SOSC 5340: Overview of Statistical Inference and Prediction

Han Zhang

Feb 8, 2021

Outline

Instructors

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Teaching Assistant: Lam, Pak Hung

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Self Introduction

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  1. Statistical estimation and inference
  2. Applied regression modeling
  3. Causal inference (second half of the semester)

Grading

Attendance	10%
Assignments	40%
Presentation of your final paper (20 min)	15%
Write-up of your final paper	35%

Attendance

- Please turn on your video

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- Online teaching can be challenging; please do ask questions whenever you are not clear!

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    - 4 times

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    - Write it down (35%)
- Treat this as a real paper that has the potential to be published at academic journals/presented at academic conferences.

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  - Hansen, Bruce. *Econometrics*, 2020. Free at the author's website `https://www.ssc.wisc.edu/~bhansen/econometrics/`

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- Examples:
  - define random variable X as gender; it can take several values from *male*, *female*, *transgender*, ...
  - define random variable X as height; it can take numeric values.

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- Cumulative distribution function (CDF): $F(X) = P(X \leq x)$
  - What is the probability that a random variable X takes a value equal to or less than x?

Normal Distribution

- Probability density function of the normal distribution

$$f(x) = P(X = x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

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- Standard normal: $\mu = 0, \sigma = 1$

Joint and Conditional Probability

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  - Probability that Y takes value y, give that X takes value x.

Probability (exercise)

- Two treatments for kidney stones

	Treatment A		Treatment B	
Kidney Stone	cured	patient	cured	patient
Small	81	87	234	270
Large	192	263	55	80
Total	273	350	289	350

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- B is more effective in the entire population, but A is more effect for both patients with small and large kidney stones.
- This is known as the Simpson's Paradox. Why?

Expected Value

- Expectation (expected value) $E(X)$:

$$E(X) = \int x \cdot f(x) dx$$

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  - $E[X] = x_1 p_1 + x_2 p_2 + \cdots + x_k p_k = \sum_{i=1}^{k} x_i \, p_i$

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- Continuous variable's expectation

$$E(X) = \int x \cdot f(x) dx$$

Expected Value (exercise)

- What is the $E(X)$ of the random variable X?

X	P(X)
0	0.8
1	0.1
2	0.06
3	0.03
4	0.04

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    1. Linearity of expectation:
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    2. Constant's expectation is constant: $E(c) = c$

Conditional Expectation

- Conditional expectation $E(Y|X = x)$:

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \begin{cases} \sum_x \mathbb{E}[Y|X = x]P(X = x) & \text{discrete } X \\ \int_{-\infty}^{\infty} \mathbb{E}[Y|X = x]f(x)dx & \text{continuous } X \end{cases} \tag{1}$$

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- Basically, this theorem says that if we have knowledge about $P(X)$, and the conditional probability of $P(Y|X)$, we can calculate the average of Y.

Variance

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- Standard deviation: $\sigma = \sqrt{V(X)}$

Variance

Definition (Alternative Formula for Variance)

$V(X) = E[X^2] - E[X]^2$

Proof.

$$
\begin{aligned}
V(X) &= E\left[\left(X - E(X)\right)^2\right] && (2)\\
&= E[X^2 - 2XE(X) + E(X)^2] && (3)\\
&= E(X^2) - 2E[XE(X)] + E[E(X)^2] && (4)\\
&= E(X^2) - 2E(X)E(X) + E(X)^2 && (5)\\
&= E(X^2) - E(X)^2 && (6)
\end{aligned}
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□

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        - They are not coming from a different distribution, say, heights of desks

I.I.D. random variables (exercise)

- When the independent assumption may be violated?

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  - e.g., population changes during the sampling process.

Sample Mean of I.I.D. random variables

- Let $X_1, \cdots, X_n$ be i.i.d. random samples of random variable X

Definition (Sample Mean)

The sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

Theorem (The Expected Value of the Sample Mean is the Population Mean)

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Sample Mean of I.I.D. random variables

The Expected Value of the Sample Mean is the Population Mean.

$$
\begin{aligned}
E(\bar{X}) &= E(\frac{1}{n}(X_1 + \cdots + X_n)) && (7)\\
&= \frac{1}{n}E(X_1 + \cdots + X_n) && (8)\\
&= \frac{1}{n}[E(X_1) + \cdots + E(X_n)] && (9)\\
&= \frac{1}{n}[E(X) + \cdots E(X)] && (10)\\
&= E(X) && (11)
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- Note that population variance $V(X)$ is an unknown but fixed quantity.

