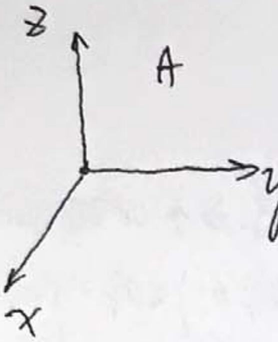
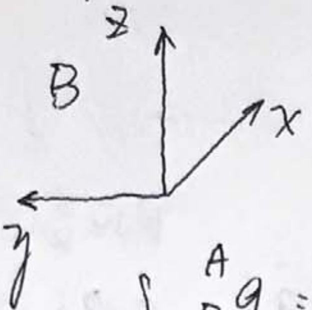
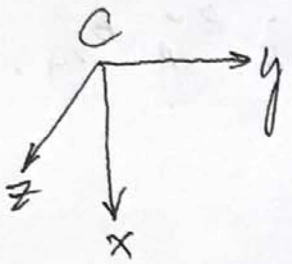
四元数与坐标转换基本运算：

给定坐标系A, B, C如下，B由A绕于其z轴旋转180°得到，

C由A关于其y轴旋转90°得到。







根据四元数轴角构造方法：

B由A绕 $^A\vec{r}$ 旋转 θ_A 得

$$\boxed{{}^A_B q = \left(\cos\frac{\theta_A}{2}, -\sin\frac{\theta_A}{2}\,{}^A\vec{r}\right)}$$

$$\begin{cases} {}^A_B q = \{{}^A\vec{z}_A, 180°\} \\ {}^A_C q = \{{}^A\vec{y}_A, 90°\} \end{cases}$$

$$\therefore\ {}^A_B q = \left(\cos\frac{180°}{2}, -\sin\frac{180°}{2}\,{}^A\vec{z}_A\right) = (0,0,0,-1)$$

$${}^A_C q = \left(\cos\frac{90°}{2}, -\sin\frac{90°}{2}\,{}^A\vec{y}_A\right) = \left(\frac{\sqrt{2}}{2},0,-\frac{\sqrt{2}}{2},0\right)$$

进而根据坐标转换公式 $\boxed{{}^B\vec{v} = {}^A_B q \otimes {}^A\vec{v} \otimes ({}^A_B q)^*}$，取 ${}^A\vec{v} = {}^A\vec{x}_A$.

则A系的x轴在B系表示 ${}^B\vec{x}_A$ 计算如下.

$${}^B\vec{x}_A = {}^A_B q \otimes {}^A\vec{x}_A \otimes ({}^A_B q)^* = (0,0,0,-1)\otimes(0,1,0,0)\otimes(0,0,0,1) = \underline{\underline{(0,-1,0,0)}}.$$

$$= (-1,0,0)^T = -{}^B\vec{x}_B$$ 即为B系x轴的负方向. ■

同理. $${}^C\vec{x}_A = {}^A_C q \otimes {}^A\vec{x}_A \otimes ({}^A_C q)^* = \left(\frac{\sqrt{2}}{2},0,-\frac{\sqrt{2}}{2},0\right)\otimes(0,1,0,0)\otimes\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2},0\right) = \underline{\underline{(0,0,0,1)}}$$

$$= (0,0,1)^T = {}^C\vec{z}_C$$，即为C系z轴方向. ■

考虑B与C之间的关系，根据四元数乘法

$$\boxed{{}^B_C q = {}^A_C q \otimes {}^B_A q} = {}^A_C q \otimes ({}^A_B q)^{-1}.$$

$$= \left(\frac{\sqrt{2}}{2},0,-\frac{\sqrt{2}}{2},0\right)\otimes(0,0,0,1) = \left(0,-\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right).$$

则B系x轴在C系表示为

$${}^C\vec{x}_B = {}^B_C q \otimes {}^B\vec{x}_B \otimes ({}^B_C q)^* = \left(0,-\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)\otimes$$
$$(0,1,0,0)\otimes\left(0,\frac{\sqrt{2}}{2},0,-\frac{\sqrt{2}}{2}\right) = \underline{\underline{(0,0,0,-1)}}$$

$$= (0,0,-1)^T = -{}^C\vec{z}_C$$，即为C系z轴负方向. ■

四元数乘法：

$a = [a_1, a_2, a_3, a_4]$

$b = [b_1, b_2, b_3, b_4]$.

