{\beta}_0 + \hat{\beta}_1 \cdot 0 + \hat{\beta}_2 \cdot 5 + \hat{\beta}_3 \cdot 2$ | 2 | 1 | 5 | 2 |
| 6 | 0 | $\hat{\beta}_0 + \hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 7 + \hat{\beta}_3 \cdot 0$ | 0 | 7 | 0 |

(21)

- Then ATE and ATT can be easily calculated as the difference in means of $Y^1$ and $Y^0$

# Inference: Neyman Variance Estimator

- If we are running experiments, $ATE$ can be estimated easily by taking the differences between $E(Y|D=1)$ and $E(Y|D=0)$
- What about statistical inference?
- Neyman Variance Estimator
  1. assume constant treatment effect
  2. Then
  $$V(ATE) = \frac{V_t}{N_t} + \frac{V_c}{N_c}$$
  3. $V_t$ is variance of $Y$ for treated users, and $N_t$ is number of treated users
- If treatment effect is not constant, true variance is usually smaller than $\frac{V_t}{N_t} + \frac{V_c}{N_c}$

# Inference: Linear regression

- If we regress $Y$ on $D$ ($Y = \alpha + \rho D$), it can be shown that the regression estimates of the standard error of regression coefficient is exactly same as the Neyman Variance estimator

$$V(\rho) = V(ATE) = \frac{V_t}{N_t} + \frac{V_c}{N_c}$$

- See Imbens and Rubin (chapter 7) for proof

# Inference: Randomization Test

- Null distribution: $D$ has no causal effect on $Y$
  - then if we shuffle the outcome, $E(Y|D=1) - E(Y|D=0) = 0$
- Randomization test
  - Calculate $ATE$ based on experimental data
  - Shuffle your observed $Y$, and recalculate $ATE_{shuffle}$ based on the shuffled data
  - Say you shuffled 1000 times, and have 1000 $ATE_{shuffle}$.
  - Then you can easily calculate 95% confidence interval/standard errors of ATE estimates
  - The $p$ value for observing $ATE$ is just the probability that your shuffled $ATE_{shuffle}$ is larger than $ATE$: $p$ -value $= P(ATE_{shuffle} > ATE)$
- Pros: do not need to assume constant treatment effect
- Cons: time consuming

# Additional Covariates

- Researchers often collect some additional covariates (i.e., pre-treatment variables)
- With additional variables, it is easier to work with regression estimator

$$Y_i = \alpha + \rho D_i + \beta X_i + \epsilon_i \tag{22}$$

- $\hat{\rho}^{adj}$:covariate-adjusted estimate of treatment effect
- $\hat{\rho}$: difference-in-means of outcome variables across treatment and control (or regression coefficient by regressing $Y$ on $D$ without covariates)
- $\hat{\rho}_X$: difference-in-means of $X$ across treatment and control

# Additional Covariates

- It can be shown that (Li and Ding, 2019, J. R. Stat. Soc, or Imbens and Rubins, Chapter 7):

$$\hat{\rho}^{adj} = \hat{\rho} - \hat{\beta}^T \hat{\rho}_X$$

- $\hat{\rho}_X$: difference-in-means of $X$ across treatment and control
  - With completely randomized experiments
    - $\hat{\rho}^{adj}$ is biased; $\hat{\rho}$ is unbiased
    - $\hat{\rho}_X$ are usually not exactly 0, especially when the data size is not that large
  - Both are consistent
    - because with more and more data, $\hat{\rho}_X$ approaches 0; this is called covariate balance
- Be careful if your treatment and control groups are not balanced; in that case, the treatment effect estimates without and with covariates can differ a lot in finite sample

# Additional Covariates

- Another classical justification to add covariates in regression is that $\hat{\rho}^{adj}$ has smaller standard error than $\hat{\rho}$
- For instance, MHE (p. 23): "Inclusion of the variable $X$... generate more precise estimates of the causal effect)"
- David A. Freedman, *On regression adjustments to experimental data*, Advances in Applied Mathematics **40** (2008), no. 2, 180–193
- It is not necessarily true!

# Additional Covariates

Winston Lin, *Agnostic notes on regression adjustments to experimental data: Reexamining Freedman's critique*, The Annals of Applied Statistics **7** (2013), no. 1, 295–318. MR3086420

- adding covariate is guaranteed to lead to smaller standard error estimates, if

1. full interaction is added; and
2. robust standard errors are used

$$Y_i = \alpha + \rho D_i + \beta X_i + \boxed{\gamma D_i X_i} + \epsilon_i$$

- Note that condition 1 is not easy to follow in practice; if you have 10 covariates, you have to add 10 interaction terms

# Recommended practice

- David A. Freedman, *On regression adjustments to experimental data*, Advances in Applied Mathematics **40** (2008), no. 2, 180–193
- Always present two treatment effects: without and with covariates
- "Regression estimates... should be deferred until rates and averages have been presented"
- Always check pre-treatment covariate balance
- Add interactions if covariance-balance is passed
- not only guarantees smaller standard error, but also detects treatment effect heterogeneity (next week)