ENGG 5501: Foundations of Optimization	2020–21 First Term

Solution to Take-Home Final Examination

Instructor: Anthony Man–Cho So — Due: 11:59pm, December 22, 2020

IMPORTANT: Please remember to observe the rules as stated on the course website. In particular, you must work out the problems on your own.

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (10pts). Let $C, A_1, \ldots, A_m \in \mathcal{S}^n$ and $b \in \mathbb{R}^m$ be given. Write down the dual of the following SDP:

$$\begin{array}{ll} \inf & \operatorname{tr}(CX) \\ \text{subject to} & \operatorname{tr}(A_iX) \le b_i \quad \text{for } i = 1, \ldots, m, \\ & X \succeq \mathbf{0}. \end{array}$$

Simplify your answer as much as possible and show all your work.

ANSWER: Let $u_i \ge 0$, where $i = 1, \ldots, m$, be the multiplier associated with the constraint $\operatorname{tr}(A_iX) \le b_i$ and $S \succeq \mathbf{0}$ be the multiplier associated with the constraint $X \succeq \mathbf{0}$. Then, the Lagrangian dual of the given SDP can be written as

$$\sup_{u \ge \mathbf{0},\, S \succeq \mathbf{0}} \inf_{X \in \mathcal{S}^n} \left\{ \operatorname{tr}(CX) + \sum_{i=1}^m u_i(\operatorname{tr}(A_iX) - b_i) - \operatorname{tr}(SX) \right\} \tag{SD}$$

Now, observe that

$$\begin{aligned} \inf_{X \in \mathcal{S}^n} \left\{ \operatorname{tr}(CX) + \sum_{i=1}^m u_i(\operatorname{tr}(A_iX) - b_i) - \operatorname{tr}(SX) \right\} &= -b^Tu + \inf_{X \in \mathcal{S}^n} \left\{ \operatorname{tr}\left[\left(C + \sum_{i=1}^m u_iA_i - S \right) X \right] \right\} \\ &= \begin{cases} -b^Tu & \text{if } C + \sum_{i=1}^m u_iA_i - S = \mathbf{0}, \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that (SD) can be written as

$$\begin{array}{ll} \sup & b^Tu \\ \text{subject to} & C - \sum_{i=1}^m u_iA_i \succeq \mathbf{0}, \\ & u \le \mathbf{0}. \end{array}$$

Problem 2 (15pts).

(a) **(5pts).** Let $P \in \mathcal{S}^n_{++}$ and $Q \in \mathcal{S}^n$ be given. Show that $P \succeq Q$ if and only if $I \succeq P^{-1/2}QP^{-1/2}$. Here, recall that $P^{1/2}$ is the positive definite square root of P; i.e., $P = P^{1/2}P^{1/2}$ with $P^{1/2} \succ \mathbf{0}$.

ANSWER: Observe that

$$\begin{aligned} P \succeq Q \quad &\iff \quad u^T(P-Q)u \succeq 0 \quad \text{for all } u \in \mathbb{R}^n \\ &\iff \quad (P^{1/2}u)^T(I - P^{-1/2}QP^{-1/2})(P^{1/2}u) \geq 0 \quad \text{for all } u \in \mathbb{R}^n \\ &\iff \quad v^T(I - P^{-1/2}QP^{-1/2})v \geq 0 \quad \text{for all } v \in \mathbb{R}^n \\ &\iff \quad I - P^{-1/2}QP^{-1/2} \succeq \mathbf{0}, \end{aligned}$$

where the second-to-last equivalence follows from the fact that $P^{1/2}$ is full rank and hence range$(P^{1/2}) = \mathbb{R}^n$.

