ENGG 5501: Foundations of Optimization	2022–23 First Term
Solution to Midterm Examination	
Time Limit: 2 Hours	November 9, 2022

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (25pts). Let $a, b \in \mathbb{R}$ be given with $a \neq 0$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |ax + b|$.

(a) **(10pts).** Show that f is convex.

ANSWER: Define the functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ by $g(x) = |x|$ and $h(x) = ax + b$, respectively. Observe that g is convex and h is affine. Moreover, the function f can be written as $f = g \circ h$. It follows from Theorem 8(c) of Handout 2 that f is convex.

(b) **(15pts).** Using the definition of the subdifferential, give an explicit expression for $\partial f(x)$, where $x \in \mathbb{R}$ is arbitrary. Show all your work.

ANSWER: Let $x \in \mathbb{R}$ be arbitrary. We consider three cases:

Case 1: $ax + b > 0$.

In this case, the function f is differentiable at x with $f'(x) = a$. Hence, by Theorem 11(b) of Handout 2, we have $\partial f(x) = \{a\}$.

Case 2: $ax + b < 0$.

In this case, the function f is again differentiable at x with $f'(x) = -a$. Hence, by Theorem 11(b) of Handout 2, we have $\partial f(x) = \{-a\}$.

Case 3: $ax + b = 0$.

We claim that $\partial f(x) = [-a, a]$. Indeed, if $s \in \partial f(x)$, then $|ay + b| \geq s(y - x)$ for all $y \in \mathbb{R}$. Taking $y = x + 1$ yields $s \leq a$; taking $y = x - 1$ yields $s \geq -a$. It follows that $s \in [-a, a]$.

Conversely, let $s \in [-a, a]$ and $y \in \mathbb{R}$ be arbitrary. If $y \geq x$, then $s(y - x) \leq a(y - x) = (ay + b) - (ax + b) = ay + b \leq |ay + b|$. On the other hand, if $y \leq x$, then $s(y - x) \leq -a(y - x) = -(ay + b) + (ax + b) \leq |ay + b|$. It follows that $s \in \partial f(x)$.

Problem 2 (10pts). Let $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ be a given function, $\mathcal{Y} \subseteq \mathbb{R}^{n_2}$ be a given set, and $\rho \in \mathbb{R}$ be a given parameter. Suppose that for each $y \in \mathcal{Y}$, the function $f_y : \mathbb{R}^{n_1} \to \mathbb{R}$ defined by $f_y(x) = f(x, y)$ is ρ-convex (recall the definition and different characterizations of ρ-convexity from Homework Set 3). Show that the function $g : \mathbb{R}^{n_1} \to \mathbb{R}$ defined by $g(x) = \sup_{y \in \mathcal{Y}} f_y(x)$ is also ρ-convex.

ANSWER: By assumption and the result in Problem 4 of Homework Set 3, the function $x \mapsto f_y(x) + \frac{\rho}{2}\|x\|_2^2$ is convex. This, together with Theorem 8(b) of Handout 2, implies that the function

$$x \mapsto \sup_{y \in \mathcal{Y}} \left\{ f_y(x) + \frac{\rho}{2}\|x\|_2^2 \right\} = \sup_{y \in \mathcal{Y}} f_y(x) + \frac{\rho}{2}\|x\|_2^2 = g(x) + \frac{\rho}{2}\|x\|_2^2$$

is convex. It follows that g is ρ-convex, as desired.

Problem 3 (25pts). Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a non-empty closed convex set and $x \in \mathcal{S}$ be arbitrary. Our goal is to show that for all $\alpha \geq 1$ and $d \in \mathbb{R}^n$,

$$\|x - \Pi_{\mathcal{S}}(x+d)\|_2 \leq \|x - \Pi_{\mathcal{S}}(x+\alpha d)\|_2. \tag{1}$$

To begin, let $v = \Pi_{\mathcal{S}}(x+d)$ and $v_\alpha = \Pi_{\mathcal{S}}(x+\alpha d)$. If $v = v_\alpha$, then there is nothing to prove. Hence, we may assume that $v \neq v_\alpha$. Now, we prove (1) by contradiction.

