# SOSC 5340: Generalized Linear Model

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# Outline


Binary Outcomes

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MLE

Interpretations

GLM

Multinomial and Ordered Logit

Poisson, Negative Binomial, and Zero-inflated Poisson

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Bootstrap

Today's Review

# Binary Outcome

- Binary outcome variable:
    - $Y_i \in \{0, 1\}$
- Examples in social science: numerous!
    - Higher education: $1 =$ has college education; $0 =$ does not have college education
    - Conflict: $1 =$ civil war; $0 =$ no civil war
    - Voting: $1 =$ vote; $0 =$ abstain

# How do we model binary outcome?

- We already know that conditional expectation $E(Y|X)$ is the best predictor
- Linear regression: with assumptions 1,2 and especially 3

$$E(Y|X) = \alpha + X\beta$$

- When $Y$ is binary:

$$E(Y|X) = P(Y = 1|X)$$

- $P(Y = 1|X)$ is the conditional probability of Y $=$ 1 given $X$
- Conditional probability must be between 0 and 1 by definition
    - But $\alpha + X\beta$ is not always between 0 and 1
    - So Assumption 3 is very likely to be violated

# How do we model binary outcome?

- Linear Probability Model (LPM)
  - Just pretend this problem does not exist; still run OLS regression with binary outcome.
- Alternatively: we can apply a function $F$ onto $\alpha + X\beta$ to ensure

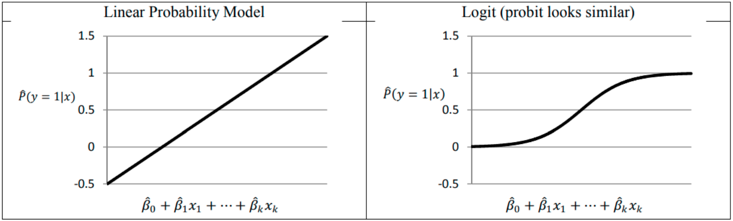
$$0 \leq E(Y|X) = F(\alpha + X\beta) \leq 1$$

# Logistic regression

- Two useful functions:
  - $logit(X) = log(\frac{X}{1-X})$
  - $logit^{-1}(X) = \frac{exp(X)}{1+exp(X)}$
- Logistic Regression
  - We use the inverse-logit function as $F$

$$E(Y|X) = logit^{-1}(\alpha + X\beta) = \frac{exp(\alpha + X\beta)}{1 + exp(\alpha + X\beta)} = \frac{1}{1 + exp(-\alpha - X\beta)}$$

# Logistic Regression vs Linear Probability Model




- inverse-logit function "squashs" $X\beta$ to $[0, 1]$

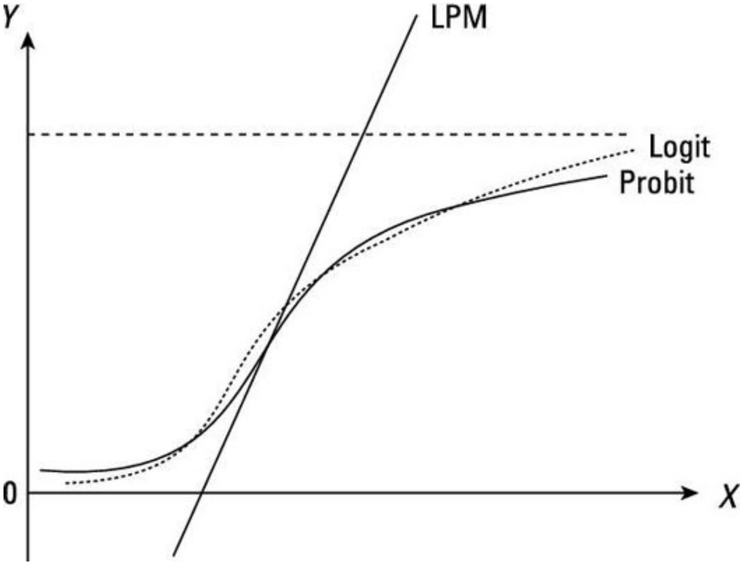
# Probit regression

- We can also "squash" $\alpha + X\beta$ using standard normal CDF (normal cumulative density function)

$$E(Y|X) = \Phi(\alpha + X\beta)$$

- Statistical model using normal CDF is known as probit regression
- In general, any CDF can be used as $F$ to squash $X\beta$ to $[0, 1]$
    - inverse-logit is the CDF of standard logistic distribution
    - $\Phi$ is the CDF of standard normal distribution

# Probit vs Logit vs Linear Probability



# More on linear probability model

- Binary data (and more general, most categorical data) always exhibit heteroscedasticity

$$\begin{aligned} V(\epsilon|X) &= V(Y - X\beta|X) \\ &= V(Y|X) \\ &= P(Y=1|X)\big[1 - P(Y=1|X)\big] \end{aligned} \tag{1}$$

- The above equation shows that variance of error changes based on the value of $X$! It is always heteroscedastic.
- So always use robust standard error if you decide to use OLS regression to model binary outcomes (linear probability model).

