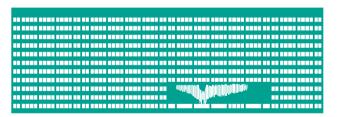
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Numerická lineární algebra 1 QR rozklad

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IT4INNOVATIONS NÁRODNÍ SUPERPOČÍTAČOVÉ CENTRUM QR rozklad

QR rozklad

Definice

Čtvercová matice Q, která splňuje $Q^TQ = I$ se nazývá ortogonální matice. Ortogonální matice má ortonormální sloupce a splňuje $Q^{-1} = Q^T$.

■ Pro sloupce Q platí

$$(oldsymbol{q}_i,oldsymbol{q}_j)=oldsymbol{q}_i^Toldsymbol{q}_j$$

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$$(oldsymbol{q}_i,oldsymbol{q}_j)=oldsymbol{q}_i^Toldsymbol{q}_j$$

Věta

Nechť $A \in \mathbb{R}^{m,n}, m \geq n$ je libovolná matice. Pak existuje ortogonální matice $Q \in \mathbb{R}^{m,n}$ a horní trojúhelníková matice $R \in \mathbb{R}^{n,n}$ takové, že platí

$$A = QR$$
.

• Uvažujme m=n. Pak $\mathsf{A}=\mathsf{QR}$ lze rozepsat jako

$$egin{aligned} ig(oldsymbol{a}_1|oldsymbol{a}_2|\cdots|oldsymbol{a}_nig) &= ig(oldsymbol{q}_1|oldsymbol{q}_2|\cdots|oldsymbol{q}_nig) egin{aligned} egin{aligned} r_{1,1} & r_{1,2} & \cdots & r_{1,n} \ & r_{2,2} & & dots \ & & \ddots & \ 0 & & & r_{n,n} \end{pmatrix} \end{aligned}$$

• Uvažujme m=n. Pak $\mathsf{A}=\mathsf{QR}$ lze rozepsat jako

$$egin{aligned} ig(oldsymbol{a}_1|oldsymbol{a}_2|\cdots|oldsymbol{a}_nig) & egin{pmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,n} \ & r_{2,2} & & dots \ & & \ddots & \ 0 & & & r_{n,n} \end{pmatrix} \end{aligned}$$

$$egin{array}{lcl} m{a}_1 &=& r_{1,1} m{q}_1 \ m{a}_2 &=& r_{1,2} m{q}_1 + r_{2,2} m{q}_2 \ &dots \ m{a}_n &=& r_{1,n} m{q}_1 + r_{2,n} m{q}_2 + \cdots + r_{n,n} m{q}_n \end{array}$$

Uvažujme m=n. Pak A=QR lze rozepsat jako

$$egin{aligned} oldsymbol{a}_1 |oldsymbol{a}_2| \cdots |oldsymbol{a}_n ig) &= oldsymbol{q}_1 |oldsymbol{q}_2| \cdots |oldsymbol{q}_n ig) egin{aligned} egin{aligned} r_{1,1} & r_{1,2} & \cdots & r_{1,n} \ & r_{2,2} & & dots \ & & \ddots & \ 0 & & & r_{n,n} \end{pmatrix} \end{aligned}$$

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angle & \langle oldsymbol{q}_1,oldsymbol{q}_2
angle \ & \langle oldsymbol{q}_1,ol$$

$$Ax = b \quad \Leftrightarrow \quad \underbrace{x_1a_1 + x_2a_2 + \dots + x_na_n = b}$$

Hledáme koeficienty rozvoje b v bázi tvořené sloupci A

$$Ax = b$$
 \Leftrightarrow $\underbrace{x_1a_1 + x_2a_2 + \cdots + x_na_n = b}_{\text{Hledáme koeficienty rozvoje } b \text{ v bázi tvořené sloupci } A}$

$$Q\underbrace{Rx}_{=y} = b$$

$$Ax = b$$
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- Existuje několik způsobů výpočtu
 - pomocí Gramova-Schmidtova procesu,
 - pomocí Givensových rotací,
 - pomocí Householderových transformací.