(b) **(10pts).** Let $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_m : \mathbb{R}^\ell \to \mathcal{S}^n$ be given affine functions with $\mathcal{A}_0(x) \succ \mathbf{0}$ for all $x \in \mathbb{R}^\ell$ and $\gamma > 0$ be a given constant. Show that the constraint

$$\sum_{i=1}^m \Big((\mathcal{A}_0(x))^{-1/2} \mathcal{A}_i(x) (\mathcal{A}_0(x))^{-1/2} \Big)^2 \preceq \gamma^2 I$$

can be formulated as a linear matrix inequality.

ANSWER: The given constraint can be written as

$$I - (\gamma \mathcal{A}_0(x))^{-1/2} \begin{bmatrix} \mathcal{A}_1(x) \\ \vdots \\ \mathcal{A}_m(x) \end{bmatrix}^T \begin{bmatrix} \gamma \mathcal{A}_0(x) & & \\ & \ddots & \\ & & \gamma \mathcal{A}_0(x) \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{A}_1(x) \\ \vdots \\ \mathcal{A}_m(x) \end{bmatrix} (\gamma \mathcal{A}_0(x))^{-1/2} \succeq \mathbf{0}.$$

By the result in (a), the above is equivalent to

$$\gamma \mathcal{A}_0(x) - \begin{bmatrix} \mathcal{A}_1(x) \\ \vdots \\ \mathcal{A}_m(x) \end{bmatrix}^T \begin{bmatrix} \gamma \mathcal{A}_0(x) & & \\ & \ddots & \\ & & \gamma \mathcal{A}_0(x) \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{A}_1(x) \\ \vdots \\ \mathcal{A}_m(x) \end{bmatrix} \succeq \mathbf{0}.$$

Using the Schur complement, the above is equivalent to

$$\begin{bmatrix} \gamma \mathcal{A}_0(x) & \mathcal{A}_1(x) & \cdots & \mathcal{A}_m(x) \\ \mathcal{A}_1(x) & \gamma \mathcal{A}_0(x) & & \\ \vdots & & \ddots & \\ \mathcal{A}_m(x) & & & \gamma \mathcal{A}_0(x) \end{bmatrix} \succeq \mathbf{0},$$

which is a linear matrix inequality because $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_m$ are affine functions.

Problem 3 (20pts). Let $g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable convex functions. Consider the problem

$$\min_{x \in \mathbb{R}^n} \ \max\{g_1(x), \ldots, g_m(x)\} \tag{1}$$

Show that $x^* \in \mathbb{R}^n$ is an optimal solution to Problem (1) if and only if there exists a vector $u^* \in \mathbb{R}^m$ such that

$$\begin{aligned} &\sum_{j=1}^m u_j^* \nabla g_j(x^*) = \mathbf{0}, \quad u^* \geq \mathbf{0}, \quad \sum_{j=1}^m u_j^* = 1, \\ &u_j^* = 0 \quad \text{if} \quad g_j(x^*) < \max\{g_1(x^*), \ldots, g_m(x^*)\}, \ \text{for } j = 1, \ldots, m. \end{aligned}$$

ANSWER: Problem (1) is equivalent to

$$\begin{array}{ll} \text{minimize} & z \\ \text{subject to} & g_j(x) \leq z \quad \text{for } j = 1, \ldots, m. \end{array} \tag{P}$$

Note that the objective function $(x, z) \mapsto z$ is convex, and for $i = 1, \ldots, m$, the function $(x, z) \mapsto g_i(x) - z$ is continuously differentiable and convex. Hence, the above formulation is a convex optimization problem. Moreover, given any $\bar{x} \in \mathbb{R}^n$, if we let $\bar{z} = \max\{g_1(\bar{x}), \ldots, g_m(\bar{x})\} + 1$, then $g_i(\bar{x}) < \bar{z}$ for $i = 1, \ldots, m$. This shows that Problem (P) satisfies the Slater condition. Hence, by Theorems 4 and 6 of Handout 7, $x^* \in \mathbb{R}^n$ is an optimal solution to Problem (1) if and only if there exists a $(u^*, z^*) \in \mathbb{R}^m \times \mathbb{R}$ such that

