SOSC 5340: Overview of Statistical Inference and Prediction

Han Zhang

Outline

Instructors

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Teaching Assistant: Li, Wei

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Self Introduction

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  1. Statistical estimation and inference
  2. Applied regression modeling
  3. Causal inference (second half of the semester)

Grading

Attendance	10%
Assignments	30%
Presentation of a published research (15 min)	10%
Presentation of your final paper (20 min)	15%
Write-up of your final paper	35%

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- Homework assignment: short coding homework to make sure that you know how to run models we covered in the lectures.

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- It is very important to write and present your own work.
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- Treat this as a real paper that has the potential to be published at academic journals/presented at academic conferences.

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- How do we use statistics to do description and prediction

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  - define random variable X as height; it can take numeric values.

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  - Probability that Y takes value y, give that X takes value x.

Probability (exercise)

- Two treatments for kidney stones

	Treatment A		Treatment B	
Kidney Stone	cured	patient	cured	patient
Small	81	87	234	270
Large	192	263	55	80
Total	273	350	289	350

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- B is more effective in the entire population, but A is more effect for both patients with small and large kidney stones.
- This is known as the Simpson's Paradox. Why?

Expected Value

- Expectation (expected value) $E(X)$:

$$E(X) = \int x \cdot f(x)dx$$

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  - Let X be a random variable with a finite number of finite outcomes $x_1, x_2, \ldots, x_k$ occurring with probabilities $p_1, p_2, \ldots, p_k$

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  - $E[X] = x_1 p_1 + x_2 p_2 + \cdots + x_k p_k = \sum_{i=1}^{k} x_i \, p_i$

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- Continuous variable's expectation

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Expected Value (exercise)

- What is the $E(X)$ of the random variable X?

X	P(X)
0	0.8
1	0.1
2	0.06
3	0.03
4	0.04

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    1. Linearity of expectation: $E(aX + bY + c) = aE(X) + bE(Y) + c$
    2. Constant's expectation is constant: $E(c) = c$

Conditional Expectation

- Conditional expectation $E(Y|X = x)$:

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \begin{cases} \sum_x \mathbb{E}[Y|X = x]P(X = x) & \text{discrete } X \\ \int_{-\infty}^{\infty} \mathbb{E}[Y|X = x]f(x)dx & \text{continuous } X \end{cases} \tag{1}$$

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  - and how X relates to Y (through $P(Y|X)$)

Conditional Expectation

- Conditional expectation $E(Y|X=x)$:
  - What is the average value of a random variable Y, when we already know that random variable X takes a fixed value x
- Useful formula 3: Law of Iterated Expectation (Law of Total Expectation)

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \begin{cases} \sum_x \mathbb{E}[Y|X=x]P(X=x) & \text{discrete } X \\ \int_{-\infty}^{\infty} \mathbb{E}[Y|X=x]f(x)dx & \text{continuous } X \end{cases} \tag{1}$$

- Basically, this theorem says that:
  - if we have knowledge about one variable X ($P(X)$)
  - and how X relates to Y (through $P(Y|X)$)
  - we can calculate the average of another variable Y.

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- The variance of a random variable X, denoted $V(Y)$, is the expected value of the square of the deviation of Y from its mean:
- $V(X) = E\left[\left(X - E(X)\right)^2\right]$
- Standard deviation: $\sigma = \sqrt{V(X)}$

Variance

Definition (Alternative Formula for Variance)

$V(X) = E[X^2] - E[X]^2$

Proof.

$$\begin{aligned}
V(X) &= E\left[\left(X - E(X)\right)^2\right] && (2)\\
&= E[X^2 - 2XE(X) + E(X)^2] && (3)\\
&= E(X^2) - 2E[XE(X)] + E[E(X)^2] && (4)\\
&= E(X^2) - 2E(X)E(X) + E(X)^2 && (5)\\
&= E(X^2) - E(X)^2 && (6)
\end{aligned}$$

$\square$

Probability and Statistics

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- Statistics (or sample statistics) is an quantity computed from samples

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        - They are not coming from a different distribution, say, heights of desks

I.I.D. random variables (exercise)

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  - e.g., population changes during the sampling process.

Sample Mean of I.I.D. random variables

- Let $X_1, \cdots, X_n$ be i.i.d. random samples of random variable X

Definition (Sample Mean)

The sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

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Mean of sample mean

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  - Expectation of the sample mean equals population mean
  - We cannot directly obtain population mean
  - But mean of sample mean is something easier to obtain

Mean of sample mean

The Mean of the Sample Mean is the Population Mean.

$$
\begin{aligned}
E(\bar{X}) &= E(\frac{1}{n}(X_1 + \cdots + X_n)) && (7)\\
&= \frac{1}{n}E(X_1 + \cdots + X_n) && (8)\\
&= \frac{1}{n}[E(X_1) + \cdots + E(X_n)] && (9)\\
&= \frac{1}{n}[E(X) + \cdots E(X)] && (10)\\
&= E(X) && (11)
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- How is the one sample mean $\bar{X}$ associates with the population mean $E(X)$?