- Gramův-Schmidtův proces
 - vstup: nezávislé vektory a_1, a_2, \dots, a_n
 - výstup: ortornormální vektory $m{q}_1,m{q}_2,\dots,m{q}_n$ takové, že $\langle m{a}_1,m{a}_2,\dots,m{a}_n
 angle = \langle m{q}_1,m{q}_2,\dots,m{q}_n
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- lacksquare První krok: $oldsymbol{v}_1=oldsymbol{a}_1,oldsymbol{q}_1=rac{oldsymbol{v}_1}{\|oldsymbol{v}_1\|},r_{11}=\|oldsymbol{v}_1\|$

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- Předpokládejme, že známe q_1, q_2, \dots, q_{i-1}

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- lacktriangle Vektor a_j můžeme vyjádřit jako

$$oldsymbol{a}_j = r_{1,j}oldsymbol{q}_1 + \dots + r_{i,j}oldsymbol{q}_i + \dots + r_{j-1,j}oldsymbol{q}_{j-1} + \underbrace{r_{j,j}oldsymbol{q}_j}_{ ext{ozn.}oldsymbol{v}_j}$$

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$$\downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow$$

$$\mathbf{v}_{j} = \mathbf{a}_{j} - r_{1,j}\mathbf{q}_{1} - \dots - r_{i,j}\mathbf{q}_{i} - \dots - r_{j-1,j}\mathbf{q}_{j-1}$$

$$\boldsymbol{v}_j = \boldsymbol{a}_j - r_{1,j}\boldsymbol{q}_1 - \cdots - r_{i,j}\boldsymbol{q}_i - \cdots - r_{j-1,j}\boldsymbol{q}_{j-1}$$

$$0 = (q_k, v_j) = (q_k, a_j - r_{1,j}q_1 - \dots - r_{i,j}q_i - \dots - r_{j-1,j}q_{j-1})$$

$$\boldsymbol{v}_i = \boldsymbol{a}_i - r_{1,i}\boldsymbol{q}_1 - \cdots - r_{i,i}\boldsymbol{q}_i - \cdots - r_{j-1,i}\boldsymbol{q}_{j-1}$$

$$0 = (\boldsymbol{q}_k, \boldsymbol{v}_j) = (\boldsymbol{q}_k, \boldsymbol{a}_j - r_{1,j} \boldsymbol{q}_1 - \dots - r_{i,j} \boldsymbol{q}_i - \dots - r_{j-1,j} \boldsymbol{q}_{j-1})$$

Protože $(\boldsymbol{q}_k, \boldsymbol{q}_i) = 0$ pro $k \neq i$, platí

$$(\boldsymbol{q}_k, \boldsymbol{a}_j - r_{k,j} \boldsymbol{q}_k) = (\boldsymbol{q}_k, \boldsymbol{a}_j) - r_{k,j} \underbrace{(\boldsymbol{q}_k, \boldsymbol{q}_k)}_{=1} = 0$$

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$$\boldsymbol{v}_{j} = \boldsymbol{a}_{j} - r_{1,j}\boldsymbol{q}_{1} - \cdots - r_{i,j}\boldsymbol{q}_{i} - \cdots - r_{j-1,j}\boldsymbol{q}_{j-1}$$

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$$\mathbf{v}_j = \mathbf{a}_j - r_{1,j}\mathbf{q}_1 - \cdots - r_{i,j}\mathbf{q}_i - \cdots - r_{j-1,j}\mathbf{q}_{j-1}$$

$$0 = (q_k, v_j) = (q_k, a_j - r_{1,j}q_1 - \dots - r_{i,j}q_i - \dots - r_{j-1,j}q_{j-1})$$

