Solution to the Final Exam

April 29, 2016

1. a) Let message $s_n = 0$ be modulated to $u_n = A$, and $s_n = 1$ be modulated to $u_n = -A$. After down conversion, the receiver should let the signal pass through a low pass filter to obtain baseband signal, and then integrate the rectangular pulse, yielding $y_n = u_n T + \int_T n(t)\,dt$, where $\eta = \int_T n(t)\,dt$ has variance $N_0 T$, and the real component has variance $\frac{N_0 T}{2}$. Since the modulation is BPSK, we only care about the real component. We have

$$p\left(Re\left(y_n\right) = x\right) = \frac{1}{\sqrt{\pi N_0 T}} e^{-\frac{(x-u_n T)^2}{N_0 T}}. \tag{1}$$

After receiving y_n, the optimal receiver is the MAP receiver:

$$\hat{u}_n = \arg\max p\left(u_n | y_n\right). \tag{2}$$

We have

$$p\left(u_n | y_n\right) = \frac{p\left(u_n, y_n\right)}{p\left(y_n\right)} = \frac{p\left(u_n\right) p\left(y_n | u_n\right)}{p\left(y_n\right)} \tag{3}$$

where $p\left(y_n\right)$ is constant for any u_n, and $p\left(u_n = A\right) = p\left(s_n = 0\right)$, $p\left(u_n = -A\right) = p\left(s_n = 1\right)$. So if

$$p\left(s_n = 0\right) \cdot p\left(y_n | s_n = 0\right) > p\left(s_n = 1\right) \cdot p\left(y_n | s_n = 1\right) \tag{4}$$

then we should demodulate y_n to $\hat{s}_n = 0$. So the decision boundary x_0 has

$$\begin{aligned}
0.3 p\left(Re\left(y_n\right) = x_0 | s_n = 0\right) &= 0.7 p\left(Re\left(y_n\right) = x_0 | s_n = 1\right) \\
0.3 e^{-\frac{(x_0 - AT)^2}{N_0 T}} &= 0.7 e^{-\frac{(x_0 + AT)^2}{N_0 T}} \\
\ln(0.3) - \frac{\left(x_0 - AT\right)^2}{N_0 T} &= \ln(0.7) - \frac{\left(x_0 + AT\right)^2}{N_0 T} \\
\frac{\left(x_0 + AT\right)^2}{N_0 T} - \frac{\left(x_0 - AT\right)^2}{N_0 T} &= \ln(0.7) - \ln(0.3) \\
\frac{\left(x_0 + AT\right)^2 - \left(x_0 - AT\right)^2}{N_0 T} &= \ln\frac{7}{3} \\
\frac{4 x_0 A T}{N_0 T} &= \ln\frac{7}{3} \\
\frac{4 x_0 A}{N_0} &= \ln\frac{7}{3}
\end{aligned} \tag{5}$$

from which we can find

$$x_0 = \frac{N_0}{4A} \cdot \ln\frac{7}{3}. \tag{6}$$

Note that if $s_n = 0$ is mapped to $u_n = -A$, then (6) should be changed to $x_0 = -\frac{N_0}{4A} \cdot \ln\left(\frac{7}{3}\right)$. If $Re\left(y_n\right) > x_0$, then demodulate to $\hat{s}_n = 0$; otherwise demodulate to $\hat{s}_n = 1$.

The error probability when transmitting $s_n = 0$ is

$$\begin{aligned} p\left(\hat{s}_n = 1|s_n = 0\right) &= p\left(Re\left(y_n\right) < x_0|s_n = 0\right) \\ &= \int_{-\infty}^{x_0} \frac{1}{\sqrt{\pi N_0 T}} e^{-\frac{(x-AT)^2}{N_0 T}} dx \\ &= 1 - \int_{x_0}^{\infty} \sqrt{\frac{2}{N_0 T}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\sqrt{\frac{2}{N_0 T}}(x-AT)\right)^2}{2}} dx \\ &= 1 - \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2}{N_0 T}}(x_0 - AT)}^{\infty} e^{-\frac{z^2}{2}} dz \;\text{ where } z \triangleq \sqrt{\frac{2}{N_0 T}}\left(x - AT\right) \\ &= 1 - Q\left(\sqrt{\frac{2}{N_0 T}}\left(x_0 - AT\right)\right) \\ &= Q\left(\sqrt{\frac{2}{N_0 T}}\left(AT - x_0\right)\right). \end{aligned} \tag{7}$$

