ENGG 5501: Foundations of Optimization	2020–21 First Term

Solution to Take-Home Midterm Examination

Instructor: Anthony Man–Cho So — Due: 11:59pm, November 13, 2020

IMPORTANT: Please remember to observe the rules as stated on the course website. In particular, you must work out the problems on your own.

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (15pts). Show that the set

$$K = \{(x,y,z) \in \mathbb{R}^3_+ : y \cdot \exp(-x/y) \le z\}$$

is a convex cone, where we adopt the convention that $\exp(-x/0) = 0$ for any $x \ge 0$.

ANSWER: It is straightforward to verify that K is a cone. Indeed, if $(x,y,z) \in K$ and $\alpha > 0$, then

$$(\alpha y) \cdot \exp(-(\alpha x)/(\alpha y)) = \alpha(y \cdot \exp(-x/y)) \le \alpha z,$$

which implies that $\alpha(x,y,z) \in K$. Now, observe that K is the epigraph of the function $\mathbb{R}^2_+ \ni (x,y) \mapsto f(x,y) = y \cdot \exp(-x/y)$. Thus, to show that K is convex, it suffices to show that f is convex on $\mathbb{R}^2_+$. Towards that end, let $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2_+$ and $\alpha \in (0,1)$ be arbitrary. If $y_1 = y_2 = 0$, then

$$f(\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2) = 0 = \alpha f(x_1,y_1) + (1-\alpha) f(x_2,y_2).$$

If one of y_1, y_2 is zero, say, $y_2 = 0$, then

$$\begin{aligned} &f(\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2) = \alpha y_1 \cdot \exp\left(-\frac{\alpha x_1 + (1-\alpha)x_2}{\alpha y_1}\right) \\ &\le \alpha y_1 \cdot \exp(-x_1/y_1) = \alpha f(x_1,y_1) + (1-\alpha) f(x_2,y_2) \end{aligned}$$

by the monotonicity of $\mathbb{R}_+ \ni x \mapsto \exp(-x)$. Otherwise, we have $y_1, y_2 > 0$. We compute

$$\begin{aligned} &f(\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2) = (\alpha y_1 + (1-\alpha)y_2) \cdot \exp\left(-\frac{\alpha x_1 + (1-\alpha)x_2}{\alpha y_1 + (1-\alpha)y_2}\right) \\ &= (\alpha y_1 + (1-\alpha)y_2) \cdot \exp\left(-\frac{\alpha y_1}{\alpha y_1 + (1-\alpha)y_2} \cdot \frac{x_1}{y_1} - \frac{(1-\alpha)y_2}{\alpha y_1 + (1-\alpha)y_2} \cdot \frac{x_2}{y_2}\right) \\ &\le (\alpha y_1 + (1-\alpha)y_2) \cdot \left[\frac{\alpha}{\alpha y_1 + (1-\alpha)y_2} \cdot y_1 \cdot \exp\left(-\frac{x_1}{y_1}\right) + \frac{1-\alpha}{\alpha y_1 + (1-\alpha)y_2} \cdot y_2 \cdot \exp\left(-\frac{x_2}{y_2}\right)\right] \\ &= \alpha f(x_1,y_1) + (1-\alpha) f(x_2,y_2), \end{aligned}$$

where the inequality follows from the convexity of $\mathbb{R}_+ \ni x \mapsto \exp(-x)$. This completes the proof.

Problem 2 (15pts). Let $C \subseteq \mathbb{R}^n$ be a convex set and $a, b \in \mathbb{R}^n$ be vectors. Suppose that $b^T x > 0$ for all $x \in C$. Show that the function $f : C \to \mathbb{R}$ defined by $f(x) = (a^T x)^2 / b^T x$ is convex.