■ Protože $(q_k, q_i) = 0$ pro $k \neq i$, platí

$$(\boldsymbol{q}_k, \boldsymbol{a}_j - r_{k,j} \boldsymbol{q}_k) = (\boldsymbol{q}_k, \boldsymbol{a}_j) - r_{k,j} \underbrace{(\boldsymbol{q}_k, \boldsymbol{q}_k)}_{=1} = 0 \Rightarrow r_{k,j} = (\boldsymbol{q}_k, \boldsymbol{a}_j)$$

lacksquare Získali jsme tedy v_j jako příslušnou lineární kombinaci. Snadno určíme q_j a $r_{j,j}$:

$$oldsymbol{q}_j = rac{oldsymbol{v}_j}{\|oldsymbol{v}_j\|} = rac{oldsymbol{v}_j}{r_{j,j}} \Rightarrow r_{j,j} = \|oldsymbol{v}_j\|$$

```
function QR GS(A \in \mathbb{R}^{m \times n})
     Q = O \in \mathbb{R}^{m \times n}, R = O \in \mathbb{R}^{n \times n}
     for j = 1 : n \text{ do}
           \boldsymbol{v} = \mathsf{A}(:,j)
           for i = 1, ..., i - 1 do
                 R(i,j) = Q(:,i)^T A(:,j)
                 \boldsymbol{v} = \boldsymbol{v} - \mathsf{R}(i,j)\mathsf{Q}(:,i)
           end for
           R(j,j) = \|\boldsymbol{v}\|
           Q(:, j) = \boldsymbol{v}/R(j, j)
      end for
end function
```

```
function QR GS(A \in \mathbb{R}^{m \times n})
      Q = Q \in \mathbb{R}^{m \times n}, R = Q \in \mathbb{R}^{n \times n}
      for i = 1 : n \text{ do}
           \mathbf{v} = \mathsf{A}(:,i)
           for i = 1, ..., i - 1 do
                 R(i,j) = Q(:,i)^T A(:,j)
                 \boldsymbol{v} = \boldsymbol{v} - \mathsf{R}(i, j)\mathsf{Q}(:, i)
            end for
            R(i, j) = ||\boldsymbol{v}||
           Q(:, j) = \boldsymbol{v}/R(j, j)
      end for
end function
```

- Algoritmus vyžaduje $2mn^2$ operací.
- Gramův-Schmidtův proces je numericky nestabilní, v praxi se proto spíše používají Givensovy rotace nebo Householderovy transformace

$$\mathsf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ 4 & 1 \end{pmatrix}$$

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lacksquare Hledáme $(oldsymbol{a}_1\,|\,oldsymbol{a}_2)=(oldsymbol{q}_1\,|\,oldsymbol{q}_2)egin{pmatrix}r_{1,1}&r_{1,2}\0&r_{2,2}\end{pmatrix}$

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ 4 & 1 \end{pmatrix}$$

lacksquare Hledáme $(m{a}_1 \,|\, m{a}_2) = (m{q}_1 \,|\, m{q}_2) egin{pmatrix} r_{1,1} & r_{1,2} \ 0 & r_{2,2} \end{pmatrix}$

$$oldsymbol{v}_1 = oldsymbol{a}_1 = egin{pmatrix} 2 \ 0 \ 4 \end{pmatrix},$$

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lacksquare Hledáme $(oldsymbol{a}_1\,|\,oldsymbol{a}_2)=(oldsymbol{q}_1\,|\,oldsymbol{q}_2)egin{pmatrix}r_{1,1}&r_{1,2}\0&r_{2,2}\end{pmatrix}$

$$v_1 = a_1 = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}, ||v|| = \sqrt{4 + 16} =$$

$$= \sqrt{20} = 2\sqrt{5} = r_{1,1},$$

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ 4 & 1 \end{pmatrix}$$

lacksquare Hledáme $(m{a}_1\,|\,m{a}_2)=(m{q}_1\,|\,m{q}_2)egin{pmatrix}r_{1,1}&r_{1,2}\0&r_{2,2}\end{pmatrix}$

$$egin{aligned} m{v}_1 = & m{a}_1 = egin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}, \| m{v} \| = \sqrt{4 + 16} = \ & = & \sqrt{20} = 2\sqrt{5} = r_{1,1}, m{q}_1 = egin{pmatrix} 2/2\sqrt{5} \\ 0 \\ 4/2\sqrt{5} \end{pmatrix} = egin{pmatrix} \sqrt{5}/5 \\ 0 \\ 2\sqrt{5}/5 \end{pmatrix} \end{aligned}$$

