Notes on Fourier Analysis

HLXY

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Abstract

Some notes after reading *Fourier Analysis: An Introduction*.

Contents

1 Preliminaries on Riemann integration

In this section, we'll take a brief review on definition and main properties of Riemann integrable functions on $\mathbb{R}$ and integration of almost everywhere continuous functions.

Firstly, we'll give the Riemann integrable theorem on the real line. Besides the classical integration theory, we also introduce null set and give the sufficient and necessary condition where a non-continuous function is integrable.

Then, we'll discuss dual and multiple integrals. Especially, we'll extend the definition of integrable Schwartz function to the whole $\mathbb{R}^d$ space.

1.1 Definition of Riemann integrable function

Let f be a real-value function on $[a, b]$, where $[a, b]$ is a bounded closed interval in $\mathbb{R}$. Importing the proportion P to separate $[a, b]$ to finite small intervals, exactly there exists finite real numbers $x_0, x_1, \ldots, x_N$, such tha

$$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b,$$

For this proportion, let I_j be the interval $[x_{j-1}, x_j]$, $|I_j|$ be the length of I_j, that $|I_j| = x_j - x_{j-1}$. Define the superior and inferior Riemann sum of f on proportion P as

$$\mathcal{U}(P, f) = \sum_{j=1}^{N} [\sup_{x \in I_j} f(x)] |I_j| \tag{1}$$

$$\mathcal{I}(P, f) = \sum_{j=1}^{N} [\inf_{x \in I_j} f(x)] |I_j| \tag{2}$$

Noting that, if f is bounded, then $\mathcal{U}(P, f)$ and $\mathcal{I}(P, f)$ exist. Obviously $\mathcal{U}(P, f)$ is greater than $\mathcal{I}(P, f)$, and if for any $\epsilon > 0$, there exists a proportion P, such that

$$\mathcal{U}(P, f) - \mathcal{I}(P, f) < \epsilon$$

then f is **Riemann integrable**, or simply f is integrable.

To define the Riemann integration of f, we need to make a short discussion, if proportion $P\prime$ is not a proportion of $[a, b]$, then P' is derived from P by adding some proportion points, then we call P' a refinement of P. While adding each points, we can get

$$\mathcal{U}(P', f) \leq \mathcal{U}(P, f) \tag{3}$$

$$\mathcal{I}(P', f) \geq \mathcal{I}(P, f) \tag{4}$$

Then we have, if P_1 and P_2 are two proportions of $[a, b]$, then

$$\mathcal{U}(P_1, f) \geq \mathcal{U}(P_2, f) \tag{5}$$

Take the union P' of P_1 and P_2, we have

$$\mathcal{U}(P_1, f) \geq \mathcal{U}(P', f) \geq \mathcal{I}(P', f) \geq \mathcal{I}(P_1, f) \tag{6}$$

We can get that from the boundedness of f,

$$U = \inf_P \mathcal{U}(P, f) \text{ and } L = \sup_P \mathcal{I}(P, f) \tag{7}$$

are both exist and $U \leq L$. Furthermore, if f is integrable, then $U = L$, define the value of its integration as $\int_a^b f(x)dx$.

Lastly, for a complex function $f = u + iv$, if its real part u and imaginary part v are both integrable, then f is integrable, and its integration is

$$\int_a^b f(x)dx = \int_a^b u(x)dx + i\int_a^b v(x)dx \tag{8}$$

1.1.1 Basic Properties

Proposition 1. *If f and g are integrable on interval $[a, b]$, then:*

1. *$f + g$ is integrable, and $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$*
2. *if $c \in \mathbb{C}$, then $\int_a^b cf(x)dx = c\int_a^b f(x)dx$*
3. *if f and g are real value functions and $f(x) \leq g(x)$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$*
4. *if $c \in [a, b]$, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$*

Proof. This proof is left to the reader as an exercise. □

Lemma 1. *If f is a real-value integrable function on $[a, b]$, ϕ is a real-value continuous function on $\mathbb{R}$, then $\phi \circ f$ is integrable.*