$$\begin{array}{rcll} \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} + \displaystyle\sum_{j=1}^m u_j^* \begin{bmatrix} \nabla g_j(x^*) \\ -1 \end{bmatrix} & = & \mathbf{0}, & (a) \\ u_j^*(g_j(x^*) - z^*) & = & 0 \quad \text{for } j = 1, \ldots, m, & (b) \\ g_j(x^*) & \leq & z^* \quad \text{for } j = 1, \ldots, m, & (c) \\ u^* & \geq & \mathbf{0}. & (d) \end{array}$$

To complete the proof, it remains to show that $z^* = \max\{g_1(x^*), \ldots, g_m(x^*)\}$. From (c), we clearly have $z^* \geq \max\{g_1(x^*), \ldots, g_m(x^*)\}$. On the other hand, using (a), (b), and (d), we obtain

$$z^* = \sum_{j=1}^m u_j^* g_j(x^*) \leq \max\{g_1(x^*), \ldots, g_m(x^*)\} \sum_{j=1}^m u_j^* = \max\{g_1(x^*), \ldots, g_m(x^*)\}.$$

Problem 4 (25pts). Let $A \in \mathcal{S}^n$ be given. Let λ_1 be the largest eigenvalue of A and v_1 be a unit-length eigenvector associated with λ_1. Consider the following problem:

$$\begin{array}{ll} \max & x^T A x \\ \text{subject to} & \|x\|_2^2 = 1, \\ & v_1^T x = 0. \end{array} \tag{2}$$

(a) **(10pts).** Write down the first-order optimality conditions of Problem (2) and explain why they are necessary for optimality.

ANSWER: Let $\theta, \gamma \in \mathbb{R}$ be the multipliers associated with the constraints $\|x\|_2^2 = 1$ and $v_1^T x = 0$, respectively. Then, the first-order optimality conditions of Problem (2) are given by

$$\begin{array}{rcll} -2Ax + 2\theta x + \gamma v_1 & = & \mathbf{0}, & (i) \\ \|x\|_2^2 & = & 1, & (ii) \\ v_1^T x & = & 0. & (iii) \end{array}$$

Suppose that $\bar{x} \in \mathbb{R}^n$ is an optimal solution to Problem (2). Note that both constraints of Problem (2) are active at $\bar{x}$, and their gradients are given by $2\bar{x}$ and v_1. Since $\bar{x} \neq \mathbf{0}$ and $v_1^T \bar{x} = 0$, the vectors $2\bar{x}$ and v_1 are linearly independent. It follows from Theorem 3 of Handout 7 that the conditions (i)–(iii) above are necessary for optimality.

(b) **(15pts).** Let λ_2 be the optimal value of and v_2 be an optimal solution to Problem (2). Using the result in (a), show that λ_2 is the second largest eigenvalue of A and v_2 is an eigenvector associated with λ_2.

ANSWER: Using (i), (iii), and the fact that $Av_1 = \lambda_1 v_1$ with $\|v_1\|_2^2 = 1$, we have

$$\gamma = \gamma v_1^T v_1 = 2v_1^T(A - \theta I)x = 2\lambda v_1^T x = 0.$$

Hence, we obtain from (i), (ii) that $Ax = \theta x$ (i.e., (θ, x) is an eigenpair of A) and $\theta = x^T Ax$. Since the eigenvectors of A form an orthonormal basis of $\mathbb{R}^n$, the solution $x = v_2$ is optimal for Problem (2) with an objective value of $\theta = \lambda_2$.

Remark: Here, by "second largest eigenvalue" we allow for the possibility that $\lambda_1 = \lambda_2$, because in this case we still have $\lambda_1 \geq \lambda_2$ and the eigenspace corresponding to λ_1 is at least 2-dimensional.