## Assumptions of OLS regression

- Assumption 1: the expected error is 0

$$E(\epsilon) = 0$$

- Assumption 2: mean independent between $X$ and the error

$$E(\epsilon | X) = 0$$

- Assumption 3 of OLS (linear model)

$$Y = X\beta + \epsilon$$

- Assumption 5: normal error (which implies Assumption 4, homoscedastic error)

$$\epsilon \sim N(0, \sigma^2)$$

# Assumptions of Logistic/Probit regressions

- Assumption 1 and 2: shared by logit/probit regressions
- Assumption 3 of logit/probit: linear model + non-linear transformation

$$
\begin{aligned}
Y^* &= X\beta + \epsilon \\
Y &= \begin{cases} 1 & \text{if } y^* > 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{2}
$$

- $Y^*$ is an unobserved latent variable
- if the latent variable is bigger than a pre-determined cutoff (here 0), we get $Y = 1$
- We only observe samples of $Y$
    - economists may say that $Y^*$ is the underlying preference, and $Y$ is revealed preference

# Assumptions about error of logit/probit

- Assumption 5 of Logistic/Probit regressions
  $\epsilon$ is distributed according to the probability density distribution of a CDF function $F$
  - $F$ is inverse-logit function; the error follows standard logistic distribution
  - $F$ is $\Phi$; the error follows standard normal distribution

Assumptions 3 and 5 together lead to

$$E(Y|X) = F(X\beta)$$

# Estimation of parameters in OLS regressions: review

- There are two ways to estimate $\beta$ in linear regression
- We can write some population equations, plug-in the sample analog, and solve these sample equations
- We can also directly minimize empirical MSE
- Both solutions result in the same $\beta$ estimate for OLS regression

$$\hat{\beta} = \left[\mathbf{X}^T\mathbf{X}\right]^{-1}\mathbf{X}^T\mathbf{Y} \tag{3}$$

# Maximum Likelihood Estimation

- There is no way to write down a closed-form solution for logistic regression coefficients.
- We use Maximum Likelihood Estimation (MLE)
- MLE is a general methods for estimating parameters in parametric statistical models and making statistical inference.
- Requirement: assumptions about functional form of conditional probability $P(Y|X)$
- Say, in logistic regression, $P(Y = 1|X) = logit^{-1}(X\beta)$, and $P(Y = 0|X) = 1 - P(Y = 1|X)$
- For a single data point, the probability we observe $Y_i$ is exactly given by $logit^{-1}(X_i\beta)$ or $1 - logit^{-1}(X_i\beta)$ (depending on observed $Y_i$)

# Maximum Likelihood Estimation

- Because we have i.i.d. samples, we can multiple these empirical probabilities together, as the probability that we observe the entire sample.
- The probability we observe the entire sample is called likelihood: $L$

$$L = \prod_{i=1}^{n} P(Y_i|X_i) \tag{4}$$

- $L$ is a function of unknown $\beta$
- Naturally, we say that a good $\beta$ is the one that makes the likelihood the largest.
  - Intuitively, it says that our chosen $\beta$ should make the probability to observe the entire sample the largest.
- Put it differently, our estimate of $\beta$ should maximize the likelihood function.

# MLE estimate

- In practice, it is easier to work with log of likelihood, called log-likelihood
- $\log L = \sum_{i=1}^{n} \log P(Y_i|X_i)$
- We try to find $\beta$ that maximize log-likelihood

$$\hat{\beta}_{MLE} = \arg\max_{\beta} \log L$$

# MLE inference

- And estimated variance of $\hat{\beta}_{MLE}$ is given by

$$\widehat{V}(\hat{\beta}_{MLE}) = diag\left(\left(\frac{\partial^2 \log L}{\partial \beta^2}\right)^{-1}\right) \tag{5}$$

- $\frac{\partial^2 \log L}{\partial \beta^2}$ is called Hessian matrix.
- $diag()$ takes out the diagonals of the matrix
- Last, we can use normal approximated intervals for confidence interval (below is an example for 95% confidence interval)

$$\left(\hat{\beta}_{MLE} - 1.96 * \hat{\sigma}(\hat{\beta}_{MLE}), \hat{\beta}_{MLE} - 1.96 * \hat{\sigma}(\hat{\beta}_{MLE})\right)$$

# MLE properties

- MLE estimate has some good properties:
- It is consistent
- It is asymptotically normal (so we can use normal-approximated confidence interval)
- Unbiaseness? No guarantee

# MLE in practice: logistic regression

- Step 1: write single point probability distribution; this case it is easy:
  - $P(Y_i = 1|X_i) = logit^{-1}(X_i\beta)$, and $P(Y_i = 0|X_i) = 1 - P(Y_i = 1|X)$
  - We can write this in a single equation:

$$P(Y_i|X_i) = \left[logit^{-1}(X_i\beta)\right]^{Y_i}\left[1 - logit^{-1}(X_i\beta)\right]^{1-Y_i} \tag{6}$$

- Step 2: for all $n$ points:

$$L = \prod_{i=1}^{n} P(Y_i|X_i) = \prod_{i=1}^{n} \left[logit^{-1}(X_i\beta)\right]^{Y_i}\left[1 - logit^{-1}(X_i\beta)\right]^{1-Y_i} \tag{7}$$