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$$a_2 = r_{1,2}q_1 + r_{2,2}q_2 \Rightarrow v_2 = a_2 - r_{1,2}q_1$$

$$0 = \mathbf{q}_1^T \mathbf{v}_2 = \mathbf{q}_1^T \mathbf{a}_2 - r_{1,2} \mathbf{q}_1^T \mathbf{q}_1$$

$$\Rightarrow r_{1,2} = \mathbf{q}_1^T \mathbf{a}_2 = \frac{\sqrt{5}}{5} \cdot 3 + 0 \cdot 1 + \frac{2\sqrt{5}}{5} \cdot 1 = \sqrt{5}$$

$$\mathsf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ 4 & 1 \end{pmatrix}$$

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$$a_2 = r_{1,2}q_1 + r_{2,2}q_2 \Rightarrow v_2 = a_2 - r_{1,2}q_1$$

$$0 = \mathbf{q}_1^T \mathbf{v}_2 = \mathbf{q}_1^T \mathbf{a}_2 - r_{1,2} \mathbf{q}_1^T \mathbf{q}_1$$

$$\Rightarrow r_{1,2} = \mathbf{q}_1^T \mathbf{a}_2 = \frac{\sqrt{5}}{5} \cdot 3 + 0 \cdot 1 + \frac{2\sqrt{5}}{5} \cdot 1 = \sqrt{5}$$

$$\boldsymbol{v}_2 = \begin{pmatrix} 3\\1\\1 \end{pmatrix} - \sqrt{5} \begin{pmatrix} \sqrt{5}/5\\0\\2\sqrt{5}/5 \end{pmatrix} = \begin{pmatrix} 2\\1\\-1 \end{pmatrix}$$

$$\mathsf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ 4 & 1 \end{pmatrix}$$

lacksquare Hledáme $(oldsymbol{a}_1\,|\,oldsymbol{a}_2)=(oldsymbol{q}_1\,|\,oldsymbol{q}_2)egin{pmatrix} r_{1,1} & r_{1,2} \ 0 & r_{2,2} \end{pmatrix}$

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$$\|\boldsymbol{v}_2\| = \sqrt{2^2 + 1 + 1} = \sqrt{6} = r_{2,2}$$

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$$egin{aligned} m{v}_1 &= & m{a}_1 = \begin{pmatrix} 2\\0\\4 \end{pmatrix}, \|m{v}\| = \sqrt{4+16} = \\ &= & \sqrt{20} = 2\sqrt{5} = r_{1,1}, m{q}_1 = \begin{pmatrix} 2/2\sqrt{5}\\0\\4/2\sqrt{5} \end{pmatrix} = \begin{pmatrix} \sqrt{5}/5\\0\\2\sqrt{5}/5 \end{pmatrix} \end{aligned}$$

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 $\|\boldsymbol{v}_2\| = \sqrt{2^2 + 1 + 1} = \sqrt{6} = r_2$

$$q_0 = v_0/||v_0|| = \begin{pmatrix} \sqrt{6}/3 \\ \sqrt{6}/6 \end{pmatrix}$$

$$|m{q}_2 = m{v}_2 / \|m{v}_2\| = egin{pmatrix} \sqrt{6}/3 \ \sqrt{6}/6 \ -\sqrt{6}/6 \end{pmatrix}$$

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■ Hledáme $(\boldsymbol{a}_1 \,|\, \boldsymbol{a}_2) = (\boldsymbol{q}_1 \,|\, \boldsymbol{q}_2) \begin{pmatrix} r_{1,1} & r_{1,2} \\ 0 & r_{2,2} \end{pmatrix}$

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$$\mathsf{Q} = \begin{pmatrix} \sqrt{5}/5 & \sqrt{6}/3 \\ 0 & \sqrt{6}/6 \\ 2\sqrt{5}/5 & -\sqrt{6}/6 \end{pmatrix}, \mathsf{R} = \begin{pmatrix} 2\sqrt{5} & \sqrt{5} \\ 0 & \sqrt{6} \end{pmatrix}$$