Problem 5 (30pts). Let $(x_i, y_i) \in \mathbb{R}^n \times \mathbb{R}$, where $i = 1, \ldots, N$, be given. Consider the following problem:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|\beta\|_2^2 + e^T\xi \\ \text{subject to} & \xi_i \geq 1 - y_i(\beta^T x_i + \beta_0) \quad \text{for } i = 1, \ldots, N, \\ & \xi \geq \mathbf{0}. \end{array} \tag{3}$$

Here, the decision variables are $\beta \in \mathbb{R}^n$, $\beta_0 \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$.

(a) **(15pts).** Write down the first-order optimality conditions of Problem (3) and explain why they are both necessary and sufficient for optimality.

ANSWER: It is easy to verify that the objective function of Problem (3) is convex in (β, β_0, ξ). Moreover, all the constraints of Problem (3) are linear in (β, β_0, ξ). It follows that (3) is a convex optimization problem. By Theorem 5 of Handout 7, the first-order conditions of Problem (3), which are given by

$$\begin{aligned} \begin{bmatrix} \beta \\ 0 \\ e \end{bmatrix} - \sum_{i=1}^N u_i \begin{bmatrix} y_i x_i \\ y_i \\ e_i \end{bmatrix} - \sum_{i=1}^N v_i \begin{bmatrix} \mathbf{0} \\ 0 \\ e_i \end{bmatrix} &= \mathbf{0}, \\ u_i(1 - y_i(\beta^T x_i + \beta_0) - \xi_i) &= 0 \quad \text{for } i = 1, \ldots, N, \\ \xi_i - 1 + y_i(\beta^T x_i + \beta_0) &\geq 0 \quad \text{for } i = 1, \ldots, N, \\ \xi, u, v &\geq \mathbf{0}, \end{aligned}$$

are necessary for optimality.

To prove sufficiency, in view of Theorem 6 of Handout 7, it remains to show that Problem (3) has an optimal solution. Towards that end, we first observe that Problem (3) is feasible (take any $\bar{\beta} \in \mathbb{R}^n$, $\bar{\beta}_0 \in \mathbb{R}$ and choose a sufficiently large $\bar{\xi}_i$ for each $i = 1, \ldots, N$ so that the constraints of Problem (3) are all satisfied). Without loss of generality, we may assume that $y = (y_1, \ldots, y_N) \neq \mathbf{0}$, for otherwise $\beta^* = \mathbf{0}$, $\beta_0^* \in \mathbb{R}$, $\xi^* = e$ is an optimal solution to Problem (3). We may also assume that

$$I_+ = \{i : y_i > 0\} \neq \emptyset, \quad I_- = \{i : y_i < 0\} \neq \emptyset.$$

Indeed, if, say, $I_+ = \emptyset$, then by taking $\beta^* = \mathbf{0}$ and a sufficiently small β_0^*, we can set $\xi_i^* = 0$ whenever $y_i < 0$ and $\xi_i^* = 1$ whenever $y_i = 0$. Note that such a choice of $(\beta^*, \beta_0^*, \xi^*)$ is optimal. A similar argument takes care of the case where $I_- = \emptyset$.

Now, let $(\bar{\beta}, \bar{\beta}_0, \bar{\xi})$ be an arbitrary feasible solution to Problem (3) and $\bar{v}$ be the corresponding objective value. Then, we only need to consider those solutions (β, β_0, ξ) that satisfy $\|\beta\|_2^2 \le 2\bar{v}$ and $\|\xi\|_1 \le \bar{v}$ when solving Problem (3). Moreover, from the constraints

$$\xi_i \ge 1 - y_i(\beta^T x_i + \beta_0) \quad \text{for } i = 1, \ldots, N,$$

we see that

$$\max_{i \in I_+} \frac{\xi_i + 1 - y_i \beta^T x_i}{y_i} \le \beta_0 \le \min_{i \in I_-} \frac{\xi_i - 1 + y_i \beta^T x_i}{|y_i|}.$$