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ANSWER: Since $b^T x > 0$ for all $x \in C$ and f is defined on C, using the appropriate Schur complement, we have

$$\operatorname{epi}(f) = \left\{ (x,t) \in C \times \mathbb{R} : \frac{(a^T x)^2}{b^T x} \le t \right\} = \left\{ (x,t) \in C \times \mathbb{R} : \begin{bmatrix} b^T x & a^T x \\ a^T x & t \end{bmatrix} \succeq \mathbf{0} \right\}.$$

Now, it is straightforward to verify the convexity of the set $\operatorname{epi}(f)$. Indeed, let $(x_1,t_1),(x_2,t_2) \in \operatorname{epi}(f)$ and $\alpha \in (0,1)$ be arbitrary. Then, we have

$$\begin{bmatrix} b^T(\alpha x_1 + (1-\alpha)x_2) & a^T(\alpha x_1 + (1-\alpha)x_2) \\ a^T(\alpha x_1 + (1-\alpha)x_2) & (\alpha t_1 + (1-\alpha)t_2) \end{bmatrix} = \alpha \begin{bmatrix} b^T x_1 & a^T x_1 \\ a^T x_1 & t_1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} b^T x_2 & a^T x_2 \\ a^T x_2 & t_2 \end{bmatrix} \succeq \mathbf{0},$$

which shows that $\alpha(x_1,t_1) + (1-\alpha)(x_2,t_2) \in \operatorname{epi}(f)$. To complete the proof, it remains to note that $\operatorname{epi}(f)$ is convex if and only if f is convex.

Problem 3 (25pts). Let $B(\mathbf{0},1) \subset \mathbb{R}^n$ be the unit Euclidean ball in $\mathbb{R}^n$ centered at the origin. For any $x \in B(\mathbf{0},1)$, consider the set

$$N(x) = \{u \in \mathbb{R}^n : u^T(y-x) \le 0 \text{ for all } y \in B(\mathbf{0},1)\}.$$

(a) **(10pts).** Show that $N(x)$ is a convex cone for any $x \in B(\mathbf{0},1)$.

ANSWER: Let $x \in \mathbb{R}^n$ be fixed. For any $\alpha > 0$ and $u \in N(x)$, we have $\alpha u^T(y-x) \le 0$ for all $y \in B(\mathbf{0},1)$. Hence, $N(x)$ is a cone. Moreover, for any $u,v \in N(x)$ and $\alpha \in (0,1)$, we have

$$(\alpha u + (1-\alpha)v)^T(y-x) = \alpha u^T(y-x) + (1-\alpha)v^T(y-x) \le 0 \text{ for all } y \in B(\mathbf{0},1).$$

It follows that $N(x)$ is convex.

(b) **(15pts).** Give an explicit description of $N(x)$. Simplify your answer as much as possible. Show all your work.

(*Hint: Consider the cases $\|x\|_2 < 1$ and $\|x\|_2 = 1$ separately. Also, recall that for any $y,z \in \mathbb{R}^n$, $|y^T z| \le \|y\|_2 \cdot \|z\|_2$ and equality holds if and only if y,z are linearly dependent.*)

ANSWER: We claim that

$$N(x) = \begin{cases} \{\mathbf{0}\} & \text{if } \|x\|_2 < 1, \\ \{\alpha x : \alpha \ge 0\} & \text{if } \|x\|_2 = 1. \end{cases}$$

Indeed, if $\|x\|_2 = \ell < 1$, then $x + (1-\ell)v \in B(\mathbf{0},1)$ for all $v \in S^{n-1}$ (recall that $S^{n-1} = \{v \in \mathbb{R}^n : \|v\|_2 = 1\}$). Hence, if $u \in N(x)$, then we must have $u^T(x + (1-\ell)v - x) = (1-\ell)u^T v \le 0$ for all $v \in S^{n-1}$. This implies that $u = \mathbf{0}$.

On the other hand, if $\|x\|_2 = 1$, then by the Cauchy-Schwarz inequality, for any $\alpha \ge 0$, we have

$$\alpha x^T(y-x) = \alpha(x^T y - \|x\|_2) \le \alpha(\|y\|_2 - 1) \le 0 \text{ for all } y \in B(\mathbf{0},1).$$

This shows that $\{\alpha x : \alpha \ge 0\} \subseteq N(x)$. Conversely, let $u \in N(x)$. By the result in (a), we may assume without loss of generality that $\|u\|_2 = 1$. Then, since $u \in B(\mathbf{0},1)$, we have $u^T(u-x) \le 0$. This implies that $u^T u = 1 \le u^T x \le \|u\|_2 \cdot \|x\|_2 = 1$, which, together with the Cauchy-Schwarz inequality, yields $u = x$. It follows that $N(x) \subseteq \{\alpha x : \alpha \ge 0\}$, as desired.