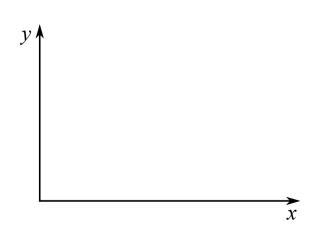
Redukovaná vs. plná QR faktorizace

Redukovaná QR faktorizace

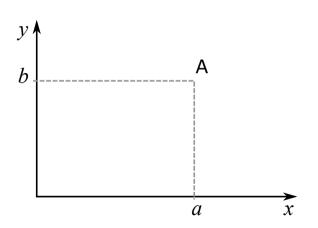
Redukovaná vs. plná QR faktorizace

Redukovaná QR faktorizace

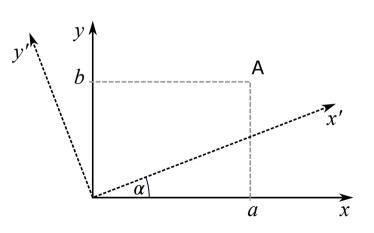
■ Plná QR faktorizace



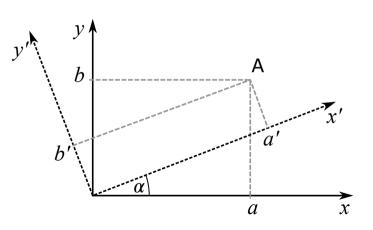
Uvažujme dva souřadné systémy: xy a x'y'



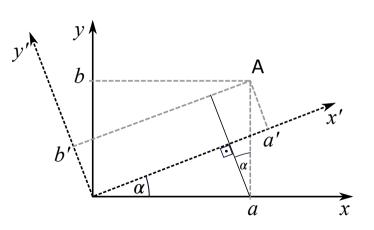
- Uvažujme dva souřadné systémy: xy a x'y'
- Bod A má souřadnice (a,b) v xy



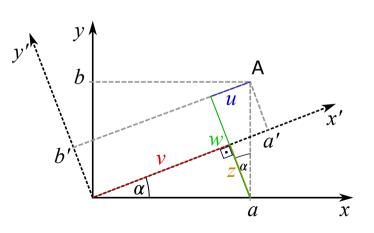
- Uvažujme dva souřadné systémy: xy a x'y'
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- Vyjádřeme jeho souřadnice v x'y'



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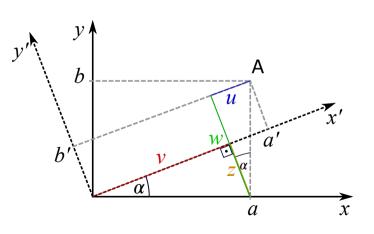


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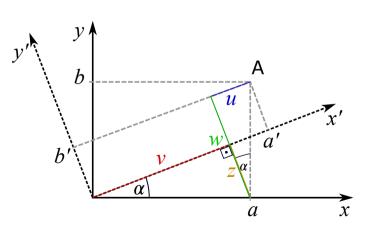
$$\sin \alpha = \frac{u}{b} \Rightarrow u = b \sin \alpha$$



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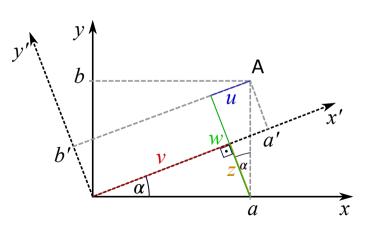


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$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}}_{=\mathsf{G}} \begin{pmatrix} a \\ b \end{pmatrix}$$

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■ Matici G nazýváme maticí rovinné rotace, která vektor (a,b) rotuje o úhel $-\alpha$ (rotujeme soustavu souřadnic o úhel $\alpha \Leftrightarrow$ rotujeme vektor o úhel $-\alpha$)

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$$\mathsf{G}^T oldsymbol{x} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sqrt{a^2 + b^2} \\ 0 \end{pmatrix}$$

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$$\begin{array}{ll}
ac - bs &=& \sqrt{a^2 + b^2} \\
as + bc &=& 0
\end{array}
\Rightarrow c = \frac{a}{\sqrt{a^2 + b^2}}, \quad s = -\frac{b}{\sqrt{a^2 + b^2}}$$