Clearly,

$$\frac{\xi_i - 1 + y_i \beta^T x_i}{|y_i|} \le \overline{M} \triangleq \frac{1}{\min_{i \in I_-} |y_i|} \left[\bar{v} - 1 + \sqrt{2\bar{v}} \left(\max_{i \in I_-} |y_i| \right) \left(\max_{i \in I_-} \|x_i\|_2 \right) \right] \quad \text{for } i \in I_-$$

and

$$\frac{\xi_i + 1 - y_i \beta^T x_i}{y_i} \ge \underline{M} \triangleq \frac{1}{\max_{i \in I_+} y_i} \left[-\bar{v} + 1 - \sqrt{2\bar{v}} \left(\max_{i \in I_+} |y_i| \right) \left(\max_{i \in I_+} \|x_i\|_2 \right) \right] \quad \text{for } i \in I_+.$$

Summarizing the above observations, we deduce that Problem (3) is equivalent to

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|\beta\|_2^2 + e^T \xi \\ \text{subject to} & \xi_i \ge 1 - y_i(\beta^T x_i + \beta_0) \quad \text{for } i = 1, \ldots, N, \\ & \xi \ge \mathbf{0}, \\ & \|\beta\|_2^2 \le 2\bar{v}, \;\; -\underline{M} \le \beta_0 \le \overline{M}, \;\; \|\xi\|_1 \le \bar{v}. \end{array} \tag{C}$$

The upshot of the above formulation is that its feasible region is compact. Hence, by Weierstrass' theorem, we conclude that Problem (C), and hence Problem (3), has an optimal solution.

(b) **(10pts).** Write down the Lagrangian dual of Problem (3) by dualizing all the constraints in (3). Simplify your answer as much as possible and show all your work.

ANSWER: The Lagrangian dual of Problem (3) is given by

$$\begin{aligned} &\sup_{u,v \ge \mathbf{0}} \inf_{\beta, \beta_0, \xi} \left\{ \frac{1}{2}\|\beta\|_2^2 + e^T \xi + \sum_{i=1}^N u_i (1 - y_i(\beta^T x_i + \beta_0) - \xi_i) - v^T \xi \right\} \\ &= \sup_{u,v \ge \mathbf{0}} \left\{ e^T u + \inf_{\beta, \beta_0, \xi} \left\{ \frac{1}{2}\|\beta\|_2^2 - \sum_{i=1}^N u_i y_i \beta^T x_i - \beta_0 y^T u + (e - u - v)^T \xi \right\} \right\}. \end{aligned}$$

Observe that

$$
\begin{aligned}
&\inf_{\beta,\beta_0,\xi}\left\{\frac{1}{2}\|\beta\|_2^2-\sum_{i=1}^N u_iy_i\beta^Tx_i-\beta_0y^Tu+(e-u-v)^T\xi\right\}\\
&=\begin{cases} -\dfrac{1}{2}\left\|\displaystyle\sum_{i=1}^N u_iy_ix_i\right\|_2^2 & \text{if } y^Tu=0 \text{ and } e-u-v=\mathbf{0},\\ -\infty & \text{otherwise.}\end{cases}
\end{aligned}
$$

It follows that the Lagrangian dual of Problem (3) can be written as

$$
\begin{aligned}
\text{maximize} \quad & e^Tu-\frac{1}{2}\left\|\sum_{i=1}^N u_iy_ix_i\right\|_2^2\\
\text{subject to} \quad & y^Tu=0,\\
& \mathbf{0}\le u\le e.
\end{aligned}
\tag{LD}
$$

(c) **(5pts).** Is the duality gap between Problem (3) and its Lagrangian dual derived in (b) zero? Explain.

ANSWER: By the result in (a), Problem (3) has an optimal solution. Hence, by Corollary 3 of Handout 7, the duality gap between (3) and (LD) is zero.