# Two class problem

$y \in \{0,1\}$ disease, $x \in \{0,1\}$ diagnostic

sample $(x_i, y_i) \overset{iid}{\sim} \Pr(X, Y)$, $i = 1 \ldots n$

population frequency

$$P(x, y) \equiv \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(x_i = x)\, \mathbb{1}(y_i = y)$$

sample frequency

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Two "models": $P(x \mid y=1)$ vs. $P(x \mid y=0)$

"Bayes factor": $\dfrac{P(x \mid y=1)}{P(x \mid y=0)} \overset{e.g.}{=} 3.0 \Rightarrow$ "x is 3x more likely under y=1"

i.e. likelihood ratio (LR)

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If LR > 1, do we conclude y = 1?

What about the "base rate" $\Pr(y=1)$?

"Prior" model: $P(y=1)$

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The correct probabilistic query:

"likelihood" "prior"

$$P(y=1 \mid x) = \frac{P(x \mid y=1)\, P(y=1)}{P(x)}$$

"posterior"

"marginal likelihood"

Bayes rule

Posterior odds

$$\frac{P(y=1 \mid x)}{P(y=0 \mid x)} = \underbrace{\frac{P(x \mid y=1)}{P(x \mid y=0)}}_{\text{LR}} \times \underbrace{\frac{P(y=1)}{P(y=0)}}_{\text{Prior odds}}$$

e.g. $P(x=1 \mid y=1) = 1.0$ true positive rate of diagnostic

$P(x=1 \mid y=0) = 0.1$ false positive rate of diagnostic

$P(y=1) = 0.001$ rare disease

↳ posterior odds $= \frac{1.0}{0.1} \times \frac{0.001}{0.999} \approx 0.01$

"data x is 100-to-1 <u>against</u> disease"

<u>Moral</u>: Priors matter! See "Lady drinking tea" (Berger, 1985)

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<u>Decision theory</u>: say posterior odds are $>1$ (e.g. 10). Do we treat the patient?

decision rule: $\hat{y}(x) \in \{0,1\}$

loss function: $\ell(y, \hat{y}(x))$

For binary classification:

| | $\hat{y}=0$ | $\hat{y}=1$ | |
|---|---|---|---|
| | $\ell(0,0)$ TN | $\ell(0,1)$ FP | $y=0$ |
| | $\ell(1,0)$ FN | $\ell(1,1)$ TP | $y=1$ |

$\ell_{TN}$ and $\ell_{TP}$ should be $\leq$ $\ell_{FN}$ and $\ell_{FP}$

but $\ell_{FN} \neq \ell_{FP}$ (not necessarily)

e.g. $\hat{y}(x) = \begin{cases} 1 & \text{if } P(y=1 \mid x) \geq 0.5 \\ 0 & \text{otherwise} \end{cases}$

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Integrated risk :

$$r(\hat{y}(\cdot)) = \mathbb{E}_{X,Y}\left[\ell(\hat{y}(X), Y)\right]$$

$$\equiv \mathbb{E}_{(X,Y)\sim \Pr}[\ldots]$$

$\hookleftarrow$ population average

Optimal rule

$$\hat{y}^*(\cdot) = \underset{\hat{y}(\cdot)}{\operatorname{argmin}}\, r(\hat{y}(\cdot))$$

$$= \operatorname{argmin} \mathbb{E}_X \mathbb{E}_{Y \mid X=x}[\ell(y(x), x)]$$

The optimal rule "always does the right thing" :

$$\hat{y}^*(x) = \underset{a}{\operatorname{argmin}}\, \mathbb{E}_{Y \mid X}[\ell(a, x)]$$

"posterior"

In other words we "only" need to know $\Pr(y \mid x)$

↳ $\hat{y}^*(x)$ is the "Bayes decision rule"

↳ $r(\hat{y}^*(\cdot))$ is the "Bayes risk"

Frequentist justification for "Bayesian inference" = Posterior computation

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Modeling: Discriminative vs. generative

We almost never know $Pr(y \mid x)$ ...