■ Postup lze zobecnit i pro vektory délky n:

$$G_{i,j} = egin{pmatrix} 1 & & & \cdots & & & 0 \\ & \ddots & & & & & \\ & & & c & & s & & \\ \vdots & & & \ddots & & & \vdots \\ & & -s & & c & & & \\ & & & & \ddots & & \\ 0 & & & \cdots & & & 1 \end{pmatrix}$$

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lacksquare i,j - osy, ve kterých se rotuje

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$$egin{aligned} oldsymbol{y} &= oldsymbol{\mathsf{G}}_{i,j}^T oldsymbol{x} \ y_k &= \left\{egin{aligned} cx_i - sx_j, & k = 1, \ sx_i + cx_j, & k = j, \ x_k, & k
eq i, j, \end{aligned}
ight.$$

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■ Givensova QR metoda:

$$G(i-1,i) = G_{i-1,i}, \quad c = \frac{x_{i-1}}{\sqrt{x_{i-1}^2 + x_i^2}}, s = \frac{-x_{i-1}}{\sqrt{x_{i-1}^2 + x_i^2}}$$

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$$\mathsf{A} = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix} \overset{\mathsf{G}(2,3)^T}{\to} \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & \times & \times \end{pmatrix} \overset{\mathsf{G}(1,2)^T}{\to} \begin{pmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix} \overset{\mathsf{G}(2,3)^T}{\to} \begin{pmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{pmatrix}$$

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Postupně tedy aplikujeme zleva jednotlivé ortogonální transformační matice G_k^T :

$$\underbrace{\mathsf{G}_p^T\mathsf{G}_{p-1}^T\cdots\mathsf{G}_2^T\mathsf{G}_1^T}_{=\mathsf{Q}^T}\mathsf{A}=\mathsf{F}$$

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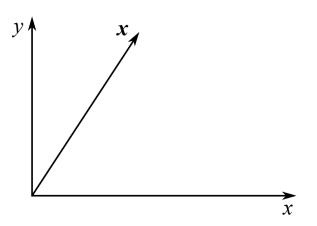
$$\underbrace{\mathsf{G}_p^T\mathsf{G}_{p-1}^T\cdots\mathsf{G}_2^T\mathsf{G}_1^T}_{=\mathsf{Q}^T}\mathsf{A}=\mathsf{R}$$

$$A = G_1G_2 \cdots G_{p-1}G_pR$$
, $A = QR$

```
function QR Givens(A \in \mathbb{R}^{m \times n})
     Q = I \in \mathbb{R}^{m \times m}. R = A
     for i = 1 : n do
          for i = m, m - 1, ..., j + 1 do
               x = \mathsf{R}(:, i)
               if \sqrt{x_{i-1}^2 + x_i^2} > 0 then
                    c = x_{i-1} / \sqrt{x_{i-1}^2 + x_i^2}
                    s = -x_i / \sqrt{x_{i-1}^2 + x_i^2}
                     G = I \in \mathbb{R}^{m \times m}
                    \mathsf{G}([i-1,i],[i-1,i]) = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}
                     R = G^T R
                     Q = QG
               end if
          end for
     end for
end function
```

Algoritmus vyžaduje přibližně $3n^2m-n^3$ operací.

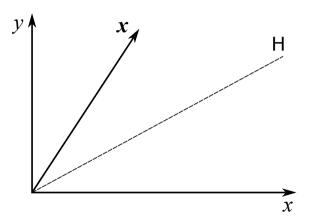
Householderovy transformace



K vektoru x sestrojíme jeho projekci do osy x stejné délky

$$oldsymbol{x} = egin{pmatrix} x_1 \ x_2 \ dots \ x_n \end{pmatrix} \stackrel{\mathsf{P}}{ o} \mathsf{P} oldsymbol{x} = egin{pmatrix} \|oldsymbol{x}\| \ 0 \ dots \ 0 \end{pmatrix} = \|oldsymbol{x}\| oldsymbol{e}_1$$