# MLE in practice: logistic regression

- Step 2 (cont'd): the log-likelihood is

$$\log L = \sum_{i=1}^{n} Y_i \log \left( logit^{-1} \left( X_i \right) + \left( 1 - Y_i \right) \right) \log \left[ 1 - logit^{-1} \left( X_i \right) \right] \tag{8}$$

- And remember that $logit^{-1}(X\beta) = \frac{exp(X\beta)}{1+exp(X\beta)}$
- With some math, you will find that

$$\frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{n} \left[ Y_i - logit^{-1}(X\beta) \right] X_i$$

# Optimization

- We want to select $\beta$ that makes $\log L$ the largest
- How? Two solutions
- Standard calculus
  - Find $\beta$ that makes the partial derivative $\frac{\partial L}{\partial \beta} = 0$.
  - For logistic regression, in general, you cannot analytically solve $\beta$ that makes the partial derivative zero.
- Numerical optimization:
  - Try many $\beta$; calculate their $\log L$
  - choose one that gives the largest $\log L$.
  - How? There are infinite number of choices of $\beta$
  - There are many mature optimization algorithms that help you find $\beta$ quicker

# Optimization

- One commonly used optimization method: gradient descent

$$\beta_{new} = \beta_{old} + \eta \cdot \frac{\partial \log L}{\partial \beta} \tag{9}$$

- $\eta$ is called learning rate; try different options
- You need to choose an starting $\beta$; try several random guess

# Optimization

- There are many other optimization methods
- They basically follow the similar idea: makes some initial guesses of $\beta$ and gradually improve on older estimates
- in R, use `optim` package

# Odds and Log Odds

- Let us move on to interpreting regression coefficients

$$X\beta = logit(E(Y|X)) = log\left[\frac{P(Y=1|X)}{1-P(Y=1|X)}\right] = log\left[\frac{P(Y=1|X)}{P(Y=0|X)}\right]$$

- $\frac{P(Y=1|X)}{P(Y=0|X)}$ is called odds; it is the ratio between two conditional probabilities: $Y = 1$ vs $Y = 0$, given $X$.
  - Odds $> 1$ means $Y = 1$ is more likely than $Y = 0$ give $X$
- $log\left[\frac{P(Y=1|X)}{P(Y=0|X)}\right]$ is the log of odds; we call it log-odds
- Following the interpretation of OLS regression, we can interpret logistic regression coefficient in this way:
  - One unit increase in $X$ will lead to $\beta$ increase in log-odds
  - Problem: it is very intuitive to think about what $\beta$ increase in log-odds means

# Logistic Regression Interpretations: Approach 1

- Example, we are interested in the effect of income and gender on whether a person vote or not. For gender, 1 is female and 0 is female. Income is in thousand dollars

$$P(Y = 1|X) = logit^{-1}(-1.92 + 0.032 * \text{income} + 0.67 * \text{gender})$$

- A simple rule of thumb (based on Gelman and Hill, *Data Analysis using Regression and Multilevel Hierarchical Models*, 2007.)
  - Divide your $\beta$ by 4, and this is roughly the upper bound of the change in probability
  - For income, we divide 0.032 by 4. It means that one unit (a thousand) increase in income predicts no more than 0.8% increase in the probability of voting.
  - For gender, $0.67/4 = 0.168$. This suggests that female's voting probability is 16.7% more than that of male's
  - Do not write this in formal paper!

# Logistic regression interpretations: Approach 2

- Remember one unit increase in $X$ lead to $\beta$ increase in log-odds.
- Write the conditional probability $P(Y = 1|X)$ before change as $p_b$, and the condition probability $P(Y = 1|X)$ after increasing $X$ for one unit as $p_a$

$$log\frac{p_a}{1-p_a} - log\frac{p_b}{1-p_b} = \beta \implies \frac{\frac{p_a}{1-p_a}}{\frac{p_b}{1-p_b}} = exp(\beta)$$

- $\frac{\frac{p_a}{1-p_a}}{\frac{p_b}{1-p_b}}$ is called odds ratio
- One unit increase in $X$ leads to $exp(\beta)$ change in odds ratio
- For income, $exp(0.032) = 1.03$
  - This means that odds is 1.03 times higher for one unit increase in income
  - Or in other words, odds ratio increase by 3%
- For gender, $exp(0.67) = 1.95$
  - This means that odds of voting is 1.95 times higher among females compared with males

# Logistic regression interpretations: Approach 3

- We can always calculate the marginal effect: how conditional probability changes for one unit increase in $X$: $\frac{\partial P(Y=1|X)}{\partial X}$
- After some calculations, you will find that;

$$\frac{\partial P(Y=1|X)}{\partial X} = \beta(logit^{-1}X\beta)(1 - logit^{-1}X\beta)$$

- In other words, one unit increase in $X$ leads to $\beta(logit^{-1}X\beta)(1 - logit^{-1}X\beta)$ changes in predicted probability
- It is easy to see that the marginal effect will change depending on exact values of $X$
- The marginal effect is generally bigger, when $X$ is around the mean