Law of Large Numbers

- Let $X_1, \cdots, X_n$ be i.i.d. random samples of random variable X

Theorem (Weak Law of Large Numbers, Jacob Bernoulli, 1713)

The sample mean $\bar{X}$ converges in probability to the population mean $E(X)$, as $n \to \infty$.

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  - And perhaps other sample quantities to population quantities?

Estimator

- Let $X_1, \cdots, X_n$ be i.i.d. random samples of random variable X

Definition (Estimate and Estimator)

Estimator of a population quantity θ is a function of the samples, $\hat{\theta} = h(X_1, \cdots, X_n)$; $\hat{\theta}$ is the estimate of θ.

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  - How can we say one estimator is better than the other? What properties should good estimators have?

Desirable Property: Unbiasedness

- For an estimator $\hat{\theta}$, its bias is $E(\hat{\theta}) - \theta$

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see proof here

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- Answer: because the expectation of sample mean equals to population mean $(E(\bar{X}) = E(X))$

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An estimator $\hat{\theta}$ is an consistent estimator if $\hat{\theta}$ converges in probability to θ, as $n \to \infty$.

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- Answer: because of the Law of Large Numbers.

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- This is called plug-in principle.

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- We want to apply the plug-in principle to estimate population variance $V(X)$.

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$\hat{V}(X) = \overline{X^2} - \overline{X}^2$

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&= \left(\mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2\right) - \frac{\mathrm{V}[X]}{n} && (14)\\
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Unbiased Estimator for Population Variance

- Plug-in population variance estimator $\hat{V}(X) = \overline{X^2} - \overline{X}^2$ is biased

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- In general, plug-in estimator is consistent, but may be biased (advanced topic).

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- $\sqrt{\hat{V}(\bar{X})}$ is called standard error.

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Central Limit Theorem

- Let $X_1, \cdots, X_n$ be i.i.d. random samples of random variable X, with finite $E(X) = \mu$ and $V(X) = \sigma^2 > 0$

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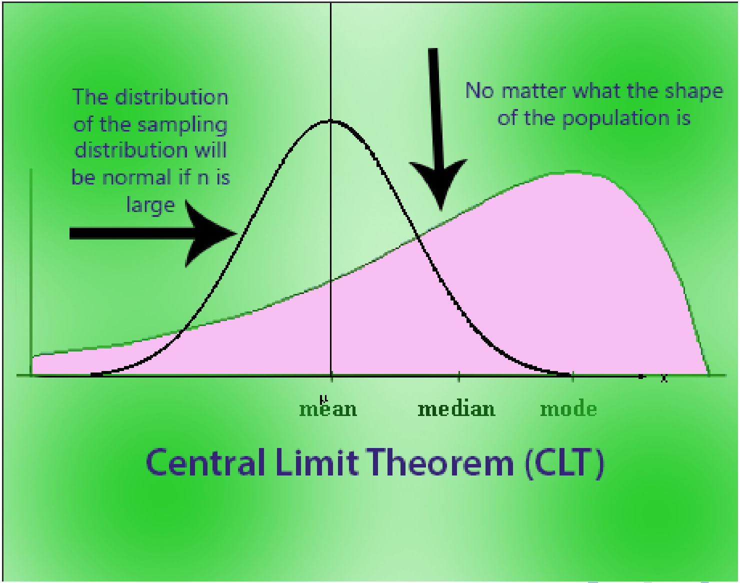

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- Sampling distribution of the sample mean will tend to be approximately normal, even when the population distribution is not distributed normally;
- Thus, central limit theorem provides a general way for us to infer the uncertainty around our estimate of sample mean (and other quantities)

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  - But not all estimators have this good property
- The good thing about asymptotically normal estimator is that we can obtain confidence interval easily

Confidence Interval

Definition (Confidence interval)

A α confidence interval for quantity of interest θ is an estimated interval that covers the true value of θ with at least α probability

- Example: in social sciences, we often uses $\alpha =$95% confidence interval that looks like $[\theta_{min}, \theta_{max}]$. The probability that the true θ falls between $[\theta_{min}, \theta_{max}]$ is at least 95%.