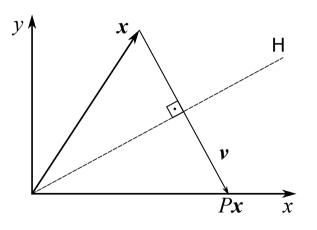
Householderovy transformace



K vektoru x sestrojíme jeho projekci do osy x stejné délky

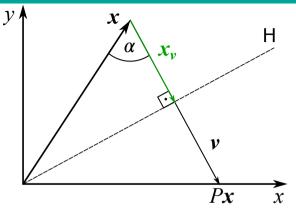
$$oldsymbol{x} = egin{pmatrix} x_1 \ x_2 \ dots \ x_n \end{pmatrix} \stackrel{\mathsf{P}}{ o} \mathsf{P} oldsymbol{x} = egin{pmatrix} \|oldsymbol{x}\| \ 0 \ dots \ 0 \end{pmatrix} = \|oldsymbol{x}\| oldsymbol{e}_1$$

Householderovy transformace

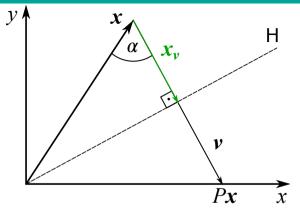


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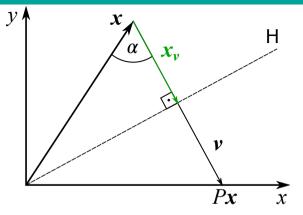
$$egin{aligned} oldsymbol{x} = egin{pmatrix} x_1 \ x_2 \ dots \ x_n \end{pmatrix} & \overset{\mathsf{P}}{ o} \ \mathsf{P} oldsymbol{x} = egin{pmatrix} \|oldsymbol{x}\| \ 0 \ dots \ 0 \end{pmatrix} = \|oldsymbol{x}\| oldsymbol{e}_1 \end{aligned}$$



lacksquare P $m{x}$ má mít stejnou délku jako $m{x}$, tzn. $\|\mathsf{P}m{x}\| = (\mathsf{P}m{x})_1 = \|m{x}\|$



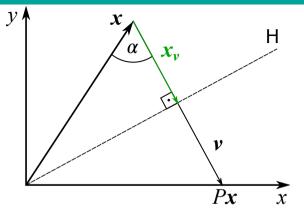
- Px má mít stejnou délku jako x, tzn. $\|\mathbf{P}x\| = (\mathbf{P}x)_1 = \|x\|$
- Px je zrcadlový obraz x s osou souměrnosti H, osa prochází počátkem a je kolmá k $v = Px x = ||x||e_1 x$



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 v = Px x = ||x||e₁ x

$$\mathsf{P}\boldsymbol{x} = \boldsymbol{x} + 2\boldsymbol{x}_v$$

$$oldsymbol{x}_v = oldsymbol{v}_n \| oldsymbol{x} \| \cos lpha, ext{ kde } oldsymbol{v}_n = rac{oldsymbol{v}}{\| oldsymbol{v} \|}$$

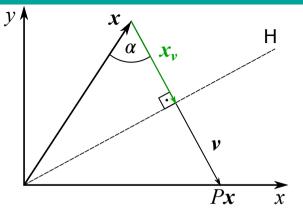


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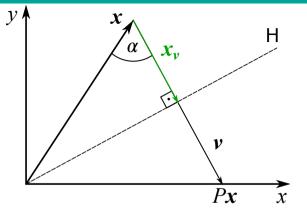
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$$\boldsymbol{x}_{v} = \boldsymbol{v}_{n} \| \boldsymbol{x} \| \cos \alpha = \frac{\boldsymbol{v}}{\| \boldsymbol{v} \|} \| \boldsymbol{x} \| \cos \alpha = \frac{\boldsymbol{v}}{\| \boldsymbol{v} \|^{2}} \underbrace{\| \boldsymbol{v} \| \| \boldsymbol{x} \| \cos \alpha}_{=-\boldsymbol{x}^{T} \boldsymbol{v}} = -\frac{\boldsymbol{v}}{\| \boldsymbol{v} \|^{2}} \boldsymbol{v}^{T} \boldsymbol{x}$$

$$\Rightarrow \mathsf{P} = \mathsf{I} - 2 \frac{\boldsymbol{v} \boldsymbol{v}^T}{\boldsymbol{v}^T \boldsymbol{v}}$$