Likewise, the error probability when transmitting $s_n = 1$ is

$$p\left(\hat{s}_n = 0|s_n = 1\right) = Q\left(\sqrt{\frac{2}{N_0 T}}\left(AT + x_0\right)\right). \tag{8}$$

So the average SER is

$$\begin{aligned} \text{SER} &= p\left(s_n = 0\right) p\left(\hat{s}_n = 1|s_n = 0\right) + p\left(s_n = 1\right) p\left(\hat{s}_n = 0|s_n = 1\right) \\ &= 0.3Q\left(\sqrt{\frac{2}{N_0 T}}\left(AT - x_0\right)\right) + 0.7Q\left(\sqrt{\frac{2}{N_0 T}}\left(AT + x_0\right)\right). \\ &= 0.3Q\left(\sqrt{\frac{2}{N_0 T}}\left(AT - \frac{N_0}{4A} \cdot \ln\frac{7}{3}\right)\right) + 0.7Q\left(\sqrt{\frac{2}{N_0 T}}\left(AT + \frac{N_0}{4A} \cdot \ln\frac{7}{3}\right)\right). \end{aligned} \tag{9}$$

Since each symbol contains information

$$0.3 \log_2 0.3 + 0.7 \log_2 0.7 = 0.881\, bits \tag{10}$$

we should normalize the average SER to obtain the average BER

$$\text{BER} = \frac{\text{SER}}{0.881} = \frac{1}{0.881}\left[0.3Q\left(\sqrt{\frac{2}{N_0 T}}\left(AT - \frac{N_0}{4A} \cdot \ln\frac{7}{3}\right)\right) + 0.7Q\left(\sqrt{\frac{2}{N_0 T}}\left(AT + \frac{N_0}{4A} \cdot \ln\frac{7}{3}\right)\right)\right]. \tag{11}$$

b) The symbol duration is $T = 10^{-2} ms = 10^{-5} s$. The received power at a point $d > d_0$ is

$$P_r\left(d\right) = P_r\left(d_0\right)\left(\frac{d_0}{d}\right)^3. \tag{12}$$

So the received amplitude is

$$A_r\left(d\right) = \sqrt{P_r\left(d\right)}. \tag{13}$$

In problem a) we considered AWGN channel without fading, while here we have fading. So we should substitute the A in problem a) with $A_r\left(d\right)$ in (13). So we can rewrite (11) as

$$\text{BER} = \frac{1}{0.881}\left[0.3Q\left(\sqrt{\frac{2}{N_0 T}}\left(A_r\left(d\right)T - \frac{N_0}{4A_r\left(d\right)} \cdot \ln\frac{7}{3}\right)\right) + 0.7Q\left(\sqrt{\frac{2}{N_0 T}}\left(A_r\left(d\right)T + \frac{N_0}{4A_r\left(d\right)} \cdot \ln\frac{7}{3}\right)\right)\right]. \tag{14}$$

From (14) and the approximation of $Q\left(x\right) \approx 0.5e^{-\frac{x^2}{2}}$, to ensure $\text{BER} \leq 10^{-4}$, we need