# Logistic regression interpretations: Approach 3

- Typically there are two ways to visualize/show marginal effect
- Marginal effect at the mean (MEM)
  - Set all other variable at their mean value
  - MEM is the change in predicted probability when the focal independent variable change for one unit
  - Cons: setting categorical variables at their means are not meaningful
    - e.g., 0 is female and 1 is male; what is gender = 0.45 means?
- Average marginal effect (AME)
  - For each observation, holding other variables at their observed value; calculate marginal effect for one focal variable
  - Take the average of marginal effects of the focal variable for each observation
- R package `margins` and stata command `margins` will return AME by default; has to explicit set parameters to calculate marginal effect at the mean
- `https://cran.r-project.org/web/packages/margins/vignettes/TechnicalDetails.pdf`

# Logistic regression interpretations: Approach 4

- Just plot predicted probability versus one focal variable you are mainly interested in
- And holding other $X$ at a fixed level.
  - say, holding others at the mean
  - or at a particular value that are theoretically interesting
- This is especially useful if you have interaction terms

# Predicted probability (example)

See RMarkdown codes and files.

# What are practical recommendations?

- Use the divide by 4 rule and make an intuitive sense of how large the effect is
- Then calculate AME or MEM
- Or plot the predicted probabilities versus the key independent variables
- You can state that
  - One unit increase in $X$ leads to $\beta$ change in log-odds
  - Or, one unit increase in $X$ leads to $exp(\beta)$ change in odds ratio
  - (but I personally find them hard to grasp; and I am sure I am not the only one)

# How to interpret probit regressions?

- No direct substantive interpretation of $\beta$ in probit regressions (it is not an odds ratio)
- Probit just makes math calculation easier, but it lacks a natural interpretation.

# Limited Dependent Variable

- Beyond binary outcomes, $Y \in \{0, 1\}$
- Categorical:
  - e.g., major choices;
- Integer (count): $Y \in \{0, 1, 2, \cdots\}$
  - e.g., event counts
- Censored: observed $Y$ is in a certain range, but we know in reality they should not be
  - e.g., US census write anyone who report their age $> 90$ as 90; so in census, age is between [0,90]
- The common problem is that the outcome $Y$ is limited to some regions, not in $(-\infty, \infty)$
  - so economists sometimes call them as limited dependent variable

# Generalized Linear Model

- To model limited dependent variables, we use generalized linear model (GLM)
- GLM looks like:
  - $h(E(Y|X)) = X\beta$
  - or, $E(Y|X) = h^{-1}(X\beta)$
- $h()$ is called link function
- Linear regression is a kind of GLM, where $h(X) = X$
- Logistic regression is a kind of GLM, where $h(X) = logit(X)$
- Other GLM choose different $h()$ to model different types of $Y$

# GLM

- In practice, scholars use MLE to make statistical estimation and inference for GLM
- Recall that to use MLE, we need to make assumptions about what $p(Y|X)$ looks like

# Estimation and Inference of MLE

- Steps are standard
  1. write down $P(Y|X)$
  2. write down $\log L$: the log-likelihood function
  3. obtain coefficient estimates that maximize log-likelihood
     - and use Hessian matrix to calculate confidence interval

# Extending Logistic Regression

- Suppose we have categorical outcome with more than two values
- Sometimes, these categories have no intrinsic orders
  - E.g., majors choices between ( Economics = 1, Political Science = 2, Sociology= 3, Public Policy = 4 )
- Other times, these categories are ordinal
  - E.g., a survey ask whether you think religion deters economic growth, on a 1-7 scale.
  - 1 means strongly disagree, and 7 means strongly agree
  - Order gives more information than pure categories
  - Why not use continuous outcome models?
    - Dont want to assume equal distances between levels
    - Say, moving from 1-4 is different from 4-7
    - Assuming continuous $Y$ does not distinguish these two

# Ordered Logit: ordered outcome

- Peter McCullough, *Regression Models for Ordinal Data*, 1980
- Recall that logistic regression assumes a generating process based on latent variables

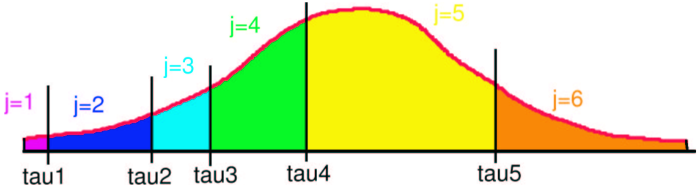
$$\begin{aligned} Y^* &= X\beta + \epsilon \\ Y &= \begin{cases} 1 & \text{if } y^* > 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{10}$$

- $Y^*$ is an unobserved latent variable
- if the latent variable is bigger than a pre-determined cutoff (here 0), $Y = 1$
- Otherwise, $Y = 0$

# Ordered Logit

- We can borrow the same intuition to derive ordered logit regression, with $J > 2$ ordinal categories
- We create $J - 1$ latent cutoffs

$$Y = \begin{cases} 1 & \text{if } Y^* \leq \tau_1 \\ 2 & \text{if } \tau_1 < Y^* \leq \tau_2 \\ 3 & \text{if } \tau_2 < Y^* \leq \tau_3 \\ . & \\ J & \text{if } \tau_{J-1} \leq Y^* \end{cases} \tag{11}$$