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- That is, variance of the sample mean decreases, as n increases.

Estimator

- Estimation: use (sample) statistics to infer population quantities

Definition (Estimate and Estimator)

Estimator of a population quantity θ is a function of the samples, $\hat{\theta} = h(X_1, \cdots, X_n)$; $\hat{\theta}$ is the estimate of θ.

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- In a nutshell, statistics uses estimator to provide estimate of population quantity

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- Example 2: linear regression coefficients can be estimated by many different estimators: OLS, MLE, GMM, Bayesian
- How can we say one estimator is better than the other?
  - What properties should good estimators have?

Desirable Property: Unbiasedness

- For an estimator $\hat{\theta}$, its bias is defined as $E(\hat{\theta}) - \theta$

Definition (Unbiased Estimator)

An estimator $\hat{\theta}$ of θ is an unbiased estimator if $E(\hat{\theta}) = \theta$ or bias is 0

see proof here

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- Answer: because the expectation of sample mean equals to population mean ($E(\bar{X}) = E(X)$)

see proof here

Desirable property: Consistency

Definition (Consistent Estimator)

An estimator $\hat{\theta}$ is an consistent estimator if $\hat{\theta}$ converges in probability to θ, as $n \to \infty$.

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  - $\lim_{n\to\infty} P(|a-b| \leq \epsilon) = 1$, for all $\epsilon > 0$.

Law of Large Numbers

- Let $X_1, \cdots, X_n$ be i.i.d. random samples of random variable X

Theorem (Weak Law of Large Numbers, Jacob Bernoulli, 1713)

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- This is called plug-in estimator

Application: plug-in estimator for population variance

- We want to apply the plug-in principle to estimate population variance $V(X)$.

Definition (Plug-in Variance Estimator)

$\hat{V}(X) = \overline{X^2} - \overline{X}^2$

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&= \mathrm{V}[X] - \frac{\mathrm{V}[X]}{n} && (15)\\
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- In general, plug-in estimator is consistent, but may be biased (advanced topic).

Theorem (Unbiased Estimator of Population Variance)

$\hat{V}(X) = \frac{n}{n-1}\left(\overline{X^2} - \overline{X}^2\right)$ *is an unbiased and consistent estimator of population variance* $V(X)$

Example 2: Estimator for Variance of the sample mean

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- $\sqrt{\hat{V}(\bar{X})}$ is called standard error.

Extensions

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- Estimation theory: theoretical properties about different families of estimators
  - i.e., are plug-in estimator always unbiased? Are MLE already unbiased?

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  - confidence interval; p-values

Confidence Interval

Definition (Confidence interval)

An α confidence interval for quantity of interest θ is an estimated interval that covers the true value of θ with at least α probability

- Example: in social sciences, we often uses $\alpha =$95% confidence interval that looks like $[\theta_{min}, \theta_{max}]$. The probability that the true θ falls between $[\theta_{min}, \theta_{max}]$ is at least 95%.

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  - relates to the concept of "Central Limit Theorem"
- Another general way is called Bootstrap

Central Limit Theorem

- Let $X_1, \cdots, X_n$ be i.i.d. random samples of random variable X, with finite $E(X) = \mu$ and $V(X) = \sigma^2 > 0$

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- *Or equivalently, $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$, as $n \to \infty$*

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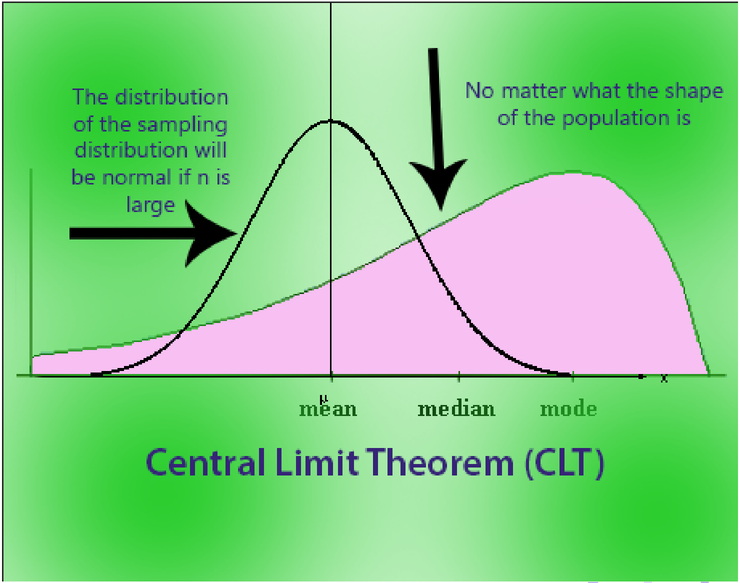
  - Variance of the sample mean equals the variance of this normal distribution
- Central limit theorem provides a general way for us to infer the uncertainty around our estimate of sample mean

