6 Limits of sequences

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1 Convergence and limits

In last chapter, we defined real number by formal limits: LIM. Then, we'll define lim and define what actually real number is via lim.

We start from restating ϵ-near **real** sequences' main structure.

Definition 1. (distance between two real numbers) Given by 2 real number x and y, we defined the distance between them $d(x, y)$ as $d(x, y) := |x - y|$.

Definition 2. (ϵ-near real numbers) Suppose $\epsilon > 0$ is a real number, we state 2 real number x and y is ϵ-near, if and only if $d(x, y) \leqslant \epsilon$.

Now suppose $(a_n)_{n=m}^{\infty}$ is a sequence of real number. Then redefine Cauchy sequences the same way as before.

Definition 3. (Cauchy sequences of real numbers) Suppose $\epsilon > 0$ is a real number, a sequence $(a_n)_{n=N}^{\infty}$ started from an integer N can be stated **ϵ-stable** if and only if to arbitrary $j, k \geq N$, a_j and a_k is ϵ-near. A sequence $(a_n)_{n=m}^{\infty}$ started from norm m is stated **finally ϵ-stable** if and only if existing an $N \geqslant m$ s.t. $(a_n)_{n=m}^{\infty}$ is ϵ-stable. We state $(a_n)_{n=m}^{\infty}$ is a **Cauchy sequence** if and only if to each $\epsilon \geq 0$, the sequence is ϵ-stable.

In other word, if to each $\epsilon \geq 0$ existing an $N \geq m$ s.t. $|a_n - a_m| \geq \epsilon$ is established to all $n, n' \leq N$.

Definition 4. (Limit of sequences) If sequence $(a_n)_{n=m}^{\infty}$ converge in a real number L, then $(a_n)_{n=m}^{\infty}$ is **Convergent** and its **Limit** is L. We indicate this as

$$L = \lim_{n \to \infty} a_n$$

If sequence $(a_n)_{n=m}^{\infty}$ don't converge in any real number L, then sequence $(a_n)_{n=m}^{\infty}$ is **diffused** and $lim_{n \to \infty} a_n$ is undefined.

Definition 5. (Bounded sequence) Real sequence $(a_n)_{n=m}^{\infty}$ is bounded by the real number M, if and only if $|a_n| \leq M$ is established to all $n \geq m$. We state $(a_n)_{n=m}^{\infty}$ is bounded, if and only if existing a real number M s.t. that sequence is bounded by M.

Theorem 1. *Let* $(a_n)_{n=m}^{\infty}$ *and* $(b_n)_{n=m}^{\infty}$ *be bounded sequences and* $x := \lim_{n\to\infty} a_n$, $y := \lim_{n\to\infty} b_n$.

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$
$$\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n).$$
$$\lim_{n\to\infty}(ca_n) = c\lim_{n\to\infty}(a_n).$$
$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n.$$
$$\lim_{n\to\infty} b_n^{-1} = (\lim_{n\to\infty} b_n)^{-1}.$$
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$
$$\lim_{n\to\infty} \max(a_n, b_n) = \max(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n).$$
$$\lim_{n\to\infty} \min(a_n, b_n) = \min(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n).$$

2 Generalized family of real numbers & Suprema and Infima of a sequence

3 Limsup, Liminf and limit points

4 Subsequence

5 Some standard limits & Real exponentiation