... especially true for higher-dimensional $x \in \mathbb{R}^p$

"making assumptions about" $P_r(y \mid x)$ vs. $P_r(x \mid y) P_r(y)$

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$X_1 \dots X_p$ are syptoms, $X_j \in \{0, 1\}$ e.g. "is coughing"

$X \equiv X_{1:p}$ takes $\underline{2^p}$ possible values

e.g. $p = 40 \Rightarrow 2^{40} \approx$ 1 trillion

no hope to estimate $P(y \mid x)\ \forall x$ without assumptions

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Discriminative example

Assume: $Pr(y \mid x) = \dfrac{1}{1 - \exp(-\beta^T x)}$ for some $\beta \in \mathbb{R}^p$

(logistic regression)

Allows us to "share information" across similar $X$.

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Generative example: Naive Bayes

Assume: $Pr(x_{1:p} \mid y) = \prod_{j=1}^{p} Pr(x_j \mid y)$

i.e. $x_j \perp\!\!\!\perp x_{j'} \mid y$ conditionally indep.

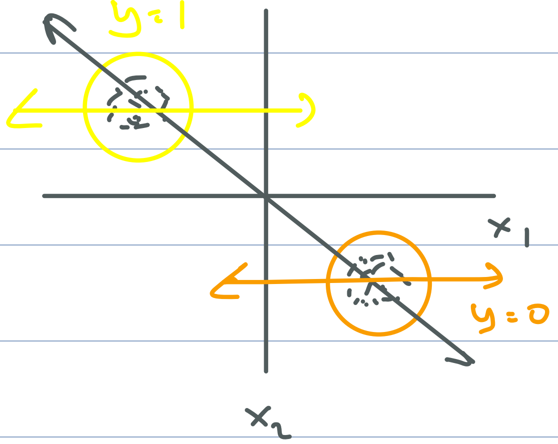
↳ $$Pr(y|x) \overset{\text{Bayes}}{=} \frac{P(y)\,P(x|y)}{P_r(x)} \overset{\text{naive}}{=} \frac{P_r(y)\prod_j P(x_j|y)}{P_r(x)}$$

what about $P(x) = P(x_1, \cdots x_p)$ ? $\neq \prod_j P_r(X_j)$

$$= \sum_y P_r(y)\, P(x|y) = \sum_y P(y) \prod_j P_r(x_j|y)$$

↳ marginalization

Note: $x_j \perp\!\!\!\perp x_{j'} \mid y \not\Rightarrow x_j \perp\!\!\!\perp x_{j'}$




related: Simpson's paradox

Takeaway: Only need to estimate $P(y)$ and $P(x_j|y)$.

Again, "information sharing". ↑ $2p$ parameters vs. $2^p$

This model is (often) "wrong" but (often) "useful". See Box.

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Data sparsity:

$$P(x_j=1 \mid y=1) = \theta_{j|1} \leftarrow \frac{\sum_i 1(x_{ij}=1,\ y_i=1)}{\sum_i 1(y_i=1)}$$

what if $\theta_{j|1} = 1$ (or $=0$)?

e.g. $P(y^{n+1}=1 \mid x_1^{n+1}=1,\ x_2^{n+1}=1, \cdots)$

$$= \frac{P(y=1)\prod_j \theta_{j|1}}{P(y=1)\prod_j \theta_{j|1} + P(y=0)\prod_j \theta_{j|0}}$$

say $\theta_{1|1} = 1$ $\quad \theta_{1|0} = 0$ $\quad \theta_{2|1} = 0$ $\quad \theta_{2|0} = 1$

$$L = \frac{0}{0}$$

Laplace smoothing

$$\theta_{j|1} \leftarrow \frac{\#(x_j = 1, y = 1) + \alpha}{\#(y=1) + 2\alpha}$$

Pierre-Simon Laplace (1700s) proposed this for the "sunrise problem"

see also: Bertrand Russell's "problem of induction" (Bertrand's chicken)