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- Note 2: confidence interval does not need to be symmetric

Normal Approximation-based Confidence Interval

Definition (Estimating Normal Approximation-based Confidence Interval)

A normal approximation-based confidence interval for θ can be estimated by:

$$\left(\hat{\theta} - z_{\frac{1+\alpha}{2}}\sqrt{\hat{V}(\hat{\theta})}, \hat{\theta} + z_{\frac{1+\alpha}{2}}\sqrt{\hat{V}(\hat{\theta})}\right)$$

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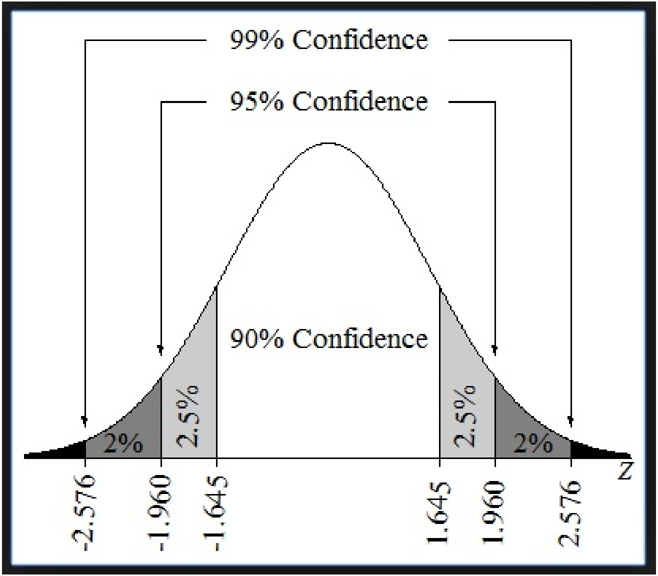
- Normal Approximation-based Confidence Interval is valid for asymptotically normal estimators

Illustration



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- E.g., for 95% confidence interval

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  - Reason 2: you have to know what $\hat{V}(\bar{X})$.
- The Bootstrap is more general method to construct confidence intervals; one of the most important modern statistical concept (Efron, 1979)
  - Drawback of Bootstrap: it's a data-driven method; slow; no analytical solutions.

Bootstrap procedures

Assume we already have $X_1, \cdots, X_n$ be i.i.d. random samples of random variable X). We are interested in estimating a α confidence interval for a population quantity θ

1. Take a with replacement sample of size n from $X_1, \cdots, X_n$
2. Calculate the sample analog of θ
3. Repeat 1 and 2 for m times. We end up having m estimates of θ, $(\hat{\theta}_1, \cdots, \hat{\theta}_m)$
4. Take the $\frac{1-\alpha}{2}$ and $\frac{1+\alpha}{2}$ quantile of the values $(\hat{\theta}_1, \cdots, \hat{\theta}_m)$. These two quantiles give us the bootstrap confidence intervals.

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  - note that in reality we only have one sample (of n units)

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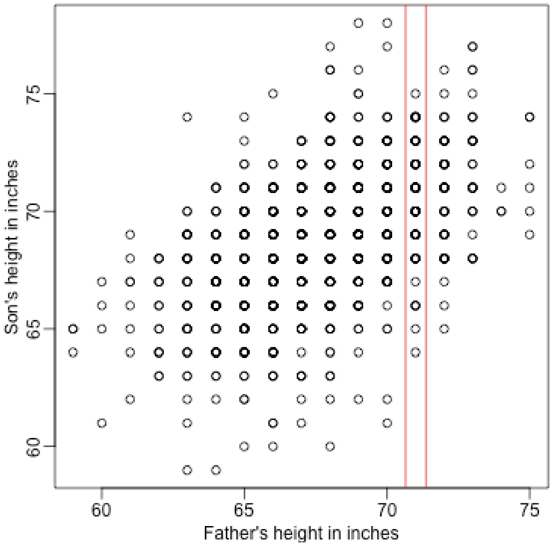
- Given two variables Y and X, and we observed X takes the value x.
- A prediction of Y given X is a function $g(X)$ that approximate Y
- For instance, $E(Y|X)$ is a prediction of Y given X
- Again, there are tons of ways to predict Y given X (e.g., median of Y given X)

Prediction (example)

- Predicting son's height with father's height



Using Conditional Expectation as Prediction

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Property 1: $E(\epsilon) = 0$.