$$a \otimes b = \begin{bmatrix} a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 \\ a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3 \\ a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2 \\ a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1 \end{bmatrix}^T$$

四元数与旋转矩阵的转换，已知 ${}^A_Bq=(q_1,q_2,q_3,q_4)$.

$$
{}^A_BM=\begin{bmatrix} 2q_1^2-1+2q_2^2 & 2(q_2q_3+q_1q_4) & 2(q_2q_4-q_1q_3)\\ 2(q_2q_3-q_1q_4) & 2q_1^2-1+2q_3^2 & 2(q_3q_4+q_1q_2)\\ 2(q_2q_4+q_1q_3) & 2(q_3q_4-q_1q_2) & 2q_1^2-1+2q_4^2 \end{bmatrix}^T.
$$

对于系 A, B, C.

$$
{}^A_BM=\begin{bmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}=\left[{}^B\vec{x}_A \,\vdots\, {}^B\vec{y}_A \,\vdots\, {}^B\vec{z}_A\right].
$$

因此 ${}^A_BM$ 即为A系三个坐标轴在B系的表示按列排列得到

$$
\boxed{{}^A_BM=\left[{}^B\vec{x}_A \,\vdots\, {}^B\vec{y}_A \,\vdots\, {}^B\vec{z}_A\right].}
$$

$$
{}^A_CM=\begin{bmatrix} 0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}=\left[{}^C\vec{x}_A \,\vdots\, {}^C\vec{y}_A \,\vdots\, {}^C\vec{z}_A\right].
$$

用旋转矩阵进行坐标转换

$$
\boxed{{}^B\vec{x}_A={}^A_BM\cdot{}^A\vec{x}_A}=\begin{bmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}-1\\0\\0\end{bmatrix}=-{}^B\vec{x}_B.
$$

$$
{}^C\vec{x}_A={}^A_CM\cdot{}^A\vec{x}_A=\begin{bmatrix} 0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\-1\end{bmatrix}={}^C\vec{z}_C. \quad =\left[{}^C\vec{x}_B \,\vdots\, {}^C\vec{y}_B \,\vdots\, {}^C\vec{z}_B\right].
$$

$$
\boxed{{}^B_CM={}^A_CM\cdot{}^B_AM}={}^A_CM\cdot({}^A_BM)^T=\begin{bmatrix} 0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\cdot\begin{bmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix} 0 & 0 & -1\\ 0 & -1 & 0\\ -1 & 0 & 0\end{bmatrix}
$$

四元数/旋转矩阵与欧拉角的转换

基本单轴旋转矩阵：

$$
R_x(\theta)=\begin{bmatrix} 1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix},\quad R_y(\theta)=\begin{bmatrix} \cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix},\quad R_z(\theta)=\begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}.
$$

若A系统其自身z轴旋转 γ，再绕新的y轴旋转 β，最后绕新的x轴转 α，则表示为 A 按照 zyx 顺序旋转 (γ,β,α) 得到 B.

$$
\boxed{{}^A_BM=\left[R_x(\alpha)\cdot R_y(\beta)\,R_z(\gamma)\right]^T.}
$$

$$
{}^B_CM=\left[R_y(90^\circ)\cdot R_z(180^\circ)\right]^T=R_y(90^\circ)^T\cdot R_z(180^\circ)^T
$$

上面定义的A, B, C系中，C系可由B系按照 zyx 顺序旋转 $(180^\circ, 90^\circ, 0^\circ)$ 得到.

$$
=\underbrace{\begin{bmatrix} 0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}}_{{}^A_CM}\underbrace{\begin{bmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{{}^B_AM}=\begin{bmatrix} 0 & 0 & -1\\ 0 & -1 & 0\\ -1 & 0 & 0\end{bmatrix}.
$$