# Ordered Logit

- So now the Assumption 3 for ordered logit becomes:

$$Y^* = X\beta + \epsilon$$

$$Y = \begin{cases} 1 & \text{if } Y^* \leq \tau_1 \\ 2 & \text{if } \tau_1 < Y^* \leq \tau_2 \\ 3 & \text{if } \tau_2 < Y^* \leq \tau_3 \\ . & \\ J & \text{if } \tau_{J-1} \leq Y^* \end{cases} \tag{12}$$

- And the error $\epsilon$ follows a standard logistic distribution (the same as logistic regression)

# Ordered Logit vs Linear Regression

- It may be easier to change from "very unlikely" (1) to "unlikely" (2), but it is more difficult to change from "unlikely" to "neutral" (3)
- For linear regression
  - It takes the same amount of changes in $X$ to turn $Y$ from 1 to 2 versus $Y$ from 2 to 3
- For ordered logit
  - $Y$ changing from 1 to 2 means latent $Y^*$ changes from below $\tau_1$ to $(\tau_1, \tau_2)$
  - $Y$ changing from 2 to 3 means latent $Y^*$ changes from $(\tau_1, \tau_2)$ to $(\tau_2, \tau_3)$
  - It often requires a different amount a change in $X$ to move $Y$ from 1 to 2 versus from 2 to 3. That's what we want to capture

# Ordered Logit

- For MLE, we have to explicitly write down $P(Y|X)$

$$\begin{aligned} \mathrm{P}(Y=1|X) &= \Pr\left(\beta X + \epsilon \leq \tau_1 | X\right) \\ &= \mathrm{P}\left(\epsilon \leq \tau_1 - \beta X | X\right) \\ &= F\left(\tau_1 - \beta X\right), \text{(definition of cumulative probability } F\text{)} \\ &= logit^{-1}(\tau_1 - \beta X) \end{aligned} \tag{13}$$

$$\begin{aligned} \mathrm{P}(Y=2|X) &= \Pr\left(\tau_1 < \beta X + \epsilon \leq \tau_2 | X\right) \\ &= \Pr\left(\tau_1 - \beta X < \epsilon \leq \tau_2 - \beta X | X\right) \\ &= F\left(\tau_2 - \beta X\right) - F\left(\tau_1 - \beta X\right) \\ &= logit^{-1}\left(\tau_2 - \beta X\right) - logit^{-1}\left(\tau_1 - \beta X\right) \end{aligned} \tag{14}$$

And so on and so forth, for $j$ up to $J-1$

# Ordered Logit

The last category $J$

$$
\begin{aligned}
\mathrm{P}(Y = J|X) &= \mathrm{P}\left(\tau_{J-1} \leq \beta X + \epsilon|X\right) \\
&= \mathrm{P}\left(\epsilon \geq \tau_{J-1} - \beta X|X\right) \\
&= 1 - \mathrm{P}\left(\epsilon < \tau_{J-1} - \beta X\right) \\
&= 1 - F\left(\tau_{J-1} - \beta X\right) \\
&= 1 - logit^{-1}\left(\tau_{J-1} - \beta X\right)
\end{aligned} \tag{15}
$$

- We have written down $P(Y|X)$ for every possible value of $Y$.
- Now we can use MLE to estimate parameters
- Now, there are regression coefficients $\beta$, as well as cutoffs $\tau$
- Statistical software will return estimates for both

# Ordered Logit

- What do the cutoffs $\tau$ mean?
- Recall that $P(Y=1|X) = logit^{-1}(\tau_1 - \beta X)$
- And $P(Y=2|X) = logit^{-1}(\tau_2 - \beta X) - logit^{-1}(\tau_1 - \beta X)$
- We add then together:

$$P(Y=1|X) + P(Y=2|X) = P(Y \leq 2|X) = logit^{-1}(\tau_2 - \beta X) \tag{16}$$

- And take the logit:

$$logit\,(P(Y \leq 2)) = \tau_2 - \beta X \tag{17}$$

- The rest is similar

$$logit\,(P(Y \leq j)) = \tau_j - \beta X$$

- In this way, $\tau$ looks like intercepts in normal regressions; so some other software (R) call them intercepts

# Multinomial Logit: categorical outcome

- Multinomial logit: for categorical outcomes that have no intrinsic order
- We extend logistic regression in a different way
- $Y$ has $J$ levels, from 0 to $J-1$
- For logistic regression, $P(Y=1|X) = logit^{-1}X\beta = \frac{exp(X\beta)}{1+exp(X\beta)}$
- For multinomial logit, we make similar assumptions about $P(Y=j|X)$

$$P(Y=j|X) = logit^{-1}X\beta_j = \frac{exp(X\beta_j)}{1+\sum_{j=1}^{J} exp(X\beta_j)} \qquad (18)$$

- And for reference group, its

$$P(Y=0|X) = logit^{-1}X\beta_j = \frac{1}{1+\sum_{j=1}^{J} exp(X\beta_j)} \qquad (19)$$

# Multinomial Logit

- For all levels except the reference group, it has its own regression coefficients
- Say we have 7 categories and 4 predictors (each of them is continuous), then in total we will have 6 * 5 = 30 coefficients
  - 6 = 7 - 1
  - 5 = 4 + 1 (plus intercepts)
- Also because we know what $P(Y = j|X)$ looks like for every possible value of $Y$, we can use MLE to estimate $\beta_j$

