ENGG 5501: Foundations of Optimization	2021–22 First Term
Solution to Final Examination	
Time Limit: 2 Hours	December 23, 2021

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (20pts). Let $A \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}^n$ be given, with $A = -A^T$. Consider the following problem:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq -c, \\ & x \geq \mathbf{0}. \end{array} \tag{1}$$

(a) **(10pts).** Write down the dual of Problem (1).

ANSWER: Using the fact that $A = -A^T$, the dual is given by

$$\begin{array}{ll} \text{maximize} & -c^T y \\ \text{subject to} & A^T y \leq c, \\ & y \geq \mathbf{0} \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \text{maximize} & -c^T y \\ \text{subject to} & Ay \geq -c, \\ & y \geq \mathbf{0}. \end{array}$$

(b) **(10pts).** Using the result in (a), or otherwise, show that Problem (1) has an optimal solution if and only if it is feasible.

ANSWER: If Problem (1) has an optimal solution, then it is trivially feasible. Conversely, if $\bar{x} \in \mathbb{R}^n$ is a feasible solution to Problem (1), then it is also a feasible solution to its dual, as the feasible sets of Problem (1) and its dual are identical. It then follows from the LP strong duality theorem (Corollary 4 of Handout 3) that Problem (1) has an optimal solution.

Problem 2 (20pts). For each of the following statements, determine if it is true or false. If it is true, then provide a proof. If it is false, then provide a counter-example.

(a) **(10pts).** Let $K \subseteq \mathbb{R}^n$ be a convex cone. Then, it is closed under addition—i.e., for any $x, y \in K$, we have $x + y \in K$.

ANSWER: True. Let $x, y \in K$ be arbitrary. Since K is a cone, we have $\frac{1}{2}x, \frac{1}{2}y \in K$. By the convexity of K, we have $\frac{1}{2}(x + y) \in K$. Again, since K is a cone, we have $x + y = 2\left(\frac{1}{2}(x + y)\right) \in K$.

(b) **(10pts).** Let $K \subseteq \mathbb{R}^n$ be a pointed cone. Then, $(K^*)^* = K$.

ANSWER: False. Take, e.g., $K = \mathbb{R}^n_{++}$. It is straightforward to verify that K is a pointed cone. Moreover, we have $K^* = \mathbb{R}^n_+$, which implies that $(K^*)^* = \mathbb{R}^n_+ \neq \mathbb{R}^n_{++} = K$.

Problem 3 (20pts). Let $A \in \mathbb{R}^{p\times n}$, $b \in \mathbb{R}^p$, $B \in \mathbb{R}^{q\times n}$, $d \in \mathbb{R}^q$, $c, u \in \mathbb{R}^n$, and $v \in \mathbb{R}$ be given. Write down the dual of the following problem:

$$\begin{array}{ll} \inf & c^T x \\ \text{subject to} & Ax \leq b, \\ & \|Bx + d\|_2 \leq u^T x + v, \\ & x \in \mathbb{R}^n. \end{array}$$

Simplify your answer as much as possible and show all your work.

ANSWER: The given problem can be expressed as

$$\begin{array}{ll} \inf & c^T x \\ \text{subject to} & -(Ax - b) \in \mathbb{R}^p_+, \\ & (u^T x + v, Bx + d) \in \mathcal{Q}^{q+1}, \\ & x \in \mathbb{R}^n. \end{array} \tag{P}$$

Let $w \in \mathbb{R}^p_+$ and $(t, y) \in \mathcal{Q}^{q+1}$ be the multipliers associated with the constraints $-(Ax - b) \in \mathbb{R}^p_+$ and $(u^T x + v, Bx + d) \in \mathcal{Q}^{q+1}$, respectively. Then, the Lagrangian dual of Problem (P) is given by

$$\sup_{w \in \mathbb{R}^p_+,\, (t,y) \in \mathcal{Q}^{q+1}} \inf_{x \in \mathbb{R}^n} \left\{ c^T x + w^T (Ax - b) - (t, y)^T (u^T x + v, Bx + d) \right\}.$$

Observe that

$$\begin{aligned} & \inf_{x \in \mathbb{R}^n} \left\{ c^T x + w^T (Ax - b) - (t, y)^T (u^T x + v, Bx + d) \right\} \\ = & -b^T w - (v, d)^T (t, y) + \inf_{x \in \mathbb{R}^n} (c + A^T w - tu - B^T y)^T x \\ = & \begin{cases} -b^T w - (v, d)^T (t, y) & \text{if } c + A^T w - tu - B^T y = \mathbf{0}, \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that the dual of Problem (P) is given by

$$\begin{array}{ll} \sup & -b^T w - (v, d)^T (t, y) \\ \text{subject to} & c + A^T w - tu - B^T y = \mathbf{0}, \\ & w \in \mathbb{R}^p_+,\ (t, y) \in \mathcal{Q}^{q+1}. \end{array}$$