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Justification:

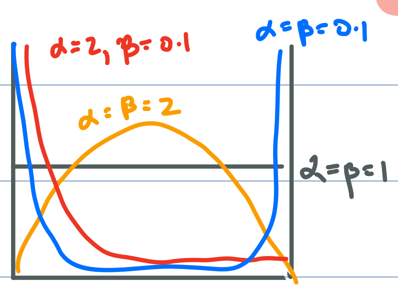
$$x^i \overset{iid}{\sim} \text{Bernoulli}(\theta) \qquad i = 1 \ldots n$$

$$\hat{\theta}_{MLE} = \arg\max_{\theta} \prod_i p(x^i \mid \theta) = \bar{x}$$

Prior: $\theta \sim \text{Beta}(\alpha, \beta)$

"normalizing constant" "kernel"

$$p(\theta \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1} \mathbb{1}(\theta \in (0,1))$$




Alternative parameterization:

$$\mu = \mathbb{E}[\theta \mid \alpha, \beta] = \frac{\alpha}{\alpha+\beta}$$

$$\kappa = \alpha + \beta \leftarrow \text{"concentration"}$$

## Posterior:

$$P(\theta \mid x_{1:n}) = \frac{P(\theta)\, P(x_{1:n} \mid \theta)}{P(x_{1:n})} \longrightarrow = \int d\theta\, P(\theta)\, P(x \mid \theta)$$

$$= \frac{\mathrm{Beta}(\theta; \alpha, \beta) \prod_i \mathrm{Bern}(x_i; \theta)}{\int \text{"} \qquad\qquad \text{"}\, d\theta}$$

$P(x_{1:n})$ is the "normalizing constant" of the posterior

$$\propto_\theta \mathrm{Beta}(\theta; \alpha, \beta) \prod_i \mathrm{Bern}(x_i; \theta)$$

$$= \cancel{\frac{1}{B(\alpha, \beta)}}\, \theta^{\alpha-1} (1-\theta)^{\beta-1} \prod_i \theta^{x_i} (1-\theta)^{1-x_i}$$

$$\propto_\theta \theta^{\alpha + \sum_i x_i - 1} (1-\theta)^{\beta + n - \sum_i x_i - 1}$$

This is the kernel of another Beta distribution.

$$\hookrightarrow P(\theta \mid x_{1:n}) = \mathrm{Beta}\left(\alpha + \sum_i x_i,\ \beta + n - \sum_i x_i\right)$$

No need to "solve" for the normalizing constant explicitly.

Why did this happen? Beta-Bernoulli conjugacy.

Beta is the conjugate prior for the Bernoulli.

# Bayesian updating

$\text{Beta}(\underbrace{\alpha + \sum_i x_i}_{\#1\text{s}}, \underbrace{\beta + n - \sum_i x_i}_{\#0\text{s}})$ $\alpha, \beta$ are "pseudocounts"

$$\mathbb{E}[\theta \mid x_{1:n}, \alpha, \beta) = \frac{\alpha + \sum_i x_i}{\alpha + \beta + n}$$ ← Laplace smoothing estimator!

$$= \frac{\kappa}{\kappa} \cdot \frac{\alpha}{\kappa + n} + \frac{\sum x_i}{\kappa + n} \cdot \frac{n}{n} = \left(\frac{\kappa}{\kappa + n}\right)\mu + \left(\frac{n}{n+\kappa}\right)\hat{\theta}_{MLE}$$

$\kappa = \alpha + \beta$ is the strength of the prior / inductive bias.

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<u>Subjectivist</u> interpretation of probability

$P(\theta) = \text{Beta}(\theta; \alpha, \beta)$ is not a "frequency"

... it is a "degree of belief"

e.g. P( Bulls win on Thursday)

P( sun comes up tomorrow)

P( Riemann's conjecture is True)

The Bayesian vs frequentist debate was over the interpretation (among other things).

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