(a) **(10pts).** Suppose that (1) does not hold. Show that

$$(v_\alpha - v)^T(x - v) > 0. \tag{2}$$

ANSWER: Suppose that $(v_\alpha - v)^T(x - v) \leq 0$. Then, we have

$$\begin{aligned} \|x - v_\alpha\|_2^2 &= \|x - v + v - v_\alpha\|_2^2 \\ &= \|x - v\|_2^2 + \|v - v_\alpha\|_2^2 + 2(x-v)^T(v - v_\alpha) \\ &\geq \|x - v\|_2^2, \end{aligned}$$

which contradicts our assumption that (1) does not hold.

(b) **(15pts).** Using (2), show that $(x + \alpha d - v)^T(v_\alpha - v) \leq 0$ for any $\alpha \geq 1$. Hence, deduce a contradiction to show that (1) holds. (*Hint: Start by showing that* $0 \geq (x+d-v)^T(v_\alpha - v) > d^T(v_\alpha - v)$.)

ANSWER: Since $v_\alpha \in \mathcal{S}$, by Theorem 3 of Handout 2 and the result in (a), we have

$$0 \geq (x+d-v)^T(v_\alpha - v) = (x-v)^T(v_\alpha - v) + d^T(v_\alpha - v) > d^T(v_\alpha - v).$$

It follows that for any $\alpha \geq 1$, we have

$$(x + \alpha d - v)^T(v_\alpha - v) = (x+d-v)^T(v_\alpha - v) + (\alpha - 1)d^T(v_\alpha - v) \leq 0.$$

However, since $v \in \mathcal{S}$, by applying Theorem 3 of Handout 2 again, we get

$$0 \geq (x + \alpha d - v_\alpha)^T(v - v_\alpha) = (x + \alpha d - v)^T(v - v_\alpha) + \|v - v_\alpha\|_2^2 > 0,$$

which is a contradiction.

Problem 4 (15pts). Construct a primal-dual pair of linear programs such that both the primal and the dual have a unique optimal solution. Justify your answer.

ANSWER: There are many possible constructions. For instance, consider the following primal-dual pair of standard-form LP:

$$\text{(P)} \quad \begin{array}{ll} \text{minimize} & x_1 \\ \text{subject to} & x_1 + x_2 = 1, \\ & x_1 - x_2 = 0, \\ & x_1, x_2 \geq 0. \end{array} \qquad \text{(D)} \quad \begin{array}{ll} \text{maximize} & y_1 \\ \text{subject to} & y_1 + y_2 \leq 1, \\ & y_1 - y_2 \leq 0. \end{array}$$

Clearly, $x^* = (1/2, 1/2)$ is the only feasible solution to Problem (P) and hence is also optimal. Now, let y^* be an optimal solution to Problem (D), whose existence is guaranteed by the LP strong duality theorem. By complementary slackness, we see that y^* must satisfy

$$y_1^* + y_2^* = 1, \qquad y_1^* - y_2^* = 0.$$

This implies that $y^* = (1/2, 1/2)$ is the unique optimal solution to Problem (D).

Problem 5 (25pts). Let $c \in \mathbb{R}^n_+$ be a given non-negative vector and $E \subseteq \{(i,j) : 1 \le i < j \le n;\ i,j \text{ integers}\}$ be a given set of index pairs. Consider the following LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_i + x_j \ge 1, \quad \text{for } (i,j) \in E, \\ & x \ge \mathbf{0}. \end{array} \tag{3}$$

(a) **(10pts).** Show that Problem (3) always has an optimal solution.

ANSWER: Observe that $\bar{x} = (1, 1, \ldots, 1)$ is feasible for Problem (3). Moreover, since $c, x \in \mathbb{R}^n_+$, the optimal value of Problem (3) is bounded below by 0. It follows from Corollary 1 of Handout 3 that Problem (3) has an optimal solution.

(b) **(15pts).** Write down the dual of Problem (3). Justify your answer.

ANSWER: Observe that the constraint matrix associated with Problem (3) has n columns, each of which corresponds to a decision variable. Moreover, each column of the constraint matrix is $|E|$-dimensional, and the (i,j)-th entry (where $(i,j) \in E$) of the k-th column equals 1 if and only if $k \in \{i,j\}$. The above argument shows that the dual of Problem (3) takes the form

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{(i,j)\in E} y_{ij} \\ \text{subject to} & \displaystyle\sum_{(i,j)\in E : k \in \{i,j\}} y_{ij} \le c_k, \quad \text{for } k = 1, \ldots, n, \\ & y \ge \mathbf{0}. \end{array}$$