$$\begin{aligned} 0.341 \times 0.5 \times e^{-\frac{1}{N_0 T}\left(A_r(d)T - \frac{N_0}{4A_r(d)} \cdot \ln\frac{7}{3}\right)^2} + 0.795 \times 0.5 \times e^{-\frac{1}{N_0 T}\left(A_r(d)T + \frac{N_0}{4A_r(d)} \cdot \ln\frac{7}{3}\right)^2} &\leq 10^{-4} \\ 0.171 e^{-\frac{1}{N_0 T}\left(A_r^2(d)T^2 + \frac{N_0^2}{16A_r^2(d)} \cdot \ln^2\left(\frac{7}{3}\right) - \frac{TN_0}{2}\ln\frac{7}{3}\right)} \\ +0.398 e^{-\frac{1}{N_0 T}\left(A_r^2(d)T^2 + \frac{N_0^2}{16A_r^2(d)} \cdot \ln^2\left(\frac{7}{3}\right) + \frac{TN_0}{2}\ln\frac{7}{3}\right)} &\leq 10^{-4} \\ e^{-\frac{1}{N_0 T}\left(A_r^2(d)T^2 + \frac{N_0^2}{16A_r^2(d)} \cdot \ln^2\left(\frac{7}{3}\right)\right)} \cdot \left(0.171 \times e^{\frac{1}{2}\ln\frac{7}{3}} + 0.398 \times e^{-\frac{1}{2}\ln\frac{7}{3}}\right) &\leq 10^{-4} \\ A_r^2\left(d\right)T^2 + \frac{N_0^2}{16A_r^2\left(d\right)} \cdot \ln^2\left(\frac{7}{3}\right) &\geq 8.56 \times 10^{-19} \end{aligned} \tag{15}$$

from which we can find

$$A_r^2(d) \leq 7 \times 10^{-11} \text{ or } A_r^2(d) \geq 8.99 \times 10^{-9}. \tag{16}$$

In (14) we need to ensure $A_r(d)T - \frac{N_0}{4A_r(d)} \cdot \ln\frac{7}{3} \geq 0$, i.e. $A_r^2(d) \geq \frac{N_0}{4T} \cdot \ln\frac{7}{3} = 2.118 \times 10^{-10}$. So we rule out $A_r^2(d) \leq 7 \times 10^{-11}$. Then $A_r^2(d)$ must satisfy

$$A_r^2(d) \geq 8.99 \times 10^{-9} \tag{17}$$

and therefore

$$P_r(d) = P_r(d_0)\left(\frac{d_0}{d}\right)^3 \geq 8.99 \times 10^{-9}. \tag{18}$$

Then we can find

$$d \leq 480.93m. \tag{19}$$

Therefore, the cell radius must not exceed $480.93m$.

{If $s_n = 0$ and $s_n = 1$ are equal-probable, then we will find that $d \leq 524.88m$. This indicates that equal-probable distribution is better.}

c) With the flat Rayleigh fading channel h, the receive power will become

$$A_f^2(d) = P_r(d)\left|h\right|^2.$$

Since $h \sim \text{Rayleigh}(1)$, we have $|h|^2 \sim \exp\left(\frac{1}{2}\right)$. The pdf function of $|h|^2$ is $f_{|h|^2}(x) = \frac{1}{2}e^{-\frac{1}{2}x}$, and its cdf function is $F_{|h|^2}(x) = 1 - e^{-\frac{1}{2}x}$. We should rewrite (14) as

$$\text{BER} = \frac{1}{0.881}\left[0.3Q\left(\sqrt{\frac{2}{N_0T}}\left(A_f(d)T - \frac{N_0}{4A_f(d)} \cdot \ln\frac{7}{3}\right)\right) + 0.7Q\left(\sqrt{\frac{2}{N_0T}}\left(A_f(d)T + \frac{N_0}{4A_f(d)} \cdot \ln\frac{7}{3}\right)\right)\right]. \tag{20}$$

Then from (20) and the approximation of $Q(x) \approx 0.5e^{-\frac{x^2}{2}}$, to ensure average BER $\leq 10^{-4}$, we need

$$\begin{aligned} E_{|h|^2}\text{BER} &\leq 10^{-4} \\ \frac{1}{0.881}E_{|h|^2}\left\{0.15e^{-\frac{1}{N_0T}\left(A_f(d)T - \frac{N_0}{4A_f(d)}\cdot\ln\frac{7}{3}\right)^2} + 0.35e^{-\frac{1}{N_0T}\left(A_f(d)T + \frac{N_0}{4A_f(d)}\cdot\ln\frac{7}{3}\right)^2}\right\} &\leq 10^{-4} \\ E_{|h|^2}\left\{e^{-\frac{1}{N_0T}\left(\frac{|h|^2}{d^3}T^2 + \frac{d^3}{|h|^2}\cdot\frac{N_0^2}{16}\cdot\ln^2\left(\frac{7}{3}\right)\right)}\right\} &\leq 1.92 \times 10^{-4}. \end{aligned} \tag{21}$$

