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Fourier Series

Fourier Coefficient

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, n = 0, 1, 2, ...$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, n = 0, 1, 2, ...$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l} dx, n = 0, 1, 2, ...$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l} dx, n = 0, 1, 2, ...$$

$a_n\, and\, b_n$ are Fourier Coefficient.
The Fourier Series is $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$.

Inner products in Hilbert Space

$$< f(x), g(x) >= \int_a^b f(x)g(x)dx$$

it seems to calculate the similarity of $g(x)\, and\, f(x)$

Complex Fourier Series

Euler's formula:
$e^{ikx} = \cos kx + i \sin kx := \Psi_k$
$e^{\frac{ik\pi x}{l}} = \cos\frac{ik\pi x}{l} + i\sin\frac{ik\pi x}{l} := \Psi_k$
$c_k = \frac{1}{2\pi}\langle f(x), \Psi_k = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-\frac{ik\pi x}{l}} dx$

$$
\begin{aligned}
f(x) &= \sum_{k=-\infty}^{\infty} c_k e^{ikx} \\
&= \sum_{k=-\infty}^{\infty} (a_k + ib_k)(\cos kx + i \sin kx) \\
&= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} < f(x), \Psi_k > \Psi_k
\end{aligned}
$$

$$
\begin{aligned}
< \Psi_j, \Psi_k > &= 0 \quad j \neq k \\
&= 2\pi \quad j = k
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= \sum_{k=-\infty}^{\infty} c_k e^{\frac{ik\pi x}{l}} \\
&= \sum_{k=-\infty}^{\infty} (a_k + ib_k)(\cos \frac{ik\pi x}{l} + i \sin \frac{ik\pi x}{l}) \\
&= \frac{1}{2l} \sum_{k=-\infty}^{\infty} < f(x), \Psi_k > \Psi_k
\end{aligned}
$$

This means that f(x) can be represented by sin and cos in any orthogonal dimensional space.

Fourier Transform

When L go to ∞ , the Fourier Series becomes Fourier Transform.
The difference is that in FT we change series into an integral. The integral is Fourier Transform Function represented by sin and cos. And it contains a variable ω. And we could restore the $f(x)$ by inverse function.

$$
\begin{aligned}
\omega_k &= \frac{k\pi}{l} \\
\Delta\omega &= \frac{\pi}{l}
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= \lim_{\Delta\omega \to 0} \sum_{k=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \int_{\frac{-\pi}{\Delta\omega}}^{\frac{\pi}{\Delta\omega}} f(\xi) e^{-ik\Delta\omega\xi} d\xi \, e^{ik\Delta\omega x} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} d\xi \, e^{i\omega x} d\omega
\end{aligned}
$$

$$\widehat{f(\omega)} = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

$$f(x) = \mathcal{F}^{-1}(\widehat{f(x)}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f(\omega)}e^{i\omega x}d\omega$$

From the formula, we can see Fourier Transform is part of Fourier Coefficient.
We can use Fourier Transform do partial differential and then recover f(x) back. change x space mode to frequency mode.

Applications with FT

1. Derivatives
2. Convolution Integrals
3. Parseval's Theorem
4. Heat Equation
5. Discrete Fourier Transform
6. Fast Fourier Transform
7. Denoising Data with FFT
8. Derivatives with FFT
9. PDEs with FFT
10. Spectrogram and Gabor Transform
11. Uncertainty Principle
12. Wavelets and Multiresolution Analysis
13. Image Compression with FFT
14. Laplace Transform

Derivatives

Prerequisite: $\lim\limits_{x\to\infty} f(x) = 0$.

$$\begin{aligned}
\mathcal{F}\left(\frac{df(x)}{dx}\right) &= \int_{-\infty}^{\infty} \frac{df(x)}{dx}e^{-i\omega x}dx \\
&= e^{-i\omega x}f(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} f(x) - i\omega e^{-i\omega x}dx \;\; \#Integration\, by\, parts \\
&= 0 + i\omega\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \\
&= i\omega\mathcal{F}(f(x))
\end{aligned}$$

Then use inverse Fourier function to calculate $f'(x)$ by $f(x)$.

Convolution Integrals

Define convolution integrals

$$(f * g) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi$$

Then $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g) = \widehat{f}\ \widehat{g}$.
Inverse Fourier will be $\mathcal{F}_{-1}(\hat{f}\ \ \hat{g})(x) = f * g$

Parseval's Theorem

$$\int_{-\infty}^{\infty} |f(\hat{x})|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Discrete Fourier Transform

Actually Discrete Fourier Series, FFT is to compute DFT.
DFT can be written by matrix multiplication. FFT is computationally effective way
Fourier coefficient is function with continuous ω. DFC is n points of different discrete ω.

$$\hat{f}_k = \sum_{j=0}^{n-1} f_j e^{\frac{-i2\pi jk}{n}}$$

$$f_j = \frac{1}{n}\left(\sum_{k=0}^{n-1} \hat{f}_k e^{\frac{i2\pi jk}{n}}\right)$$

Vector $\hat{f}$ = DFT matrix $*$ Vector f
Value of function in x space map to value of function in frequency (amplitude).

$$\{f_0, f_1, f_2 \cdots, f_n\} \overset{DFT}{\Longrightarrow} \{\hat{f}_0, \hat{f}_1, \hat{f}_2 \cdots, \hat{f}_n\}$$

$$\omega_n = e^{\frac{-2\pi i}{n}}$$

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{(n-1)} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_n(n-1)^2 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

Fast Fourier Transform

The time complexity is $O(nlog(n))$

$$D_{512} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \omega^{511} \end{bmatrix}$$

$$\hat{f} = F_{1024} f = \begin{bmatrix} I_{512} & -D_{512} \\ I_{512} & -D_{512} \end{bmatrix} \begin{bmatrix} F_{512} & 0 \\ 0 & F_{512} \end{bmatrix} \begin{bmatrix} f_{even} \\ f_{odd} \end{bmatrix}$$

Do this $F_{1024} \to F_{512} \to F_{256} \to \cdots \to F_4 \to F_2$ recursively to reduce the computation.