Problem 4 (20pts). Let $f_i : \mathbb{R} \to \mathbb{R}$ be continuously differentiable concave functions and $C > 0$ be a given constant. Consider the problem:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^n f_i(x_i) \\ \text{subject to} & \displaystyle\sum_{i=1}^n x_i \le C. \end{array} \tag{2}$$

Show that $x^* \in \mathbb{R}^n$ is an optimal solution to Problem (2) if and only if $\sum_{i=1}^n x_i^* \le C$ and there exists a scalar $\lambda^* \ge 0$ such that $x_i^* = \arg\max_{t \in \mathbb{R}} \{ f_i(t) - \lambda^* t \}$ for $i = 1, \ldots, n$ and $\lambda^* \left(\sum_{i=1}^n x_i^* - C \right) = 0$. (*Remark: If you use first-order optimality conditions to tackle this problem, remember to justify their necessity and sufficiency.*)

ANSWER: Since (2) is a linearly constrained concave maximization problem, by Theorems 5 and

6 of Handout 7, the KKT conditions, which are given by

$$\begin{aligned}
\sum_{i=1}^{n} x_i &\leq C, && (a)\\
-f_i'(x_i) + \lambda &= 0 \quad \text{for } i = 1,\ldots,n, && (b)\\
\lambda &\geq 0 && (c)\\
\lambda\left(\sum_{i=1}^{n} x_i - C\right) &= 0, && (d)
\end{aligned}$$

are necessary and sufficient for optimality. Since the function $t \mapsto -f_i(t) + \lambda t$ is convex for any given $\lambda \geq 0$, condition (b) is equivalent to $x_i = \arg\max_{t\in\mathbb{R}}\{f_i(t) - \lambda t\}$. This completes the proof.

Problem 5 (20pts). Consider the following problem:

$$\begin{array}{ll}
\text{minimize} & -x_1 + x_2 \\
\text{subject to} & x_1^2 + x_2^2 - 2x_1 = 0, \\
& (x_1, x_2) \in \mathcal{X} \triangleq \operatorname{conv}\{(-1,0),(0,1),(1,0),(0,-1)\}.
\end{array} \tag{3}$$

(a) **(10pts).** Deduce the optimal solution $x^* \in \mathbb{R}^2$ to Problem (3) graphically.

ANSWER: Note that $x_1^2 + x_2^2 - 2x_1 = (x_1 - 1)^2 + x_2^2 - 1$. Figure 1 shows the geometry of Problem (3). From the figure, it is clear that $x^* \in \mathbb{R}^2$ satisfies $(x_1^*)^2 + (x_2^*)^2 - 2(x_1^*) = 0$ and $(x_1^*) - (x_2^*) = 1$ with $x_1^* \geq 0$ and $x_2^* \leq 0$. Solving this system gives $x_1^* = 1 - \frac{1}{\sqrt{2}}$ and $x_2^* = -\frac{1}{\sqrt{2}}$.

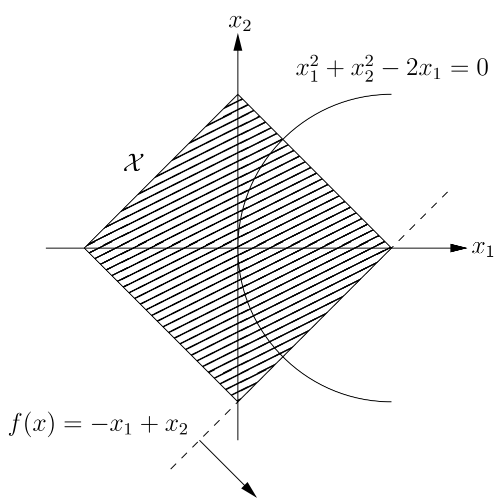

Figure 1: Geometry of Problem (3)

(b) **(10pts).** By expressing $\mathcal{X}$ as a set of inequality constraints, determine whether the KKT conditions associated with Problem (3) hold at the optimal solution x^* found in (a). Justify your answer.

ANSWER: The active constraints at $x^* \in \mathbb{R}^2$ are $h(x) = x_1^2 + x_2^2 - 2x_1 = 0$ and $g(x) = x_1 - x_2 - 1 \leq 0$. Since $\nabla g(x^*) = (1,-1)$ and $\nabla h(x^*) = -\sqrt{2}(1,1)$, we see that $\{\nabla g(x^*), \nabla h(x^*)\}$ are linearly independent. It follows from Theorem 3 of Handout 7 that the KKT conditions associated with Problem (3) hold at x^*.