Denote $\alpha = -\frac{T}{N_0d^3}$, and $\beta = -\frac{d^3N_0}{16T}\ln^2\left(\frac{7}{3}\right)$. Then

$$\begin{aligned} & E_{|h|^2}\left\{e^{-\frac{1}{N_0T}\left(\frac{|h|^2}{d^3}T^2 + \frac{d^3}{|h|^2}\cdot\frac{N_0^2}{16}\cdot\ln^2\left(\frac{7}{3}\right)\right)}\right\} \\ = \;& E_{|h|^2}\left\{e^{\alpha|h|^2 + \frac{\beta}{|h|^2}}\right\} \\ = \;& \int_0^\infty e^{\alpha x + \frac{\beta}{x}} \cdot f_{|h|^2}(x)\,dx \\ = \;& \int_0^\infty e^{\alpha x + \frac{\beta}{x}} \cdot \frac{1}{2}e^{-\frac{1}{2}x}dx \\ = \;& \frac{1}{2}\int_0^\infty e^{\left(\alpha - \frac{1}{2}\right)x + \frac{\beta}{x}}dx \\ \overset{(a)}{=} \;& \sqrt{\frac{\beta}{\alpha - \frac{1}{2}}}K_1\left(\sqrt{4\beta\left(\alpha - \frac{1}{2}\right)}\right) \\ = \;& \sqrt{\frac{\frac{d^3N_0}{16T}\ln^2\left(\frac{7}{3}\right)}{\frac{T}{N_0d^3} + \frac{1}{2}}}K_1\left(\sqrt{\frac{N_0}{4T}\ln^2\left(\frac{7}{3}\right)\left(\frac{T}{N_0} + \frac{1}{2}d^3\right)}\right) \end{aligned} \tag{22}$$

where the equality (a) follows from (3.324) in [1], and $K_1(\cdot)$ is Bessel function with parameter 1. Fig. 1 plots the value of (22) versus d. From (21) and Fig. 1, we can find

$$d \leq 163.6m.$$

Note that as $\frac{T}{N_0} \gg d^3$ for $d < 500m$, then in (22), we can approximate $\sqrt{\frac{N_0}{4T}\ln^2\left(\frac{7}{3}\right)\left(\frac{T}{N_0}+\frac{1}{2}d^3\right)}$ by a constant

$$\sqrt{\frac{N_0}{4T}\ln^2\left(\frac{7}{3}\right)\frac{T}{N_0}} = \sqrt{\frac{1}{4}\ln^2\left(\frac{7}{3}\right)} = \frac{1}{2}\ln\left(\frac{7}{3}\right) = 0.424. \quad (23)$$

Likewise we can approximate $\frac{T}{N_0 d^3} + \frac{1}{2}$ by $\frac{T}{N_0 d^3}$ in (22). Then we will see that the BER is proportional to d^3, i.e., the BER is inversely proportional to $P_r(d)$.

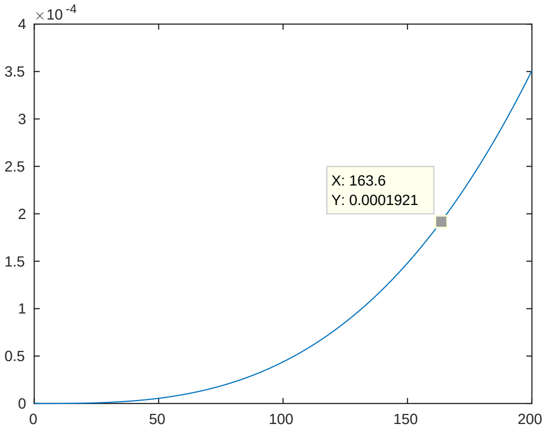

Figure 1: $E_{|h|^2}\left\{e^{-\frac{1}{N_0 T}\left(\frac{|h|^2}{d^3}T^2+\frac{d^3}{|h|^2}\cdot\frac{N_0^2}{16}\cdot\ln^2\left(\frac{7}{3}\right)\right)}\right\}$

2. a) Since α_i follows Rayleigh distribution, α_i^2 follows exponential distribution, with PDF as $f_i(x) = \lambda_i e^{-\lambda_i x}, x \geq 0$, and CDF as $F_i(x) = 1 - e^{-\lambda_i x}$. The mean of the exponential distribution is $\frac{1}{\lambda_i}$.