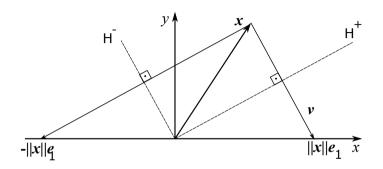
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 \Rightarrow P = I - $2\frac{vv^T}{v^Tv}$ matice zrcadlení (Householderova matice)



- Projekce do osy x není definována jednoznačně $(\pm ||x||e_1)$.
- lacksquare Pro lepší numerickou stabilitu se volí projekce volí jako P $m{x} = -\mathrm{sign}(x_1)m{e}_1$

■ P je symetrická ortogonální matice:

$$\mathsf{P}^T = (\mathsf{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}})^T = \mathsf{I} - 2\frac{(\boldsymbol{v}^T)^T\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}} = \mathsf{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}}$$

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$$\mathsf{PP}^T = \mathsf{PP} = (\mathsf{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}})(\mathsf{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}}) = \mathsf{I} - 4\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}} + 4\frac{\boldsymbol{v}\boldsymbol{v}^T\boldsymbol{v}\boldsymbol{v}^T}{(\boldsymbol{v}^T\boldsymbol{v})^2} = \mathsf{I} - 4\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}} + 4\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}} = \mathsf{I}$$

Householderova QR dekompozice

$$\begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix} \xrightarrow{\mathbf{P}_1} \begin{pmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix} \xrightarrow{\mathbf{P}_2} \begin{pmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \\ 0 & 0 & \times \end{pmatrix} \xrightarrow{\mathbf{P}_3} \begin{pmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \\ 0 & 0 & 0 \end{pmatrix}$$

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$$\underbrace{\mathsf{P}_n\mathsf{P}_{n-1}\cdots\mathsf{P}_1}_{=\mathsf{Q}^T}\mathsf{A}=\mathsf{R}\quad\Rightarrow\mathsf{A}=\mathsf{Q}\mathsf{R},\mathsf{Q}=\mathsf{P}_1\cdots\mathsf{P}_{n-1}\mathsf{P}_n$$

```
function QR Householder(A \in \mathbb{R}^{m \times n})
      Q = I \in \mathbb{R}^{m \times m}, R = A
      for j = 1 : n do
            \boldsymbol{x} = \mathsf{R}(j:m,j)
            \boldsymbol{v} = -\operatorname{sign}(x_1) \|\boldsymbol{x}\| \operatorname{eye}(m-j+1,1) - \boldsymbol{x}
            oldsymbol{v} = oldsymbol{v}/\|oldsymbol{v}\|
            P = I \in \mathbb{R}^{m \times m}
            P(j:m,j:m) = P(j:m,j:m) - 2\boldsymbol{v}\boldsymbol{v}^T
            R = PR
            Q = QP
      end for
end function
```

Algoritmus vyžaduje přibližně $2mn^2 - \frac{2}{3}n^3$ operací.

- Aplikace QR rozkladu
 - výpočet ortonormální báze z množiny vektorů
 - řešení soustav lineárních rovnic
 - řešení problému nejmenších čtverců (přeurčených soustav)
 - hledání vlastních čísel (spektrální rozklad)
 - výpočet pseudoinverze (Moore-Penroseovy):

$$\begin{aligned} \mathsf{A}\mathsf{A}^+\mathsf{A} &= \mathsf{A},\\ \mathsf{A}^+\mathsf{A}\mathsf{A}^+ &= \mathsf{A},\\ (\mathsf{A}\mathsf{A}^+)^T &= \mathsf{A}\mathsf{A}^+\\ , (\mathsf{A}^+\mathsf{A})^T &= \mathsf{A}^+\mathsf{A} \end{aligned}$$

Děkuji za pozornost

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20. dubna 2023

