ENGG 5781: Matrix Analysis and Computations	**2021-22 First Term**
Final Examination	
Time Limit: 2.5 hours	December 14, 2021

Note:

- This is an open-book examination.
- For theorems and properties that appear in the course notes, you can use them directly and without proof. Exceptions are when you are asked to prove those theorems and properties.
- Theorems and properties outside the scope of the course notes *may not* be applied directly. Exceptions are when those results can be straightforwardly deduced, and/or your answer also provide the proofs of those results.
- Answer any **four** of the five problems. **No** bonus will be granted if you answer five problems.
- On your answer sheet, write down which four problems you solved.
- The problems are equally weighted, each with 25%.

Problem 1

(a) Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$. Show that $\operatorname{rank}(\mathbf{A} + \mathbf{B}) \leq \operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B})$. (8.33%)

(b) Let $\mathbf{R} \in \mathbb{S}^n$ be a positive definite matrix. Verify that $f(\mathbf{x}) = \sqrt{\mathbf{x}^T \mathbf{R} \mathbf{x}}$ is a norm. (8.33%)

(c) Let $\mathbf{A} \in \mathbb{R}^{m \times k}$. Suppose that there exists a full column rank $\mathbf{B} \in \mathbb{R}^{m \times l}$ such that $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{B})$. Show that $k < l$ must not be true. Do not use the subspace dimension and basis results in Lecture 1, page 19–21, e.g., $l = \dim \mathcal{R}(\mathbf{B}) = \dim \mathcal{R}(\mathbf{A}) \leq k$, to solve this problem. (8.33%)

Problem 2

(a) Let $\mathbf{A} \in \mathbb{C}^{n\times n}$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A}$. Show that $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$ and $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$. (8.33%)

(b) Let

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix} \in \mathbb{C}^{2\times 2}.$$

Show that if $\mathbf{B}\mathbf{B}^H = \mathbf{B}^H\mathbf{B}$ is true, then $\mathbf{B}$ is diagonal. (8.33%)

(c) Suppose that we have the following result: Let $\mathbf{B} \in \mathbb{C}^{n\times n}$ be upper triangular. If $\mathbf{B}\mathbf{B}^H = \mathbf{B}^H\mathbf{B}$ is true, then $\mathbf{B}$ must be diagonal. Use the above result to aid you to show how the following result: Let $\mathbf{A} \in \mathbb{C}^{n\times n}$, and suppose that $\mathbf{A}\mathbf{A}^H = \mathbf{A}^H\mathbf{A}$ is true. Then $\mathbf{A}$ always admits an eigendecomposition $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$ where $\mathbf{V} \in \mathbb{C}^{n\times n}$ is unitary; $\mathbf{\Lambda} \in \mathbb{C}^{n\times n}$ is diagonal. (8.33%)

Problem 3

(a) Let $\mathbf{X} \in \mathbb{S}^{m+n}$. Partition

$$\mathbf{X} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix},$$

where $\mathbf{A} \in \mathbb{S}^m$, $\mathbf{B} \in \mathbb{R}^{m\times n}$, $\mathbf{C} \in \mathbb{S}^n$, and suppose that $\mathbf{C}$ is invertible. Show that

$$\det(\mathbf{X}) = \det(\mathbf{C}) \det(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T).$$

(8.33%)

(b) Let $\mathbf{A} \in \mathbb{S}^n$. Show that

$$\begin{aligned} &\lambda_{\min}(\mathbf{A}) \le a_{ii}, \quad \text{for all } i, \\ &\lambda_{\max}(\mathbf{A}) \le \|\mathbf{A}\|_F \end{aligned}$$

(8.33%)

(c) Let $\mathbf{Y} \in \mathbb{S}^n$. Consider the problem below

$$\begin{aligned} \min_{\mathbf{X}\in\mathbb{S}^n} \ & \|\mathbf{Y} - \mathbf{X}\|_F^2 \\ \text{s.t. } & |\lambda_i(\mathbf{X})| = 1, \quad i = 1, \ldots, n. \end{aligned}$$