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Conditional Expectation as Prediction (cont'd)

Property 2: $E(\epsilon|X) = 0$.

$$\begin{aligned}
E(\epsilon|X) &= E\left[Y - E(Y|X)|X\right] && (21)\\
&= E(Y|X) - E\left[E(Y|X)|X\right] && (22)\\
&= E(Y|X) - E(Y|X), \text{(Law of Iterated Expectation)} && (23)\\
&= 0 && (24)
\end{aligned}$$

$\square$

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  - Recall independence means that $P(\epsilon|X) = P(\epsilon)$

Independent, mean independent, and uncorrelated

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- Mean independent: $P(Y|X) = E(Y)$
- Uncorrelated: $E(XY) = E(X)E(Y)$
- In general, we have the following relationship (the reverse is not true):

X, Y are independent $\implies$ X, Y are mean independent $\implies$ X, Y are uncorrelated.

Using Conditional Expectation as Prediction

- Property 3 says $g(X)$ and error is uncorrelated; it can be derived from Property 2 (mean independence) and Property 1 ($E(\epsilon) = 0$)

Property 3: $E\big(g(X)\epsilon\big) = 0$, for any $g(X)$.

$$
\begin{aligned}
E\big[g(X)\epsilon\big] &= E\big[g(X)\big(Y - E(Y|X)\big)\big] && (25)\\
&= E\big[g(X)Y - g(X)E(Y|X)\big] && (26)\\
&= E\big[g(X)Y\big] - E\big[g(X)E(Y|X)\big] && (27)\\
&= E\big[g(X)Y\big] - E\big[E\big(g(X)Y|X\big)\big], \text{(g(X) is a constant given } X\text{)} && (28)\\
&= E\big[g(X)Y\big] - E\big[g(X)Y\big], \text{(Law of Iterated Expectation)} && (29)\\
&= 0 && (30)
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- MSE make sure that you get penalized more, if the absolute error is large.
    - MSE is perhaps the most widely used error metric
- Both MAE and MSE ≥ 0; a good estimation thus should minimize MAE or MSE

Conditional Expectation As the Best Predictor

- There are some even better properties of $E(Y|X)$ that make it the best predictor, given Mean Squared Error

Theorem (Conditional Expectation as the Best Predictor)

Conditional Expectation Function $E(Y|X)$ is the best predictor of Y because it minimizes Mean Squared Error

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- Hint: use the conditional expectation error $\epsilon = Y - E(Y|X)$

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$$
\begin{aligned}
E\left[\left(Y - g(X)\right)^2\right] &= E\left[\left(\epsilon + E(Y|X) - g(X)\right)^2\right] && (31)\\
&= E\left[\epsilon^2 + 2\epsilon\left(E(Y|X) - g(X)\right) + \left(E(Y|X) - g(X)\right)^2\right] && (32)\\
&= E\left[\epsilon^2\right] + 2E\left[\epsilon\left(E(Y|X) - g(X)\right)\right] + E\left[\left(E(Y|X) - g(X)\right)^2\right] && (33)\\
&= E\left[\epsilon^2\right] + E\left[\left(E(Y|X) - g(X)\right)^2\right], \text{(Property 3)} && (34)\\
&\geq E\left[\epsilon^2\right] && (35)\\
&= E\left[\left(Y - E(Y|X)\right)^2\right] && (36)
\end{aligned}
$$

$\square$

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- Note that this says that the conditional expectation gives an upper bound on how well we can make a guess of Y based on X, if we want to minimize MSE
- If the conditional expectation itself is not a very good predictor, we can still make lots of errors
  - But in this case, other predictions can only be worse

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  - Motivated differently from Aronow and Miller