The BER of BPSK is $P_b = Q\left(\sqrt{2\text{SNR}}\right) = Q\left(\sqrt{2\alpha_0^2}\right)$. To have $P_b < 10^{-3}$, we should ensure

- **Q-function table**: $\alpha_0^2 > \tilde{\alpha}^2 = 4.775$.
- **Q-function approximation**: $\alpha_0^2 > \tilde{\alpha}^2 = 6.215$.

The following will use $\tilde{\alpha}^2 = 4.775$.

The outage probability is

$$\Pr\left\{\alpha_0^2 < \tilde{\alpha}^2\right\} = F_0\left(\tilde{\alpha}^2\right) = 1 - e^{-\lambda_0\tilde{\alpha}^2}. \quad (24)$$

When $E\left[\alpha_0^2\right] = 5$, we can find $\lambda_0 = \frac{1}{5}$. When $E\left[\alpha_0^2\right] = 10$, we can find $\lambda_0 = \frac{1}{10}$. So the outage probability is

$$\begin{aligned} p_{\text{out},0} &= \Pr\left\{E\left[\alpha_0^2\right]=5\right\}\left(1-e^{-\frac{1}{5}\tilde{\alpha}^2}\right) + \Pr\left\{E\left[\alpha_0^2\right]=10\right\}\left(1-e^{-\frac{1}{10}\tilde{\alpha}^2}\right) \\ &= 0.5\left(1-e^{-\frac{1}{5}\tilde{\alpha}^2}\right) + 0.5\left(1-e^{-\frac{1}{10}\tilde{\alpha}^2}\right) \\ &= 0.497. \end{aligned} \quad (25)$$

b) Using selection combining, the outage probability is

$$\begin{aligned} p_{\text{out}} &= \Pr\left\{\alpha_0^2 < \tilde{\alpha}^2, \alpha_1^2 < \tilde{\alpha}^2, \alpha_2^2 < \tilde{\alpha}^2\right\} \\ &= \Pr\left\{\alpha_0^2 < \tilde{\alpha}^2\right\}\Pr\left\{\alpha_1^2 < \tilde{\alpha}^2\right\}\Pr\left\{\alpha_2^2 < \tilde{\alpha}^2\right\} \\ &= p_{\text{out},0}\cdot p_{\text{out},1}\cdot p_{\text{out},2} \end{aligned} \quad (26)$$

where $p_{\text{out},0} = 0.204$,

$$\begin{aligned} p_{\text{out},1} &= \Pr\left\{E\left[\alpha_1^2\right]=0\right\} + \Pr\left\{E\left[\alpha_1^2\right]=20\right\}\left(1-e^{-\frac{1}{20}\tilde{\alpha}^2}\right) \\ &= 0.5 + 0.5\left(1-e^{-\frac{1}{20}\tilde{\alpha}^2}\right) \\ &= 0.606 \end{aligned} \quad (27)$$

and

$$\begin{aligned} p_{\text{out},2} &= \Pr\left\{E\left[\alpha_2^2\right]=5\right\}\left(1-e^{-\frac{1}{5}\tilde{\alpha}^2}\right)+\Pr\left\{E\left[\alpha_2^2\right]=10\right\}\left(1-e^{-\frac{1}{10}\tilde{\alpha}^2}\right) \\ &= 0.75\left(1-e^{-\frac{1}{5}\tilde{\alpha}^2}\right)+0.25\left(1-e^{-\frac{1}{10}\tilde{\alpha}^2}\right) \\ &= 0.556. \end{aligned} \tag{28}$$