Central Limit Theorem



Desirable Property: asymptotically normal

- Central Limit Theorem means that sampling distribution of the sample mean will tend to be approximately normal

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An estimator $\hat{\theta}$ is an asymptotically normal estimator, if $\sqrt{n}(\hat{\theta} - \theta) \sim N(0, \phi^2)$ for finite $\phi > 0$, as $n \to \infty$.

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  - But not all estimators have this good property
- The good thing about asymptotically normal estimator is that we can obtain confidence interval easily

Normal Approximation-based Confidence Interval

Definition (Estimating Normal Approximation-based Confidence Interval)

A normal approximation-based confidence interval for θ can be estimated by:

$$\left(\hat{\theta} - q_{\frac{1+\alpha}{2}}\sqrt{\hat{V}(\hat{\theta})}, \hat{\theta} + q_{\frac{1+\alpha}{2}}\sqrt{\hat{V}(\hat{\theta})}\right)$$

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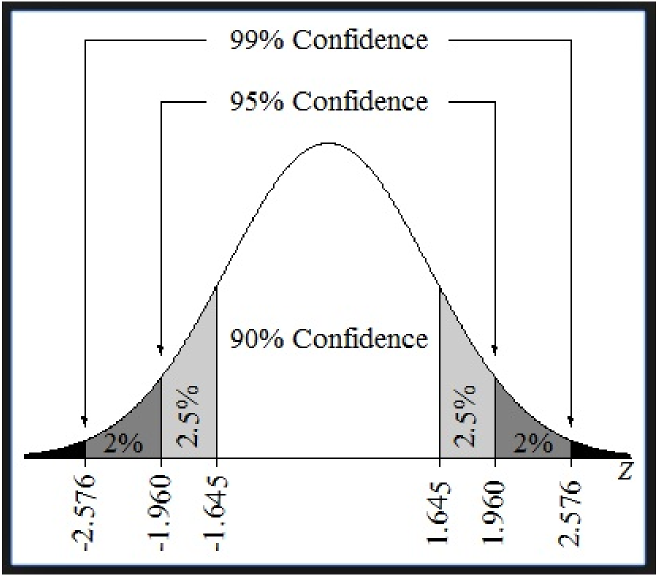
- q is the quantile function of a standard normal distribution
  - $\alpha = 0.95$; $q_{0.975} = 1.96$
  - $\alpha = 0.99$; $q_{0.995} = 2.58$
- Normal Approximation-based Confidence Interval is valid for asymptotically normal estimators

Illustration



Steps to estimate the Normal Approximation-based Confidence Interval for sample mean in a given sample

- Step 1: calculate sample mean $\bar{X}$ and sampling variance of the sample mean $\hat{V}(\bar{X})$

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- E.g., for 95% confidence interval

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  - E.g., the first proof of asymptotic normality of matching estimator is in 2006
- The Bootstrap is more general method to construct confidence intervals; one of the most important modern statistical concept (Efron, 1979)
  - Drawback of Bootstrap: it's a data-driven method; slow; no analytical solutions.

Bootstrap procedures

Assume we already have $X_1, \cdots, X_n$ be i.i.d. random samples of random variable X). We are interested in estimating a α confidence interval for a population quantity θ

1. Take a with replacement sample of size n from $X_1, \cdots, X_n$
2. Calculate the sample analog of θ
3. Repeat 1 and 2 for m times. We end up having m estimates of θ, $(\hat{\theta}_1, \cdots, \hat{\theta}_m)$
4. Take the $\frac{1-\alpha}{2}$ and $\frac{1+\alpha}{2}$ quantile of the values $(\hat{\theta}_1, \cdots, \hat{\theta}_m)$. These two quantiles give us the bootstrap confidence intervals.

