3.F Duality

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November 20, 2023

1 Dual space and dual map

Maps to $\mathbf{F}$ play an important role in linear algebra, so they have a special name and note.

Definition 1. (Linear functional) A **Linear functional** on V is a map from V to F, in other words, Linear maps are elements of $\mathcal{L}(V, \mathbf{F})$.

Example 1. **Linear functional**

- Define $\phi : \mathbf{R}^3 \to \mathbf{R}$ as $\phi(x, y, z) = 4x - 5y + 2z$, then ϕ is a linear functional on $\mathbf{R}^3$
- Pick $(c_1, \ldots, c_n) \in \mathbf{F}^n$, Define $\phi : \mathbf{F}^n \to \mathbf{F}$ as $\phi(x_i, \ldots, x_n) = c_1x_1 + \cdots + c_nx_n$, then ϕ is a linear functional on $\mathbf{F}^n$.
- Define $\phi : \mathcal{P}(\mathbf{R}) \to \mathbf{R}$ as $\phi(p) = 3p^n(5) + 7p^n(4)$, then ϕ is a linear functional on $\mathcal{P}(\mathbf{R})$.
- Define $\phi : \mathcal{P}(\mathbf{R}) \to \mathbf{R}$ as $\phi(p) = \int_0^1 p(x)\,dx$, then ϕ is a linear functional on $\mathcal{P}(\mathbf{R})$

And there's also a special name and note for vector space $\mathcal{L}(V, \mathbf{F})$.

Definition 2. (Dual space, V') The vector space consisting of all linear functional on V is stated as V's **dual space**, noted as V'. In other words, $V' \in \mathcal{L}(V, \mathbf{F})$