# Interpreting multinomial logit

- Based on the assumptions of multinomial, it is easy to see:

$$\frac{P(Y=j|X)}{P(Y=0|X)} = exp(X\beta_j) \tag{20}$$

- Therefore, one unit increase in $X$ leads to $exp(\beta_j)$ increase in odds ratio of $Y=j$ occurring, relative to $Y=0$

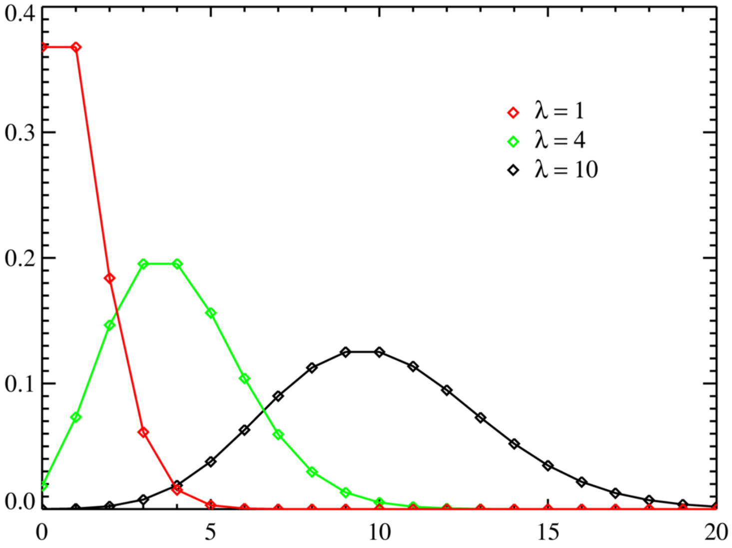
# Poisson Distribution

- Example: $Y$ is event count
  - e.g., number of times each person visit a physician)
  - Number of new born / decease in a country
  - Usually small counts are more likely than large counts
- Key difference: $Y$ are non-negative integers; in linear regression $Y$ is assumed to be continuous variable between $(-\infty, \infty)$
- Event count usually follows Poisson distribution

$$Pr(X = k) = \frac{\tau^k e^{-\tau}}{k!}$$

- $k! = k(k-1)(k-2)\cdots 1$ is factorial
- Property: $E(X) = V(X) = \tau$

# Poisson Distribution



# Poisson Regression

- The conditional probability $P(Y|X)$ is assumed to be distributed according to Poisson:

$$P\left(Y=y|X\right)=\frac{\exp\left(-\tau\right)\tau^{y}}{y!},\quad y=0,1,2,\ldots \qquad (21)$$
$$\tau=\exp\left(X\beta\right)$$

- And the conditional expectation $E(Y|X)$ is given by:

$$E(Y|X)=\tau=exp(X\beta) \qquad (22)$$

## Poisson Regression (cont'd)

- Why don't we explicitly write $E(\epsilon) = 0$ and $E(\epsilon X) = 0$ as in the Assumption 1 and 2 of linear, logistic and probit regressions?
  - Hint: our assumption of the form of $P(Y|X)$ is very strong
  - It directly gives what $E(Y|X)$ should look like
  - And $E(\epsilon) = 0$ and $E(\epsilon X) = 0$ are essentially the property of $\epsilon = Y - E(Y|X)$
  - So in many textbooks, when introducing generalized linear models, they will omit Assumptions 1 and 2, since it is implied by the assumption of the function form of $P(Y|X)$
- Poisson assumption implies that the data is heteroskedastic:

$$\begin{aligned} V(\epsilon|X) &= V(Y - E(Y|X)|X) \\ &= V(Y|X) \\ &= exp(X\beta) \end{aligned} \tag{23}$$

# Poisson and Log-Linear model

- Poisson regressions:

$$E(Y|X) = \tau = exp(X\beta)$$

- An alternative way is to take log at both side of the equation

$$\log E(Y|X) = log(\tau) = X\beta$$

- It means that the link function of Poisson regression is log
- Sociologists and demographers call $\log E(Y|X) = log(\tau) = X\beta$ as log-linear model

# MLE for Poisson regression

1. $P\left(Y=y|X\right)=\frac{\exp(-\tau)\tau^{y}}{y!}$; $\tau=exp(X\beta)$
2. Likelihood is: $L=\prod_{i=1}^{N}\frac{\exp(-\tau_i)\tau_i^{y}}{y!}$
   - and log-likelihood is :

$$\sum_{i=1}^{n} y_i X_i'\beta-\exp(X_i'\beta)-\log y_i!$$

3. try to maximize by setting the derivative to be 0

$$\frac{\partial l}{\partial\beta}=\sum_{i=1}^{n}\left(y_i-\exp(X_i'\beta)\right)X_i=0$$

- There is no closed-form solution, unfortunately. Numerical optimization is required.