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  - note that in reality we only have one sample (of n units)

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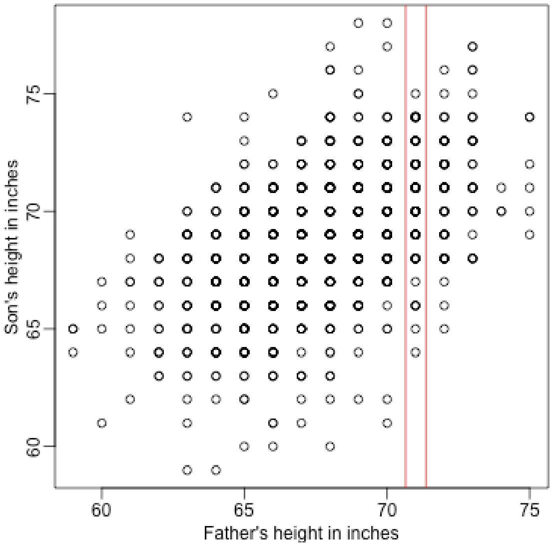
- Given two variables Y and X, and we observed X takes the value x.
- A prediction of Y given X is a function $g(X)$ that approximate Y
- For instance, $E(Y|X)$ is a prediction of Y given X
- Again, there are tons of ways to predict Y given X (e.g., $median(Y|X)$)

Prediction (example)

- Predicting son's height with father's height



Using Conditional Expectation as Prediction

- If $g(X) = E(Y|X)$, that is, we use the conditional expectation as the prediction

Property 1: $E(\epsilon) = 0$.

$$\begin{aligned} E(\epsilon) &= E\big[Y - E(Y|X)\big] && (17)\\ &= E(Y) - E\big[E(Y|X)\big] && (18)\\ &= E(Y) - E(Y), (\text{Law of Iterated Expectation}) && (19)\\ &= 0 && (20) \end{aligned}$$

□

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Conditional Expectation as Prediction (cont'd)

Property 2: $E(\epsilon|X) = 0$.

$$\begin{aligned}
E(\epsilon|X) &= E\left[Y - E(Y|X)|X\right] && (21)\\
&= E(Y|X) - E\left[E(Y|X)|X\right] && (22)\\
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  - That is, we only assume that on average X and error are independent
  - Recall independence means that $P(\epsilon|X) = P(\epsilon)$

Independent, mean independent, and uncorrelated

- Independent: $P(XY) = P(X)P(Y)$

X, Y are independent $\implies$ X, Y are mean independent $\implies$ X, Y are uncorrelated.

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- Independent: $P(XY) = P(X)P(Y)$
- Mean independent: $E(Y|X) = E(Y)$
- Uncorrelated: $E(XY) = E(X)E(Y)$
- In general, we have the following relationship (the reverse is not true):

X, Y are independent $\implies$ X, Y are mean independent $\implies$ X, Y are uncorrelated.

Using Conditional Expectation as Prediction

- Property 3 says error is uncorrelated with any other $g(X)$

Property 3: $E\big(g(X)\epsilon\big) = 0$, for any $g(X)$.

$$
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Using Conditional Expectation as Prediction

- Property 3 says error is uncorrelated with any other $g(X)$
- Derived from Property 2 (mean independence) and Property 1 $(E(\epsilon) = 0)$

Property 3: $E\big(g(X)\epsilon\big) = 0$, for any $g(X)$.

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  - MSE is perhaps the most widely used error metric
- Both MAE and MSE ≥ 0; a good estimation thus should minimize MAE or MSE

Conditional Expectation As the Best Predictor

- There are some even better properties of $E(Y|X)$ that make it the best predictor, given Mean Squared Error

Theorem (Conditional Expectation as the Best Predictor)

Conditional Expectation Function $E(Y|X)$ *is the best predictor of* Y *because it minimizes Mean Squared Error*

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- Hint: use the conditional expectation error $\epsilon = Y - E(Y|X)$

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&= E\left[\epsilon^2\right] + E\left[(E(Y|X)-g(X))^2\right], \text{(Property 3)} && (34)\\
&\geq E\left[\epsilon^2\right] && (35)\\
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- $E(Y|X)$ is the best predictor of Y, if the criterion is to minimize MSE
  - There are other criteria to evaluation predictions
- Our next half semester is devoted onto understanding how to estimate $E(Y|X)$

Bias-variance trade-off

- Let us compare the MSE under $g(X)$ and under the best prediction $E(Y|X)$

Bias Variance Decomposition.

$$E\left[(Y - g(X))^2\right] = E\left[\epsilon^2\right] + E\left[(E(Y|X) - g(X))^2\right] \tag{37}$$

$$= \left(E(\epsilon^2) + V(\epsilon)\right) + E\left[(E(Y|X) - g(X))^2\right. \tag{38}$$

$$= V(\epsilon) + E(\text{bias}^2) \tag{39}$$

□

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- Let us compare the MSE under $g(X)$ and under the best prediction $E(Y|X)$
- Go back to equation (19),

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  - That is, this is unrelated to the mode you are going to use
- $E(\text{bias}^2)$ is the mean of the (squared) bias
  - Bias can be improved by using a better approximation (the ideal case: just let $g(X) = \text{E(Y|X)\$}$)

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  - Discuss Conditional expectation is the best predictor (Chapter 3.1)
  - Motivated differently from Aronow and Miller