Let $\mathbf{Y} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^T$ be the eigendecomposition of $\mathbf{Y}$, with $\boldsymbol{\Lambda} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \ge \cdots \ge \lambda_n$. Show that the solution to the above problem is $\mathbf{X} = \mathbf{V}\mathbf{D}\mathbf{V}^T$ where $\mathbf{D}$ is diagonal with the diagonal elements given by

$$d_{ii} = \begin{cases} 1, & \lambda_i \ge 0 \\ -1, & \lambda_i < 0 \end{cases},$$

(8.33%)

Problem 4

(a) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. From the definition of pseudo-inverse $\mathbf{A}^{\dagger} = \mathbf{V}_1 \tilde{\boldsymbol{\Sigma}}^{-1} \mathbf{U}_1^T$, verify that the pseudo-inverse of a full row rank $\mathbf{A}$ is $\mathbf{A}^{\dagger} = \mathbf{A}^T (\mathbf{A}\mathbf{A}^T)^{-1}$. (8.33%)

(b) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Show that

$$\sigma_1(\mathbf{A}) = \max_{\substack{\mathbf{x} \in \mathbb{R}^n,\ \mathbf{y} \in \mathbb{R}^m \\ \|\mathbf{x}\|_2 \le 1, \|\mathbf{y}\|_2 \le 1}} \mathbf{y}^T \mathbf{A} \mathbf{x}$$

(8.33%)

(c) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{B} \in \mathbb{R}^{m \times n}$.

i. Based on what we know in variational characterization of eigenvalues of symmetric matrices (Lecture 4), argue why it is true that

$$\sigma_k(\mathbf{A}) = \max_{\substack{\mathcal{S} \subseteq \mathbb{R}^n, \\ \dim \mathcal{S} = k}} \min_{\mathbf{x} \in \mathcal{S}, \|\mathbf{x}\|_2 = 1} \|\mathbf{A}\mathbf{x}\|_2.$$

ii. Show that $\sigma_k(\mathbf{A} + \mathbf{B}) \ge \sigma_k(\mathbf{A}) - \sigma_1(\mathbf{B})$.

(8.33%)

Problem 5 Let

(a) Let $\mathbf{y} \in \mathbb{R}^m$, let $\mathbf{A} \in \mathbb{R}^{m\times n}$, and let $\mathbf{W} \in \mathbb{S}^n$ be positive definite. Derive the solution to the following problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \ \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \mathbf{x}^T\mathbf{W}\mathbf{x}.$$

(8.33%)

(b) Consider the problem below:

$$\min_{\mathbf{x}\in\mathbb{R}^n,\mathbf{z}\in\mathbb{R}^k} \|\mathbf{y} - \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{z}\|_2^2,$$

where $\mathbf{y} \in \mathbb{R}^m, \mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{B} \in \mathbb{R}^{m\times k}$. Suppose that we are only interested in the solution to $\mathbf{x}$. Show that $\mathbf{x} = (\mathbf{P}_{\mathbf{B}}^{\perp}\mathbf{A})^{\dagger}\mathbf{P}_{\mathbf{B}}^{\perp}\mathbf{y}$ is a solution to $\mathbf{x}$. (8.33%)

(c) Let $\mathbf{z} \in \mathbb{R}^m$, $\mathbf{z} \neq 0$. Let $\mathbf{x} \in \mathbb{R}^n$. Let $\mathcal{U} = \{\boldsymbol{\Delta} \in \mathbb{R}^{m\times n} \mid \|\boldsymbol{\delta}_i\|_2 \leq 1,\ i = 1,\ldots,n\}$. Show that

$$\max_{\boldsymbol{\Delta}\in\mathcal{U}} \|\mathbf{z} - \boldsymbol{\Delta}\mathbf{x}\|_2 = \|\mathbf{z}\|_2 + \|\mathbf{x}\|_1.$$

(8.33%)

The End