# Interpretation of Poisson Regression

- In log-linear model format:

$$\log E(Y|X) = log(\tau) = X\beta$$

- One unit increase in $X$ leads to $\beta$ increase of the average of $y$ in its log scale
- In Poisson regression format:

$$E(Y|X) = exp(X\beta)$$

- One unit increase in $X$ leads to $exp(\beta) - 1$ increase in $Y$
- One unit increase in $X$ multiplies the mean of $Y$ by a factor $exp(\beta)$
- The ratio between the new $Y$ and old $Y$ is $exp(\beta)$, on average

# Over-dispersion of Count Data

- Poisson regression assumes that $P(Y|X)$ follows a Poisson distribution
- Recall that Poisson distribution assumes that the mean and the variance is the same
- Sometimes we have data whose variance is bigger than mean
- E.g., Long, J. Scott. 1990. *The Origins of Sex Differences in Science*. Social Forces. 68(3):1297-1316.
- The outcome is the number of published articles by a Ph.D. student in biochemistry
- The mean number of articles is 1.69 and the variance is 3.71, a bit more than twice the mean.
- Why? There are always super-starts :) and people who publish nothing : (

# Zero-inflated Poisson Regression

- One common situation of over-dispersion: there are a lot of zeros in the outcome $Y$ and a few big values, which boosts the variance of outcome
- Example: civil war as outcome.
- Zero-inflated Poisson Regression is designed to address this issue
- It assumes that data has two generating processes
  1. With probability $1-\lambda$, the data is generated according to Poisson with mean $\tau$
  2. With probability $\lambda$, we generate excess zeros.
- The final conditional probability is

$$P\left(Y=y|X\right)=\lambda+\left(1-\lambda\right)\frac{\exp\left(-\tau\right)\tau^{y}}{y!}$$

# Zero-inflated Poisson Regression (cont'd)

- With the assumptions in the previous slide
- $E(Y|X) = (1-\lambda)\tau$
- $V(Y|X) = (1-\lambda)\tau(1+\tau\lambda)$
- $V$ is bigger than $E$, of a ratio of $1+\tau\lambda$
- Essentially, zero-inflated Poissin regression is the mix of two regressions:
    - One Poisson regression, with prob $1-\lambda$
    - One logistic regressions (0 and all others), with prob $\lambda$
    - Each regression has its own coefficients
- So it is a more complex model than negative binomial regression, which adds only one additional parameter

# Negative binomial regression

- Another way to deal with over-dispersion: choose a different functional form about $P(Y|X)$

$$P(Y = y|X) = \frac{\Gamma(\alpha + y)}{y!\Gamma(\alpha)(\tau + \alpha)^{\alpha+y}} \tag{24}$$

- And $\tau = exp(X\beta)$
- $\Gamma$ is Gamma function, an extension of factorial
- With this more complex parametric assumption
- $E(Y|X) = \tau$ (similar to Poisson regression)
- $V(Y|X) = \tau(1 + \frac{1}{\alpha}\tau)$
- Positive $\alpha$ ensures that variance is bigger than the mean

# Other count data model

- Zero truncated regressions
  - Say, the outcome of the length of stay in a hospital, which is at least 1 day
  - Zero-truncated Poisson:
    - Remove the probability $P(y = 0)$ because it's not possible)
    - Re-scale the rest of the probability distribution to make it sums to 1

# How do we choose between models?

- Let us use our example of number of published articles by Ph.D. biochemists
- We can choose between three models:
  - Poisson regression
  - Negative binomial regression
  - Zero-inflated Poisson regression
- Decide whether or not to use Poisson regression is relative easier: (Cameron and Trivedi, "Regression-based tests for overdispersion in the Poisson model", *Journal of Econometrics*, 1990)
- Assume $E(Y|X) = \tau$, then
- Null Hypothesis: $V(Y|X) = E(Y|X) = \tau$
- Alternative Hypothesis: $V(Y|X) = \tau + c\tau$
- Cameron and Trivedi's overdispersion test just seeks to examine whether $c = 0$
- (For R users: `dispersiontest` in AER package)

# Use Likelihood for Hypothesis Testing

- But how can we compare negative binomial regression vs zero-inflated Poisson regression?
- We can compare Likelihood among similar models to choose the best one
- Intuition:
  - Likelihood $L$ represents the joint probability that we observe the entire data, given our parameters
  - Assume we have two models
  - A better model should have larger likelihood

# Likelihood Ratio Test

- Define Likelihood Ratio Test Statistics $D$ as:

$$
\begin{aligned}
D &= -2\log\frac{L_{\text{null}}}{L_{\text{alternative}}} \\
&= 2(\log L_{\text{alternative}} - \log L_{\text{null}})
\end{aligned} \tag{25}
$$

- For comparing models, null model is often the simpler model, and alternative model is often the more complex model
- Null Hypothesis: $D = 0$
- Alternative Hypothesis: $D > 0$
- The bigger the $D$, the more evidence for the alternative model

## Likelihood Ratio Test (cont'd)

- Wilk's Theorem (1938): $D$ has an $\chi^2$ -distribution, with degrees of freedom equal to the difference in number of parameters between alternative model and the null model, if the null model is nested within the alternative model
- Nested basically means that the null model can be viewed as a simple case of the alternative model
  - e.g., null is logistic regression with 5 variables; alternative adds another variable
  - null is Poisson; alternative is negative binomial or zero-inflated Poisson
- For non-nested models, Wilk's Theorem does not hold; we need something else (shortly)