So we can find

$$p_{\text{out}} = p_{\text{out},0}\cdot p_{\text{out},1}\cdot p_{\text{out},2} = 0.497\times 0.606\times 0.556 = 0.168. \tag{29}$$

3. a) When the transmitter does not know the channel, it should always transmit with the same power. For channel gain $g_1 = 0.5$, $g_2 = 0.4$, $g_3 = 0.2$, and $g_4 = 0.1$, the capacities are respectively

$$C_1 = \log_2(1+g_1\text{SNR}) = 2.585 \tag{30}$$
$$C_2 = \log_2(1+g_2\text{SNR}) = 2.322 \tag{31}$$
$$C_3 = \log_2(1+g_3\text{SNR}) = 1.585 \tag{32}$$
$$C_4 = \log_2(1+g_4\text{SNR}) = 1.000. \tag{33}$$

The ergodic capacity is

$$C_e = \sum_i p_i\cdot C_i = 1.889. \tag{34}$$

b) The outage probability at rate R is

$$p_{\text{out}}(R) = \Pr\left\{\log_2(1+g\text{SNR}) < R\right\}. \tag{35}$$

From (30) to (33), we plot the outage capacity versus outage probability in Fig. 2.

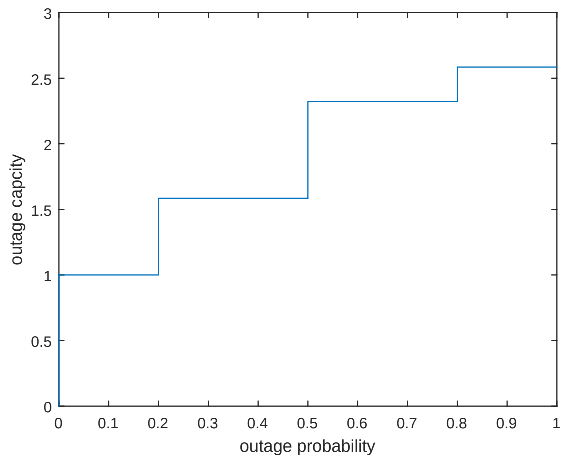

Figure 2: Outage capacity versus outage probability.

c) By water-filling algorithm we let

$$0.2\left(\mu-\frac{1}{0.5}\right)^+ + 0.3\left(\mu-\frac{1}{0.4}\right)^+ + 0.3\left(\mu-\frac{1}{0.2}\right)^+ + 0.2\left(\mu-\frac{1}{0.1}\right)^+ = 10 \tag{36}$$

from which we find $\mu = 14.650$ which yields the optimal power allocation scheme

$$P_1^* = \left(\mu-\frac{1}{0.5}\right)^+ = 12.65 \tag{37}$$
$$P_2^* = \left(\mu-\frac{1}{0.4}\right)^+ = 12.15 \tag{38}$$
$$P_3^* = \left(\mu-\frac{1}{0.2}\right)^+ = 9.65 \tag{39}$$
$$P_4^* = \left(\mu-\frac{1}{0.1}\right)^+ = 4.65. \tag{40}$$

The corresponding channel capacity is

$$C = \sum_{i=1}^{4} p_i \log_2 \left(1 + P_i^* g_i\right) = 1.915. \tag{41}$$

4. a) Denote

$$\begin{aligned}
\mathbf{U} &= \begin{bmatrix} -0.5946 & -0.8041 \\ -0.8041 & 0.5946 \end{bmatrix} \\
\mathbf{S} &= \begin{bmatrix} 0.87 & \\ & 0.33 \end{bmatrix} \\
\mathbf{V} &= \begin{bmatrix} -0.8507 & 0.5257 \\ -0.5257 & -0.8507 \end{bmatrix}
\end{aligned}$$

and $\sigma_1^2 = 0.87^2 = 0.757$, $\sigma_2^2 = 0.33^2 = 0.109$. Since both the transmitter and the receiver know $\mathbf{H}$, the transmitter can precode with $\mathbf{V}^{-1}$, the receiver can receive with beamforming matrix $\mathbf{U}^{-1}$, and the transmitter can allocate its power over the two streams by water-filling algorithm. Normalize the noise power $10^{-4}W$ to 1, then the transmit power $10mW$ becomes $\frac{10mW}{10^{-4}W} = 100$. By water-filling algorithm, we let