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Problem 4 (10pts). Let $A \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^n$ be given. Show that $\{x \in \mathbb{R}^n : Ax \le \mathbf{0}\} \subseteq \{x \in \mathbb{R}^n : c^T x \le 0\}$ if and only if $A^T y = c$ for some $y \ge \mathbf{0}$.

ANSWER: By Farkas' lemma, exactly one of the following systems is solvable:

$$\begin{array}{cl} \text{(I)} & A^T y = c,\ y \ge \mathbf{0}. \\ \text{(II)} & Ax \le \mathbf{0},\ c^T x > 0. \end{array}$$

It follows that (I) is solvable if and only if $c^T x \le 0$ whenever $x \in \mathbb{R}^n$ satisfies $Ax \le \mathbf{0}$. However, this is precisely the statement we need to prove.

Problem 5 (20pts). Let $A \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^n$ be given. Let $v : \mathbb{R}^m \to \mathbb{R}$ be the function defined by

$$\begin{array}{rcll} v(b) & = & \text{maximize} & c^T x \\ & & \text{subject to} & Ax \le b, \\ & & & x \ge \mathbf{0}. \end{array} \tag{1}$$

In other words, $v(b)$ is the optimal value of the LP (1) when the right-hand side of the first inequality constraint is b.

(a) **(10pts).** Let $b \in \mathbb{R}^m$ be fixed. Find the dual of Problem (1).

ANSWER: The given LP can be written in the standard dual form

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & \begin{bmatrix} A \\ -I \end{bmatrix} x \le \begin{bmatrix} b \\ \mathbf{0} \end{bmatrix}. \end{array}$$

It follows that the dual is given by

$$\begin{array}{ll} \text{minimize} & b^T z \\ \text{subject to} & A^T z - w = c, \\ & z, w \ge \mathbf{0}, \end{array} \quad \text{or equivalently,} \quad \begin{array}{ll} \text{minimize} & b^T z \\ \text{subject to} & A^T z \ge c, \\ & z \ge \mathbf{0}. \end{array}$$

(b) **(10pts).** Using the result in (a), or otherwise, show that the function $v(\cdot)$ is concave.

ANSWER: For any $b \in \mathbb{R}^m$ such that $v(b)$ is finite, we have

$$\begin{array}{rcll} v(b) & = & \text{minimize} & b^T z \\ & & \text{subject to} & A^T z \ge c, \\ & & & z \ge \mathbf{0} \end{array}$$

by the LP Strong Duality Theorem. Since $v(\cdot)$ is a pointwise infimum of the collection $\{b \mapsto b^T z : A^T z \ge c,\ z \ge \mathbf{0}\}$ of linear functions, we conclude that $v(\cdot)$ is concave.

Problem 6 (15pts). Let $A \in \mathcal{S}^n$ be a symmetric matrix and $c \in \mathbb{R}^n$ be a vector. Consider the following LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \ge c, \\ & x \ge \mathbf{0}. \end{array} \tag{2}$$

Show that if $\bar{x} \in \mathbb{R}^n$ satisfies $A\bar{x} = c$ and $\bar{x} \ge \mathbf{0}$, then $\bar{x}$ is an optimal solution to Problem (2).

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ANSWER: Since A is symmetric, we see from Problem 5 that the dual of (2) is given by

$$\begin{array}{rl} \text{maximize} & c^T y \\ \text{subject to} & Ay \leq c, \\ & y \geq \mathbf{0}. \end{array} \tag{3}$$

If $\bar{x} \in \mathbb{R}^n$ satisfies $A\bar{x} = c$ and $\bar{x} \geq \mathbf{0}$, then $\bar{x}$ is feasible for both (2) and its dual (3). Moreover, when evaluated at $\bar{x}$, the objective functions of both (2) and (3) have the same value. It follows from the LP Strong Duality Theorem that $\bar{x}$ is optimal for Problem (2).

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