# Likelihood Ratio Test (cont'd)

- How do express Wilk's Theorem in the p-value language?
  - Say we get a $D = 12$, and the degree of freedom is 2
  - Definition: the probability of obtaining a test statistics that equals to $D$ or higher is approximately $p \iff$ p-value is $p$
  - $P(D < 12, d.f. = 2) = 0.9975$
    - in R, just type `pchisq(12, 2)`, which is the cumulative probability distribution of $D$
    - It means that the probability of observing a $D$ smaller than 12 is 0.9975
    - So the probability we observe a $D$ equal to or larger than 12 is 1 - 0.9975 $=$ 0.0025, which is our p-value)

# Bias-Variance Trade-Off and Likelihood Ratio Test

- But, a more complex model (adding more parameters) usually can predict more accurately and thus often always have larger likelihood
- AIC: Akaike information criterion (named after Hirotugu Akaike, 1974); reaching balance between predictive power and model complexity
- $k$ is the number of parameters in a model

$$\text{AIC} = 2k - 2\log L \tag{26}$$

# Bootstrap

- So far we have used normal confidence interval to obtain confidence intervals and p-values
- These calculations requires asymptotically normal estimator
- The Bootstrap is an alternative approach to construct confidence intervals; one of the most important modern statistical concept (Efron, 1979)
    - reply on computer resampling; no math formula needed

*Principle:*

1. *use the sample as if it is the population*
2. *draw samples from the (pseudo) population, and calculate quantity of interest*
3. *repeate 2 multiple times; we have a sampling distribution of the quantity of interest*

# Bootstrap example: confidence interval for mean

Assume we already have $X_1, \cdots, X_n$ be i.i.d. random samples of random variable $X$). We are interested in estimating 95% confidence interval for sample mean $\bar{X}$

1. Take a with replacement sample of size $n$ from $X_1, \cdots, X_n$
2. Calculate the sample mean of the new sample, $\bar{X}_1$
3. Repeat 1 and 2 for $m$ times. We end up having $m$ estimated means
4. Take the 2.5% and 97.5% quantile of the $m$ estimated means. These two quantiles give us the bootstrap confidence intervals.

# Bootstrap example: logistic regression coefficients

- Example for calculating the confidence interval for logistic regression coefficients
  1. Take a with replacement sample of size $n$ of the original data
  2. Estimate regression coefficient with this sample, and save it
  3. Repeat 1 and 2 for $m$ times. We end up having $m$ estimates of $\beta$, $(\hat{\beta}_1, \cdots, \hat{\beta}_m)$
- With $m$ estimated $\beta$, we can essentially approximate its probability density. It becomes easier to calculate every quantity:
  - $E(\hat{\beta})$ is approximated by the mean of $(\hat{\beta}_1, \cdots, \hat{\beta}_m)$
  - Standard error of $\hat{\beta}$ is approximated by the standard error of $(\hat{\beta}_1, \cdots, \hat{\beta}_m)$
  - Take the $\frac{1-\alpha}{2}$ and $\frac{1+\alpha}{2}$ quantile of the values $(\hat{\theta}_1, \cdots, \hat{\theta}_m)$. These two quantiles give us the bootstrap confidence intervals.

## Bootstrap example: more complex properties

- Sometimes we want to calculate confidence interval for more general quantity of interests that are a function of $\beta$
- E.g., we want to get confidence interval for predicted probability: $\hat{P}(Y=1|X) = \frac{\exp(X\hat{\beta})}{1+\exp(X\hat{\beta})}$
- We know
  - lower bound of $\beta$ is $\hat{\beta}_{lower} = \hat{\beta} - 1.96\hat{se}$
  - upper bound of $\beta$ is $\hat{\beta}_{upper} = \hat{\beta} + 1.96\hat{se}$
- Is the confidence interval for predicted probability given by the normal approximated confidence interval?

$$\left(\frac{\exp\left(X\hat{\beta}_{lower}\right)}{1+\exp\left(X\hat{\beta}_{lower}\right)}, \frac{\exp\left(X\hat{\beta}_{upper}\right)}{1+\exp\left(X\hat{\beta}_{upper}\right)}\right)$$

- NO

# Bootstrap simulations

See lec4_codes.Rmd and PDF

# When do we use bootstrap?

- Bootstrap method is a very general method, can be used to calculate confidence intervals for most models you have seen in this class
- Bootstrap is often your last resorts
  - It is slow (not possible until 80s)
  - The estimates are random
  - Less theoretical guarantee
  - But it nearly always work if you do not know how to calculate standard errors for some quantity of interests

# Today's Review

| Type of $Y$ | Regression to use |
|---|---|
| Continuous | linear |
| Binary | logit/probit |
| Categorical | multinomial logit / ordered logit |
| Count (integer) | Poisson, negative binomial and zero-inflated |

# Recommended Readings

- More proofs
  - Wooldridge, *Introductory Econometrics: A Modern Approach*, 2015. Chapter 17
  - Hansen, *Econometrics*, 2020. Chapter 4, 5, 23. Free at the author's website `https://www.ssc.wisc.edu/~bhansen/econometrics/`
- There are many other GLMs (e.g., censored outcome).
  - `https://data.princeton.edu/wws509`, Generalized Linear Models course by Germán Rodríguez
  - Powers, Daniel, and Yu Xie. *Statistical methods for categorical data analysis*. Emerald Group Publishing, 2008.