$$\left(\mu - \frac{1}{\sigma_1^2}\right)^+ + \left(\mu - \frac{1}{\sigma_2^2}\right)^+ = 100 \tag{42}$$

from which we find $\mu = 55.248$. Thus the optimal power allocation scheme is

$$P_1^* = \left(\mu - \frac{1}{\sigma_1^2}\right)^+ = 53.927 \tag{43}$$

$$P_2^* = \left(\mu - \frac{1}{\sigma_2^2}\right)^+ = 46.073. \tag{44}$$

So the channel capacity is

$$C = \log_2\left(1 + P_1^* \sigma_1^2\right) + \log_2\left(1 + P_2^* \sigma_2^2\right) = 7.976. \tag{45}$$

b) The transmitter should allocate all its power on the strongest stream. So the transmitter's beamforming vector should be

$$\mathbf{V}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.851 & -0.526 \\ 0.526 & -0.851 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.851 \\ 0.526 \end{bmatrix}. \tag{46}$$

The receiver should receive with

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{U}^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -0.595 & -0.804 \\ -0.804 & 0.595 \end{bmatrix} = \begin{bmatrix} -0.595 & -0.804 \end{bmatrix}. \tag{47}$$

c) If the transmitter does not know the channel, the it should use space-time block code (STBC), as shown in Tab. 1.

time \ space	A1	A2
t	x_1	x_2
$t+1$	$-x_2^*$	x_1^*

Table 1: Alamouti code.

Assume that the channel does not change across two consecutive transmissions, and denote by $h_{i,j}$ the (i,j)-th element in $\mathbf{H}$ (i.e., the channel from the j-th transmit antenna to the i-th receive antenna). The receiver will receive $y_i^{(j)}$ where j is time index and i is the receive antenna index:

$$y_1^{(1)} = h_{1,1}x_1 + h_{1,2}x_2 + n_1^{(1)} \tag{48}$$

$$y_2^{(1)} = h_{2,1}x_1 + h_{2,2}x_2 + n_2^{(1)} \tag{49}$$

$$y_1^{(2)} = -h_{1,1}x_2^* + h_{1,2}x_1^* + n_1^{(2)} \tag{50}$$

$$y_2^{(2)} = -h_{2,1}x_2^* + h_{2,2}x_1^* + n_2^{(2)}. \tag{51}$$

Then the receiver should combine the received signals by

$$\begin{aligned}
\tilde{x}_1 &= h_{1,1}^* y_1^{(1)} + h_{1,2}\left(y_1^{(2)}\right)^* + h_{2,1}^* y_2^{(1)} + h_{2,2}\left(y_2^{(2)}\right)^* \\
&= \left(|h_{1,1}|^2 + |h_{1,2}|^2 + |h_{2,1}|^2 + |h_{2,2}|^2\right) x_1 + h_{1,1}^* n_1^{(1)} + h_{1,2}\left(n_1^{(2)}\right)^* + h_{2,1}^* n_2^{(1)} + h_{2,2}\left(n_2^{(2)}\right)^* \qquad (52) \\
\tilde{x}_2 &= h_{1,2}^* y_1^{(1)} + h_{1,1}\left(y_1^{(2)}\right)^* + h_{2,2}^* y_2^{(1)} + h_{1,1}\left(y_2^{(2)}\right)^* \\
&= \left(|h_{1,1}|^2 + |h_{1,2}|^2 + |h_{2,1}|^2 + |h_{2,2}|^2\right) x_2 + h_{1,2}^* n_1^{(1)} + h_{1,1}\left(n_1^{(2)}\right)^* + h_{2,2}^* n_2^{(1)} + h_{1,1}\left(n_2^{(2)}\right)^*. \qquad (53)
\end{aligned}$$

5. a) As shown in Fig. 3. Solid lines indicate input 0; dash lines indicate input 1.

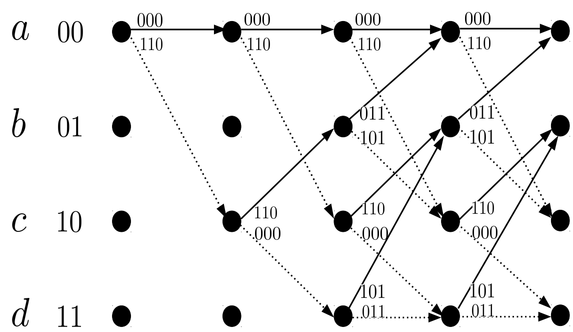

Figure 3: Trellis diagram for the convolutional code.

b) Fig. 4 plots the signal flow graph, from which we can write

$$\begin{aligned}
c &= a_0 D^2 NL + b D^2 NL \qquad (54) \\
b &= c D^2 L + d D^2 L \qquad (55) \\
d &= c NL + d D^2 NL \qquad (56) \\
a_1 &= b D^2 L. \qquad (57)
\end{aligned}$$

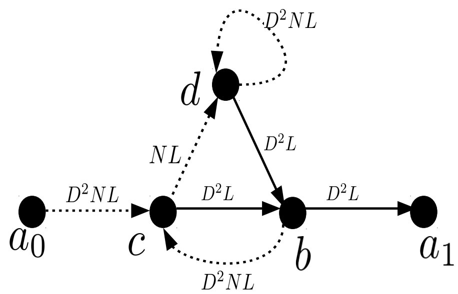

Figure 4: Signal flow graph.

Canceling out b, c, and d from (54) to (57), we can find

$$a_1 = a_0 \frac{D^6NL^3 + D^6N^2L^4 - D^8N^2L^4}{1 - (D^2NL + D^4NL^2 + D^4N^2L^3 - D^6N^2L^3)}. \qquad (58)$$

So the transfer function is

$$\begin{aligned}
T(D,N,L) &= \frac{a_1}{a_0} \\
&= \frac{D^6NL^3 + D^6N^2L^4 - D^8N^2L^4}{1 - (D^2NL + D^4NL^2 + D^4N^2L^3 - D^6N^2L^3)} \\
&= \left(D^6NL^3 + D^6N^2L^4 - D^8N^2L^4\right) \sum_{n=0}^{\infty} \left(D^2NL + D^4NL^2 + D^4N^2L^3 - D^6N^2L^3\right)^n. \qquad (59)
\end{aligned}$$

c) From the transfer function in (59), we can see that the minimum hamming distance is 6.

d) Using the Viterbi algorithm, calculate and plot the cumulative hamming distance on the trellis diagram in Fig. 5. The first three bits of **R** is 001. From the first branch of the trellis diagram in Fig. 3, the branch with input 0 and output 000 has hamming distance 1 with the input data 001, and the branch with input 1 and output 110 has hamming distance 3 with the input data 001. So on and so forth. Finally the Viterbi algorithm chooses the path with the smallest cumulative hamming distance with the received data as the final survival path, and the path corresponds to the decoded message. From Fig. 5 we see that we have two (bold) paths whose first branches both have input 0. We don't care about the later two bits since they are padding zeros. So the sequence is decoded as 0.

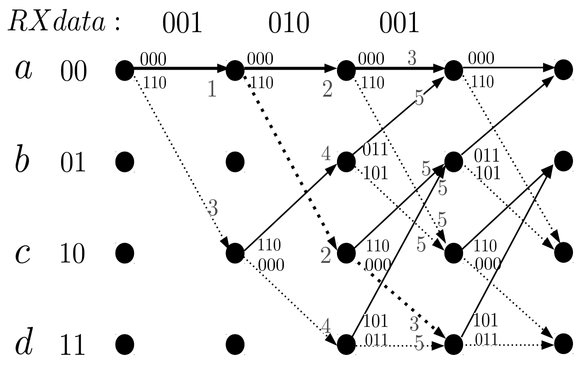

Figure 5: Viterbi algorithm.

References

[1] D. Zwillinger, *Table of integrals, series, and products*, 7th ed. Elsevier, 2014.